# LINEAR OPERATORS ON $L_{p}$ FOR $0<p<1$ <br> BY <br> N. J. KALTON ${ }^{1}$ 


#### Abstract

If $0<p<1$ we classify completely the linear operators $T: L_{p} \rightarrow X$ where $X$ is a $p$-convex symmetric quasi-Banach function space. We also show that if $T: L_{p} \rightarrow L_{0}$ is a nonzero linear operator, then for $p<q<2$ there is a subspace $Z$ of $L_{p}$, isomorphic to $L_{q}$, such that the restriction of $T$ to $Z$ is an isomorphism. On the other hand, we show that if $p<q<\infty$, the Lorentz space $L(p, q)$ is a quotient of $L_{p}$ which contains no copy of $l_{p}$.


1. Introduction. The aim of this paper is to study and classify operators on the spaces $L_{p}(0,1)$ for $0<p<1$ into other spaces of measurable functions. The underlying theme is the idea that operators on $L_{p}$ cannot be "small" when $p<1$. Historically the first result of this type was obtained by Day [4] in 1940 who showed that there is no nonzero operator of finite rank on $L_{p}$. Later, completing a partial result of Williamson [34], Pallaschke [26] and Turpin [31] showed that there is no compact endomorphism $T: L_{p} \rightarrow L_{p}$ other than zero. Recently the author [10] has shown that there is no nonzero compact operator on $L_{p}$ with any range space. In fact, if $T$ is a nonzero operator $T: L_{p} \rightarrow X$ (where $X$ is any topological vector space) there is a subspace $H$ of $L_{p}$ isomorphic to $l_{2}$ such that $T \mid H$ is an isomorphism.

It is quite possible that this last result can be improved quite substantially. To be precise we may ask the question for $0<p<1$ :

Question. Suppose $T$ : $L_{p} \rightarrow X$ is a nonzero operator and $p<q \leqslant 2$. Does there exist a subspace $Y$ of $L_{p}$ such that $Y \cong L_{q}$ and $T \mid Y$ is an isomorphism?

We do not know the answer to this or the weaker question with $l_{q}$ replacing $L_{q}$. However in [11] we showed that if $T: L_{p} \rightarrow L_{p}$ is nonzero we can even obtain such a subspace $Y \cong L_{p}$. Of course, in general there is no hope of a result of this strength; consider the inclusion map $L_{p} \rightarrow L_{0}$ (that this does not preserve a copy of $L_{p}$ is well known; it can be deduced easily from Example 9.9 below since $L_{p} \subset L(p, q) \subset L_{0}$ for $\left.p<q<\infty\right)$.

Our main result (Theorem 7.2) here will provide an affirmative answer to the question when $X=L_{0}$. Of course this also implies an affirmative answer for any space of functions densely embedded in $L_{0}$, or more generally any space $X$ such

[^0]that the space of linear operators $\mathcal{E}\left(X, L_{0}\right)$ separates the points of $X$. Unfortunately, there are spaces $X$ which are not too artificial such that $\mathcal{L}\left(X, L_{0}\right)=\{0\}$; an example was constructed by Christensen and Herer [2], but we also observe in $\S 10$ that $L_{p}(\Gamma) / H_{p}(0<p<1)$ is an example where $\Gamma$ is the unit circle and $H_{p}$ is the usual Hardy space.

Our other result related to the question above is a negative one. We show that a quotient of $L_{p}$ need not contain $l_{p}(0<p<1)$. We remark that a result of Turpin [32, p. 94] together with one of the author [15] shows that $l_{p}$ must be finitely represented in any quotient of $L_{p}$. Indeed in [13] we defined a $p$-Banach space $X$ to be $p$-trivial if $\mathcal{L}\left(L_{p}, X\right)=\{0\}$ and showed that this is an appropriate generalization of the Radon-Nikodym property. If $X$ is not $p$-trivial, the $l_{p}$ is finitely represented in $X$. The example in question here is the Lorentz space $L(p, q)$ where $p<q<\infty$.

