CONTINUOUS FUNCTIONS ON COUNTABLE COMPACT ORDERED SETS AS SUMS OF THEIR INCREMENTS

BY

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ABSTRACT. Every continuous function from a countable compact linearly ordered set A into a Banach space V (vanishing at the least element of A) admits a representation as a sum of a series of its increments (in the topology of uniform convergence). This series converges to no other sum under rearrangements of its terms. A uniqueness result to the problem of representation of a regulated real function on the unit interval as a sum of a continuous and a steplike function is derived.

0. Introduction. Let A be a compact linearly ordered set, V a (real or complex) Banach space. Let $C(A, V)$ denote the Banach space of continuous functions from A to V with the supremum norm. The right neighbour a' (the left neighbour $'a$) of $a \in A$ is defined as the maximal (minimal) $b \in A$ such that $(a, b) = \emptyset$ ($(b, a) = \emptyset$). With $f: A \rightarrow V$ associate $f^\dagger: A \rightarrow V$ defined by

$$f^\dagger(a) = f(a') - f(a).$$

With $a \in A$, $v \in V$ associate a function $J_a^v: A \rightarrow V$ —the v -jump at a —defined by

$$J_a^v(b) = \begin{cases} 0, & b < a, \\ v, & a < b. \end{cases}$$

Then $J_a^{f^\dagger(a)}$ is called the increment of f at a . Note that if f is in $C(A, V)$ so is every increment of f . Let m_A (M_A) denote the minimal (maximal) element of A . The purpose of this article is to prove

THEOREM 1. *Let A be a countable compact ordered set, V a Banach space and let $f \in C(A, V)$ satisfy $f(m_A) = 0$. Then:*

(a) *There is an enumeration $(J_n)_n$ of the increments of f such that $f = \sum_n J_n$ holds in $C(A, V)$.*

(b) *If $(\bar{J}_n)_n$ is any enumeration of the increments of f for which $\sum_n \bar{J}_n$ converges in $C(A, V)$, then $\sum_n \bar{J}_n = f$.*

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In the special case where $\nu = \rho + 1$ is a compact countable ordinal Theorem 1(a) implies: Let $f: \nu \rightarrow V$ be continuous. Then there is an enumeration $(\mu_n)_n$ of the ordinals smaller than ν such that for every $\mu < \nu$, $f(\mu) = f(0) + \sum_{\mu_n < \mu} (f(\mu_n + 1) - f(\mu_n))$. Moreover, the convergence is uniform in μ (in a sense made precise in §§2, 3). Similarly, there is an enumeration $\bar{\mu}_n$ of the ordinals smaller than ν so that the identity $f(\mu) = f(\rho) + \sum_{\mu < \bar{\mu}_n} (f(\bar{\mu}_n) - f(\bar{\mu}_n + 1))$ holds uniformly for $\mu < \nu$.

Conditionally convergent series, every convergent rearrangement of which have the same sum, are known to exist in any infinite dimensional Banach space [Ha], [McA]. Theorem 1 provides a host of natural examples of this phenomenon. The simplest of those is the following one: Let $A = \omega + 1 = \{0, 1, 2, \dots, \omega\}$ and let $V = R$, the reals. Let $(a_n)_n$ be any sequence of reals such that $\sum_n a_n$ is conditionally convergent, and $\sum_n a_n = a$. Define $f \in C(A, R)$ by $f(n) = \sum_{i < n} a_i$, $f(\omega) = a$. It is straightforward to check that $(J_n^{a_n})_n$ is an enumeration of f 's increments, $\sum_n J_n^{a_n} = f$, and that $\sum_n J_n^{a_n}$ is convergent in $C(A, R)$ iff $\sum_n a_n$ is convergent to a . Thus, $\sum_n J_n^{a_n}$ is a conditionally convergent series in $C(A, R)$, every convergent rearrangement of which has the sum f .

Although V is assumed to be Banach space in Theorem 1, the theorem is true in a wider context. We may assume V to be arbitrary complete normed abelian group.¹ (A typical example of such a V which is not the additive group of a Banach space is the p -adic ring Q_p , see e.g. [F].)

Let A be arbitrary ordered set. Call $a \in A$ *right isolated* (*left isolated*) iff $a < a'$ ($a' < a$). Call a a *core point* iff it is neither right nor left isolated. Let $f: A \rightarrow V$. We write $f(a+) = v$ iff $\forall \varepsilon > 0 \exists b > a [a < c < b \Rightarrow \|f(c) - v\| < \varepsilon]$. Define $f(a-)$ similarly. Note that $f(a) = f(a+)$ ($f(a) = f(a-)$) whenever a is right (left) isolated. Let $f(m_A-) = f(m_A)$, $f(M_A+) = f(M_A)$. We call $f: A \rightarrow V$ a *regulated function* iff:

- (i) $f(m_A+)$, $f(M_A-)$ exist, as do $f(a+)$, $f(a-)$ for $m_A < a < M_A$.
- (ii) $f(a) = f(a-)$ whenever a is right isolated,
 $f(a) = f(a+)$ whenever a is left isolated,
 $f(a) = f(a-)$ whenever a is a core point.

The reader will note that a regulated function is continuous if and only if it satisfies $f(a-) = f(a+)$ for every core point a .

If A is compact, the family of all regulated functions from A to V with the supremum norm is again a Banach space $\text{Reg}(A, V)$ containing $C(A, V)$. For $f \in \text{Reg}(A, V)$ define $f^*: A \rightarrow V$ by $f^*(a) = f(a+) - f(a)$ if a is a core point, and $f^*(a) = f^\dagger(a)$ otherwise. The *increment of f at a* is redefined to be $J_a^{f^*(a)}$. Theorem 1 is equivalent to

¹By a norm on an additive group V we mean a function $\|\cdot\|$ from V into the nonnegative real numbers, satisfying $\|v\| = 0$ iff $v = 0$, $\|v\| = \|-v\|$ and $\|v + w\| \leq \|v\| + \|w\|$.

