ON OSCILLATORY ELLIPTIC EQUATIONS ON MANIFOLDS

BY

A. BAIDER AND E. A. FELDMAN¹

ABSTRACT. In this note we investigate the possibility of oscillatory behavior for a second-order selfadjoint elliptic operators on noncompact Riemannian manifolds (M, g) . Let A be such an operator which is semibounded below and symmetric on $C_0^\infty(M) \subseteq L^2(M, d\mu)$ where $d\mu$ is a volume element on M . If φ is a C^∞ function such that $A\varphi = \lambda\varphi$, we would naively say that φ is oscillatory (and by extension λ is oscillatory if it possesses such an eigenfunction φ) if $M - \varphi^{-1}(0)$ has an infinite number of bounded connected components. For technical reasons this is not quite adequate for a definition. However, in §1 we give the usual definition of oscillatory which is a slight generalization of the one above. Let Λ_0 be the number below which this phenomenon cannot occur; Λ_0 is the oscillatory constant for the operator A . In that A is semibounded and symmetric on $C_0^\infty(M) \subseteq L^2(M, d\mu)$, A has a Friedrichs extension. Let Λ_c be the bottom of the continuous spectrum of the Friedrichs extension of A . Our main result is $\Lambda_0 = \Lambda_c$.

This is the exact analogue of a theorem first proved by Hartman and Putnam [6], when M is the real line. More recently, John Piepenbrink [7] generalized [6] to $\mathbf{R}^n$, requiring in the process, a technical condition (see [7, (3.1)]), which is, roughly, a restriction on the growth at infinity of the coefficients of the principal part of A . For example, if

$$A = - \sum \frac{\partial}{\partial x_i} \left[(1 + |x|^2)^{1/2} \right]^\alpha \frac{\partial}{\partial x_i},$$

then Piepenbrink's condition (3.1) holds if and only if $\alpha < 2$. In our generalization to arbitrary noncompact manifolds, no restrictions are placed upon the behavior of A at large distances. More significantly, the equality $\Lambda_0 = \Lambda_c$ is valid regardless of the "amount of topology" that M may possess. In [7] this type of difficulty is obviated by the fact that $\mathbf{R}^n$ is contractible.

The proof depends upon two theorems, Theorem A and Theorem B. Theorem B is a geometric characterization of Λ_c due to one of the authors in [1], while Theorem A is a deformation of domain argument in the spirit of [3], [8], [9]. The paper is arranged in the following manner. In §1 we give the necessary definitions, state Theorems A and B and show how the equality follows from Theorems A and B. The remainder of the paper is a proof of Theorem A. §2 is the analytic part of the proof of Theorem A. The core of the argument is the "crushed ice" estimates of Rauch and Taylor [8], [9], with which we prove Theorem A, provided we have a fine enough "even spacing lemma". This "even spacing lemma", is the geometric-

Received by the editors May 31, 1978 and, in revised form, April 10, 1979.

AMS (MOS) subject classifications (1970). Primary 58G99, 35J20.

¹This paper was partially supported by NSF grant #MCS-77-02757.

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0002-9947/80/0000-0160/\$03.50

topological part of the argument, as well as the most technical, and is exposed in §3. An interesting aspect of this lemma, is the need to use a triangulation of the manifold, which is a good approximation to the Riemannian structure.

Finally, we would like to thank I. Chavel for reading parts of the manuscript, and making many helpful suggestions.

1. Definitions and the statements of the results. Let M be a manifold possibly with boundary. Let g be a Riemannian metric on M , and let μ be a smooth volume element on M , i.e., $\mu = f dv$ where f is a smooth positive function, and where dv is the volume element g induces on M . For each smooth vector field v on M , we define the smooth functions $\operatorname{div}_\mu v$, the divergence of v with respect to μ , by the formula $d(i_v \mu) = (\operatorname{div}_\mu v)\mu$. If f is a C^∞ function defined on M , then

$$\operatorname{div}_\mu f v = f \operatorname{div}_\mu v + df(v).$$

Finally, if f is a smooth function we define the elliptic operator

$$\Delta_\mu f = -\operatorname{div}_\mu \operatorname{grad} f,$$

which is symmetric because

$$\int_M \varphi \Delta_\mu \psi d\mu = \int_M \Delta_\mu \varphi \psi d\mu = \int_M \operatorname{grad} \varphi \cdot \operatorname{grad} \psi d\mu,$$

where $\varphi, \psi \in C_0^\infty(\operatorname{Int} M)$ are smooth functions with compact support in the interior of M , and $\cdot$ is the fiberwise inner product given by g .