Our main theorem is made possible by two other results. The first is Nikisin's theorem [21] that every linear operator $T \in \mathcal{L}\left(L_{p}, L_{0}\right)$ may be factored $L_{p} \rightarrow$ $L(p, \infty) \rightarrow L_{0}$ where the second operator is a multiplication operator [here $L(p, \infty)$ is the weak space $L_{p}$ ]. The second is Theorem 6.1 that if $p<1, L(p, \infty)$ is $p$-convex. We give in Theorem 6.4 a complete characterization of operators $T$ : $L_{p} \rightarrow L[p, \infty]$ where $0<p<1$. Such an operator is of the form

$$
T f(t)=\sum_{n=1}^{\infty} a_{n}(t) f\left(\sigma_{n} t\right)
$$

where $a_{n}:(0,1) \rightarrow \mathbf{R}$ and $\sigma_{n}:(0,1) \rightarrow(0,1)$ are Borel maps and if

$$
\alpha_{x}(B)=\sum_{n=1}^{\infty} m\left(\left(\left|a_{n}\right|>x\right) \cap \sigma_{n}^{-1} B\right)
$$

for $B$ a Borel set then

$$
\alpha_{x}(B) \leqslant C x^{P} m(B), \quad B \in \mathscr{B}
$$

for some constant $C$. Conversely, any such $\left\{a_{n}\right\},\left\{\sigma_{n}\right\}$ defines an operator $T$ : $L_{p} \rightarrow L(p, \infty)$. This result is analogous to the result for endomorphisms of $L_{p}$ given in [11].

We also initiate in $\S \S 8$ and 9 a general study of $p$-Banach function spaces where $0<p<1$. There are certain differences from the theory of Banach function spaces, which give this study a distinctive flavour. One example is the fact noted above that if $p<1, L(p, \infty)$ is $p$-convex; thus there are symmetric $p$-Banach function spaces which are strictly larger than $L_{p}$, while if $p=1, L_{1}$ is the largest symmetric Banach function space and $L(1, \infty)$ is not locally convex. Lotz [18] and the author [14] have' shown that if $l_{1}$ and $L_{1}$ embed in a Banach lattice with order-continuous norm then they embed as a sublattice. Here we give in $\$ 8$ similar but slightly weaker results for $l_{p}$ or $L_{p}$ embedding in a $p$-Banach function space. The difficulty, as will be seen from the proofs, lies in the fact that we may not suppose a $p$-Banach function space densely embedded in $L_{p}$.

In §9 we study symmetric $p$-Banach function spaces ( $0<p<1$ ) and introduce the class of totally symmetric $p$-Banach function spaces. Totally symmetric spaces lie between $L_{p}$ and $L(p, \infty)$ and hence have no analogues for $p=1$. We classify
completely operators from $L_{p}$ into symmetric $p$-Banach function spaces. We show in particular that if $X$ is a separable, $\sigma$-complete symmetric $p$-Banach function space, then $X$ is isomorphic to a quotient of $L_{p}$ if and only if $X$ is totally symmetric.

We note at this point that the study of $p$-Banach lattices where $0<p<1$ is complicated by the fact that we cannot essentially reduce them to the study of function spaces; such a technique is available for Banach lattices (e.g., as in [14]). The possibility of such a reduction for $p$-Banach lattices with order-continuous quasinorm is closely related to Maharam's problem on the existence of a control measure for an order-continuous submeasure (cf. [2]).

We conclude with a note on the organization of the paper. $\S 2$ is purely to introduce notation. In §3, we give a treatment of Nikisin's theorem which is not essentially original. However we believe it may be useful to give a self-contained treatment. We also prove it in much stronger form than is required (the notion of type is not needed in this paper). §§4 and 5 develop some routine techniques. In §6 we prove our main representation theorem for operators from $L_{p}$ into $L(p ; \infty)$, and this leads in $\S 7$ to our main result on operators from $L_{p}$ into $L_{0}$. In §8 we study embeddings of $l_{p}$ and $L_{p}$ in $p$-Banach function spaces. In $\S 9$ we study symmetric $p$-Banach function spaces and operators as $L_{p}$. Finally, in $\S 10$ we give some results on translation-invariant operators on function spaces on compact groups and an example of a quotient of $L_{p}$ which admits no nonzero operators into a $p$-Banach lattice with order-continuous quasinorm.