THEOREM 2. *Let A be a compact countable ordered set, V a Banach space, and let $f \in \text{Reg}(A, V)$ satisfy $f(m_A) = 0$. Then:*

(a) *There is an enumeration $(J_n)_n$ of the increments of f such that $f = \sum_n J_n$ holds in $\text{Reg}(A, V)$.*

(b) *If $(\bar{J}_n)_n$ is any enumeration of the increments of f for which $\sum_n \bar{J}_n$ converges in $\text{Reg}(A, V)$, then $\sum_n \bar{J}_n = f$.*

Indeed, $f^* = f^\dagger$ for $f \in C(A, V)$ and so Theorem 2 clearly implies Theorem 1. Conversely, assume Theorem 1 and let A be a countable compact ordered set. Replacing each core point $a \in A$ by a pair of points $\bar{a} < \bar{a}^+$, we obtain another compact countable ordered set $\hat{A}$ with no core points. Define $g: \hat{A} \rightarrow A$ by $g(\bar{a}) = g(\bar{a}^+) = a$ for a core point $a \in A$, $g(a) = a$ for a noncore point $a \in A$. For $f \in \text{Reg}(A, V)$ let $\hat{f} = f \circ g$. Then $\hat{f} \in C(\hat{A}, V)$ and it is easily checked that Theorem 1 for $\hat{f}$ implies Theorem 2 for f (compare Proposition 3.1).

We apply Theorem 2 to obtain a theorem on $\text{Reg}(I, V)$, where $I = [0, 1]$ is the closed unit interval (see [GMW], [M]). $s \in \text{Reg}(I, V)$ is called *steplike* iff there is an enumeration $(J_n)_n$ of its increments (only countably many of which are nonzero) so that $s = \sum_n J_n$ holds in $\text{Reg}(I, V)$. The separation of discontinuities of an $f \in \text{Reg}(I, V)$ by means of a steplike function dates back essentially to Lebesgue's Theorem on monotone real functions: every monotone real function has a unique representation $f = g + s$, where g is a monotone continuous function and s is a monotone steplike function. Similarly, every $f \in \text{Reg}(I, V)$ of bounded variation has a unique representation $f = g + s$, where $g \in C(I, V)$ and s is steplike. In general, however, such a representation need not exist, and when it exists need not be unique [GMW], [M]. Call $f \in \text{Reg}(I, V)$ *representable* if f has a representation $f = g + s$, where $g \in C(I, V)$ and $s \in \text{Reg}(I, V)$ is steplike, and *uniquely representable* if f has precisely one such representation. Obviously, the representability and unique representability of $f \in \text{Reg}(I, V)$ depend only on f^* . Now $f \rightarrow f^*$ is a continuous linear mapping of $\text{Reg}(I, V)$ onto $C_0(I, V) = \{h: I \rightarrow V \mid \{t: \|h(t)\| > \varepsilon\} \text{ is finite for every } \varepsilon > 0\}$ with the supremum norm. Call $h \in C_0(I, V)$ *summable* (*uniquely summable*) iff some—or equivalently, any— f in $\text{Reg}(I, V)$ with $f^* = h$ is representable (uniquely representable).

The complete characterization of summable or uniquely summable members of $C_0(I, R)$ is still open. The methods of this paper suffice to characterize, however, those summable members of $C_0(I, R)$ whose support has a countable closure (to appear elsewhere). We shall address ourselves here only to the uniqueness problem.

In [M] it is shown that an $h \in C_0(I, R)$ exists, such that every $f \in \text{Reg}(I, R)$ with $f(0) = 0$ and $f^* = h$ is steplike. Obviously, the support of such an h is necessarily dense in I , and so has I for its closure. In a similar way, it

can be shown that given any countable set A in I of uncountable closure, there is an $h \in C_0(I, R)$ whose support is A such that the family $\{s \in \text{Reg}(I, R): s \text{ is steplike and } s^* = h\}$ is of the cardinality of the continuum. This is not anymore possible if h 's support is of countable closure. The following fact was stated in [M] for the case $V = R$:

THEOREM 3. *Let A be a countable closed subset of I . Let $f \in \text{Reg}(I, V)$ satisfy $\{t: f^*(t) \neq 0\} \subseteq A$. If f is representable, then f is uniquely representable.*

PROOF. Let $W \subseteq \text{Reg}(I, V)$ be the closed subspace of those $f \in \text{Reg}(I, V)$ that vanish at 0 and are constant on every component of $I - A$. For $f: I \rightarrow V$ let $f_A: A \rightarrow V$ denote the restriction of f to A . Clearly, $f_A \in \text{Reg}(A, V)$ whenever $f \in W$, and the mapping $Tf = f_A$ is a linear isometry of W onto $\text{Reg}_0(A, V) = \{f \in \text{Reg}(A, V): f(m_A) = 0\}$.

We show now that $W = \{s \in \text{Reg}(I, V): s \text{ is steplike and } \{t: s^*(t) \neq 0\} \subseteq A\}$. Clearly $s \in W$ whenever s is steplike and $\{t: s^*(t) \neq 0\} \subseteq A$. Conversely, let $f \in W$. By Theorem 2(a), there is an enumeration $(J_n)_n$ of f_A 's increments so that $f_A = \sum_n J_n$. Apply T^{-1} and obtain $f = \sum_n T^{-1}J_n$ in $\text{Reg}(I, V)$. But $(T^{-1}J_n)_n$ is obviously an enumeration of f 's increments, and so f is steplike.

Now assume that $f \in \text{Reg}(I, V)$ is representable, and that $\{t: f^*(t) \neq 0\} \subseteq A$. Let $f = g_1 + s_1 = g_2 + s_2$ where $g_1, g_2 \in C(I, V)$ and s_1, s_2 steplike. By $f^* = s_1^* = s_2^*$ we have $s_1, s_2 \in W$, and $(Ts_1)^* = (Ts_2)^* = f_A^*$. Hence $Ts_1, Ts_2 \in \text{Reg}(A, V)$ have the same increments and vanish at m_A . By Theorem 2(b), $Ts_1 = Ts_2$, whence $s_1 = s_2$ and $g_1 = g_2$. $\square$

Theorem 1 is proved essentially by induction on the Cantor rank of the scattered space A . In §1 the class of countable compact order types is characterized as the smallest class of order types including $\bar{0}$ and closed under one infinitary operation, the compact-limit operation (Definition 1.1). This yields a useful induction principle for countable compact order types. §2 is devoted to the proof of a rather technical summing lemma, which is the core of the inductive proof of Theorem 1, given in §3.