The general operator we want to consider is of the form $A = \Delta_\mu + c$ where c is a continuous function, such that $(A\varphi, \varphi) \geq l(\varphi, \varphi)$ for some $l > -\infty$, where $(\cdot, \cdot)$ denotes the inner product on $L^2(M, d\mu)$, and $\varphi \in C_0^\infty(M)$. Thus A is semibounded and symmetric.

Let U be an open set in M and let $C_0^\infty(U)$ be the C^∞ functions with compact support in U . We then define

$$\lambda_A(U) = \inf_{\varphi \in C_0^\infty(U), \varphi \neq 0} \frac{(A\varphi, \varphi)}{(\varphi, \varphi)}.$$

Note that if $U \subset V$, then $\lambda_A(U) \geq \lambda_A(V)$.

THEOREM A. *Let M_0 be a compact manifold with boundary $\partial M_0 = N$. Let g be a Riemannian metric on M_0 and $d\mu$ be a volume element on M_0 . Let c be a continuous function on M_0 , and set $A = -\operatorname{div}_\mu \operatorname{grad} + c$. Let $\Lambda = \lambda_A(\operatorname{int} M_0)$, and pick any number $l > \Lambda$. Then we can find $M_1 \subseteq M_0$, a compact submanifold with boundary, and a smooth 1-parameter family of diffeomorphism $h_t: M_1 \rightarrow M_0$ such that*

- (i) $\lambda_A(\operatorname{int} M_1) > l$,
- (ii) $h_0 = \operatorname{id}$, $h_1(M_1) = M_0$,
- (iii) $h_t(M_1) \subseteq h_{t'}(M_1)$ if $t \leq t'$, and thus $\lambda(t) = \lambda_A(\operatorname{int}(h_{1-t}(M_1)))$ is monotone increasing and continuous.

Let M be a smooth manifold without boundary. Let $g, d\mu$ and A be as above. Then A defines a symmetric linear map on $C_0^\infty(M) \subseteq L^2(M, d\mu)$ which is semi-bounded below. We can then form the Friedrichs extension [10, p. 329] which we will also denote by A . The Friedrichs extension is a selfadjoint operator on its

domain D_A and possesses a spectral decomposition. Thus we can speak of the discrete and the continuous part of the spectrum. In particular, let $\lambda_c(A)$ denote the bottom of the continuous spectrum of A , i.e., $\lambda_c(A) = \inf\{\lambda: \lambda \text{ is in the continuous spectrum of } A\}$. Baider [1] has given the following geometric characterization of $\lambda_c(A)$.

THEOREM B. $\lambda_c(A) = \sup_{K \text{ compact}} \lambda_A(M - K)$.

We will now begin to discuss oscillatory behavior. Let M be a smooth manifold, $\partial M = \emptyset$. Let $g, d\mu$ and A be as before. An open connected set in M will be referred to as a *domain*. If the boundary of a domain Ω , $\partial\Omega = \bar{\Omega} - \Omega$ is a smooth $n - 1$ dimensional manifold, then we call Ω a *regular domain*. A domain is *bounded* if $\bar{\Omega}$ is compact.

DEFINITION. Let λ be a real number. $\Omega \subseteq M$ a domain. We call Ω a λ -*domain* for A if Ω is bounded and if $\lambda_A(\Omega) = \lambda$. Notice that if Ω is a regular λ -domain (this clearly can be weakened some), then $\lambda_A(\Omega)$ is the lowest eigenvalue for A , with Dirichlet boundary data on $\partial\Omega$.

PROPOSITION. Let $\lambda \in \mathbf{R}$. The following two statements are equivalent.

- (a) For each compact subset $K \subseteq M$, we can find a regular λ -domain Ω such that $\bar{\Omega} \subseteq M - K$.
- (b) For each compact subset $K \subseteq M$, we can find a $\tilde{\lambda}$ -domain $\tilde{\Omega}$, such that $\bar{\tilde{\Omega}} \subseteq M - K$ and $\tilde{\lambda} < \lambda$.

PROOF. It is trivial to see that (a) implies (b).