Our approach is not always economical. Results on $L(p, \infty)$ in $\S 6$ are special cases of results on symmetric spaces in $\S 9$ (Theorem 6.1 is a special case of Theorem 9.4; Corollary 6.5 is a special case of Theorem 9.6). However we felt there was an advantage in developing the theory for the main results rapidly before moving to a more general theory.
2. Prerequisites. A quasinorm on a real vector space $X$ is a map $x \rightarrow\|x\|$ ( $X \rightarrow \mathbf{R}$ ) such that

$$
\begin{gather*}
\|x\|>0 \quad \text { if } x \neq 0  \tag{2.0.1}\\
\|t x\|=|t|\|x\|, \quad x \in X, t \in \mathbf{R},  \tag{2.0.2}\\
\|x+y\| \leqslant k(\|x\|+\|y\|), \quad x, y \in X, \tag{2.0.3}
\end{gather*}
$$

where $k$ is a constant independent of $x$ and $y$. The best such constant $k$ is called the modulus of concavity of the quasinorm. If $k=1$, then the quasinorm is called a norm.

The sets $\{x:\|x\|<\varepsilon\}$ for $\varepsilon>0$ form a base of neighborhoods for a Hausdorff vector topology on $X$. This topology is (locally) $p$-convex where $0<p \leqslant 1$ if for some constant $A$ and any $x_{1}, \ldots, x_{n} \in X$,

$$
\begin{equation*}
\left\|x_{1}+\cdots+x_{n}\right\| \leqslant A\left(\left\|x_{1}\right\|^{p}+\cdots+\left\|x_{n}\right\|^{p}\right)^{1 / p} \tag{2.0.4}
\end{equation*}
$$

In this case we may endow $X$ with an equivalent quasinorm,

$$
\begin{equation*}
\|x\|^{*}=\inf \left\{\left(\sum_{i=1}^{n}\left\|x_{i}\right\|^{p}\right)^{1 / p}: x_{1}+\cdots+x_{n}=x\right\} \tag{2.0.5}
\end{equation*}
$$

and then

$$
\|x\| \geqslant\|x\|^{*} \geqslant A^{-1}\|x\|, \quad x \in X
$$

and $\|\cdot\|^{*}$ is $p$-subadditive, i.e.

$$
\begin{equation*}
\left\|x_{1}+\cdots+x_{n}\right\|^{*} \leqslant\left(\sum_{i=1}^{n}\left\|x_{i}\right\|^{* P}\right)^{1 / p} \tag{2.0.6}
\end{equation*}
$$

A theorem of Aoki and Rolewicz [28, p. 57] asserts that every quasinormed space is $p$-convex for some $p>0$. A complete quasinormed space is called a quasiBanach space, and if it is equipped with a $p$-subadditive quasinorm it is called a p-Banach space. If $X$ and $Y$ are quasi-Banach spaces then the space $\mathcal{L}(X, Y)$ of (bounded) linear operators $T: X \rightarrow Y$ is also a quasi-Banach space under the quasinorm

$$
\|T\|=\sup (\|T x\|:\|x\| \leqslant 1)
$$

Suppose $X$ is a vector lattice; then $\|\cdot\|$ is a lattice quasinorm on $X$ if whenever $|x| \leqslant|y|,\|x\| \leqslant\|y\|$. A complete quasinormed lattice is called a quasi-Banach lattice. If $X$ is a $p$-convex quasi-Banach lattice, then, as in Equation (2.0.5), $X$ may be requasinormed with a $p$-subadditive lattice quasinorm; with such a quasinorm it is termed a $p$-Banach lattice.

