PARAMETRIZATIONS OF G_δ -VALUED MULTIFUNCTIONS

BY

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ABSTRACT. Let T, X be Polish spaces, $\mathcal{T}$ a countably generated sub- σ -field of $\mathcal{B}_T$, the Borel σ -field of T , and $F: T \rightarrow X$ a multifunction such that $F(t)$ is a G_δ in X for each $t \in T$. F is $\mathcal{T}$ -measurable and $\text{Gr}(F) \in \mathcal{T} \otimes \mathcal{B}_X$, where $\text{Gr}(F)$ denotes the graph of F . We prove the following three results on F .

(I) There is a map $f: T \times \Sigma \rightarrow X$ such that for each $t \in T$, $f(t, \cdot)$ is a continuous, open map from Σ onto $F(t)$ and for each $\sigma \in \Sigma$, $f(\cdot, \sigma)$ is $\mathcal{T}$ -measurable, where Σ is the space of irrationals.

(II) The multifunction F is of Souslin type.

(III) If X is uncountable and $F(t)$, $t \in T$, are all dense-in-itself then there is a $\mathcal{T} \otimes \mathcal{B}_X$ -measurable map $f: T \times X \rightarrow X$ such that for each $t \in T$, $f(t, \cdot)$ is a Borel isomorphism of X onto $F(t)$.

1. Introduction. The object of this paper is to study G_δ -valued multifunctions. We take T, X to be Polish spaces, $\mathcal{T}$ a countably generated sub- σ -field of $\mathcal{B}_T$, the Borel σ -field of T , and $F: T \rightarrow X$ a multifunction such that F is $\mathcal{T}$ -measurable, $\text{Gr}(F) \in \mathcal{T} \otimes \mathcal{B}_X$ and $F(t)$ is a G_δ in X for each $t \in T$. Definitions and notation are given in §2. G_δ -valued multifunctions arise in the study of C^* -algebras, group representations, etc. ([5], [12]).

In [15], the existence of a $\mathcal{T}$ -measurable selector for F is established and this article can be viewed as a sequel to [15]. Having proved the existence of a measurable selector for F , several questions arise. Can we express $\text{Gr}(F)$ as a union of the graphs of measurable selectors for F ? If yes, can we get these graphs to be, moreover, disjoint? Naturally, for the second problem, $F(t)$, $t \in T$, must all be of the same cardinality.

We approach the first problem in more than one way. In §3, we prove a representation theorem for such multifunctions of the kind recently obtained by Ioffe [4] and Srivastava [14] for closed valued multifunctions. In §4, we prove that these multifunctions are of Souslin type in the sense of Leese [8]. This gives us a very important relationship between F and closed valued multifunctions and enables us to answer our question in the affirmative.

We consider the second problem in §5. By a very old and classical result of Luzin ([9, p. 252], [10]), the answer to this question is "yes" for countable-valued F . In the case, $F(t)$, $t \in T$, are all uncountable, we prove a parametrization theorem, analogous to the one recently obtained by Mauldin [11], for F .

2. Preliminaries. The set of positive integers will be denoted by N . S will denote the set of all finite sequences of positive integers, including the empty sequence e .

Received by the editors November 15, 1978 and, in revised form, May 11, 1979.

AMS (MOS) subject classifications (1970). Primary 04A15, 54C60, 54C65; Secondary 02K30.

Key words and phrases. Multifunctions, selectors, parametrizations, representations, uniformizations.

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0002-9947/80/0000-0158/\$03.50

For each $k \geq 0$, we denote by S_k the set of elements of S of length k . For $s \in S$, $|s|$ will denote the length of s and if $i \leq |s|$ is a positive integer, s_i will denote the i th co-ordinate of s . If $n \in N$, sn will denote the catenation of s and n . We put $\Sigma = N^N$. Endowed with the product of discrete topologies on N , Σ becomes a homeomorph of the space of irrationals. For $\sigma \in \Sigma$ and $k \in N$, σ_k will denote the k th co-ordinate of σ and $\sigma|k = (\sigma_1, \dots, \sigma_k)$. If $k = 0$, $\sigma|k = e$. If $s \in S$, Σ_s will denote the set $\{\sigma \in \Sigma: \sigma|k = s\}$.

D will denote the set of all finite sequences of 0's and 1's, including the empty sequence e . C will denote the set $\{0, 1\}^N$. Endowed with the product of discrete topologies on $\{0, 1\}$, it becomes a homeomorph of the Cantor set. For $d \in D$, $k \geq 0$, $i \in \{0, 1\}$ and $\varepsilon \in C$, d_k , ε_k , $\varepsilon|k$, $|d|$ and d_i are similarly defined.