Let $M = \bigcup_{i=1}^{\infty} U_i$, U_i open, regular bounded domains where $\bar{U}_i \subseteq U_{i+1}$. For each compact set K , we can find $i(K)$ such that $K \subseteq \bar{U}_{i(K)}$ and hence $M - \bar{U}_{i(K)} \subseteq M - K$. If (b) holds, we can find a $\tilde{\lambda}$ -domain Ω such that $\bar{\Omega} \subseteq V \subseteq M - \bar{U}_{i(K)}$, where V is the connected component of $M - \bar{U}_{i(K)}$ containing $\bar{\Omega}$ and $\tilde{\lambda} < \lambda$. If V is bounded, it is therefore a regular bounded domain, such that $\partial V \cap \bar{\Omega} = \emptyset$. Thus we can find a bounded, regular domain W , $\bar{\Omega} \subseteq W \subseteq \bar{W} \subseteq V$. Therefore, W is a regular λ' -domain, $\lambda' < \lambda$. (a) then follows from Theorem A. Finally if V is not bounded, we can clearly find a regular bounded domain W , such that $\bar{\Omega} \subseteq W \subseteq \bar{W} \subseteq V$. (a) then follows as above.

DEFINITION. We say λ is *oscillatory* for A if λ satisfies (a) or (b) of the proposition. If λ is oscillatory and if $\lambda < \mu$, then (b) of the proposition shows that μ is also oscillatory. Thus we can define the *oscillation constant* $\Lambda_0(A)$ of A to be

$$\Lambda_0(A) = \inf\{\lambda: \lambda \text{ is oscillatory}\}.$$

THEOREM. $\Lambda_c(A) = \Lambda_0(A)$.

PROOF. Let $M = \bigcup_{i=1}^{\infty} U_i$, U_i all regular bounded domains, such that $\bar{U}_i \subseteq U_{i+1}$. Thus $\lambda_A(M - \bar{U}_i) < \lambda_A(M - \bar{U}_{i+1})$ and it follows easily from Theorem B that

$$\Lambda_c(A) = \sup \lambda_A(M - \bar{U}_i) = \lim_{i \rightarrow \infty} \lambda_A(M - \bar{U}_i).$$

Assume λ is oscillatory. For each i , we can find a regular λ -domain in $M - \bar{U}_i$ and

thus a regular λ -domain in $U_k - \bar{U}_i$ for some $k > i$. Hence $\lambda_A(M - \bar{U}_i) = \inf_{I \supset k} (U_I - \bar{U}_i) \leq \lambda$, which implies that $\lambda_c(A) \leq \lambda_0(A)$. Conversely, let $\lambda < \lambda_0(A)$. Choose an $\varepsilon > 0$ such that $\lambda + \varepsilon < \lambda_0(A)$. We can then find some U_i such that $M - \bar{U}_i$ contains no $\tilde{\lambda}$ -domains, $\tilde{\lambda} \leq \lambda + \varepsilon$. Hence $\lambda_A(M - \bar{U}_i) > \lambda$ which implies $\lambda_c(A) > \lambda_0(A)$.

2. Crushed ice. For the remainder of the note let (M, g) be a compact, connected m -dimensional Riemannian manifold and let $M_0 \subseteq M$ be a compact m -dimensional manifold with $\partial M_0 = N$. Let $d\mu$ be a volume element on M , let c be a continuous function on M_0 , let $A = -\operatorname{div}_\mu \operatorname{grad} + c$, and let $\Delta_g = -\operatorname{div}_g \operatorname{grad}$ be the usual Laplacian. $d\mu = f dv$ where $f > 0$ and dv is the Riemannian volume element. Then we can find a real number $K > 0$ such that $(1/K)d\mu \leq dv \leq K d\mu$ and let $\alpha = \inf c(x)$. If ψ is a smooth function on M_0 , then $\int \|\operatorname{grad} \psi\|^2 d\mu \geq (1/K) \int \|\operatorname{grad} \psi\|^2 dv$, and $\int \psi^2 d\mu \leq K \int \psi^2 dv$. Thus if ψ is the first nontrivial eigenfunction for the Dirichlet problem for $M_0 \subseteq M$, we see that

$$\lambda_A(M_0) \geq \frac{1}{K^2} \frac{\int \|\operatorname{grad} \psi\|^2 dv}{\int \psi^2 dv}.$$

Therefore, $\lambda_A(M_0) \geq (1/K^2)\lambda_\Delta(M_0) + \alpha$, and it suffices to prove Theorem A in the case where the operator $A = \Delta_g = \Delta$. We will denote $\lambda_A(\cdot)$ by $\lambda(\cdot)$, and furthermore denote by $B(x, r)$, $x \in M$ and r a positive real, the open metric ball centered at x of radius r .