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1. An induction principle for countable compact order types. We specify some notation first. An ordinal number is identified with the set of smaller ordinals and so $\mu < \nu$ and $\mu \in \nu$ are interchangeable. A cardinal number is an ordinal not equivalent to a smaller ordinal. ω denotes the first infinite cardinal. $|A|$ denotes the cardinality of A , that is, the cardinal number equivalent to the set A . An *order* on A (or an *ordering* of A) is a total irreflexive and transitive relation on A . Let $<$ be an order on A , $B, C \subseteq A$, then $B < C$ means $b < c$ for all $b \in B, c \in C$. Note that $\emptyset < B < \emptyset$ holds for every $B \subseteq A$. $B < x$ ($x < B$) stands for $B < \{x\}$ ($\{x\} < B$). The order

type of A is denoted by $\bar{A}$. The reader is referred to [E] for the definitions of an order type and of arithmetical operations on ordered sets and order types. Intervals in an ordered set are denoted in the usual manner, e.g. $[a, b] = \{c \in A: a \leq c \leq b\}$, $(a, b) = \{c \in A: a < c < b\}$. Ordinals are considered as ordered sets, being well ordered by the membership relation. An ordered set is considered as a topological space, with the topology generated by the sets $\{b: b < a\}$ and $\{b: a < b\}$. The following proposition will be useful ([HK], see e.g. [Ke, p. 162]):

PROPOSITION 1.0. *Let A be a nonempty ordered set. Then A is compact if and only if:*

- (i) A has a smallest element,
- (ii) every nonempty subset of A has a least upper bound in A .

DEFINITION 1.1. Let A_n be an ordered set and let $\alpha_n = \bar{A}_n$ ($n \in \omega$). Assume further that $A_n \cap A_m = \emptyset$ for $n \neq m$, $x \notin \bigcup_{n \in \omega} A_n$ and denote by $<_n$ the ordering of A_n . Define the x -compact-limit-sum of $(A_n)_{n \in \omega}$, denoted $x\text{-CL}_n A_n$, as follows. The domain of $x\text{-CL}_n A_n$ is $\{x\} \cup \bigcup_{n \in \omega} A_n$. The ordering on $x\text{-CL}_n A_n$ is defined by:

- (1) $a < b$ iff $a <_n b$ for $a, b \in A_n$.
- (2) $A_{2n} < A_{2n+2} < x < A_{2n+3} < A_{2n+1}$, $n \in \omega$.

Let α be the order type of $x\text{-CL}_{n \in \omega} A_n$. Then α is defined as the compact-limit-sum of $(\alpha_n)_{n \in \omega}$, denoted by $\alpha = \text{CL}_n \alpha_n$. The rest of this section is devoted to proving

THEOREM 1.2. *Let $\mathcal{C}$ denote the class of countable compact order types. Then $\mathcal{C}$ is the smallest class of order types including $\bar{0}$ and closed under the compact-limit-sum operation.*

As a corollary we have the following

INDUCTION PRINCIPLE. Let $P(A)$ be a statement on ordered sets. If $P(\emptyset)$ is true, and $P(x\text{-CL}_{n \in \omega} A_n)$ is true whenever $P(A_n)$ is true for every $n \in \omega$, then $P(A)$ is true for every countable compact ordered set.

Let $\mathcal{C}_0$ denote the smallest class of order types including $\bar{0}$ and closed under the compact-limit-sum operation. Since $x\text{-CL}_n A_n$ is countable whenever each A_n is, and by Proposition 1.0 it is compact whenever each A_n is, we have $\mathcal{C}_0 \subseteq \mathcal{C}$.

The reverse inclusion is proved essentially by induction on the Cantor rank $\text{rk}(\alpha)$ of α . We first resume some notation and facts (see e.g. [Ke], [Ku]). For a topological space, X , X' denotes the set of nonisolated points of X . Define by induction $X^{(\mu)}$ for every ordinal μ as follows. $X^{(0)} = X$, $X^{(\mu+1)} = X^{(\mu)'}$, and $X^{(\nu)} = \bigcap_{\mu < \nu} X^{(\mu)}$ when ν is a limit ordinal. Then $X^{(\nu)} \subseteq X^{(\mu)}$ for $\mu < \nu$, and $\text{rk}(X)$ is defined as the least ordinal μ for which $X^{(\mu)} = X^{(\mu+1)}$. X is called scattered iff $X^{(\nu)} = \emptyset$ for $\nu > \text{rk}(X)$.

Let X be a scattered compact nonempty Hausdorff space. Then $\text{rk}(X) = \mu + 1$ for some ordinal μ , and $X^{(\mu)}$ is finite. Define $\text{ch}(X) = (\mu, m)$ where $\text{rk}(X) = \mu + 1$ and $|X^{(\mu)}| = m > 0$. Also let $\text{ch}(\emptyset) = (0, 0)$.

Every countable compact set of reals X is scattered, its induced topology coincides with its order topology, and its rank is countable. Every countable compact ordered set is order isomorphic with a compact countable set of reals, and so $\mathcal{C}$ is actually the class of order types of countable compact sets of reals.

We define $\text{ch}(\alpha)$ for a compact order type α by $\text{ch}(\alpha) = \text{ch}(A)$, where A is an ordered set of type α . In view of the previous remarks the following is clear.

PROPOSITION 1.3. *Let $\alpha \in \mathcal{C}$, $\text{ch}(\alpha) = (\mu, m)$. Then there are α_i , $i < m$ such that $\alpha = \sum_{i < m} \alpha_i$ and $\text{ch}(\alpha_i) = (\mu, 1)$.*

PROPOSITION 1.4. *Let A be a scattered ordered set and let x be a nonisolated point of A . Then x is an accumulation point of isolated points. Moreover, if $m_A < x < M_A$ is a core point, then for every $a, b \in A$ with $a < x < b$ there are isolated points $\bar{a}, \bar{b} \in A$ such that $a < \bar{a} < x < \bar{b} < b$.*

PROOF. Let $\mu(x)$ be the ordinal μ satisfying $x \in A^{(\mu)} - A^{(\mu+1)}$. Then x is nonisolated iff $\mu(x) > 0$. Also, if $\mu(x) > 0$ then there is an interval around x containing in addition only y 's with $\mu(y) < \mu(x)$. The proposition follows by a straightforward induction. $\square$