Let $(X, \mathfrak{A})$ and $(Y, \mathfrak{B})$ be measurable spaces. We denote by $\mathfrak{A} \otimes \mathfrak{B}$ the product of the σ -fields $\mathfrak{A}$ and $\mathfrak{B}$. We say that $\mathfrak{A}$ is *countably generated* if there exist subsets A_n , $n \geq 1$, of X such that $\mathfrak{A}$ is generated by $\{A_n: n \geq 1\}$. A nonempty set $A \in \mathfrak{A}$ is called an $\mathfrak{A}$ -atom if $A \supseteq B \in \mathfrak{A} \Rightarrow B = A$ or $B = \emptyset$. If X is a metric space, $\mathfrak{B}_X$ will denote the Borel σ -field of X . If $E \subseteq X \times Y$ and $x \in X$, E^x will denote the set $\{y \in Y: (x, y) \in E\}$ and will be called the *section* of E at x . The projection maps from $X \times Y$ to X and from $X \times Y$ to Y will be denoted respectively by $\Pi_X^{X \times Y}$ and $\Pi_Y^{X \times Y}$, or simply by Π_X and Π_Y if there is no ambiguity.

A *multifunction* $F: T \rightarrow X$ is a function whose domain is T and whose values are nonempty subsets of X . A function $f: T \rightarrow X$ is called a *selector* for F if $f(t) \in F(t)$ for each $t \in T$. The set $\{(t, x) \in T \times X: x \in F(t)\}$ is denoted by $\text{Gr}(F)$ and is called the *graph* of F . If X is a metric space and $\mathfrak{T}$ is a σ -field on T , we say that F is $\mathfrak{T}$ -measurable if the set $\{t \in T: F(t) \cap V \neq \emptyset\} \in \mathfrak{T}$ for every open set V in X . If M is a subset of $T \times X$, we say that $C \subseteq M$ *uniformizes* M if sections of C are at most a singleton and $\Pi_T(C) = \Pi_T(M)$.

Let X, Y be topological spaces. We say that a function $f: X \rightarrow Y$ is *open* (resp. *closed*) if for every open (resp. closed) set W in X , $f(W)$ is open (resp. closed) in the range of f .

The rest of our terminology is from [6].

We now state some known results without proof which will be frequently used in the sequel.

LEMMA 2.1 ([2]). *Let T be a Polish space and $\mathfrak{T}$ a countably generated sub- σ -field of $\mathfrak{B}_T$. Let $A \in \mathfrak{B}_T$ be a union of $\mathfrak{T}$ -atoms. Then $A \in \mathfrak{T}$.*

LEMMA 2.2. *Let T, X be Polish spaces and $\mathfrak{T}$ a countably generated sub- σ -field of $\mathfrak{B}_T$. Suppose G is a subset of $T \times X$ such that $G \in \mathfrak{T} \otimes \mathfrak{B}_X$ and G' is a G_δ in X for every $t \in T$. Then for every closed subset A of X the set $\{t \in T: A \subseteq G'\} \in \mathfrak{T}$.*

PROOF. Let Y be a metric compactification of X . By a well-known result of Alexandrov and Hausdorff, X is a G_δ in Y . Consequently, $G \in \mathfrak{T} \otimes \mathfrak{B}_Y$ and G' is a G_δ in Y for each $t \in T$. Let $A \subseteq X$ be closed. Then it is easily verified that

$$\{t \in T: A \subseteq G'\} = T \setminus \Pi_T((T \times A) \cap ((T \times Y) \setminus G)).$$

By a result of Arsenin and Kunugui [1] (see also [13]) it follows that the set $\{t \in T:$

$A \subseteq G'\} \in \mathfrak{B}_T$. Further, this set is a union of $\mathfrak{T}$ -atoms. The result now follows from Lemma 2.1.

We now state a very useful result, which is proved in [15], for G_δ -valued multifunctions.

LEMMA 2.3. *Let T, X be Polish spaces and $\mathfrak{T}$ a countably generated sub- σ -field of $\mathfrak{B}_T$. Let $G \in \mathfrak{T} \otimes \mathfrak{B}_X$ and G^t be a G_δ in X for each $t \in T$. Then there exist sets $G_n \in \mathfrak{T} \otimes \mathfrak{B}_X$ such that G_n^t is open in X for $t \in T$ and $n \geq 1$ and $G = \bigcap_{n=1}^{\infty} G_n$.*

In the rest of the paper, T, X will denote arbitrary Polish spaces and $\mathfrak{T}$ a countably generated sub- σ -field of $\mathfrak{B}_T$. X will be given a complete metric such that $\text{diam}(X) < 1$. $\{V_n: n \geq 1\}$ will be a base for the topology of X such that $V_1 = X$. $F: T \rightarrow X$ will denote a multifunction such that F is $\mathfrak{T}$ -measurable, $\text{Gr}(F) \in \mathfrak{T} \otimes \mathfrak{B}_X$ and $F(t)$ is a G_δ in X for each $t \in T$. G will denote the graph of F and $G_n, n \geq 1$, will be a nonincreasing sequence of sets in $\mathfrak{T} \otimes \mathfrak{B}_X$ such that G_n^t is open for $t \in T$ and $n \geq 1$ and $G = \bigcap_{n=1}^{\infty} G_n$. The existence of such a sequence of sets is ensured by Lemma 2.3.

