THE INTERIOR OPERATOR LOGIC AND PRODUCT TOPOLOGIES

BY

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ABSTRACT. In this paper we present a model theory of the interior operator on product topologies with continuous functions. The main results are a completeness theorem, an axiomatization of topological groups, and a proof of an interpolation and definability theorem.

0. Introduction and basic results. This paper develops a model theory for $L(I^n)_{n \in \omega}$, the interior operator logic, which is analogous to the author's development for $L(Q^n)_{n \in \omega}$, the "open set" quantifier logic, in [16]. The significant difference is that $L(I^n)_{n \in \omega}$ satisfies an interpolation and definability theorem in marked contrast to $L(Q^n)_{n \in \omega}$. Another contrasting result is an omitting types theorem for $L(I^n)_{n \in \omega}$.

These results were obtained by the author shortly after the results contained in [16] on $L(Q^n)_{n \in \omega}$ were proved. More recently, J. A. Makowsky and M. Ziegler have obtained similar results for $L(I)$ using proof theory.

In §1 we prove a completeness theorem for $L(I^n)_{n \in \omega}$. This proof uses the ideas developed for $L(Q^n)_{n \in \omega}$. We also show that if an $L(I^n)_{n \in \omega}$ theory has a Hausdorff topological model then it has a 0-dimensional normal topological model. Another result presented in this section is an axiomatization for the $L(I)$ theory of topological groups.

We conclude this paper by presenting a proof of a Robinson joint-consistency theorem for $L(I^n)_{n \in \omega}$. This result implies both interpolation and definability. Thus $L(I^n)_{n \in \omega}$ has a more "smooth" model theory than $L(Q^n)_{n \in \omega}$. Another result is the omitting types theorem.

We will assume that the reader is familiar with the basic results of model theory.

DEFINITION 1. If we take a first order model $\mathfrak{A}$ and $q_i \subseteq \mathcal{P}(A)$ then we call $(\mathfrak{A}, q_1, q_2, q_3, \dots)$ a *weak model* for $L(I^n)_{n \in \omega}$. If each q_i is a topology on A then $(\mathfrak{A}, q_1, q_2, q_3, \dots)$ is called *topological*. A topological model $(\mathfrak{A}, q_1, q_2, \dots)$ is called *complete* if each q_k is the k th topological product of q_1 on A . We will abbreviate $(\mathfrak{A}, q_1, q_2, q_3, \dots)$ by $(\mathfrak{A}, \mathbf{q})$.

The formulas of $L(I^n)_{n \in \omega}$ are defined analogously to first order logic with the additional clause:

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If φ is a formula and $x_1, \dots, x_n$ are distinct variables, then $I^n x_1, \dots, x_n \varphi$ is also a formula ($x_1, \dots, x_n$ occur free in $I^n x_1, \dots, x_n \varphi$). Again we abbreviate $I^n x_1, \dots, x_n \varphi$ by $Ix\varphi$.

Since the language $L(I^n)$ is not closed under substitution of free variables, we introduce the following notation to take the place of substitution.

DEFINITION. Given n -tuples of variables $\mathbf{x}$ and substitutable terms $\mathbf{k}$, $\varphi[\mathbf{k}]$ denotes the formula $\exists \mathbf{x}(\bigwedge_{i=1}^n x_i = k_i \wedge \varphi)$.

The notion of an n -tuple $a_1, \dots, a_n \in A$ satisfying a formula $\varphi(v_1, \dots, v_n)$ of $L(I^n)_{n \in \omega}$ in a weak model $(\mathfrak{A}, \mathbf{q})$ is defined in the usual manner by induction on the complexity of φ and is denoted by $(\mathfrak{A}, \mathbf{q}) \models \varphi[\mathbf{a}]$. The only difficult case is the $Ix\varphi$ case:

$$(\mathfrak{A}, \mathbf{q}) \models Ix\varphi[\mathbf{a}] \quad \text{iff} \quad \mathbf{a} \in \emptyset \subseteq [\varphi(\mathbf{x})]^{(\mathfrak{A}, \mathbf{q})} \quad \text{for some } \emptyset \in q_n$$

($[\varphi(\mathbf{x})]^{(\mathfrak{A}, \mathbf{q})}$ denotes the set $\{\mathbf{a} \in A^n : (\mathfrak{A}, \mathbf{q}) \models \varphi[\mathbf{a}]\}$).

It can be shown by induction that for $\mathbf{a} \in A^n$, a sentence $\varphi[\mathbf{a}]$ holds in $(\mathfrak{A}, \mathbf{a}, \mathbf{q})$ iff $(\mathfrak{A}, \mathbf{q}) \models \varphi[\mathbf{a}]$ as previously defined.

REMARK. Notice that by our definition of $\models$, $Ix\varphi$ is the interior of the set defined by φ in a topological model.

The axioms for $L(I^n)_{n \in \omega}$ are the standard first order ones where we replace $\varphi(c)$ by $\varphi[\mathbf{c}]$, e.g.

$$\bigwedge_{i=1}^n x_i = y_i \rightarrow (Ix\varphi[\mathbf{x}] \leftrightarrow Iy\varphi[\mathbf{y}])$$

and

$$Ax: \forall \mathbf{x}(\varphi \leftrightarrow \psi) \rightarrow (Ix\varphi \leftrightarrow Ix\psi).$$

The rules of inference are:

- (i) Modus ponens, i.e. from $\varphi, \varphi \rightarrow \psi$ infer ψ ,
- (ii) Generalization, i.e. from φ infer $\forall \mathbf{x}\varphi$.

We can now prove the following completeness theorem for topological models.

THEOREM 2. An $L(I^n)_{n \in \omega}$ theory Σ has a topological model $(\mathfrak{A}, \mathbf{q})$, if and only if Σ is consistent with

- (A0) axioms for $L(I^n)_{n \in \omega}$,
- (A1) $Ix(y = y) \leftrightarrow y = y$,
- (A2) $Ix\varphi \rightarrow \varphi$,
- (A3) $Ix\varphi \rightarrow IxIx\varphi$,
- (A4) $(Ix\varphi \wedge Ix\psi) \leftrightarrow Ix(\varphi \wedge \psi)$.

PROOF (ONLY IF). Straightforward because the axioms hold in every topology.

(IF) Let Σ^* be a maximal consistent extension of Σ with witnesses using $\varphi[\mathbf{c}]$ instead of $\varphi(c)$. As in the proof of the completeness theorem for first order logic take $\mathfrak{A}$ to be the model generated by Σ^* .

Define $q_k = \{\{\mathbf{a} | Ix\varphi[\mathbf{a}] \in \Sigma^*\} | \varphi \text{ is a formula of } L(I^n)_{n \in \omega}(A)\}$.

We claim $(\mathfrak{A}, \mathbf{q})$ models Σ^* . We prove this by induction on the I complexity of φ that, for every formula $\varphi(\mathbf{x})$ and n -tuple $\mathbf{a}$, $\varphi[\mathbf{a}] \in \Sigma^*$ iff $(\mathfrak{A}, \mathbf{q}) \models \varphi[\mathbf{a}]$. The only difficult clause is $Ix\varphi[\mathbf{a}]$.

If $Ix\varphi[b] \in \Sigma^*$ then $a \in \{b \in A^n \mid Ix\varphi[b] \in \Sigma^*\} \in q_n$. Hence $(\mathfrak{A}, \mathbf{q}) \models Ix\varphi[a]$ using (A2).

Assume $(\mathfrak{A}, \mathbf{q}) \models Ix\varphi[a]$. That is, there is an $\emptyset \in q_n$ such that $a \in \emptyset \subseteq [\varphi(x)]^{(\mathfrak{A}, \mathbf{q})}$. By the definition of q_n there is an $Ix\psi$ such that

$$a \in \emptyset = [Ix\psi]^{(\mathfrak{A}, \mathbf{q})} \subseteq [\varphi(x)]^{(\mathfrak{A}, \mathbf{q})}.$$

Using (A4), (A3) and the fact that $\forall x((Ix\psi) \wedge \varphi \leftrightarrow (Ix\psi)) \in \Sigma^*$ we obtain that $\forall x(Ix\psi \rightarrow Ix\varphi) \in \Sigma^*$. Thus we have $Ix\varphi[a] \in \Sigma^*$.