PROPOSITION 2.1. *Let (M, g) , $M_0 \subseteq M$, $\partial M_0 = N_0$ be as before and let $S = (x_1, \dots, x_K)$ be a finite set of points in M_0 and let $0 < r_0 < R_0$ be real numbers such that $2r_0 < R_0$ and*

- (i) $B(x_i, r_0) \subseteq \operatorname{Int} M_0$,
- (ii) $\{B(x_i, R_0)\}$ cover M_0 ,
- (iii) $R_0 < \text{the injectivity radius of } M$,
- (iv) *no point of M_0 can be in more than N of sets $B(x_i, R_0)$.*

Set $M_1 = M_0 - \bigcup_{i=1}^K B(x_i, r_0)$.

Then we can find a constant $c > 0$ which depends only on (M, g) such that

$$\lambda(M_1) \geq \begin{cases} \frac{cKr_0^{m-2}}{Nd \operatorname{vol}(M_0)} & \text{if } m = \dim M > 2, \\ \frac{cK}{Nd \operatorname{vol}(M_0) |\log r_0|} & \text{if } \dim M = 2, \end{cases}$$

where d is a number such that $KR_0^m = d \operatorname{vol}(M_0)$.

NOTE. The expression "crushed ice" for this sort of situation is due to Rauch and Taylor [8], [9], where they view the $B(x_i, r_0)$ as K little coolers (round ice cubes) and λ represents the "rate" of cooling. This part of the proof is essentially theirs, our contribution being the refined "even spacing" lemma.

PROOF. Let $\psi: M_1 \rightarrow \mathbf{R}$ be such that $\psi|_{\partial M_1} = 0$ and $\Delta\psi = \lambda(M_1)\psi$. Let $A_i = \{x \in M | r_0 \leq d(x, x_i) < R_0\}$ and let $M_2 = \bigcup A_i$. Extend ψ to a continuous function with square integrable weak differential on M_2 , by making it zero on $M_2 - M_1$. Denote this also by ψ .

$$\int_{M_1} |\text{grad } \psi|^2 = \int_{M_2} |\text{grad } \psi|^2 > \frac{1}{N} \sum_i \int_{A_i} |\text{grad } \psi|^2.$$

If on the set A_i we introduce Riemann normal coordinates centered at x_i , and if r is the radial coordinate, we see

$$\int_{A_i} |\text{grad } \psi|^2 > \int_{A_i} \left(\frac{\partial \psi}{\partial r} \right)^2 dv.$$

Let $V = \{v \in TM | \|v\| \leq R_0\}$ and let $E: V \rightarrow M \times M$ be $E(v) = (p(v), \exp_{p(v)} v)$, where p is the bundle projection. If dv_E is the Euclidean volume element on the tangent spaces and if dv is the Riemannian volume element, then

$$E^*(dv(E(x, v))) = f(x, v) dv_E, \quad v \in T_x M.$$

Thus f is a smooth positive function defined on a compact set and we can find some number $c > 0$ which depends on (M, g) alone such that

$$\int_{A_i} |\text{grad } \psi|^2 > c \int_{A_i} \left(\frac{\partial \psi}{\partial r} \right)^2 r^{m-1} dr d\Sigma,$$

where $d\Sigma$ is the volume element of the unit $m - 1$ sphere in $\mathbf{R}^m$. We can then apply the argument in [9, p. 45] to get the estimate

$$\int_{A_i} |\text{grad } \psi|^2 > c' \left(\int_{A_i} \psi^2 dv \right) \frac{r_0^{m-2}}{R_0^m} \quad \text{for } m > 2,$$

and a slightly different estimate involving logs for $m = 2$. Therefore,

$$\int_{M_1} |\text{grad } \psi|^2 > \frac{c'}{N} \frac{r_0^{m-2}}{R_0^m} \sum_i \int_{A_i} \psi^2 > \frac{c' K r_0^{m-2}}{R_0^m K} \int_{M_1} \psi^2,$$

and hence, $\lambda(M_1) > c' r_0^{m-2} / Nd \text{ vol}(M_0)$ because $R_0^m K = d \text{ vol}(M_0)$. The case $m = 2$ follows in a similar way.