COROLLARY 1.5. *Let A be a countable scattered ordered set and let x be a core point in A satisfying $m_A < x < M_A$. Then there are sequences $(a_n)_{n \in \omega}$ and $(b_n)_{n \in \omega}$ of isolated points in A such that $a_n < a_{n+1} < x < b_{n+1} < b_n$, and $x = \sup_n a_n = \inf_n b_n$.*

Order the class $\{(\mu, m): \mu \text{ an ordinal, } m \in \omega\}$ lexicographically: $(\mu, m) < (\bar{\mu}, \bar{m})$ if $\mu < \bar{\mu}$ or $\mu = \bar{\mu}$ and $m < \bar{m}$. A proof by induction on $\text{ch}(A)$ or $\text{ch}(\alpha)$ will refer to this well-ordering. We prove now $\mathcal{C} \subseteq \mathcal{C}_0$. Let $\alpha \in \mathcal{C}$, let $\text{ch}(\alpha) = (\mu, m)$ and proceed by induction. Let $\mu = 0$ and proceed by induction on m . Since $\bar{0} \in \mathcal{C}_0$ the case $m = 0$ is done. Now $\text{ch}(\alpha) = (0, m)$ iff $\alpha = \bar{m}$. Assume $\bar{m} \in \mathcal{C}_0$. Let $\alpha_0 = \bar{m}$, $\alpha_n = \bar{0}$ for $n > 0$. Then $\text{CL}_n \alpha_n = \bar{m} + 1$, and so $\bar{m} + 1 \in \mathcal{C}_0$.

Assume now $\mu > 0$. Let first $m = 1$ and let A be an ordered set of order type α . Let $A^{(\mu)} = \{x\}$. By Corollary 1.5 pick a monotone sequence $(a_n)_{n \in \omega}$ of isolated points converging to x , say $a_n < a_{n+1} < x$. Define A_{2n} by $A_0 = [m_A, a_0]$, $A_{2n+2} = [a'_n, a_n]$. Since a_n is isolated and $a_n \neq M_A$, $a_n < a'_n$ and so $A_{2n} < A_{2n+2}$ and $\bigcup_{n \in \omega} A_{2n} = [m_n, x)$. In a similar way define a sequence of (possibly empty) closed intervals A_{2n+1} in A so that $A_{2n+3} < A_{2n+1}$ and

$\bigcup_{n \in \omega} A_{2n+1} = (x, M_A]$. Now, by $A_n^{(\mu)} \subseteq A^{(\mu)} = \{x\}$, $x \notin A_n$ we have $A_n^{(\mu)} = \emptyset$. Thus A_n is a compact countable ordered set with $\text{ch}(A_n) = (\mu_n, m_n)$, where $\mu_n < \mu$. By the induction hypothesis, $\bar{A}_n \in \mathcal{C}_0$, and so by $A = x\text{-CL}_n A_n$, $\alpha \in \mathcal{C}_0$.

Assume next that $\text{ch}(\alpha) = (\mu, m+1)$. By Proposition 1.3 let $\alpha = \beta + \tilde{\alpha}$, where $\text{ch}(\beta) = (\mu, m)$ and $\text{ch}(\tilde{\alpha}) = (\mu, 1)$. By the previous case, $\tilde{\alpha} = \text{CL}_n \tilde{\alpha}_n$ where $\tilde{\alpha}_n \in \mathcal{C}_0$. By induction, $\beta \in \mathcal{C}_0$. Let $\alpha_0 = \beta$, $\alpha_{2n} = \tilde{\alpha}_{2n+2}$ and $\alpha_{2n+1} = \tilde{\alpha}_{2n+1}$. Then $\alpha_n \in \mathcal{C}_0$ for all n , and $\alpha = \text{CL}_n \alpha_n$, whence $\alpha \in \mathcal{C}_0$.

This completes the proof of Theorem 1.2.

2. The Summing Lemma. We say that a set A is *countable* if $|A| < \omega$. An ordering $<$ of A is called an $|A|$ -*ordering* if A ordered by $<$ is isomorphic to $|A|$. Let $<$ be an $|A|$ -ordering of a countable set A , and let $a_0, a_1, \dots$ be the enumeration of A in the order $<$. Define ${}_m[A]_n^<$ for $m, n \in \omega$ as follows. ${}_m[A]_n^< = \{a_i : m < i < n\}$ if $m, n < |A|$. ${}_m[A]_n^< = \emptyset$ if $|A| < m$. ${}_m[A]_n^< = {}_m[A]_{|A|}^<$ if $|A| < n$. For $B \subseteq A$, $m, n \in \omega$ let ${}_m[B]_n^< = B \cap {}_m[A]_n^<$. We abbreviate by omitting $<$ when no confusion is possible, and write $[B]_n^<$ for ${}_0[B]_n^<$, ${}_n[B]^<$ for ${}_n[B]_{|A|}^<$.

Let h be a function from the countable set A into a Banach space V . For a finite $B \subseteq A$ let $h(B) = \sum_{a \in B} h(a)$. Let $<$ be an $|A|$ -ordering of A , and let $B \subseteq A$. Whenever $v = \lim_n h([B]_n)$ exists we write $v = \sum_B^< h$, and we say that $<$ *sums h to v over B* . We say that $<$ *sums h over B* whenever $<$ sums h over B to some v . If $\mathfrak{B}$ is a family of subsets of A , we say that $<$ *sums h over $\mathfrak{B}$* if $<$ sums h over B for every $B \in \mathfrak{B}$. We say that $<$ *sums h uniformly over $\mathfrak{B}$* if for every $\varepsilon > 0$ there is an $N \in \omega$ such that for every $B \in \mathfrak{B}$ and $k, l \in \omega$ with $N < k, l$, we have $\|h_k([B]_l)\| < \varepsilon$.