Let q_k^* be the topology generated by q_k . By axioms (A1), (A2), (A4) we know that q_k is a basis for q_k^* . We claim

$$(\mathfrak{A}, \mathbf{q}) \equiv_{L(I^n)_{n \in \omega}(A)} (\mathfrak{A}, \mathbf{q}^*).$$

We will prove this by showing that, for each $\psi(x_1, \dots, x_n)$,

$$[\psi(x)]^{(\mathfrak{A}, \mathbf{q})} = [\psi(x)]^{(\mathfrak{A}, \mathbf{q}^*)}.$$

Again this is proved by induction on the complexity of ψ . The only difficult case is $Ix\varphi$. Since q_n is a basis for q_n^* and from the definition of $\models$ we easily obtain

$$[Ix\varphi]^{(\mathfrak{A}, \mathbf{q})} \subseteq [Ix\varphi]^{(\mathfrak{A}, \mathbf{q}^*)}.$$

Let $a \in [Ix\varphi]^{(\mathfrak{A}, \mathbf{q}^*)}$. Then $a \in \emptyset \subseteq [\varphi(x)]^{(\mathfrak{A}, \mathbf{q}^*)}$ where $\emptyset \in q_n^*$. Since q_n is a basis for q_n^* , there is an $\emptyset^\# \in q_n$ such that $a \in \emptyset^\# \subseteq \emptyset$.

Using the induction hypothesis we see that $a \in \emptyset^\# \subseteq [\varphi(x)]^{(\mathfrak{A}, \mathbf{q})}$ whence $a \in [Ix\varphi]^{(\mathfrak{A}, \mathbf{q})}$.

DEFINITION 3 (TARSKI AND VAUGHT). $(\mathfrak{B}, \mathbf{r})$ is said to be an *elementary extension* of $(\mathfrak{A}, \mathbf{q})$, in symbols $(\mathfrak{B}, \mathbf{r}) < (\mathfrak{A}, \mathbf{q})$, if and only if $A \subseteq B$ and for all formulas $\varphi(x_1, \dots, x_n)$ of $L(I^n)_{n \in \omega}$ and all $a_1, \dots, a_n \in A$ we have $(\mathfrak{A}, \mathbf{q}) \models \varphi[a]$ iff $(\mathfrak{B}, \mathbf{r}) \models \varphi[a]$. A sequence $(\mathfrak{A}_\alpha, \mathbf{q}^\alpha)$, $\alpha < \gamma$, of weak models is said to be an *elementary chain* if and only if we have $(\mathfrak{A}_\alpha, \mathbf{q}^\alpha) < (\mathfrak{A}_\beta, \mathbf{q}^\beta)$ for all $\alpha < \beta < \gamma$.

The union of an elementary chain $(\mathfrak{A}_\alpha, \mathbf{q}^\alpha)$, $\alpha < \gamma$, is the weak model $(\mathfrak{A}, \mathbf{q}) = \bigcup_{\alpha < \gamma} (\mathfrak{A}_\alpha, \mathbf{q}^\alpha)$ such that $\mathfrak{A} = \bigcup_{\alpha < \gamma} \mathfrak{A}_\alpha$ and $q_n = \{S \subset A^n \mid \text{for some } \beta < \gamma, \beta \leq \alpha < \gamma \text{ implies } S \cap A_\alpha^n \in q_n^\alpha\}$.

These definitions enable us to state:

THEOREM 4. *Let $(\mathfrak{A}_\alpha, \mathbf{q}^\alpha)$, $\alpha < \gamma$, be an elementary chain of weak models of (A0)–(A4) and let $(\mathfrak{A}, \mathbf{q})$ be the union. Then, for all $\alpha < \gamma$, $(\mathfrak{A}_\alpha, \mathbf{q}^\alpha) < (\mathfrak{A}, \mathbf{q})$.*

We conclude this introduction by stating an omitting types theorem whose proof is similar to the one found in Keisler [7].

THEOREM 5. *Let Γ be a set of sentences of $L(I^n)_{n \in \omega}$ containing (A0)–(A4) and $\Sigma_n(y_{n_1}, \dots, y_{n_n})$, $n \in \omega$, be sets of formulas of $L(I^n)_{n \in \omega}$. If Γ is consistent and omits each Σ_n then Γ has a weak model which omits each Σ_n .*

REMARK. In this section we could have developed a theory of weak models using fewer axioms than (A0)–(A4) for our definition of $\models$. For brevity and directness we restricted the scope of the results to interior operators.

1. Main Theorem and applications. We will show that the $L(I^n)_{n \in \omega}$ theory of continuous functions (relations) on product spaces has the following axiomatization:

- (A0), . . . , (A4),
 (A5) $Ixy\varphi \rightarrow Ix\varphi$,
 (A6) $Ix_1, \dots, x_k(\varphi(\sigma \circ \mathbf{x})) \leftrightarrow Ix(\varphi(\sigma \circ \mathbf{x}))$ where $\sigma: n+1 \rightarrow n+1$ and range $\sigma = \{i_1 < \dots < i_k\}$,
 (A7) $Ix\varphi \wedge Iy\psi \leftrightarrow Ixy(\varphi \wedge \psi)$,
 (A8 $_{\varphi}$) $Iy\psi(\mathbf{y}) \wedge \varphi(\mathbf{x}, \mathbf{y} \upharpoonright k) \rightarrow Ixy_{k+1}, \dots, y_m \exists \mathbf{z}(\varphi(\mathbf{x}, \mathbf{z}) \wedge \psi(\mathbf{z}, y_{k+1}, \dots, y_m))$
 where $\text{lh } \mathbf{y} = m$ and $k \leq m$.

That is, the inverse image of a segment of an interior point concatenated with its tail is an interior point which is stronger than the usual definition of continuous but is equivalent to the usual definition of a continuous function (relation) on product spaces. Here, if $\mathbf{x} = \langle x_1, \dots, x_n \rangle$ and $\sigma: n+1 \rightarrow n+1$, we have that $\sigma \circ x(i) = x_{\sigma(i)}$.

Suppose we have a topological model $(\mathfrak{A}, \mathbf{q})$ satisfying (A0)–(A8 $_{\varphi_\alpha}$), $\alpha \in J$. Let $\{\check{V}_\beta\}_{\beta \in D}$ be a collection of subsets of A . If we add the $\{\check{V}_\beta\}_{\beta \in D}$ to q_1 and still expect to have a model satisfying (in the expanded language with a V_β for each $\check{V}_\beta$) (A0)–(A8 $_{\varphi_\alpha}$), $\alpha \in J$, what do we need to add to q_n , $n \in \omega$?

Let φ_α , $\alpha \in J$, be a collection of (n_α, m_α) -ary relations which satisfy (A8 $_{\varphi_\alpha}$), $\alpha \in J$. There is trouble later on if the φ_α are formulas of $L(I^n)$ since $L(I^n)$ is not closed under substitution. We will restrict the φ_α to be relations defined by formulas of L for convenience and clarity.