Now let $K$ be a compact metric space, and let $\mathscr{B}=\mathscr{B}(K)$ be the $\sigma$-algebra of Borel subsets on $K$. Suppose $\lambda$ is a probability measure on $K$ (i.e. a positive Borel measure of total mass one), with no atoms. Then $L_{0}(K)=L_{0}(K, \mathscr{B}, \lambda)$ denotes the space of all Borel measurable real functions on $K$, where functions equal $\lambda$-almost everywhere are identified. This is an $F$-space (complete metric linear space) under the topology of convergence in measure. Then a quasi-Banach function space $X$ on $K$ is a subspace of $L_{0}(K)$ containing the simple functions such that if $g \in X$, $f \in L_{0}$ and $|f| \leqslant|g|$ ( $\lambda$-a.e.) then $f \in X$, and equipped with a complete lattice quasinorm so that the inclusion map $X \hookrightarrow L_{0}$ is continuous. As usual if the associated quasinorm is $p$-subadditive then $X$ is a $p$-Banach function space.
A quasi-Banach lattice $X$ is $\sigma$-complete if whenever $\left\{x_{n}\right\}$ is a bounded increasing sequence of positive elements of $X$ then $\left\{x_{n}\right\}$ has a supremum in $X$. If $X$ is a $\sigma$-complete quasi-Banach function space then it may be requasinormed to have the Fatou property

$$
\begin{equation*}
\left\|\sup _{n} x_{n}\right\|=\sup _{n}\left\|x_{n}\right\| \tag{2.0.7}
\end{equation*}
$$

for every such sequence; note that in this case $\sup _{n} x_{n}$ in $X$ must coincide with the usual pointwise supremum in $L_{0}$. We shall always assume the Fatou property is satisfied as in quasi-Banach function space.

Throughout this paper, $K$ will denote an arbitrary, but fixed, compact metric space and we may suppress mention of the underlying probability measure space ( $K, \mathscr{B}, \lambda$ ). Some examples of quasi-Banach function spaces on $K$ are given by: (1) the spaces $L_{p}(0<p<\infty)$ of functions $f$ such that

$$
\|f\|_{p}=\left\{\int_{K}|f(t)|^{p} d \lambda(t)\right\}^{1 / p}<\infty,
$$

(2) the weak $L_{p}$-spaces $L(p, \infty), 0<p<\infty$ of functions $f$ such that

$$
\begin{equation*}
\|f\|_{p, \infty}=\sup _{0<x<\infty} x(\lambda(|f|>x))^{1 / p}<\infty, \tag{2.0.8}
\end{equation*}
$$

and (3) the Lorentz spaces $L(p, q), 0<p<\infty, 0<q<\infty$ of functions $f$ such that

$$
\begin{equation*}
\|f\|_{p, q}=\left\{\frac{q}{p} \int_{0}^{1} t^{q / p-1} f^{*}(t)^{q} d t\right\}^{1 / q}<\infty \tag{2.0.9}
\end{equation*}
$$

where $f^{*}$ is the decreasing rearrangement of $|f|$ on $(0,1)$, i.e.

$$
f^{*}(t)=\inf _{\lambda(E)=t} \sup _{s \in K-E}|f(s)|
$$

(see Hunt [8], Saghar [29]).
Note that

$$
\begin{aligned}
\|f\|_{p, q}^{q} & =q / p \int_{0}^{\infty} t^{q / p-1} f^{*}(t)^{q} d t \\
& =\left[t^{q / p} f^{*}(t)^{q}\right]_{0}^{\infty}-\int_{0}^{\infty} t^{q / p} d f^{*}(t)^{q} \\
& =q \int_{0}^{\infty} F(x)^{q / p} x^{q-1} d x
\end{aligned}
$$

where $F(x)=\lambda(|f|>x)$ is the distribution of $f$ (strictly, of course $1-F$ is the distribution of $f$ as a random variable). Hence

$$
\begin{equation*}
\|f\|_{p, q}=\left\{q \int_{0}^{\infty} F(x)^{q / p} x^{q-1} d x\right\}^{1 / q} \tag{2.0.10}
\end{equation*}
$$

A quasi-Banach function space is called symmetric if it is quasinormed such that $\|f\|=\|g\|$ whenever $f$ and $g$ have the same distribution. Note that each of the above examples is symmetric.