THE SUMMING LEMMA. Let $A = \bigcup_{i \in \omega} A_i$, where $\{A_i : i \in \omega\}$ is a disjointed family of countable sets. For each $i \in \omega$ let $\mathfrak{B}_i$ be a family of subsets of A_i , and let $\mathfrak{B} = \{B \subseteq A : B \cap A_i \in \mathfrak{B}_i \text{ for all } i \in \omega\}$. Let V be a Banach space and let $h : A \rightarrow V$. For $i \in \omega$ let c_i be a positive real number and let $<_i$ be an $|A_i|$ -ordering of A_i so that:

$$(2.0) \sum_{i \in \omega} c_i < \infty.$$

$$(2.1) <_i \text{ sums } h \text{ uniformly over } \mathfrak{B}_i.$$

$$(2.2) \|h_k([B]_l^<)\| < c_i \text{ for } B \in \mathfrak{B}_i, k, l \in \omega.$$

Then there is an $|A|$ -ordering $<$ of A such that:

$$(2.3) a < b \text{ iff } a <_i b \text{ for } a, b \in A_i.$$

$$(2.4) < \text{ sums } h \text{ uniformly over } \mathfrak{B}.$$

$$(2.5) \text{ Let } B \in \mathfrak{B} \text{ and } B_i = B \cap A_i. \text{ Then}$$

$$\sum_B^< h = \sum_{i \in \omega} \left(\sum_{B_i}^{<_i} h \right).$$

PROOF. Define by induction on $s \in \omega$, $m_s, n_{is} \in \omega$ ($i \in \omega$) as follows. Let $m_0 = n_{i0} = 0$ ($i \in \omega$).

Assume that m_{s-1}, n_{is-1} are already defined for $i \in \omega$. By (2.0) we may choose m_s so that $m_{s-1} < m_s$ and

$$(2.6) \sum_{m_s < i} c_i < 1/2s.$$

By (2.2) we may further choose n_{is} for $i < m_s$ so that $n_{is-1} < n_{is}$ and

$$\|h(k[B]_i^{\prec'})\| < \frac{1}{2 \cdot m_s \cdot s} \quad \text{for } B \in \mathfrak{B}_i, n_{is} < k, l. \quad (2.7)$$

Let $n_{is} = 0$ for $m_s < i$.

Define A_{is} by

$$A_{is} =_{n_{is}} [A_i]_{n_{is}+1}^{\prec'}. \quad (2.8)$$

By construction, $A_i = \bigcup_{s \in \omega} A_{is}$ and $A_{is} <_i A_{is+1}$.

Let $<$ be any $|A|$ -ordering of A satisfying (2.3) and the condition

$$D_s = \bigcup_{\substack{i \leq s \\ r \leq s}} A_{ir} \text{ is an initial segment of } <. \quad (2.9)$$

(For example, $A_{00} < A_{01} < A_{10} < A_{11} < A_{02} < A_{12} < A_{20} < A_{21} < A_{22} < \dots$, i.e. $A_{ir} < A_{\tilde{i}\tilde{r}}$ iff $\max(i, r) < \max(\tilde{i}, \tilde{r})$ or $\max(i, r) = \max(\tilde{i}, \tilde{r})$ and $i < \tilde{i}$ or $\max(i, r) = \max(\tilde{i}, \tilde{r})$, $i = \tilde{i}$ and $r < \tilde{r}$.)

We show next that (2.4) holds. Let $s \in \omega$, $0 < s$. Let $N_s = |D_s|$, let $B \in \mathfrak{B}$, $N_s < k, l$ be given, and we demonstrate that $\|h(k[B]_l^{\prec'})\| < 1/s$. Let $B_i = B \cap A_i$, $B'_i =_k [B_i]_l^{\prec'}$. By (2.9) $[A]_{N_s}^{\prec'} = D_s$ and so, by $N_s < k$, $B'_i \cap D_s = \emptyset$. Hence there are $k_i, l_i \in \omega$ such that $n_{is} < k_i, l_i$ and $B'_i =_{k_i} [B_i]_{l_i}^{\prec'}$. Thus by (2.7) $\|h(B'_i)\| < 1/(2 \cdot m_s \cdot s)$ for $i < m_s$. Also, by (2.2) $\|h(B'_i)\| < c_i$. Thus we have by $h(k[B]_l^{\prec'}) = \sum_{i \in \omega} h(B'_i)$ and (2.6)

$$\|h(k[B]_l^{\prec'})\| < \sum_{i < m_s} \|h(B'_i)\| + \sum_{m_s < i} c_i < m_s \cdot \frac{1}{2 \cdot m_s \cdot s} + \frac{1}{2s} = \frac{1}{s}.$$

Finally, we prove (2.5). Let $B \in \mathfrak{B}$, $B_i = B \cap A_i$ and let $v_i = \sum_{B'_i}^{\prec'} h$. By (2.2) $\|v_i\| < c_i$ and so $\sum_{i \in \omega} v_i$ is an absolutely convergent series in V . Thus it is convergent to some $v \in V$ unconditionally, i.e. also under arbitrary rearrangement of its terms. Let $v' = \sum_B^{\prec'} h$ and let s be a positive integer. We establish (2.5) by showing $\|v - v'\| < 3/s$.

Let $B'_i = [B_i]_{n_{is}}^{\prec'}$, $B''_i =_{n_{is}} [B_i]^{\prec'}$. Then $v_i = h(B'_i) + \sum_{B''_i}^{\prec'} h$. By (2.7),

$$\left\| \sum_{B''_i}^{\prec'} h \right\| < \frac{1}{2 \cdot s \cdot m_s} \quad \text{for } i < m_s$$

and so

$$\|v_i - h(B'_i)\| < \frac{1}{2 \cdot s \cdot m_s}, \quad i < m_s.$$

Let $D = B \cap D_s$. By (2.9) we have $h(D) = \sum_{i < m_s} h(B'_i)$. Hence

$$\left\| \sum_{i < m_s} v_i - h(D) \right\| = \left\| \sum_{i < m_s} (v_i - h(B'_i)) \right\| < m_s \cdot \frac{1}{2 \cdot m_s \cdot s} = \frac{1}{2s}.$$

Also, by proof of (2.4)

$$\|v' - h(D)\| < 1/s.$$

By (2.2) and (2.6)

$$\left\| \sum_{m_s < i} v_i \right\| < \sum_{m_s < i} \|v_i\| < \frac{1}{s}.$$

Hence

$$\|v - v'\| = \left\| \left(\sum_{i < m_s} v_i - h(D) \right) - (v' - h(D)) + \sum_{m_s < i} v_i \right\| < \frac{3}{s},$$

and the proof of the Summing Lemma is complete. $\square$

3. Proof of Theorem 1. Let Δ denote the class of compact ordered sets for which Theorem 1 is true, and let $\mathfrak{Q} = \{\bar{A} : A \in \Delta\}$. Theorem 1 states $\mathcal{C} \subseteq \mathfrak{Q}$.² We first list two obvious properties of $\mathfrak{Q}$.