Let $\varphi_\alpha^{-1} = \{(\mathbf{b}, \mathbf{a}) | (\mathbf{a}, \mathbf{b}) \in \varphi_\alpha\}$ be the inverse relation of φ_α , $\alpha \in J$. We then define a collection of (definable) relations as follows (cf. [16]): $WT_0 = \{\varphi_\alpha^{-1}\}_{\alpha \in J} \cup \{\text{identity relation on each } n \in \omega\}$, $WT_{n+1} = WT_n \cup \{\varphi(\sigma \circ \mathbf{x}, \mathbf{y}) \text{ where } \sigma \text{ maps } n_\varphi \text{ into } n_\varphi \text{ where } \varphi \in WT_n\} \cup \{\varphi(\mathbf{x}, \sigma \circ \mathbf{y}) \text{ where } \sigma \text{ maps } m_\varphi \text{ into } m_\varphi \text{ where } \varphi \in WT_n\} \cup \{\varphi(x_1, \dots, x_{n_\varphi}, y_1, \dots, y_{m_\varphi}) \wedge \psi(z_1, \dots, z_{n_\psi}, t_1, \dots, t_{m_\psi}) \text{ where } \varphi, \psi \in WT_n\} \cup \{\varphi(x_1, \dots, c, \dots, x_{n_\varphi}, y_1, \dots, y_{m_\varphi}) \text{ where } \varphi \in WT_n \text{ and } c \text{ is an individual constant symbol}\} \cup \{\varphi(x_1, \dots, x_{n_\varphi}, y_1, \dots, c, \dots, y_{m_\varphi}) \text{ where } \varphi \in WT_n \text{ and } c \text{ is an individual constant symbol}\} \cup \{(\exists \mathbf{y}(\varphi(x_1, \dots, x_{n_\varphi}, y_1, \dots, y_{m_\varphi}) \wedge \psi(y_1, \dots, y_{n_\psi}, z_1, \dots, z_{m_\psi})), k \leq m_\varphi, k \leq n_\psi, \text{ i.e. the composition of the two relations, where } \varphi, \psi \in WT_n\}$.

Let $WT = \bigcup_{n \in \omega} WT_n$.

The intuitive meaning of WT is that it is the smallest collection of definable relations containing WT_0 and closed under composition, projection, products and mappings of the variables. Hence, since each φ_α satisfies (A8 $_{\varphi_\alpha}$), $\alpha \in J$, and $(\mathfrak{A}, \mathbf{q})$ models (A0)–(A8 $_{\varphi_\alpha}$), $\alpha \in J$, we have that each $\varphi \in WT$ takes definable open sets to definable open sets.

Define q_n^* , $n \in \omega$, as follows: q_n^* is the topology generated by $\{\varphi(\prod_{i=1}^k B_j) | \text{each } B_j \in q_k \text{ or } B_j \in \{\mathcal{Q}_{\beta_j}\}_{\beta \in D} \text{ for } 1 \leq j \leq k, \varphi \in WT \text{ and } \varphi \text{ maps into } A^n\}$. $\varphi(C)$ means $\{\mathbf{b} | (\mathfrak{A}, \mathbf{q}) \models \varphi(\mathbf{c}, \mathbf{b}) \text{ and } \mathbf{c} \in C\}$.

We now state the following important lemma whose proof is analogous to Lemma 2.2 in [16].

LEMMA 6. Let $(\mathfrak{A}, \mathbf{q}^*)$ be as above. Then $(\mathfrak{A}, \mathbf{q}^*)$ models $(A0) - (A8_{\varphi_\alpha})$, $\alpha \in J$, in the expanded language containing a V_β for each ∇_β .

Now we will proceed to prove the main completeness theorem by presenting the following lemma which tells us that for each $\mathbf{c} \in \mathcal{O}$ (an open set in the I^n interpretation) we can add a $\prod_{i=1}^n o_i$ (an open n -box) to the I^n interpretation such that $\mathbf{c} \in \prod_{i=1}^n o_i \subseteq \mathcal{O}$ and still keep $(A0) - (A8_{\varphi_\alpha})$, $\alpha \in J$.

LEMMA 7. Let Σ be an $L(I^n)_{n \in \omega}$ theory consistent with $(A0) - (A8_{\varphi_\alpha})$, $\alpha \in J$. If $\mathbf{c} = \langle c_1, \dots, c_n \rangle$ is an n -tuple so that $(Ix\varphi)[\mathbf{c}]$ is consistent with Σ then $V_i(c_i)$, $\forall x(IxV_i(x) \leftrightarrow V_i(x))$, $1 \leq i \leq n$, and $\forall x(\bigwedge_{i=1}^n V_i(x_i) \rightarrow \varphi(\mathbf{x}))$ is consistent with Σ and $(A0) - (A8_{\varphi_\alpha})$, $\alpha \in J$. Here the $V_i(x_i)$, $1 \leq i \leq n$, are new one-place predicate symbols.

PROOF. We need only prove this for countable Σ since then by using Theorem 2 we have it for all Σ and $(A0) - (A8_{\varphi_\alpha})$, $\alpha \in J$. Let $(\mathfrak{A}, \mathbf{q})$ be a topological model of Σ . This is possible since Σ is consistent with $(A0) - (A4)$ and Theorem 2.

We will assume that we have some countable enumeration of the "potential" basic open sets of $(\mathfrak{A}, \mathbf{q}^*)$, i.e., $\mathcal{O}_\beta^* = \bigcap \varphi_\beta(\prod B_k)$ where without loss of generality we can assume that $B_l = \nabla_l$ for $1 \leq l \leq n$ and $B_l = \mathcal{O}_{\delta_l} \in q_{\gamma_l}$ for $k > l > n$. Also we take an enumeration of $\sigma_k(x_1, \dots, x_{m_k})$ of the formulas of $L(I^n)_{n \in \omega}(A)$.

We want to define $o_i \subseteq A$, $1 \leq i \leq n$, such that

$$\mathbf{c} \in \prod_{i=1}^n o_i \subseteq [Ix\varphi]^{(\mathfrak{A}, \mathbf{q})}$$

and forming $(\mathfrak{A}, \mathbf{q}^*)$ from the $\{o_i\}_{1 \leq i \leq n}$ we have that $(\mathfrak{A}, \mathbf{q}) < (\mathfrak{A}, \mathbf{q}^*)$. To do this we will construct the o 's by induction.

Suppose we have picked $b_i^1, \dots, b_i^{f(k)}$ for each o_i so that

$$\prod_{i=1}^n \{b_i^1, \dots, b_i^{f(k)}\} \subseteq [Ix\varphi]^{(\mathfrak{A}, \mathbf{q})}$$

and $\mathbf{c} = \mathbf{b}^1$. Now to pick the b_i^m , $1 \leq i \leq n$, $m \leq f(k+1)$, we want to insure that

$$\prod_{i=1}^n \{b_i^1, \dots, b_i^k, b_i^{f(k+1)}\} \subseteq [Ix\varphi]^{(\mathfrak{A}, \mathbf{q})}$$

and also to somehow guarantee that if $\sigma(\mathbf{x})$ is a formula of $L(I^n)_{n \in \omega}$, $\mathbf{a} \in A^m$, and $(\mathfrak{A}, \mathbf{q}) \models \neg Ix\sigma(\mathbf{x})[\mathbf{a}]$ then we do not get $(\mathfrak{A}, \mathbf{q}^*) \models Ix\sigma(\mathbf{x})[\mathbf{a}]$. This is equivalent to $[\sigma(\mathbf{x})]^{(\mathfrak{A}, \mathbf{q}^*)} = [\sigma(\mathbf{x})]^{(\mathfrak{A}, \mathbf{q})}$ for every $\sigma(\mathbf{x})$.

We claim that the $b_i^{k+1}, \dots, b_i^{f(k+1)}$ can be picked such that

$$\prod_{i=1}^n \{b_i^1, \dots, b_i^{f(k+1)}\} \subseteq [Ix\varphi]^{(\mathfrak{A}, \mathbf{q}^*)} \quad (*)$$

and $(**)$ if $\mathcal{O}_\beta^* \in q_m^*$ and $\mathcal{O}_\beta^*$ is the $(k+1)$ th basic open set $\mathcal{O}_\beta^* \subseteq [\sigma_{k+1}(\mathbf{x})]^{(\mathfrak{A}, \mathbf{q}^*)}$ then there is an $\mathcal{O}_\beta \in q_m$ such that $\mathcal{O}_\beta^* \subseteq \mathcal{O}_\beta \subseteq [\sigma(\mathbf{x})]^{(\mathfrak{A}, \mathbf{q})}$.