Proposition 2.1 is our main analytic lemma; we now need the even spacing notions.

DEFINITION 2.2. Let (M, g) be a Riemannian manifold possibly with boundary and let d be the induced metric. We say a sequence of finite subsets of points $S_n = \{x_1, \dots, x_{K_n}\}$, $x_i \in \text{Int } M$, are *evenly spaced* if for each n we can find a number $R(n)$ such that the balls $B(x_i, R(n))$, $1 \leq i \leq K_n$, cover M and such that we can find a number N independent of n , such that each point of M is in no more than N of the $B(x_i, R(n))$ for each n .

THEOREM 2.3 (EVEN SPACING LEMMA). *Let M be a compact connected m -dimensional Riemannian manifold and let $M_0 \subseteq M$ be a compact connected m -dimensional submanifold with boundary $\partial M_0 = N$. Then we can find sets of points $S_n = \{x_1, \dots, x_{K_n}\}$ in M_0 which are evenly spaced in M_0 . Let $R(n)$ be the radius in the*

definition. We can also find numbers $r(n) > 0$, $r(n) < R(n)/2$ such that the sets $B(x_i, r(n))$ satisfy

- (a) $B(x_i, r(n)) \subseteq \text{Int}(M_0)$, $1 \leq i \leq K_n$,
- (b) $B(x_i, r(n))$ and $B(x_i, R(n))$ are diffeomorphic to smooth Euclidean balls,
- (c) $B(x_i, r(n))$ are pairwise disjoint,
- (d) $K_n = K_0 2^{mn}$, $R(n) = R_0/2^n$ and $r(n) = r_0/2^n$.

We can now apply the Even Spacing Lemma to Proposition 2.1. We can assume the number R_0 in 2.3 is less than the injectivity radius of M . Let $M_n = M_0 - \bigcup_{i=1}^{K_n} B(x_i, r(n))$ and we see

$$\lambda(M_n) \geq \begin{cases} \frac{CK_n r(n)^{m-2}}{Nd_n \text{vol}(M_0)}, & m > 2, \\ \frac{C}{Nd_n \text{vol}(M_0)} \cdot \frac{K_n}{|\log r(n)|}, & \end{cases}$$

where $d_n \text{vol}(M_0) = R(n)^m K_n = K_0 2^{mn} R_0^m / 2^{mn} = K_0 R_0^m = d_0 \text{vol}(M_0)$ and is independent of n . Hence $\lambda(M_n) \rightarrow \infty$ as $n \rightarrow \infty$ and we get the following theorem.

THEOREM 2.4. *Let M , M_0 and $\partial M_0 = N$ be as above. Then for each number $K > 0$ we can find $r_0 > 0$ and a finite set of points $x_1, \dots, x_s, x_i \in M_0$, such that*

- (i) $B(x_i, r_0) \subseteq \text{Int } M_0$;
- (ii) the $B(x_i, r_0)$ are pairwise disjoint;
- (iii) if $M_1 = M_0 - \bigcup_{i=1}^s B(x_i, r_0)$ then $\lambda(M_1) > K$.

REMARK. This discussion also holds for the case where we have Neumann boundary data on $N = \partial M_0$ and zero boundary data on the boundaries of the $B(x_i, r_0)$. One follows [8], [9] as we just did and uses the Lions' reflection to extend ψ from M_0 to all of M without significantly changing its H_1 norm.

COROLLARY 2.5. *Theorem A for the Laplacian follows from Theorem 2.4.*

PROOF. Pick $K > 0$. Let us apply Theorem 2.4 and find a finite number of points $x_1, \dots, x_s$ and a number $r > 0$, $r < \text{injectivity radius of } M$ such that the geodesic balls $B(x_i, r) = B_i$, $1 \leq i \leq s$, are pairwise disjoint in $\text{Int } M_0$ and $\lambda(M_0 - \bigcup B(x_i, r)) > K$. For each B_i let γ_i be a smooth curve joining ∂B_i to ∂M_0 , which meets these boundaries perpendicularly. Thicken γ_i a bit and smooth the edges at the boundaries to form a tube denoted by τ_i . Let $O_i = B_i \cup \tau_i$ and we can assume we have chosen the γ_i and τ_i such that the O_i are pairwise disjoint. Then $\lambda(M_0 - \bigcup O_i) > \lambda(M_0 - \bigcup B_i) > K$. We can now define our diffeomorphism h_t by eliminating the O_i 's one at a time, by first shrinking the ball and then withdrawing the tube as in the pictures.