PROPOSITION 3.0. (i) $\bar{m} \in \mathfrak{Q}$ for every $m \in \omega$.

(ii) $\alpha, \beta \in \mathfrak{Q} \Rightarrow \alpha + \beta \in \mathfrak{Q}$.

Call $f \in C(A, V)$ *representable (uniquely representable)* iff there is an enumeration $(J_n)_n$ of f 's increments so that $f = \sum_n J_n$ holds in $C(A, V)$ (and whenever $(\bar{J}_n)_n$ is another enumeration of f 's increments for which $\sum_n \bar{J}_n$ converges, the identity $f = \sum_n \bar{J}_n$ holds in $C(A, V)$).

PROPOSITION 3.1. Let A, B be ordered sets, and let g be a continuous mapping of A onto B satisfying $g(a_1) < g(a_2)$ whenever $a_1 < a_2$. For $f \in C(B, V)$ define $\hat{f} \in C(A, V)$ by $\hat{f} = f \circ g$. Then:

(i) f is representable (uniquely representable) iff $\hat{f}$ is representable (uniquely representable),

(ii) if $A \in \Delta$ then $B \in \Delta$.

PROOF. Let $\hat{C}(B, V) = \{\hat{f} : f \in C(B, V)\}$. Clearly, $f \rightarrow \hat{f}$ is an isometric isomorphism of $C(B, V)$ onto $\hat{C}(B, V)$. For $b \in B$, $g^{-1}(b)$ is a compact interval of A . Let $a_b = \max g^{-1}(b)$. Let $f \in C(B, V)$, $b \in B$, $v \in V$. Then $J_b^v \in C(B, V)$ is an increment of f iff $J_{a_b}^v = \hat{J}_b^v$ is an increment of $\hat{f}$. Let $(J_n)_n$ be an enumeration of f 's increments. Then $(\hat{J}_n)_n$ is an enumeration of $\hat{f}$'s increments, and $\sum_{n \in \omega} J_n$ converges to $h \in C(B, V)$ iff $\sum_{n \in \omega} \hat{J}_n$ converges to $\hat{h} \in \hat{C}(B, V)$. (i) follows, and (ii) follows trivially from (i). $\square$

²In fact, $\mathfrak{Q}$ is the class of scattered compact order types [M1].

Let A be a countable compact ordered set, and let $f, h: A \rightarrow V$. Let $\mathfrak{B}_A = \{[m_A, a): a \in A\}$. We say that an $|A|$ -ordering $<$ sums h (uniformly) to f iff $<$ sums h (uniformly) over $\mathfrak{B}_A$ and $f(a) = \sum_{[m_A, a)} h$ for every $a \in A$.

We prove Theorem 1 rephrased as follows:

THEOREM 1'. *Let A be a countable compact ordered set, V a Banach space, and let $f: A \rightarrow V$ be a continuous function satisfying $f(m_A) = 0$. Then: (a') There is an $|A|$ -ordering $<$ of A that sums $f^\dagger$ uniformly to f .*

(b') *If $<$ is any $|A|$ -ordering of A that sums $f^\dagger$ uniformly to $\tilde{f}$, then $\tilde{f} = f$.*

PROOF OF (a'). We prove (a') by induction on $\text{ch}(A) = (\mu, m)$. By Proposition 3.0(i) we may assume $\mu > 0$ and by Proposition 3.0(ii) and Proposition 1.3 it is enough to prove (a') for $m = 1$. So assume $m = 1$ and let $A^{(\mu)} = \{x\}$.

We may further assume that $m_A < x < M_A$ and x is a core point of A . (Otherwise, $x = M_A$, $x < x'$, $x = m_A$ or $x' < x$. Let $\hat{A} = A \cup \{x_n: n \in \omega\}$ where $x < x_{n+1} < x_n$ (and $x_n < x'$) if $x = M_A$ (if $x < x'$) and $x_n < x_{n+1} < x$ (and $x' < x_n$) if $x = m_A$ (if $x' < x$). Then $\text{ch}(\hat{A}) = \text{ch}(A) = (\mu, 1)$, $\hat{A}^{(\mu)} = \{x\}$ and x is a core point of $\hat{A}$. Define $g: \hat{A} \rightarrow A$ by $g(a) = a$, $a \in A$ and $g(x_n) = x$. By Proposition 3.1(i), (a') for $f \in C(A, V)$ follows from (a') for $\hat{f} = f \circ g \in C(\hat{A}, V)$.)

Choose $M_n \in A$ so that the following conditions hold:

- (1) $M_{2n} < M_{2n+2} < x < M_{2n+3} < M_{2n+1} < M_1 = M_A$ ($n \in \omega$).
- (2) $M_n < M'_n$, $n \neq 1$.
- (3) $x = \sup\{M_{2n}: n \in \omega\} = \inf\{M_{2n+1}: n \in \omega\}$.
- (4) $M_{2n} < a, b < M_{2n+3}$ implies $\|f(b) - f(a)\| < 2^{-(2n+3)}$.

This is possible by continuity of f and Corollary 1.5. Let $m_0 = m_A$, $m_{2n+2} = M'_{2n}$ and $m_{2n+1} = M'_{2n+3}$, $n \in \omega$. Define $A_n = [m_n, M_n]$. Then $A = x\text{-CL}_n A_n$ and A_n is a compact ordered set satisfying $A_n^{(\mu)} = \emptyset$, since $A_n^{(\mu)} \subseteq A^{(\mu)} = \{x\}$ and $x \notin A_n \supseteq A_n^{(\mu)}$. Thus $\text{ch}(A_n) = (\mu_n, m_n)$ where $\mu_n < \mu$.

We may further assume

- (5) $f(M_{2n}) = f(m_{2n+2})$, $f(M_{2n+3}) = f(m_{2n+1})$, $n \in \omega$.
(Otherwise, let $\hat{A}_n = A_n \cup \{c_n\}$ where $A_n < c_n$, $c_n \notin A$, and let $\hat{A} = x\text{-CL}_n \hat{A}_n$. Define $g: \hat{A} \rightarrow A$ by $g(a) = a$ for $a \in A$, $g(c_{2k}) = m_{2k+2}$, $g(c_{2k+3}) = m_{2k+1}$, $g(c_1) = M_1$, and let $\hat{f} = f \circ g$. Let $\hat{M}_n = M_{\hat{A}_n} = c_n$, $\hat{m}_n = m_{\hat{A}_n} = m_n$. Then $\hat{f}(\hat{M}_{2n}) = \hat{f}(\hat{m}_{2n})$, and (a') for $\hat{f}$ implies (a') for f by Proposition 3.1(i).)