This is done as follows. Define $(\mathbf{x}/\mathbf{b})$ to be $\prod_{i=1}^n \{x_i, b_i\}$ and let

$$\mathbf{C} = \bigcap_{\mathbf{b} \in \prod_{i=1}^n \{b_i^1, \dots, b_i^k\}} \bigcap_{\mathbf{t} \in (\mathbf{x}/\mathbf{b})} \{Ix\varphi[\mathbf{t}]\}^{(\mathfrak{A}, \mathbf{q})}$$

where $\{I\mathbf{x}\varphi[t]\}^{(\mathfrak{A}, \mathfrak{Q})} = \{(a_1, \dots, a_n) | (\mathfrak{A}, \mathfrak{Q}) \models I\mathbf{x}\varphi[k_1, \dots, k_n] \text{ where } k_i = a_i \text{ if } t_i = x_i, k_i = b_i \text{ otherwise}\}$. (Notice that in using axioms (A5)–(A7) that C is a definable open set.)

Taking $\mathcal{O}_{\beta_{k+1}}^* = \bigcap_{j=1}^{n_\beta} \varphi_{\beta_j}(\prod_{k=1}^{n_\beta} B_k)$ we define

$$C^{n_\beta} = \{\mathbf{c} \in A^{n \cdot n_\beta} | \langle c(1 \cdot k), \dots, c(n \cdot k) \rangle \in C \text{ for each } k, 1 \leq k \leq n_\beta\},$$

C^{n_β} is open because C is and by axiom (A5).

Let $\sigma_i: n_\beta \rightarrow n_\beta$ be a permutation for each i , $1 \leq i \leq n$. Then define $\sigma = \langle \sigma_1, \dots, \sigma_n \rangle * \mathbf{c}$ where $\mathbf{c} \in C^{n_\beta}$ by $\sigma * \mathbf{c}(i) = \mathbf{c}(\sigma_k^{-1}(i))$ if $i \equiv k \pmod{n}$. That is, we look at the coordinates of the vector $\mathbf{c}$ as a collection of n_β vectors of length n and permute each of its i th coordinates by σ_i . Define $\sigma * C^{n_\beta}$ to be $\{\sigma * \mathbf{c} | \mathbf{c} \in C^{n_\beta}\}$ and again we have that $\sigma * C^{n_\beta}$ is open by the axioms. Hence $C^* = \bigcap_{\sigma} \sigma * C^{n_\beta}$ is open because the finite intersection of open sets is open.

We claim that if $\mathbf{b} \in C^*$ then

$$\prod_{i=1}^n \{b'_i, \dots, b_i^{k+n_\beta}\} \subseteq [I\mathbf{x}\varphi]^{(\mathfrak{A}, \mathfrak{Q})}$$

where $b_i^{k+l} = \mathbf{b}(i \cdot n + l)$.

If $\mathbf{t} \in \prod_{i=1}^n \{b'_i, \dots, b_i^{k+n_\beta}\}$ then it is straightforward to see that there is a σ such that

$$\langle \sigma * \mathbf{b}(n - i + 1), \dots, \sigma * \mathbf{b}(n - (i + k)) \rangle = \mathbf{t}$$

for some $1 \leq i \leq n_\beta$. Then $\sigma * \mathbf{b} \in C^{n_\beta}$ since $\text{id} * C^{n_\beta} = C^{n_\beta}$. Thus $\mathbf{t} \in C$ which implies that $\mathbf{t} \in [I\mathbf{x}\varphi]^{(\mathfrak{A}, \mathfrak{Q})}$.

Remembering that

$$\mathcal{O}_{\beta_{k+1}}^* = \bigcap_{j=1}^{n_\beta} \varphi_{\beta_j} \left(\prod_{i=1}^n \mathcal{V}_i \times \prod_{l=1}^k \beta_k \right),$$

define $C^\# = \{\mathbf{c} \in A^{n_\beta(n+k)} | \mathbf{c} \upharpoonright n \cdot n_\beta \in C^* \text{ and } \langle \mathbf{c}(n_\beta \cdot n + k \cdot i + 1), \dots, \mathbf{c}(n_\beta \cdot n + k \cdot (i + 1)) \rangle \in \prod_{l=1}^k B_l \text{ for each } 1 \leq i \leq n_\beta\}$.

Take $\mathbf{c} \in C^\#$ and define $\varphi * \mathbf{c}$ to be the vector defined by

$$\varphi * \mathbf{c}(m \cdot i + l) = \varphi_{\beta_i}(\langle \mathbf{c}(n \cdot i + 1), \dots, \mathbf{c}(n \cdot (i + 1)), \mathbf{c}(n \cdot n_\beta + n \cdot i + 1), \dots, \mathbf{c}(n \cdot n_\beta + n(i + 1)) \rangle)(l).$$

By axiom (A8) and the other axioms we know that $\varphi * C^\#$ is open where $\varphi * C^\# = \{\varphi * \mathbf{c} | \mathbf{c} \in C^\#\}$.

Define $\bigcap \varphi * C^\# = \{\mathbf{c} \in A^m | \text{there is a } \mathbf{b} \in C^\# \text{ such that } \mathbf{c}(i) = \varphi * \mathbf{b}(k \cdot i) \text{ for each } 1 \leq k \leq n_\beta\}$. Again by the axioms $\bigcap \varphi * C^\#$ is an open subset of A^m . Now if

$$\bigcap \varphi * C^\# - [\sigma_{k+1}(\mathbf{x})]^{(\mathfrak{A}, \mathfrak{Q})} \neq \emptyset$$

then there is a $\mathbf{b}^{k+1}, \dots, \mathbf{b}^{k+n_\beta} \in C^*$ such that

$$\bigcap_{j=1}^{n_\beta} \varphi_{\beta_j}(\mathbf{b}^{k+j} \times \prod \beta_k) - [\sigma_{k+1}(\mathbf{x})]^{(\mathfrak{A}, \mathfrak{Q})} \neq \emptyset.$$

Otherwise let $\mathbf{b}^{k+i} = \mathbf{b}^k$ for $1 \leq i \leq n_\beta$. Set $f(k + 1) = k + n_\beta$ and we are done.

Note. The proof of the corresponding result in [16] contains an error at this point. The correct proof is obtained by a direct adaptation of this one to $L(Q^n)_{n \in \omega}$.

Now we will show that if $\mathfrak{o}_i = \{b_i^k \mid k \in \omega\}$ then we have the conclusion to the lemma. $\mathbf{c} \in \prod_{i=1}^n \mathfrak{o}_i \subseteq [Ix\varphi]^{(\mathfrak{A}, \mathbf{q})}$ by (*). To show $(\mathfrak{A}, \mathbf{q}) < (\mathfrak{A}, \mathbf{q}^*)$ we use induction. The difficult case is the Ix clause.

Since $q_i \subseteq q_i^*$ for all i we have that if

$$(\mathfrak{A}, \mathbf{q}) \models Ix\chi(\mathbf{x})[\mathbf{a}]$$

then

$$(\mathfrak{A}, \mathbf{q}^*) \models Ix\chi(\mathbf{x})[\mathbf{a}]$$

for arbitrary $\mathbf{a} \in A^m$.

Suppose that $(\mathfrak{A}, \mathbf{q}) \models \neg Ix\chi(\mathbf{x})[\mathbf{a}]$ and $(\mathfrak{A}, \mathbf{q}^*) \models Ix\chi(\mathbf{x})[\mathbf{a}]$. $\mathbf{a} \in \mathfrak{O}_\beta^* \subseteq [\chi(\mathbf{x})]^{(\mathfrak{A}, \mathbf{q}^*)}$ is a basic open set of q_m^* . Thus by (**)

$$\mathfrak{O}_\beta^* \subseteq \mathfrak{O}_\beta \subseteq [\chi(\mathbf{x})]^{(\mathfrak{A}, \mathbf{q})}$$

for some $\mathfrak{O}_\beta \in q_m$. Whence $(\mathfrak{A}, \mathbf{q}) \models Ix\chi(\mathbf{x})[\mathbf{a}]$ which is a contradiction. Thus the lemma is shown.