3. The proof of Theorem 2.3 (the Even Spacing Lemma). We will first discuss the case where $M_0 = M$ and $\partial M_0 = \emptyset$. We will then show how to alter the proof for the general case.

Let us isometrically embed M in $\mathbf{R}^s$ for s sufficiently large. Let $\pi: U_\eta \rightarrow M$ be the tubular neighborhood of radius η of M in $\mathbf{R}^s$ ($U_\eta = \{x \in \mathbf{R}^s \mid \|x - M\| \leq \eta$ and π is the projection back onto M)). Following Whitney [12], for any $\eta > 0$ we can find a simplicial complex K , $K \subseteq U_\eta$ (geometrically in $\mathbf{R}^s$, not just abstract) with the following properties.

- (a) K is itself a topological m -dimensional manifold.
- (b) $\pi|_K: K \rightarrow M$ is a homeomorphism of K onto M .
- (c) Each m -simplex $\sigma \subseteq K$ is almost parallel to the tangent plane $T_{\pi(x)}(M)$ for each $x \in \sigma$. More precisely for each m -simplex $\sigma \subseteq K$, there exists a constant $c > 0$, such that $|\det(T_\pi|_\sigma)| \geq c > 0$.

The locally affine structure in K allows us to speak of a distance d_L between two points (the inf of the arclengths of continuous piecewise affine paths joining two points) and an m -volume v_L of measurable subsets of K . Let d and v be the corresponding distance and volume defined by the Riemannian structure on M . It is easy to see from the properties of our triangulation and the compactness of M that we can find positive numbers c_1, c_2, d_1 and d_2 such that

(A) for $x, y \in K$, $c_1 d(\pi(x), \pi(y)) \leq d_L(x, y) \leq c_2 d(\pi(x), \pi(y))$;

(B) for any measurable $S \subseteq K$, $d_1 v(\pi(S)) \leq v_L(S) \leq d_2 v(\pi(S))$.

Let σ be an m -simplex of K , $\sigma = p_0, \dots, p_m$. Set $v_i = p_i - p_0$. Then σ is congruent to an m -simplex with 0 as a vertex and $v_1, \dots, v_m$ as edges. For each σ we can then form $2^m - 1$ other m -simplices $\sigma(i_1, \dots, i_m)$ where 0 is a vertex and the edges are $i_j v_j$, $i_j = \pm 1$ ($\sigma \neq \sigma(1, \dots, 1)$).

In each $\sigma \subseteq K$, there is a point $x_\sigma \in \sigma$ which is the center of the largest inscribed ball $B_L(x_\sigma, r_\sigma)$ of radius $r(\sigma)$. Let $r_\sigma(i_1, \dots, i_m)$ be the radius of the corresponding inscribed ball in $\sigma(i_1, \dots, i_m)$, and set

$$\tilde{r}_0 < \inf_{\sigma} (r_\sigma(i_1, \dots, i_m)/2).$$

Let $p_\sigma = \pi(x_\sigma)$, pick $r_0 < \tilde{r}_0/c_2$ and note that the ball

$$B(p_\sigma, r_0) = \{y \mid d(y, p_\sigma) < r_0\} \subseteq \pi(B_L(x_\sigma, \tilde{r}_0)).$$

Hence the balls $B(p_\sigma, r_0)$ are disjoint.