3. A representation theorem.

LEMMA 3.1. *Let X be compact. Then for each $t \in T$, there is a system $\{n_s^t: s \in S\}$ of positive integers such that for $s \in S_k, k \geq 0$, and $t \in T$,*

- (i) *the map $t' \rightarrow n_s^{t'}$, defined on T , is $\mathfrak{T}$ -measurable,*
- (ii) *$\text{diam}(V_{n_s^t}) < 2^{-k}$,*
- (iii) *$\bar{V}_{n_m^t} \subseteq G_{k+1}^t \cap V_{n_s^t}, m \geq 1$,*
- (iv) *$G^t \cap V_{n_s^t} \neq \emptyset$,*
- (v) *$G^t \subseteq V_{n_s^t}$,*
- (vi) *$G^t \cap V_{n_s^t} \subseteq \bigcup_{m=1}^{\infty} V_{n_m^t}$.*

PROOF. For each $t \in T$, we define $n_s^t, s \in S$, by induction on $|s|$. We define $n_e^t = 1, t \in T$. The above conditions are clearly satisfied for $s = e$. Suppose $n_s^t, t \in T$, are defined satisfying (i)–(vi) for every $s \in \bigcup_{i \leq k} S_i$, for some $k \geq 0$. Fix an $s \in S_k$. We define $n_{sm}^t, t \in T, m \in N$, by induction on m . We first make a simple observation. Let $W \subseteq X$ be closed and $t \in T$. Then

$$W \subseteq G_{k+1}^t \cap V_{n_s^t} \Leftrightarrow (\exists l \in N) (n_s^t = l \text{ and } W \subseteq G_{k+1}^t \cap V_l).$$

By the induction hypothesis and Lemma 2.2, it follows that the set

$$\{t \in T: W \subseteq G_{k+1}^t \cap V_{n_s^t}\} \in \mathfrak{T}.$$

For $m \geq 1$, let

$$T_m^0 = \emptyset \quad \text{if } \text{diam}(V_m) \geq 2^{-(k+1)};$$

$$= \{t \in T: G^t \cap V_m \neq \emptyset, \bar{V}_m \subseteq G_{k+1}^t \cap V_{n_s^t} \text{ and}$$

$$(\forall l < m) (\text{diam}(V_l) < 2^{-(k+1)} \Rightarrow (G^t \cap V_l = \emptyset \text{ or } \bar{V}_l \not\subseteq G_{k+1}^t \cap V_{n_s^t}))\}$$

if $\text{diam}(V_m) < 2^{-(k+1)}$.

As F is $\mathfrak{T}$ -measurable, by the above observation, it follows that the sets $T_m^0, m \geq 1$,

belong to $\mathfrak{T}$. Also, these are pairwise disjoint and $T = \bigcup_{m=1}^{\infty} T_m^0$. We define $n_{s1}' = m$ if $t \in T_m^0$. Clearly, the map $t \rightarrow n_{s1}'$ is $\mathfrak{T}$ -measurable. Suppose for some $p \in N$, maps $t \rightarrow n_{si}', i \leq p$, have been defined to be $\mathfrak{T}$ -measurable. For $m \geq 1$, let

$$T_m^p = \emptyset \quad \text{if } \text{diam}(V_m) \geq 2^{-(k+1)};$$

$$= \left\{ t \in T: n_{sp}' < m, G^t \cap V_m \neq \emptyset, \bar{V}_m \subseteq G_{k+1}' \cap V_{n_s'}, \text{ and } \right.$$

$$\left. (\forall l < m) \left(\text{diam}(V_l) < 2^{-(k+1)} \Rightarrow \left(n_{sp}' \geq l \text{ or } G^t \cap V_l = \emptyset \right. \right. \right.$$

$$\left. \left. \left. \text{or } \bar{V}_l \not\subseteq G_{k+1}' \cap V_{n_s'} \right) \right) \right\}$$

$$\text{if } \text{diam}(V_m) < 2^{-(k+1)}.$$

It is easily checked that the sets T_m^p , $m \geq 1$, belong to $\mathfrak{T}$ and are pairwise disjoint. We define

$$n_{s,p+1}' = m \quad \text{if } t \in T_m^p,$$

$$= n_{sp}' \quad \text{if } t \in T \setminus \bigcup_{m=1}^{\infty} T_m^p.$$

The definition of n_s' , $s \in S$, $t \in T$, is complete. That the conditions (i)–(v) are satisfied follows immediately from the definitions of n_s' , $s \in S$, $t \in T$. To check (vi) note that $G^t \cap V_{n_s'} \subseteq G_{k+1}' \cap V_{n_s'}$ and $G_{k+1}' \cap V_{n_s'}$ is open.