Now we are able to prove the main completeness theorem.

THEOREM 8. *Let Σ be an $L(I^n)_{n \in \omega}$ theory. Then Σ is consistent with (A0)–(A8 $_{\varphi_\alpha}$), $\alpha \in J$, if and only if Σ has a complete topological model $(\mathfrak{B}, \mathbf{r})$ such that each φ_α , $\alpha \in J$, is continuous.*

PROOF (IF). Straightforward since (A0)–(A8 $_{\varphi_\alpha}$), $\alpha \in J$, are true in every complete topological model where the φ_α , $\alpha \in J$, are continuous.

(ONLY IF) Assume Σ is consistent with (A0)–(A8 $_{\varphi_\alpha}$), $\alpha \in J$. Then by Theorem 2 we have a topological model $(\mathfrak{A}, \mathbf{q})$ of Σ and (A0)–(A8 $_{\varphi_\alpha}$), $\alpha \in J$, which is generated by the definable open set. By repeated applications of Lemma 7 we obtain a topological model $(\mathfrak{B}, \mathbf{r})$ of Σ , (A0)–(A8 $_{\varphi_\alpha}$), $\alpha \in J$, and if $\mathbf{b} \in [\varphi(\mathbf{x})]^{(\mathfrak{B}, \mathbf{r})} \in r_n$ then there is a $\mathfrak{o}_1, \dots, \mathfrak{o}_n \in r_1$ such that $\mathbf{b} \in \prod_{i=1}^n \mathfrak{o}_i \subseteq [\varphi(\mathbf{x})]^{(\mathfrak{B}, \mathbf{r})}$. Hence $(\mathfrak{B}, \mathbf{r})$ is complete.

If we omit (A8 $_{\varphi_\alpha}$), $\alpha \in J$, then we obtain the following interesting corollary.

COROLLARY 9. *Let Σ be an $L(I^n)_{n \in \omega}$ theory. Then Σ is consistent with (A0)–(A7) if and only if Σ has a complete topological model.*

PROOF. This is a direct application of Theorem 8.

COROLLARY 10 (COMPACTNESS THEOREM). *Let Σ be an $L(I^n)_{n \in \omega}$ theory. Then Σ has a complete topological model where each φ_α , $\alpha \in J$, is continuous if and only if every finite subset of Σ has a complete topological model where φ_α , $\alpha \in J$, is continuous.*

PROOF. An easy application of the main completeness theorem.

COROLLARY 11. *The set of $L(I^n)_{n \in \omega}$ sentences valid in every complete topological model (with φ_α , $\alpha \in J$, continuous) is recursively enumerable in the signature of L and the set $\{\varphi_\alpha \mid \alpha \in J\}$.*

PROOF. Theorem 8 shows that a sentence is provable from $(A0)-(A8_{\varphi_\alpha})$, $\alpha \in J$, if and only if it is valid, so we are done.

We can now state a Löwenheim-Skolem theorem for complete topological models with continuous functions whose proof is analogous to the author's for $L(Q^n)_{n \in \omega}$ in [16].

THEOREM 12. (a) Let $(\mathfrak{U}, \mathbf{q})$ be an infinite complete topological model where each φ_α , $\alpha \in J$, is continuous. Then for any $\aleph \geq |L| + |A|$ there is a complete topological model $(\mathfrak{B}, \mathbf{r})$ such that $(\mathfrak{U}, \mathbf{q}) < (\mathfrak{B}, \mathbf{r})$, $|B| = \aleph$, and each φ_α is continuous in $(\mathfrak{B}, \mathbf{r})$.

(b) Let $(\mathfrak{U}, \mathbf{q})$ be a complete topological model where each φ_α , $\alpha \in J$, is continuous. Then for any $|L| \leq \aleph \leq |A|$ there is a complete topological model $(\mathfrak{B}, \mathbf{r}) < (\mathfrak{U}, \mathbf{q})$ such that $|B| = \aleph$, and each φ_α , $\alpha \in J$, is continuous in $(\mathfrak{B}, \mathbf{r})$.

We finish the applications by presenting two other theorems whose proofs are adaptations to $L(I^n)_{n \in \omega}$ (as in the main completeness theorem) of the proof for $L(Q^n)_{n \in \omega}$ in [16].

THEOREM 13. Let Σ be an $L(I^n)_{n \in \omega}$ theory and κ an infinite regular cardinal. Then Σ is consistent with $(A0)-(A8_{\varphi_\alpha})$, $\alpha \in J$, and $Ixy(x \neq y) \leftrightarrow x \neq y$ if and only if Σ has a 0-dimensional normal complete topological model of cardinality κ where each φ_α , $\alpha \in J$, is continuous (complete in the model theoretic sense).

The proof is a straightforward adaptation of the proof in [16] using the following lemma.

LEMMA. Let Σ be an $L(I^n)$ theory consistent with $(A0)-(A8_{\varphi_\alpha})$, $\alpha \in J$. Then if Σ is consistent with $\forall xy(Ixy(x \neq y) \leftrightarrow x \neq y)$, $\forall x(\neg \psi(x) \leftrightarrow Ix \neg \psi(x))$, $\forall x(\neg \varphi(x) \leftrightarrow Ix \neg \varphi(x))$, and $\neg \exists x(\psi(x) \wedge \varphi(x))$, i.e., ψ and φ define disjoint closed sets, then $\forall x(\psi(x) \rightarrow U^{\psi, \varphi}(x)) \forall x(\varphi(x) \rightarrow U^{\psi, \varphi}(x))$, $\forall x(Ix U^{\psi, \varphi}(x) \leftrightarrow U^{\psi, \varphi}(x))$, and $\forall x(Ix \neg U^{\psi, \varphi}(x) \leftrightarrow U^{\psi, \varphi}(x))$ are consistent with Σ and $(A0)-(A8_{\varphi_\alpha})$, $\alpha \in J$. Here $U^{\psi, \varphi}(x)$ is a new one-place predicate symbol. The conclusion means that $U^{\psi, \varphi}$ and $\neg U^{\psi, \varphi}$ define complete open sets which separate ψ and φ .

PROOF. We need only show the lemma for countable Σ ; then using the compactness theorem we obtain it for all Σ . Let $(\mathfrak{U}, \mathbf{q})$ be a countable topological model of Σ and $(A0)-(A8_{\varphi_\alpha})$, $\alpha \in J$, where the q_i are generated by the definable open sets.

As in the proof of Lemma 7 we want to obtain $(\mathfrak{U}, \mathbf{q}^*)$ from $\mathfrak{U}$, $A - \mathfrak{U}$ and $(\mathfrak{U}, \mathbf{q})$ such that $[\psi]^{(\mathfrak{U}, \mathbf{q}^*)} \subseteq \mathfrak{U}$ and $[\varphi]^{(\mathfrak{U}, \mathbf{q}^*)} \subseteq A - \mathfrak{U}$ and $(\mathfrak{U}, \mathbf{q}) < (\mathfrak{U}, \mathbf{q}^*)$.

We will define, as in Lemma 7, $\mathfrak{U}$ and $A - \mathfrak{U}$ by induction. To do this, suppose we have defined $r_1, \dots, r_{f(k)}$ for $\mathfrak{U}$ and $s_1, \dots, s_{f(k)}$ for $A - \mathfrak{U}$ up to stage k . Now we will define $r_{f(k)+1}, \dots, r_{f(k+1)}$ for $\mathfrak{U}$ and $s_{f(k)+1}, \dots, s_{f(k+1)}$ for $A - \mathfrak{U}$.