Let K_1 be the standard subdivision of K . For all the properties of K_1 and its definition see [12]; we will need the following facts. Each m -simplex σ is subdivided into 2^m m -simplices of K_1 . Furthermore if $\sigma = p_0, \dots, p_m$ is an m -simplex of K ,

form the congruent m -simplex with edges $v_i = p_i - p_0$ and a vertex at 0. Then each of the m -simplices σ^1 of K_1 into which σ has been subdivided is congruent to one of the simplices with a vertex 0 and edges $i_j v_j$, $i_j = \pm \frac{1}{2}$. Thus if $x_{\sigma(1)}$ are the centers of the inscribed balls of the m -simplices $\sigma(1)$ of K_1 , then the balls $B_L(x_{\sigma(1)}, \tilde{r}_0/2)$ are disjoint, as are

$$B(p_{\sigma(1)}, r_0/2) = \{y \in M \mid d(p_{\sigma(1)}, y) < r_0/2\} \quad \text{where } p_{\sigma(1)} = \pi(x_{\sigma(1)}).$$

Let R_σ be the radius of the ball centered at x_σ which contains σ , $\sigma \subseteq K$ on m -simplex and let $R_\sigma(i_1, \dots, i_m)$ be the corresponding radius for the simplex $\sigma(i_1, \dots, i_m)$. Set $\tilde{R}_0 = \sup R_\sigma(i_1, \dots, i_m)$. The balls $B_L(p_\sigma, \tilde{R}_0)$ cover K and if we choose $R_0 > c_1 \tilde{R}_0$ and $R_0 > 2r_0$, we see that the balls $B(p_\sigma, R_0)$ cover M . Let us pass to the standard subdivision K_1 . If $\sigma(1)$ is an m -simplex in K_1 , then the balls $B(p_{\sigma(1)}, R_0/2)$ must cover K .

Let $\tilde{\delta}_0$ be the supremum of the diameters of m -simplices $\sigma(i_1, \dots, i_m)$ as σ runs through the m -simplices of K . Let $\delta_0 = \tilde{\delta}_0/c_1$. Hence $\text{diameter}(\pi(\sigma)) \leq \delta_0$.

If we replace K by K_1 , then $\tilde{\delta}_0$ and δ_0 are replaced by $\tilde{\delta}_0/2$ and $\delta_0/2$. Let $d(p)$ = distance from p to its first cut point and let $\gamma = \min d(p)$. Replacing K by $K_{(n)}$ (n th standard subdivision) if need be, we can assume $2R_0 + \delta_0 < \gamma$.

Let us collect what we have been able to show. Let us pick K as above, r_0 and R_0 such that $r_0 < 2R_0$ and $2R_0 + \delta_0 < \gamma$. Let $K_{(n)}$ be the n th standard subdivision of K . Let $\{\sigma(n)\}$ be the set of m -simplices of $K_{(n)}$. Let $\{x(\sigma(n))\}$ be the "centers" of the $\{\sigma(n)\}$ as constructed above and let $p(\sigma(n)) = \pi(x(\sigma(n)))$. If we set $S_n = \{(p(\sigma(n)))\}$, $r(n) = r_0/2^n$ and $R(n) = R_0/2^n$, we see that

- (i) the balls $B(p(\sigma(n)), R(n))$ cover M ;
- (ii) the balls $B(p(\sigma(n)), r(n))$ are disjoint;
- (iii) these balls are diffeomorphic to smooth euclidean balls, and
- (iv) $r(n) < R(n)/2$.

If $L(n)$ is the number of points in S_n , we then see that $L(n) = L(0) \cdot 2^{mn}$ and therefore

$$L(n)r(n)^{m-2} = L(0)2^{mn} \left(\frac{r_0^{m-2}}{2^n} \right)^{m-2} = \frac{L(0)r_0^{m-2}}{4^n}.$$

Thus it remains to show the even spacing. Let N_σ = the number of p_σ in $B(p_\sigma, 2R_0)$. Let $N_0 = \sup_{\sigma \subseteq K} N_\sigma$ where the σ 's are m -simplices. Then $N_0 >$ largest number of sets $B(p_\sigma, R_0)$ any point in M may touch. We wish to estimate N_0 . We know if $p_{\sigma'} \in B(p_\sigma, 2R_0)$ then $\pi(\sigma') \subseteq B(p_\sigma, 2R_0 + \delta_0)$. Hence

$$v(B(p_\sigma, 2R_0 + \delta_0)) > \sum_{p_{\sigma'} \in B(p_\sigma, 2R_0)} v(\pi(\sigma')).$$