THEOREM 3.2. *There is a map $f: T \times \Sigma \rightarrow X$ such that for each $t \in T$, $f(t, \cdot)$ is a continuous, open map from Σ onto $F(t)$ and for each $\sigma \in \Sigma$, $f(\cdot, \sigma)$ is $\mathfrak{T}$ -measurable.*

PROOF. Without loss of generality, we assume that X is a compact metric space. For each $t \in T$, we get a system $\{n_s': s \in S\}$ of positive integers satisfying conditions (i)–(vi) of Lemma 3.1.

Let $f(t, \sigma)$ be the unique point of $\bigcap_k \bar{V}_{n_{\sigma|k}}'$, $t \in T$, $\sigma \in \Sigma$. By conditions (iii)–(vi) of Lemma 3.1, $f(t, \Sigma) = F(t)$, $t \in T$. By standard arguments we show that for each $t \in T$, $f(t, \cdot)$ is continuous and open. Let $U \subseteq X$ be open, $\sigma \in \Sigma$ and $t \in T$. Then

$$f(t, \sigma) \in U \Leftrightarrow \bigcap_k \bar{V}_{n_{\sigma|k}}' \subseteq U$$

$$\Leftrightarrow (\exists k \geq 1) (\exists l \geq 1) (n_{\sigma|k}' = l \text{ and } \bar{V}_l \subseteq U).$$

Therefore,

$$f(\cdot, \sigma)^{-1}(U) = \bigcup_k \bigcup_l \{t \in T: n_{\sigma|k}' = l\} \in \mathfrak{T},$$

where the inner union is taken over all l such that $\bar{V}_l = U$ and the outer union is over all k . It follows that $f(\cdot, \sigma)$ is $\mathfrak{T}$ -measurable for each $\sigma \in \Sigma$.

COROLLARY 1. *F admits a $\mathfrak{T}$ -measurable selector.*

COROLLARY 2. *There exist $\mathfrak{T}$ -measurable selectors $f_1, f_2, \dots$ for F such that for each $t \in T$, $\{f_n(t): n \geq 1\}$ is dense in $F(t)$.*

PROOF. Let $\sigma^1, \sigma^2, \dots$ be a countable dense set in Σ . Then, for each $t \in T$, $\{f(t, \sigma^n): n \geq 1\}$ is dense in $F(t)$. Put $f_n = f(\cdot, \sigma^n)$, $n \geq 1$.

REMARK 1. In [16], it is proved that Theorem 3.2 remains valid if the condition “ $f(t, \cdot)$ is open” is replaced by “ $f(t, \cdot)$ is closed”.

REMARK 2. Let Y be a Polish space and $h: T \times Y \rightarrow X$ be a map such that for each $t \in T$, $h(t, \cdot)$ is continuous and open and for each $y \in Y$, $h(\cdot, y)$ is $\mathfrak{T}$ -measurable. Define a multifunction $H: T \rightarrow X$ by $H(t) = h(t, Y)$, $t \in T$. By a result of Hausdorff [3], $H(t)$ is a G_δ in X for each $t \in T$. Let $\{y_n: n \geq 1\}$ be a countable dense set in Y . For $n \geq 1$, define $f_n: T \rightarrow X$ by $f_n(t) = h(t, y_n)$, $t \in T$. Let $V \subseteq X$ be open. Then

$$\{t \in T: H(t) \cap V \neq \emptyset\} = \bigcup_{n \geq 1} f_n^{-1}(V).$$

It follows that the multifunction H is $\mathfrak{T}$ -measurable. The question now arises: Is $\text{Gr}(H) \in \mathfrak{T} \otimes \mathfrak{B}_X$? We do not know the answer. In [16], it is proved that the answer to this question is ‘yes’ if the condition “ $h(t, \cdot)$ is open” is replaced by “ $h(t, \cdot)$ is closed”.

4. Multifunctions of Souslin type.

DEFINITION. Let $(L, \mathfrak{L})$ be a measurable space and Z a metric space. A multifunction $H: L \rightarrow Z$ is said to be of *Souslin type* if there is a Polish space P , a continuous map $\beta: P \rightarrow Z$ and a $\mathfrak{L}$ -measurable, closed-valued multifunction $W: L \rightarrow P$ such that $H(t) = \beta(W(t))$, for each $t \in L$.

REMARK 1. Our definition of multifunctions of Souslin type is slightly different from the one given in [8].