If $B \subset K$ is a Borel set, then $1_{B}$ denotes the indicator function of $B$ and $P_{B}$, the natural projection of $L_{0}$ onto $L_{0}(B)$,

$$
\begin{aligned}
P_{B} f(s) & =f(s), \quad s \in B, \\
& =0, \quad s \notin B .
\end{aligned}
$$

The space $\mathfrak{\Re}(K)$ is the space of signed Borel measures, so that $\mu(K) \simeq C(K)^{*}$. If $\mu \in \mathfrak{R}(K)$, then $|\mu|$ is its total variation and if $0<p<1,|\mu|_{p}$ is its $p$-variation (cf. [11]). Then $\|\mu\|=|\mu|(K)$ and $\|\mu\|_{p}=\left(|\mu|_{p}(K)\right)^{1 / p}$; if $\|\mu\|_{p}<\infty$ then $\mu$ is purely atomic.
3. Nikisin's theorem. In this section we give a self-contained treatment of Nikisin's theorem on the factorization of operators into $L_{0}$. In fact, Nikisin [21] proves a rather more general result for superlinear operators, but we shall specialize to the case which concerns us. We note that an earlier version of Nikisin's theorem is given in [19] and [20].

Suppose $X$ is a quasi-Banach space. Then we say $X$ is type $p$ for $0<p<2$, if for some constant $C$ and every $x_{1}, \ldots, x_{n} \in X$,

$$
\int_{0}^{1}\left\|\sum r_{i}(t) x_{i}\right\|^{p} d t \leqslant C \sum_{i=1}^{n}\left\|x_{i}\right\|^{p}
$$

where $r_{1}, \ldots, r_{n}$ are the Rademacher functions on $(0,1), r_{n}(t)=\operatorname{sgn}\left(\sin 2^{n} \pi t\right)$. If $X$ is $p$-convex then it is also of type $p(0<p<1)$.

Theorem 3.1. Suppose $X$ is a quasi-Banach space of type $p(0<p<2)$ and that $T: X \rightarrow L_{0}(K)$ is a linear operator. Then given $\varepsilon>0$ there exists $E \in \mathscr{B}$ with $\lambda(E) \geqslant 1-\varepsilon$ and such that $P_{E} T \in \mathcal{L}(X, L(p, \infty))$ where

$$
\begin{aligned}
P_{E} f(t) & =f(t), \quad t \in E, \\
& =0, \quad t \notin E .
\end{aligned}
$$

Proof. First we observe that since $T$ is continuous, for every $\varepsilon>0$ there exists $R(\varepsilon)<\infty$ such that if $\|x\| \leqslant 1$,

$$
\begin{equation*}
\lambda(|T x|>R(\varepsilon)) \leqslant \varepsilon \tag{3.1.1}
\end{equation*}
$$

We now establish:

$$
\begin{align*}
& \text { Given any } \varepsilon>0 \text { and } x_{1}, \ldots, x_{n} \in X \text { there exists } \\
& E=E\left(x_{1}, \ldots, x_{n}\right) \in \mathscr{B} \text { such that } \lambda(E) \geqslant 1-\varepsilon \text { and }  \tag{3.1.2}\\
& \left(\sum_{i=1}^{n}\left|T x_{i}(s)\right|^{2}\right)^{1 / 2} \leqslant A(\varepsilon)\left(\sum_{i=1}^{n}\left\|x_{i}\right\|^{p}\right)^{1 / p} \quad(s \in E)
\end{align*}
$$

where

$$
A(\varepsilon)=4\left(\frac{16 C}{\varepsilon}\right)^{1 / p} R\left(\frac{\varepsilon}{16}\right)
$$