Let $\Sigma = \inf\{v_L(\sigma(i_1, \dots, i_m)) \mid \sigma \text{ an } m\text{-simplex} \subseteq K\}$. We can find a constant e depending on the metric g above such that

$$N_0 < d_2 e (2R_0 + \delta_0)^m / \Sigma'.$$

If we replace K by K_1 the standard subdivision, we are interested in the number N_1 of $p_{\sigma(1)}$ in a ball radius R_0 about a given $p_{\sigma(1)}$. Here δ_0 is replaced by $\delta_1 = \delta_0/2$. d_2

and e remain unchanged. Each m -simplex σ splits into 2^m m -simplices of equal volume, so the number Σ' in the denominator is replaced by $\Sigma'/2^m$ and we see that

$$N_1 \leq \frac{d_2 e (R_0 + \delta_0/2)^m}{(1/2)^m \Sigma'} = N_0,$$

which completes our proof in the case where $M = M_0$.

Before we proceed to the case of a manifold with boundary we need a definition.

DEFINITION. Let $x \in \partial M_0$, let $v \in T(M_0)_x$ be a unit vector perpendicular to ∂M_0 which points into M_0 and let $\gamma_x(t)$ ($t \geq 0$) be the unique geodesic, parametrized by arclength such that $\gamma_x(0) = x$ and $\dot{\gamma}_x(0) = v$. For small $t > 0$, $\gamma_x[0, t] \subseteq M_0$ and x is the unique point on ∂M_0 which is closest to $\gamma_x(t)$. Let t_x be the supremum of all $t > 0$ such that the above properties hold. Thus the paths $\gamma_x[0, t]$, $t > t_x$, no longer minimize the distance from $\gamma_x(t)$ to ∂M_0 . The point $\gamma_x(t_x)$ is called the *focal cut point* of ∂M_0 along γ_x and the set of points $\{\gamma_x(t_x) | x \in \partial M_0\}$ is called the *focal cut locus* of ∂M_0 .

We now proceed to the general case. Let δ_0 and γ be as before. Let K be the complex. Let

$$\begin{aligned} A_n &= \{m\text{-simplices in } K_n \text{ such that } \pi(\sigma) \subseteq \text{Int } M_0\}, \\ B_n &= \{m\text{-simplices in } K_n \text{ such that } \pi(\sigma) \cap N = \partial M_0 \neq \emptyset\}, \\ C_n &= \{m\text{-simplices in } K_n \text{ such that } \pi(\sigma) \subseteq M - M_0\}, \end{aligned}$$

where K_n is the n th standard subdivision of K . We assume that our initial triangulation K and our initial R_0 are so chosen that

- (i) $R_0 > 3\delta_0$,
- (ii) $2R_0 + \delta_0 < \gamma$,
- (iii) A_0 , B_0 and C_0 are all nonempty,
- (iv) $2\delta_0 < \text{distance from } \partial M_0 \text{ to its focal cut locus}$.

By successively subdividing K if necessary we can clearly arrange this situation. Let $S_n = \{p_{\sigma(n)} | B(p_{\sigma(n)}, r(n)) \subseteq \text{Int } M_0\}$. The only thing left to show is that $\{B(p(\sigma(n)), R(n))\}$ cover M . If $x \in \pi(\sigma(n))$, $\sigma(n) \in A_n$, then $x \in B(p(\sigma(n)), R(n))$. Say $x \in \pi(\sigma(n)) \cap M_0$, $\sigma(n) \in B_n$. Then we can find $y \in N$ such that $d(x, y) < \delta_0/2^n$. Let $T_n = \{z \in M_0 | d(z, 2M_0) = \delta_0/2^n\}$. It is easy to see that each $x \in T_n$ is in $\pi(\sigma(n))$ for some $\sigma(n) \in A_n$. Hence we can find some $\sigma(n) \in A_n$ such that $d(y, p(\sigma(n))) < 2\delta_0/2^n$ and thus $d(x, p(\sigma(n))) < 3\delta_n/2^n < R_0/2^n$.

REMARK. A careful accounting of this construction shows the following result. For each $\varepsilon > 0$ we can choose the original r_0 so small that $\text{vol}(M_0 - M_n) < \varepsilon$ for all n , where $M_n = M_0 - \bigcup_{\sigma(n) \in K_n} B(x_{\sigma(n)}, r_0/2^n)$.

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DEPARTMENT OF MATHEMATICS, HUNTER COLLEGE (CUNY), NEW YORK, NEW YORK 10021

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL (CUNY), NEW YORK, NEW YORK 10036