REMARK 2. By a representation theorem for closed valued multifunctions proved in [14], we get the following. *If $(L, \mathfrak{L})$ is a measurable space, Z a metric space and $H: L \rightarrow Z$ a multifunction of Souslin type then there is a map $h: L \times \Sigma \rightarrow Z$ such that for each $t \in L$, $h(t, \cdot)$ is a continuous map from Σ onto $H(t)$ and for each $\sigma \in \Sigma$, $h(\cdot, \sigma)$ is $\mathfrak{L}$ -measurable.*

Now we prove the main result of this section.

THEOREM 4.1. *The multifunction F is of Souslin type.*

PROOF. We first assume that X is a compact, zero-dimensional metric space and basic open sets $V_1, V_2, \dots$ are clopen. By Lemma 2.2, the sets $T_{nm} = \{t \in T: V_m \subseteq G'_n\}$, $m \geq 1$, $n \geq 1$, belong to $\mathfrak{T}$. As G'_n is open for $t \in T$ and $n \geq 1$, $G_n = \bigcup_{m=1}^{\infty} (T_{nm} \times V_m)$. Put $P = X \times \Sigma$ and $\beta = \Pi_X$. Let

$$B = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} (T_{nm} \times V_m \times \Sigma_m^n),$$

where $\Sigma_m^n = \{\sigma \in \Sigma: \sigma_n = m\}$, $n \geq 1$, $m \geq 1$. Define $W: T \rightarrow P$ by $W(t) = B'$, $t \in T$, so that $W(t) = \bigcap \bigcup (V_m \times \Sigma_m^n)$, where the inner union is taken over all $m \geq 1$ such that $t \in T_{nm}$ and the outer intersection is over all n . For each $n \geq 1$, $\{V_m \times \Sigma_m^n: m \geq 1\}$ is a discrete family of closed sets in P . It follows that the inner union is closed in P for each n . Therefore, $W(t)$ is closed in P for each $t \in T$. Also, it is easily checked that $\beta(W(t)) = F(t)$, $t \in T$.

We now check that W is $\mathfrak{T}$ -measurable. Let $i \in N$ and $s \in S_k$, $k > 0$. It is enough to show that

$$\Pi_T^{T \times P}((T \times V_i \times \Sigma_s) \cap B) \in \mathcal{T}.$$

Now,

$$\begin{aligned} (T \times V_i \times \Sigma_s) \cap B &= \bigcap_{n \geq 1} \bigcup_{m \geq 1} (T_{nm} \times (V_m \cap V_i) \times (\Sigma_m^n \cap \Sigma_s)) \\ &= \left(\bigcap_{j=1}^k (T_{js_j} \times (V_{s_j} \cap V_i) \times \Sigma_s) \right) \\ &\quad \cap \left(\bigcap_{n \geq k} \bigcup_{m \geq 1} (T_{nm} \times (V_m \cap V_i) \times (\Sigma_m^n \cap \Sigma_s)) \right) \\ &= \left(\bigcap_{j=1}^k (T_{js_j} \times P) \right) \\ &\quad \cap \left(\bigcap_{n \geq k} \bigcup_{m \geq 1} \left(T_{nm} \times \left(V_m \cap V_i \cap \bigcap_{j=1}^k V_{s_j} \right) \times (\Sigma_m^n \cap \Sigma_s) \right) \right). \end{aligned}$$

Hence,

$$\begin{aligned} \Pi_T^{T \times P}((T \times V_i \times \Sigma_s) \cap B) &= \Pi_T^{T \times P} \left[\left(\bigcap_{j=1}^k (T_{js_j} \times P) \right) \right. \\ &\quad \left. \cap \left(\bigcap_{n \geq k} \bigcup_{m \geq 1} \left(T_{nm} \times \left(V_m \cap V_i \cap \bigcap_{j=1}^k V_{s_j} \right) \times (\Sigma_m^n \cap \Sigma_s) \right) \right) \right] \\ &= \bigcap_{j=1}^k T_{js_j} \cap \Pi_T^{T \times X} \left(\bigcap_{n \geq k} \bigcup_{m \geq 1} \left(T_{nm} \times \left(V_m \cap V_i \cap \bigcap_{j=1}^k V_{s_j} \right) \right) \right) \\ &= \bigcap_{j=1}^k T_{js_j} \cap \Pi_T^{T \times X} \left(\left(T \times \left(V_i \cap \bigcap_{j=1}^k V_{s_j} \right) \right) \cap \left(\bigcap_{n \geq k} \bigcup_{m \geq 1} (T_{nm} \times V_m) \right) \right) \\ &= \bigcap_{j=1}^k T_{js_j} \cap \Pi_T^{T \times X} \left(\left(T \times \left(V_i \cap \bigcap_{j=1}^k V_{s_j} \right) \right) \cap G \right). \end{aligned}$$

As F is $\mathcal{T}$ -measurable, it follows that W is $\mathcal{T}$ -measurable.