Again without loss of generality we have $\mathfrak{O}_{\beta_{k+1}}^* = \bigcap_{j=1}^n \varphi_{\beta_j}(\prod B_k)$ where $B_1 = \mathfrak{U}$ and $B_2 = A - \mathfrak{U}$. We will define the r 's and s 's such that $r_i \neq s_j$ for $1 \leq i \neq j \leq f(k+1)$ and such that if

$$\mathfrak{O}_{\beta_{k+1}}^* \subseteq [\sigma_{k+1}(\mathbf{x})]^{(\mathfrak{U}, \mathbf{q}^*)}$$

then there is a $\mathcal{O}_\beta \in q_m$ such that

$$\mathcal{O}_{\beta_{k+1}}^* \subseteq \mathcal{O}_\beta \subseteq [\sigma(x)]^{(\mathfrak{A}, \mathbf{q}^*)}.$$

Define $\Delta = \{c \in A^{2n_\beta} \mid c(i) = c(j) \text{ for some } 1 \leq i \leq n_\beta \text{ and } n_\beta < j \leq 2n_\beta\}$. Δ is closed in A^{2n_β} by the axiom $\forall xy(Ixy(x \neq y) \leftrightarrow (x \neq y))$. Hence $A^{2n_\beta} - \Delta \cup [\varphi \vee \psi]^{(\mathfrak{A}, \mathbf{q})}$ is an open subset of A^{2n_β} . As in Lemma 7 take

$$C^* = A^{2n_\beta} - \Delta \cup [\varphi \cup \psi]^{(\mathfrak{A}, \mathbf{q})}$$

and form C^* , $\varphi * C^*$ and $\bigcap \varphi * C^*$ which are open subsets of A^{2n_β} and A , respectively. (If C^* is finite the proof is trivial, so we will assume it is infinite.)

If

$$\bigcap \varphi * C^* - [\sigma_{k+1}(x)]^{(\mathfrak{A}, \mathbf{q})} \neq \emptyset$$

then pick $r_{f(k)+1}, \dots, r_{f(k)+n_\beta}$ and $s_{f(k)+1}, \dots, s_{f(k)+n_\beta}$ so that

$$\bigcap_{i=1}^{n_\beta} \varphi_{\beta_i}(\langle r_{f(k)+i}, s_{f(k)+i} \rangle x \prod B_k) - [\sigma_{k+1}(x)]^{(\mathfrak{A}, \mathbf{q})} \neq \emptyset.$$

Otherwise let $r_{f(k)+i} = r_{f(k)}$ and $s_{f(k)+i} = s_{f(k)}$ for each $1 \leq i \leq n_\beta$. Set $f(k+1) = f(k) + n_\beta$ and we are done.

Let $\mathcal{U} = [\varphi]^{(\mathfrak{A}, \mathbf{q})} \cup \{r_i\}_{i \in \omega}$, and using the fact that $\mathcal{U} \cap \{s_i\}_{i \in \omega} = \emptyset$ we can easily prove as in Lemma 3.1.5 in [16] that $(\mathfrak{A}, \mathbf{q}) < (\mathfrak{A}, \mathbf{q}^*)$.

Note. Again the proof of Lemma 3.1.5 in [16] should be corrected along the lines of this proof since it is incorrect as published. The verification of $(\mathfrak{A}, \mathbf{q}) < (\mathfrak{A}, \mathbf{q}^*)$, however, is correct.

COROLLARY 14. *Let Σ be a countable $L(I^n)_{n \in \omega}$ theory. Then Σ is consistent with (A0)–(A8 $_{\varphi_\alpha}$), $\alpha \in I$, and $Ixy(x \neq y) \leftrightarrow x \neq y$ if and only if Σ has a second countable 0-dimensional metrizable complete topological model where each φ_α is continuous.*

PROOF. Use the fact that a second countable, regular and Hausdorff space is metrizable.

Let $L(I)$ be the sublanguage of $L(I^n)_{n \in \omega}$ in which the interior operator Ix is only applied to the single variable x .

We now study the interrelation of $L(I)$ theories and $L(I^n)_{n \in \omega}$ theories. The reason for this is that in $L(I^n)_{n \in \omega}$ we have a method of expressing the fact that a function is continuous in a product topology. It is thus natural to ask what conditions on functions (or relations) in an $L(I)$ theory Σ are necessary to insure that they can be interpreted as continuous functions in some $L(I^n)_{n \in \omega}$ theory extending Σ .

The following definition and theorem formalize this.

DEFINITION 15. $\varphi_\alpha(x_1, \dots, x_n, y_1, \dots, y_m)$, $\alpha \in J$, a collection of (n, m) -ary relations is called *$L(I)$ -continuous* (in Σ) if and only if

$$\bigwedge_{i=1}^m Iy_i \psi_i(y_i) \wedge \theta(t, y) \rightarrow Iz \exists y \left(\bigwedge_{i=1}^m \psi_i(y_i) \wedge \theta(t, y) \right)$$

is consistent with Σ , where the $\varphi_i(y_i)$ are arbitrary formulas of $L(I)$, $\mathbf{t} \in ((\sigma \circ \mathbf{x})/z)$ and $\sigma: n+1 \rightarrow n+1$ (where θ is an arbitrary composition of the φ_α , i.e., $\theta \in WT$, and $((\sigma \circ \mathbf{x})/z)$ is the collection of k -tuples which are permuted by σ and then any number of them are replaced by z).

THEOREM 16. *Let T be an $L(I)$ theory and let $\varphi_\alpha(x_1, \dots, x_{n_\alpha}, y_1, \dots, y_{m_\alpha})$, $\alpha \in J$, be (n_α, m_α) -ary $L(I)$ -continuous relations. Then there is an $L(I^n)_{n \in \omega}$ theory T^* , such that $T \subseteq T^*$ and $(A0)-(A8_{\varphi_\alpha})$, $\alpha \in J$, are in T^* . (This is to say that we can find a complete topological model $(\mathfrak{A}, \mathbf{q})$ of T where each φ_α is continuous in the product topology.)*

Consider a group $(G, \cdot)$. Now take a topology τ on G . We call $(G, \cdot, \tau)$ a *topological group* if $^{-1}$ and $\cdot$ are continuous maps into G . Other definitions of topological-algebraic structures, e.g. a topological vector space, often appear in mathematics. Using Theorem 16 we are now able to give an $L(I)$ axiomatization of their $L(I)$ theories. For more details on topological groups, etc., see [5].

We formalize these comments in the following corollary.

COROLLARY 17. *Let T be an $L(I)$ theory. Then T has a topological group model if and only if T is consistent with the basic $L(I)$ axioms, group axioms, and $Iy\psi(y)[t] \rightarrow Ix\psi(t)$ where*

$$\mathbf{t} \in \left(\frac{y_{\sigma(1)}^{\varepsilon(1)} \cdot y_{\sigma(2)}^{\varepsilon(2)} \cdot \dots \cdot y_{\sigma(k)}^{\varepsilon(k)}}{x} \right),$$

$\sigma: k+1 \rightarrow k+1$ and $\varepsilon: k+1 \rightarrow \{1, -1\}$.

PROOF. These axioms for topological groups are just the definition of $L(I)$ -continuity for x^{-1} and

COROLLARY 18. *Let T be an $L(I)$ theory. Then T has a topological abelian group model if and only if T is consistent with the basic $L(I)$ axioms, abelian group axioms, $Iy\psi(y)[x^{-1}] \rightarrow Ix\psi(x^{-1})$, and $Iy\psi(y)[x^n \cdot y] \rightarrow Ix\psi(x^n \cdot y)$.*

2. Interpolation, definability and omitting types. In this section we will prove a Robinson-type joint-consistency theorem and an omitting types theorem for $L(I^n)_{n \in \omega}$.