Define $f_n \in C(A_n, V)$ by $f_n(a) = f(a) - f(m_n)$, $n \in \omega$. Then $f_n^\dagger = f^\dagger|_{A_n}$ (by (5) this holds true also at M_n). By the induction hypothesis, there is an $|A_n|$ -ordering $<'_n$ of A_n that sums $f_n^\dagger$ (hence $f^\dagger$) uniformly to f_n . Let $\mathfrak{B}_n = \{[m_n, a): a \in A_n\} \cup \{A_n\}$. Let $K_n \in \omega$ satisfy

$$\|f^\dagger([B]_i^{<'_n})\| < 2^{-n} \quad \text{for } B \in \mathfrak{B}_n, K_n < k, l. \quad (6)$$

Let $[A_n]_{K_n}^{<'_n} = \{a_i: 0 < i < K_n\}$ where $a_i < a_j$ for $i < j$. Modify the order $<'_n$ into an $|A|$ -ordering $<_n$ by reordering $[A_n]_{K_n}^{<'_n}$ in the ordering inherited from

A , that is, let $a_i <_n a_j$ for $0 \leq i < j < K_n$, and $a <_n b$ iff $a <'_n b$ for $a \in A_n$, $b \in \kappa_n[A_n]^{<_n}$. We shall show that $<_n$ satisfies (2.1) and (2.2) (with $h = f^\dagger$ and $c_n > 0$ that satisfy (2.0)).

Now (2.1) is obvious, since $<_n$ differs from $<'_n$ only on a finite set. We prove (2.2). Let $a_{-1} = m_n$, $a_{K_n} = M_n$. Let $B_j = \{a_i: 0 \leq i < j\}$; and let $a \in (a_{j-1}, a_j]$. Denote $B_a = [m_n, a)$. Then

$$[B_a]_{\kappa_n}^{<_n} = B_j \quad (0 \leq j < K_n). \quad (7)$$

Now, by hypothesis

$$f_n(a_j) = \sum_{B_{a_j}}^{<_n} f_n^\dagger = f_n^\dagger(B_j) + \sum_{\kappa_n[B_{a_j}]^{<_n}}^{<_n} f_n^\dagger = f^\dagger(B_j) + \sum_{\kappa_n[B_{a_j}]^{<_n}}^{<_n} f^\dagger.$$

Now,

$$\sum_{\kappa_n[B_{a_j}]^{<_n}}^{<_n} f^\dagger = \lim_l f_n^\dagger(\kappa_n[B_{a_j}]_l^{<_n}), \quad \kappa_n[B_{a_j}]_l^{<_n} = \kappa_n[B_{a_j}]_l^{<'_n}$$

and so by (6) we have

$$\|f^\dagger(\kappa_n[B_{a_j}]_l^{<_n})\| < 2^{-n}. \quad (8)$$

Hence

$$\left\| \sum_{\kappa_n[B_{a_j}]^{<_n}}^{<_n} f^\dagger \right\| < 2^{-n}.$$

Also, by (4),

$$\|f_n(a_j)\| = |f(a_j) - f(m_n)| < 2^{-n} \quad (n > 1).$$

Thus

$$\|f^\dagger(B_j)\| < 2 \cdot 2^{-n} \quad (n > 1). \quad (9)$$

Now let $a \in (a_{j-1}, a_j]$ ($0 \leq j < K_n$), and let $k, l \in \omega$. Then

$$f_n^\dagger([B_a]_l^{<_n}) = f_n^\dagger([B_a]_{\kappa_n}^{<_n}) + f_n^\dagger(\kappa_n[B_a]_l^{<_n}) \quad (j < l).$$

Hence, by (7), (8) and (9)

$$\|f_n^\dagger([B_a]_l^{<_n})\| < 3 \cdot 2^{-n} \quad (n > 1).$$

But $f^\dagger(\kappa[B_a]_l^{<_n}) = f_n^\dagger([B_a]_l^{<_n}) - f_n^\dagger([B_a]_k^{<_n})$ ($k < l$). It follows that

$$\|f^\dagger(\kappa[B_a]_l^{<_n})\| < 6 \cdot 2^{-n} \quad (n > 1).$$

Let $c_n = 6 \cdot 2^{-n}$ for $n > 1$. Then c_0, c_1 can be properly chosen so that $\sum_n c_n < \infty$ and for every $B \in \mathfrak{B}_n$, $k, l \in \omega$ we have $\|f^\dagger(\kappa[B]_l^{<_n})\| < c_n$. Thus (2.2) holds.

Let $A' = \bigcup_{n \in \omega} A_n$ and let $<'$ be an $|A'|$ -ordering of A' so that (2.3)–(2.5) hold.

Let $<$ be any $|A|$ -ordering of $A = A' \cup \{x\}$ whose restriction to A' is $<'$. We show that $<$ sums $f^\dagger$ uniformly to f .

Indeed, by (2.4) $<$ sums $f^\dagger$ uniformly over $\mathfrak{B} = \{B \subseteq A: B \cap A_n \in \mathfrak{B}_n\}$. Now for all $a \in A$, $[m_0, a) \cap A_n \in \mathfrak{B}_n$ and so $<$ sums $f^\dagger$ uniformly over $\{[m_0, a): a \in A\}$. We shall show that $\sum_{[m_0, a)}^\dagger f^\dagger = f(a)$ for each $a \in A$. By (2.3), (5) we have

$$\sum_{A_n}^\dagger f^\dagger = \sum_{[m_n, M_n)}^\dagger f_n^\dagger = f_n(M_n),$$

so

$$\sum_{A_n}^\dagger f^\dagger = f(M_n) - f(m_n) \quad (10)$$

and for $a \in A_n$, by (2.3)

$$\sum_{[m_n, a)}^\dagger f^\dagger = f(a) - f(m_n). \quad (11)$$

Now, by (2.5) (10), (11) and by $f^\dagger(x) = 0$ we have for every $a \in A$:

$$\sum_{[m_0, a)}^\dagger f^\dagger = \sum_{n \in \omega} \left(\sum_{[m_0, a) \cap A_n}^\dagger f^\dagger \right) = \sum_{A_k < A_n} (f(M_k) - f(m_k)) + f(a) - f(m_n).$$