Now, let X be a zero-dimensional Polish space. Let Y be a zero-dimensional compact metric space containing a homeomorph of X . We consider F as a multifunction with values subsets of Y . By the previous case, we get a Polish space Q , a continuous map $g: Q \rightarrow Y$ and a $\mathcal{T}$ -measurable, closed set-valued multifunction $H: T \rightarrow Q$ such that $F(t) = g(H(t))$, $t \in T$. Put $P = g^{-1}(X)$ and β the restriction of g to P . As X is a G_δ in Y , P is Polish. Note that $H(t) \subseteq P$, $t \in T$. Put $W = H$.

Finally, let X be an arbitrary Polish space. Let $g: \Sigma \rightarrow X$ be a continuous, open and onto map. Define a multifunction $H: T \rightarrow \Sigma$ by $H(t) = g^{-1}(F(t))$, $t \in T$. Clearly, $H(t)$ is a G_δ in Σ for each $t \in T$. Let $U \subseteq \Sigma$ be open, then

$$\{t \in T: H(t) \cap U \neq \emptyset\} = \{t \in T: F(t) \cap g(U) \neq \emptyset\}.$$

As F is $\mathfrak{T}$ -measurable and g open, H is $\mathfrak{T}$ -measurable. To show $\text{Gr}(H) \in \mathfrak{T} \otimes \mathfrak{B}_\Sigma$, we define $h: T \times \Sigma \rightarrow T \times X$ by

$$h(t, \sigma) = (t, g(\sigma)), \quad t \in T, \sigma \in \Sigma.$$

Then h is continuous and $\text{Gr}(H) = h^{-1}(G)$ so that $\text{Gr}(H)$ is Borel in $T \times \Sigma$. Note that whenever t and t' belong to the same $\mathfrak{T}$ -atoms, $G' = G'$ and consequently, $(\text{Gr}(H))' = (\text{Gr}(H))'$. This implies that $\text{Gr}(H)$ is a union of $\mathfrak{T} \otimes \mathfrak{B}_\Sigma$ -atoms. Therefore, by Lemma 2.1, $\text{Gr}(H) \in \mathfrak{T} \otimes \mathfrak{B}_\Sigma$. By the previous case, we get a Polish space P , a closed set-valued, $\mathfrak{T}$ -measurable multifunction $W: T \rightarrow P$ and a continuous map $f: P \rightarrow \Sigma$ such that $H(t) = f(W(t))$, $t \in T$. Put $\beta = g \circ f$. The desired properties are easy to verify.

REMARK. A close examination of the various cases in the above proof reveals that the map $\beta: P \rightarrow X$ is obtained to be continuous, open and onto.

5. Decomposition of $\text{Gr}(F)$ into graphs of measurable selectors. In this section, X and $F(t)$, $t \in T$, are all assumed to be uncountable. We prove

THEOREM 5.1. *If for every $t \in T$, $F(t)$ is dense-in-itself then there is a $\mathfrak{T} \otimes \mathfrak{B}_X$ -measurable map $f: T \times X \rightarrow X$ such that for each $t \in T$, $f(t, \cdot)$ is a Borel isomorphism of X onto $F(t)$.*

This result is analogous to a result of Mauldin [11] and we follow some of his ideas. We first show by an example that the condition “ $F(t)$ is dense in itself” cannot be dropped from Theorem 5.1.

EXAMPLE. Let X be an uncountable Polish space containing a countable, dense, open set U . (Union of the Cantor set and the mid-points of the removed intervals is such a Polish space.) Let $T = \Sigma$, $\mathfrak{T} = \mathfrak{B}_T$ and let $Y = X \setminus U$. Let E be a G_δ set in $\Sigma \times Y$ such that $\Pi_\Sigma(E) = \Sigma$ and E does not admit a Borel uniformization [9, p. 265]. Let $G: Y \rightarrow Y$ be a continuous, onto map such that $g^{-1}(y)$ is uncountable for each $y \in Y$. Let $B = \{(t, y) \in T \times Y: (t, g(y)) \in E\}$. Then B is a G_δ set in $\Sigma \times Y$ such that every section of B is uncountable and B does not admit a Borel uniformization. Let $F: T \rightarrow X$ be defined by $F(t) = B' \cup U$, $t \in T$. It is clear that $F(t)$ is a G_δ in X for each $t \in T$ and $\text{Gr}(F) \in \mathfrak{T} \otimes \mathfrak{B}_X$. As U is dense in X , F is $\mathfrak{T}$ -measurable. If F satisfies the conclusions of Theorem 5.1, by a result of Mauldin [11], there is a Borel set $M \subseteq \text{Gr}(F)$ such that for each $t \in T$, M' is nonempty and perfect. Let $H = M \setminus (T \times U) \subseteq B$. The sections of H are nonempty and compact so that H , and therefore, B admits a Borel uniformization.