THEOREM 19. *Let T_1, T_2 be $L_1(I^n)_{n \in \omega}, L_2(I^n)_{n \in \omega}$, theories, respectively, consistent with $(A0)-(A8_\varphi)$. Then if $T_1 \cap T_2$ is a complete $L_1 \cap L_2(I^n)_{n \in \omega}$ theory consistent with $(A0)-(A8_\varphi)$ then $T_1 \cup T_2$ is $L_1 \cup L_2(I^n)_{n \in \omega}$ consistent with $(A0)-(A8_\varphi)$.*

PROOF. Denote $L_1 \cap L_2$ by L and $T_1 \cap T_2$ by T . Let $(\mathfrak{A}, \mathbf{q})$ be a topological model of T_1 . Assume $\sigma(\mathbf{x}), \psi(\mathbf{x})$ are formulas of $L_1(I^n)_{n \in \omega}(\mathcal{A})$ such that σ defines an open set and ψ does not. Let $\mathbf{b} \in [\psi(\mathbf{x})]^{(\mathfrak{A}, \mathbf{q})} - [Ix\psi]^{(\mathfrak{A}, \mathbf{q})}$ and $\varphi(\mathbf{b}) \in \text{Th}_{L_1}((\mathfrak{A}, \mathbf{q}))$. Take $\chi(\mathbf{x})$ to be a formula of $L_2(I^n)_{n \in \omega}$ such that $\forall \mathbf{x}(Ix\chi(\mathbf{x}) \leftrightarrow \chi(\mathbf{x})) \in T_2$, i.e., χ is open, $\chi[\mathbf{b}] \in T_2 \cup T_L((\mathfrak{A}, \mathbf{q}))$, and take $a_1, \dots, a_n$ to be new constant symbols. Notice that $T_2' = T_2 \cup T_L((\mathfrak{A}, \mathbf{q}))$ is consistent since T is complete and consistent.

Form

$$\begin{aligned} T_1^\# &= \text{Th}_{L_1}((\mathfrak{A}, \mathbf{q})) \cup \{ \neg \psi[\mathbf{a}] \wedge \sigma[\mathbf{a}] \}, \\ T_2^\# &= T_2' \cup \{ \chi[\mathbf{a}] \}. \end{aligned}$$

Let $\text{Cn}(T_i^\#)$ be the set of consequences of $T_i^\#$ in $L(I^n)_{n \in \omega}(\mathcal{A} \cup \{a_1, \dots, a_n\})$.

We claim that $\text{Cn}(T_1^\#) \cup \text{Cn}(T_2^\#)$ is consistent. To prove this, suppose not; then for some θ we have that $\theta \in \text{Cn}(T_1^\#)$ and $\neg \theta \in \text{Cn}(T_2^\#)$.

That is to say,

$$\text{Th}((\mathfrak{A}, \mathbf{q})) \vdash (\neg \psi[\mathbf{a}] \wedge \sigma[\mathbf{a}]) \rightarrow \theta$$

and

$$T_2' \vdash \chi[\mathbf{a}] \rightarrow \neg \theta.$$

Replacing $a_1, \dots, a_n$ by $z_1, \dots, z_n$ and generalizing we obtain

$$\text{Th}_L((\mathfrak{A}, \mathbf{q})) \vdash \forall \mathbf{z}((\neg \psi[\mathbf{z}] \wedge \sigma[\mathbf{z}]) \rightarrow \theta[\mathbf{z}]),$$

or equivalently

$$\text{Th}_L((\mathfrak{A}, \mathbf{q})) \vdash \forall \mathbf{z}(\neg \theta[\mathbf{z}] \rightarrow (\psi[\mathbf{z}] \vee \neg \sigma[\mathbf{z}])),$$

and

$$T_2' \vdash \forall \mathbf{z}(\chi[\mathbf{z}] \rightarrow \neg \theta[\mathbf{z}]).$$

Using the monotonicity of the interior operator we show

$$\text{Th}_L((\mathfrak{A}, \mathbf{q})) \vdash \forall \mathbf{z}(I\mathbf{z} \neg \theta[\mathbf{z}] \rightarrow I\mathbf{z}(\psi \vee \neg \sigma)[\mathbf{z}])$$

and

$$T_2' \vdash \forall \mathbf{z}(I\mathbf{z}\chi[\mathbf{z}] \rightarrow I\mathbf{z} \neg \theta[\mathbf{z}]).$$

But $I\mathbf{z}\chi[\mathbf{z}] \rightarrow \chi[\mathbf{z}]$, so $T_2 \vdash \chi[\mathbf{z}] \rightarrow I\mathbf{z} \neg \theta[\mathbf{z}]$, and so

$$T_2' \vdash \chi[\mathbf{b}] \rightarrow (I\mathbf{z} \neg \theta[\mathbf{z}])[\mathbf{b}].$$

Using the consistency and completeness of $T_2' \cap L(I^n)_{n \in \omega}(\mathcal{A}) = \text{Th}_L((\mathfrak{A}, \mathbf{q}))$ we have that

$$\text{Th}_L((\mathfrak{A}, \mathbf{q})) \vdash (I\mathbf{z}\theta[\mathbf{z}])[\mathbf{b}] \rightarrow (I\mathbf{z})(\psi \vee \neg \sigma)[\mathbf{z}][\mathbf{b}]$$

so

$$\text{Th}_L((\mathfrak{A}, \mathbf{q})) \vdash (I\mathbf{z}(\psi \vee \neg \sigma)[\mathbf{z}])[\mathbf{b}].$$

Thus, using (A4), we get $\text{Th}_L((\mathfrak{A}, \mathbf{q})) \vdash (I\mathbf{z}(\sigma \wedge (\psi \vee \neg \theta))[\mathbf{z}])[\mathbf{b}]$. However, $\vdash \sigma \wedge (\psi \vee \neg \varphi) \leftrightarrow \psi$, hence $\text{Th}_L((\mathfrak{A}, \mathbf{q})) \vdash (I\mathbf{z}\psi[\mathbf{z}])[\mathbf{b}]$, a contradiction since $[I\mathbf{z}\psi[\mathbf{z}]]^{(\mathfrak{A}, \mathbf{q})} = [I\mathbf{x}\psi]^{(\mathfrak{A}, \mathbf{q})}$.

Thus $\text{Cn}(T_1^\#) \cup \text{Cn}(T_2^\#)$ is consistent. From this we easily conclude that $\Delta = \text{Cn}(T_1^\#) \cup \text{Cn}(T_2^\#) \cup T_2$ is consistent.

Let $(\mathfrak{B}, \mathbf{r})$ be a topological model of Δ generated by the definable open sets. Then $(\mathfrak{A}, \mathbf{q}) <_{L(I^n)_{n \in \omega}} (\mathfrak{B}, \mathbf{r})$ and $(\mathfrak{B}, \mathbf{r}) \models T_2$.

Notice for the following induction that $T_1^\# \cup \text{Th}_L((\mathfrak{B}, \mathbf{r}))$, $\text{Th}_L((\mathfrak{B}, \mathbf{r}))$ are consistent and $\text{Th}_L((\mathfrak{B}, \mathbf{r}))$ is complete.

Interchanging T_1 and T_2 in the construction and iterating ω times through all the triples of L_1 definable open and nonopen sets and the definable L_2 open sets and vice versa we produce an elementary chain $(\mathfrak{A}_i, \mathfrak{q}^i)$, $i \in \omega$, of models of T_1 , $(\mathfrak{B}_i, \mathfrak{r}^i)$, $i \in \omega$, of models of T_2 whose topologies are generated by definable open sets such that

$$\dots (\mathfrak{A}_i \upharpoonright L, \mathfrak{q}^i) < (\mathfrak{B}_i \upharpoonright L, \mathfrak{r}^i) < (\mathfrak{A}_{i+1} \upharpoonright L, \mathfrak{q}^{i+1}) < (\mathfrak{B}_{i+1} \upharpoonright L, \mathfrak{r}^{i+1}), \dots$$

forms an elementary chain. We also have that if $\psi(\mathbf{x})$ is a formula of $L_1(I^n)_{n \in \omega}(A_k)$ such that $[\psi(\mathbf{x})]^{(\mathfrak{A}_k, \mathfrak{q}^k)}$ is not open in q_n^k , $\emptyset \in q_n^k$ and $\emptyset^\# \in r_n^k$ then either

$$\emptyset \cap \emptyset^\# \subseteq [I\mathbf{x}\psi]^{(\mathfrak{A}_k, \mathfrak{q}^k)} \quad \text{or} \quad \emptyset \cap \emptyset^\# - [\psi(\mathbf{x})]^{(\mathfrak{A}_k, \mathfrak{q}^k)} \neq \emptyset.$$

Similarly for $L_2(I^n)_{n \in \omega}$ in place of $L_1(I^n)_{n \in \omega}$.