By $f(m_0) = 0$ we deduce for $a \in A_{2n}$:

$$\sum_{[m_0, a)}^\dagger f^\dagger = \sum_{k < n} (f(m_{2k+2}) - f(m_{2k})) + f(a) - f(m_{2n}) = f(a).$$

By $f(x) = \lim_n f(m_{2n})$ we have

$$\sum_{[m_0, x)}^\dagger f^\dagger = \sum_{n \in \omega} (f(m_{2n+2}) - f(m_{2n})) = \lim_n f(m_{2n}) = f(x).$$

Finally, let $a \in A_{2n+1}$. Then

$$\begin{aligned} \sum_{[m_0, a)}^\dagger f^\dagger &= \sum_{k \in \omega} (f(m_{2k+2}) - f(m_{2k})) \\ &\quad + \sum_{j > n} (f(m_{2j+1}) - f(m_{2j+3})) + f(a) - f(m_{2n+1}) \\ &= f(x) + \sum_{j > n} (f(m_{2j+1}) - f(m_{2j+3})) + f(a) - f(m_{2n+1}). \end{aligned}$$

But $\sum_{n < j < l} (f(m_{2j+1}) - f(m_{2j+3})) = f(m_{2n+1}) - f(m_{2l+1})$. Also, $\lim_l f(m_{2l+1}) = f(x)$, so

$$\sum_{[m_0, a)}^\dagger f^\dagger = f(x) + (f(m_{2n+1}) - f(x)) + (f(a) - f(m_{2n+1})) = f(a).$$

PROOF OF (b').³ We shall rather prove

(b'') Let $<'$ be any $|A|$ -ordering of A that sums $f^\dagger$ to a continuous function $\tilde{f}$. Then $\tilde{f} = f$.

((b') follows, as any $|A|$ -ordering $<'$ that sums $f^\dagger$ uniformly over $\{[m_0, a) : a \in A\}$ sums it to a continuous function.)

(b'') is proved using the Induction Principle (§1). It is obvious for a finite A , and so we have to show that if $A = x\text{-CL}_{n \in \omega} A_n$ and (b'') is true of A_n , $n \in \omega$, it is also true of A . As before, we may assume $A_n \neq \emptyset$ for all n , and setting $m_n = m_{A_n}$, $M_n = M_{A_n}$ we may assume (5).

Let $a \in A_n$ and define $f_n(a) = f(a) - f(m_n)$, $\tilde{f}_n(a) = \tilde{f}(a) - \tilde{f}(m_n)$. Then, $f_n, \tilde{f}_n \in C(A_n, V)$ and $f_n^\dagger = f^\dagger|_{A_n}$. By assumption, $<'$ sums $f^\dagger$ over $[m_0, a)$ to $\tilde{f}(a)$ and over $[m_0, m_n)$ to $\tilde{f}(m_n)$, hence $<'$ sums $f_n^\dagger$ over $[m_n, a) = [m_0, a) - [m_0, m_n)$ to $\tilde{f}_n(a) = \tilde{f}(a) - \tilde{f}(m_n)$. By the induction hypothesis, $\tilde{f}_n = f_n$. Hence $\tilde{f}(a) = \tilde{f}(m_n) + f_n(a)$ and in particular

$$\tilde{f}(M_n) - \tilde{f}(m_n) = f(M_n) - f(m_n). \quad (12)$$

It is left to show that $\tilde{f}(m_n) = f(m_n)$ and that $\tilde{f}(x) = f(x)$. Now

$$\tilde{f}(m_0) = \sum_{[m_0, m_0)}^{<} f^\dagger = 0 = f(m_0).$$

Assuming $\tilde{f}(m_{2n}) = f(m_{2n})$ we have

$$\tilde{f}(m_{2n+2}) = \sum_{[m_0, m_{2n+2})}^{<} f^\dagger = \sum_{[m_0, m_{2n})}^{<} f^\dagger + \sum_{[m_{2n}, m_{2n+2})}^{<} f^\dagger = f(m_{2n}) + \sum_{[m_{2n}, m_{2n+2})}^{<} f^\dagger.$$

By (5), (12):

$$\sum_{[m_{2n}, m_{2n+2})}^{<} f^\dagger = f(m_{2n+2}) - f(m_{2n})$$

hence $\tilde{f}(m_{2n+2}) = f(m_{2n+2})$. Thus $\tilde{f}(m_{2n}) = f(m_{2n})$, $n \in \omega$. Also $\tilde{f}(x) = \lim_n \tilde{f}(m_{2n})$ by continuity of $\tilde{f}$, hence $\tilde{f}(x) = \lim_n f(m_{2n}) = f(x)$. Finally, for $l \in \omega$ we have by (5), (12)

$$\begin{aligned} \tilde{f}(m_{2n+1}) - \tilde{f}(m_{2n+2l+1}) &= \sum_{0 \leq i < l} (\tilde{f}(m_{2n+2i+1}) - \tilde{f}(m_{2n+2i+3})) \\ &= \sum_{0 \leq i < l} (f(m_{2n+2i+1}) - f(m_{2n+2i+3})) \\ &= f(m_{2n+1}) - f(m_{2n+2l+1}). \end{aligned}$$

³(b) is actually a consequence of (a), see [M1]. We use the Induction Principle (§1) to give an independent proof of (b'').

By continuity of $\tilde{f}$, $\lim_l \tilde{f}(m_{2n+2l+1}) = \tilde{f}(x)$. Hence, by $\tilde{f}(x) = f(x)$ and $f(x) = \lim_l f(m_{2n+2l+1})$:

$$\begin{aligned}\tilde{f}(m_{2n+1}) &= \tilde{f}(m_{2n+1}) - \tilde{f}(x) + f(x) \\ &= \lim_l (\tilde{f}(m_{2n+1}) - \tilde{f}(m_{2n+2l+1})) + f(x) \\ &= \lim_l (f(m_{2n+1}) - f(m_{2n+2l+1})) + f(x) = f(m_{2n+1}).\end{aligned}$$

This completes the proof of Theorem 1.

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