LEMMA 5.2. *Let X be compact. Then for each $t \in T$, there is a system $\{n'_d: d \in D\}$ of positive integers such that for $d \in D_k$, $k \geq 0$, and $t \in T$,*

- (i) *the map $t' \rightarrow n'_d$ is $\mathfrak{T}$ -measurable,*
- (ii) *$\text{diam}(V_{n'_d}) < 2^{-k}$,*
- (iii) *$d' \in D_k, d \neq d' \Rightarrow \bar{V}_{n'_d} \cap \bar{V}_{n'_{d'}} = \emptyset$,*
- (iv) *$F(t) \cap V_{n'_d} \neq \emptyset$,*
- (v) *$\bar{V}_{n'_d} \subseteq G'_{k+1} \cap V_{n'_d}, i = 0$ or $i = 1$.*

PROOF. We use induction on $|d|$. Let $n'_e = 1$ for all t . (i)–(v) are satisfied for

$d = e$. Suppose for some $k > 0$, n'_d is defined for all $d \in \bigcup_{l < k} D_l$ and for all $t \in T$ satisfying (i)–(v). Fix a $d \in D_k$. Put

$$T^m = \{t \in T: n'_d = m\}, \quad m \geq 1.$$

By the induction hypothesis, the sets T^m , $m \geq 1$, belong to $\mathfrak{T}$, are pairwise disjoint and $T = \bigcup_{m \geq 1} T^m$. Now, for any pair (u, v) of positive integers define T^m_{uv} , $m \geq 1$, as follows.

If $\text{diam}(V_u) < 2^{-(k+1)}$, $\text{diam}(V_v) < 2^{-(k+1)}$, $\bar{V}_u \cap \bar{V}_v = \emptyset$, $\bar{V}_u \subseteq V_m$, $\bar{V}_v \subseteq V_m$ then

$$T^m_{uv} = \{t \in T^m: \bar{V}_u \subseteq G'_{k+1}, \bar{V}_v \subseteq G'_{k+1}, V_u \cap F(t) \neq \emptyset \text{ and } V_v \cap F(t) \neq \emptyset\};$$

$$= \emptyset \quad \text{otherwise.}$$

As F is $\mathfrak{T}$ -measurable, by Lemma 2.2, $T^m_{uv} \in \mathfrak{T}$. Also $T^m = \bigcup_{(u,v)} T^m_{uv}$.

Let $\alpha: N \rightarrow N \times N$ be a one-one, onto function.

Put

$$S_i^m = T^m_{\alpha(i)} \quad \text{if } i = 1,$$

$$= T^m_{\alpha(i)} \setminus \bigcup_{j < i} T^m_{\alpha(j)} \quad \text{if } i > 1.$$

The sets S_i^m , $i \geq 1$, belong to $\mathfrak{T}$, are pairwise disjoint and $T^m = \bigcup_{i \geq 1} S_i^m$. We define

$$\left. \begin{aligned} n'_{d0} &= (\alpha(j))_1 \\ n'_{d1} &= (\alpha(j))_2 \end{aligned} \right\} \quad \text{if } t \in S_j^m \text{ for any } m.$$

This completes the definition of $\{n'_d: d \in D_{k+1}\}$, $t \in T$. It is easy to check that (i)–(v) are satisfied.

LEMMA 5.3. *There is a map $g: T \times C \rightarrow X$ such that for each $t \in T$, $g(t, \cdot)$ is a homeomorphism from C into $F(t)$ and for each $\varepsilon \in C$, $g(\cdot, \varepsilon)$ is $\mathfrak{T}$ -measurable.*

PROOF. Without loss of generality, we assume that X is a compact metric space. For each $t \in T$, we get a system $\{n'_d: d \in D\}$ of positive integers satisfying (i)–(v) of Lemma 5.2. Let $g(t, \varepsilon)$ be the unique point in $\bigcap_k \bar{V}_{n'_{t|k}}$, $t \in T$, $\varepsilon \in C$. By standard arguments, we show that for each $t \in T$, $g(t, \cdot)$ is a homeomorphism defined on C and by (v), it is into $F(t)$. Let $t \in T$, $\varepsilon \in C$ and $U \subseteq X$ be open. Then

$$g(t, \varepsilon) \in U \Leftrightarrow \bigcap_k \bar{V}_{n'_{t|k}} \subseteq U$$

$$\Leftrightarrow (\exists k > 1) (\bar{V}_{n'_{t|k}} \subseteq U)$$

$$\Leftrightarrow (\exists k > 1) (\exists l > 1) (\bar{V}_l \subseteq U) (n'_{t|k} = l)$$

so that

$$g(\cdot, \varepsilon)^{-1}(U) = \bigcup \bigcup (\{t \in T: n'_{t|k} = l\}) \in \mathfrak{T},$$

where the inner union is taken over all l such that $\bar{V}_l \subseteq U$ and the outer union is over all k . It follows that $g(\cdot, \epsilon)$ is $\mathcal{T}$ -measurable for every $\epsilon \in C$.