Let $(\mathfrak{A}^\omega, \mathfrak{q}^\omega)$ be the topological model generated by $\bigcup_{i \in \omega} (\mathfrak{A}_i, \mathfrak{q}^i)$ and $(\mathfrak{B}^\omega, \mathfrak{r}^\omega)$ be the topological model generated by $\bigcup_{i \in \omega} (\mathfrak{B}_i, \mathfrak{r}^i)$.

Because the q_n^i and r_n^i are generated by the definable open sets we see that $(\mathfrak{A}^\omega, \mathfrak{q}^\omega) \models T_1$, $(\mathfrak{B}^\omega, \mathfrak{r}^\omega) \models T_2$. We also know that

$$\mathfrak{A}^\omega \upharpoonright L = \mathfrak{B}^\omega \upharpoonright L. \quad (*)$$

Define $(\mathfrak{Q}, \mathfrak{p}) = (\mathfrak{A}^\omega \cup \mathfrak{B}^\omega, (r_1^\omega \cup q_1^\omega)^*, (r_2^\omega \cup q_2^\omega)^*, \dots)$ where $(r_i^\omega \cup q_i^\omega)^*$ is the topology generated by the definable sets of $r_i^\omega \cup q_i^\omega$ and $\mathfrak{A}^\omega \cup \mathfrak{B}^\omega$ is the $L_1 \cup L_2$ model formed from $\mathfrak{A}^\omega$ and $\mathfrak{B}^\omega$ using $(*)$.

We claim that

$$(\mathfrak{A}^\omega, \mathfrak{q}^\omega) < (\mathfrak{Q}, \mathfrak{p})$$

and

$$(\mathfrak{B}^\omega, \mathfrak{r}^\omega) < (\mathfrak{Q}, \mathfrak{p}).$$

This implies that $(\mathfrak{Q}, \mathfrak{p}) \models T_1 \cup T_2$.

We will show $(\mathfrak{A}^\omega, \mathfrak{q}^\omega) < (\mathfrak{Q}, \mathfrak{p})$ and the other equivalence follows by analogy. This assertion follows easily from the claim that

$$[\psi(\mathbf{x})]^{(\mathfrak{A}^\omega, \mathfrak{q}^\omega)} = [\psi(\mathbf{x})]^{(\mathfrak{Q}, \mathfrak{p})}$$

for each $\psi(\mathbf{x})$ a formula of $L_1(I^n)_{n \in \omega}(A^\omega)$ which we prove by induction on the complexity of $\psi(\mathbf{x})$.

The only difficult case is the $I\mathbf{x}$ case. Suppose $\psi(\mathbf{x}) = I\mathbf{x}\varphi$. Since $q_i^\omega \subseteq p_i$ we obtain that $[\psi(\mathbf{x})]^{(\mathfrak{A}^\omega, \mathfrak{q}^\omega)} \subseteq [\psi(\mathbf{x})]^{(\mathfrak{Q}, \mathfrak{p})}$. Take $\mathbf{a} \in [\psi(\mathbf{x})]^{(\mathfrak{Q}, \mathfrak{p})}$. By definition there are $\emptyset \in q_n^*$, $\emptyset^\# \in r_n^*$ such that $\mathbf{a} \in \emptyset \cap \emptyset^\# \subseteq [\varphi(\mathbf{x})]^{(\mathfrak{Q}, \mathfrak{p})}$.

By the construction of q_n^ω , r_n^ω we know that there are a χ of $L_1(I^n)_{n \in \omega}(A^\omega)$ and a δ of $L_2(I^n)_{n \in \omega}(A^\omega)$ such that

$$\mathbf{a} \in [\chi(\mathbf{x})]^{(\mathfrak{A}^\omega, \mathfrak{q}^\omega)} \cap [\delta(\mathbf{x})]^{(\mathfrak{B}^\omega, \mathfrak{r}^\omega)} \subseteq \emptyset \cap \emptyset^\#.$$

Hence, we know that by our construction

$$\mathbf{a} \in [\chi(\mathbf{x})]^{(\mathfrak{A}^\omega, \mathfrak{q}^\omega)} \cap [\varphi(\mathbf{x})]^{(\mathfrak{B}^\omega, \mathfrak{r}^\omega)} \subseteq [I\mathbf{x}\varphi(\mathbf{x})]^{(\mathfrak{A}^\omega, \mathfrak{q}^\omega)}$$

since otherwise by the construction of the elementary chain, $[\chi(\mathbf{x})]^{(\mathfrak{A}^\omega, \mathfrak{q}^\omega)} \cap [\delta(\mathbf{x})]^{(\mathfrak{B}^\omega, \mathfrak{r}^\omega)}$ would not be a subset of $[\delta(\mathbf{x})]^{(\mathfrak{A}^\omega, \mathfrak{q}^\omega)} = [\varphi(\mathbf{x})]^{(\mathfrak{Q}, \mathfrak{p})}$. Thus we are done because $(\mathfrak{Q}, \mathfrak{p}) \models T_1 \cup T_2$ and (A0)–(A8 _{φ}).

COROLLARY 20. $L(I^n)_{n \in \omega}$ with (A0)–(A8_φ) has an interpolation theorem.

PROOF. Straightforward since the Robinson joint-consistency theorem implies interpolation for compact logics.

COROLLARY 21. $L(I^n)_{n \in \omega}$ with (A0)–(A8_φ) has a Beth definability theorem.

PROOF. Again the Robinson joint-consistency theorem implies the definability theorem.

THEOREM 22. $L(I)$ with (A0)–(A4) has a Robinson joint-consistency theorem.

PROOF. The proof is the same as the $L(I^n)_{n \in \omega}$ case. In place of I^n just use I^1 .

COROLLARY 23. $L(I)$ with (A0)–(A4) has an interpolation and a Beth definability theorem.

PROOF. Use Theorem 22.

REMARK. In [15] we show an interpolation theorem for a system weaker than $L(I)$. Also our methods apply to give an interpolation theorem for “ideal models” (see [0]) which was done independently by S. Shelah.

We now will state and prove an omitting types theorem for $L(I^n)_{n \in \omega}$ with (A0)–(A8_φ).

DEFINITION 24. A set of sentences Γ topologically omits $\Sigma(x_1, \dots, x_n)$ if and only if $\Gamma \cup \{(A0)–(A8_\varphi)\}$ has a model which omits Σ .

THEOREM 25. Let Γ be a countable set of sentences of $L(I^n)_{n \in \omega}$ and $\Sigma_n(y_{n_1}, \dots, y_{n_{r_n}})$, $n \in \omega$, be sets of formulas of $L(I^n)_{n \in \omega}$. If Γ is consistent with (A0)–(A8_φ) and Γ topologically omits each Σ_n , the Γ has a topological model where each φ is continuous and omits each Σ_n .

PROOF. By Theorem 5 we obtain a countable weak model $(\mathfrak{A}, q)$ of Γ and (A0)–(A8_φ) which omits each Σ_n . If we take $q_k^\#$ to be the topology generated by the definable subsets of q_k , it is straightforward to see as in Theorem 2 that $(\mathfrak{A}, q) < (\mathfrak{A}, q^\#)$.

We actually proved in the proof of the main completeness theorem that for a countable topological model $(\mathfrak{A}, q^\#)$ there is a complete model $(\mathfrak{A}, r)$ such that $(\mathfrak{A}, r) < (\mathfrak{A}, q^\#)$. Hence $(\mathfrak{A}, r)$ models Γ , each φ is continuous, and it omits each Σ_n , $n \in \omega$.

REMARK. We would like to point out that we showed in [14] that $L(I^n)_{n \in \omega}$ cannot have an isomorphic ultrapowers theorem so our Theorem 19 is sharp.

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