PROOF OF THEOREM 5.1. By Lemma 5.3, we get a map $g: T \times C \rightarrow X$ such that for each $t \in T$, $g(t, \cdot)$ is a homeomorphism from C into $F(t)$ and for all $\epsilon \in C$, $g(\cdot, \epsilon)$ is $\mathcal{T}$ -measurable. In particular, g is $\mathcal{T} \otimes \mathcal{B}_C$ -measurable [6, p. 378]. As X and C are Borel isomorphic we get a $\mathcal{T} \otimes \mathcal{B}_X$ -measurable map $h: T \times X \rightarrow X$ such that for each $t \in T$, $h(t, \cdot)$ is a Borel isomorphism from X into $F(t)$. Let $k: T \times X \rightarrow T \times X$ be defined by

$$k(t, x) = (t, h(t, x)), \quad t \in T, x \in X,$$

and let

$$B = \{(t, x) \in T \times X: x \in h(t, X)\}.$$

Then, $B \subseteq G$ and as k is one-one, Borel, B is Borel in $T \times X$. By Lemma 2.1, $B \in \mathcal{T} \otimes \mathcal{B}_X$. Also, $k: (T \times X, \mathcal{T} \otimes \mathcal{B}_X) \rightarrow (T \times X, \mathcal{T} \otimes \mathcal{B}_X)$ is a measurable map such that for each $t \in T$, $k(t, \cdot)$ is a Borel isomorphism from X into $\{t\} \times F(t)$. Now, we do a Schroeder-Bernstein type argument as done by Mauldin [11] and get a measurable map $\alpha: (T \times X, \mathcal{T} \otimes \mathcal{B}_X) \rightarrow (T \times X, \mathcal{T} \otimes \mathcal{B}_X)$ such that for each $t \in T$, $\alpha(t, \cdot)$ is a Borel isomorphism from X onto $\{t\} \times F(t)$. Put $f = \Pi_X \circ \alpha$.

COROLLARY 1. Let $M \subseteq T \times X$ be a Borel set such that for every $t \in \Pi_T(M)$, M' is dense in itself, and both a K_σ and a G_δ set in X . Then M is a union of $2^{\aleph_0}$ disjoint Borel uniformizations.

Under the hypothesis of the above corollary, Larman [7] proved that M contains $2^{\aleph_0}$ disjoint Borel uniformizations. The problem of the existence of $2^{\aleph_0}$ disjoint Borel uniformizations of M when its sections are not assumed to be dense in itself remains open.

ACKNOWLEDGEMENT. We are deeply grateful to Professor A. Maitra for his great help.

REFERENCES

1. W. Arsenin and A. Ljapunov, *Theory of A sets*, Uspehi Mat. Nauk (N.S), **5**, no. 5 (39) (1950), 45–108.
2. D. Blackwell, *On a class of probability spaces*, Proc. 3rd Berkeley Sympos. Math. Statist. and Prob., vol. 2, Univ. of California Press, Berkeley, Calif., 1956, pp. 1–6.
3. F. Hausdorff, *Über innere Abbildungen*, Fund. Math. **23** (1934), 279–291.
4. A. D. Ioffe, *Representation theorems for multifunctions and analytic sets*, Bull. Amer. Math. Soc. **84** (1978), 142–144.
5. R. R. Kallman and R. D. Mauldin, *A cross section theorem and an application to C^* -algebras*, Proc. Amer. Math. Soc. **69** (1978), 55–61.
6. K. Kuratowski, *Topology*. I, Academic Press, New York and London; Państwowe Wydawnictwo Naukowe, Warsaw, 1966.
7. D. G. Larman, *Projecting and uniformizing Borel sets with K_σ sections*. II, Mathematika **20** (1973), 233–246.
8. S. J. Leese, *Multifunctions of Souslin type*, Bull. Austral. Math. Soc. **11** (1974), 395–411.
9. N. Luzin, *Leçons sur les ensembles analytiques et leurs applications*, Gauthiers-Villars, Paris, 1930.
10. A. Maitra, *Analytic sets with countable sections*, Fund. Math. (to appear).

11. R. D. Mauldin, *Borel parametrizations*, Trans. Amer. Math. Soc. **250** (1979), 223–234.
12. D. Miller, *A selector for equivalence relations with G_δ orbits*, Proc. Amer. Math. Soc. **72** (1978), 365–369.
13. J. Saint-Raymond, *Boreliens A coupes K_σ* , Bull. Soc. Math. France **104** (1976), 389–400.
14. S. M. Srivastava, *A representation theorem for closed valued multifunctions*, Bull. Acad. Polon. Sci. (to appear).
15. _____, *Selection theorems for G_δ -valued multifunctions*, Trans. Amer. Math. Soc. **254** (1979), 283–293.
16. _____, *A representation theorem for G_δ -valued multifunctions*, Amer. J. Math. (to appear).

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