To prove (3.1.2) let us assume $\varepsilon>0$ and that $x_{1}, \ldots, x_{n} \in X$. Define $B \subset[0,1]$ to be the set of $t$ such that

$$
\left\|\sum_{i=1}^{n} r_{i}(t) x_{i}\right\|>\left(\frac{16 C}{\varepsilon}\right)^{1 / p}\left(\sum\left\|x_{i}\right\|^{p}\right)^{1 / p}
$$

Then $m(B) \leqslant \varepsilon / 16$. However if

$$
\left\|\sum r_{i}(t) x_{i}\right\| \leqslant\left(\frac{16 C}{\varepsilon}\right)^{1 / p}\left(\sum\left\|x_{i}\right\|^{p}\right)^{1 / p}
$$

then by (3.1.1) the set of $s \in K$ such that

$$
\left|\sum r_{i}(t) T x_{i}(s)\right|>\frac{1}{4} A(\varepsilon)\left(\sum\left\|x_{i}\right\|^{p}\right)^{1 / p}
$$

has measure at most $\varepsilon / 16$.
Now let $Q \subset K \times[0,1]$ be the set of $(s, t)$ such that

$$
\left|\sum r_{i}(t) T x_{i}(s)\right| \leqslant \frac{1}{4} A(\varepsilon)\left(\sum\left\|x_{i}\right\|^{p}\right)^{1 / p}
$$

Then $(\lambda \times m)(Q) \geqslant 1-\varepsilon / 8$. Now, by Fubini's theorem, if for $s \in K$,

$$
Q_{s}=\{t:(s, t) \in Q\} \quad \text { and } \quad E=\left\{s: m\left(Q_{s}\right) \geqslant \frac{7}{8}\right\}
$$

then $\lambda(E) \geqslant 1-\varepsilon$.

Suppose $s \in E$. Then

$$
m\left(t:\left|\sum r_{i}(t) T x_{i}(s)\right| \leqslant \frac{1}{4}\left(\sum\left|T x_{i}(s)\right|^{2}\right)^{1 / 2}\right) \leqslant \frac{3}{4}
$$

by the Paley-Zygmund inequality [9, p. 24], but $m\left(Q_{s}\right) \geqslant 7 / 8$. Hence (3.1.2) follows.
To complete the proof of the theorem we shall suppose that whenever $E \in \mathscr{B}$ and $\lambda(E) \geqslant 1-\varepsilon$, then there exists $x \in X$ with $\|x\| \leqslant 1$ and $\left\|P_{E} T x\right\|_{p, \infty}>A$. We shall deduce a contradiction and this will show that for some $E,\left\|P_{E} T\right\|_{p, \infty} \leqslant A$, and prove the theorem.

For each $E \in \mathscr{B}$, let $\Gamma[E]$ be the set of pairs $(x, \xi)$ such that $x \in X,\|x\|<1$, $1 \leqslant \xi<\infty$ and $\lambda(E \cap(|T x|>A \xi))>\xi^{-p}$. Let

$$
\begin{aligned}
\alpha_{E} & =\inf (\xi: \text { there exists } x \text { with }(x, \xi) \in \Gamma[E]), \\
& =\infty \text { if } \Gamma[E]=\varnothing
\end{aligned}
$$

By hypothesis, $\alpha_{E}<\infty$ if $\lambda(E) \geqslant 1-\varepsilon$. Note that $\alpha_{E} \geqslant \alpha_{F}$ whenever $E \subset F$.
Let $E_{0}=K$ and choose $\left\{E_{n}: n=1,2, \ldots\right\},\left\{x_{n}: n=1,2, \ldots\right\}$ and $\left\{\xi_{n}\right.$ : $n=1,2, \ldots\}$ by induction as follows. Suppose $E_{n-1}$ is given. If $\Gamma\left[E_{n-1}\right]=\varnothing$, let $x_{n}=0, \xi_{n}=\infty$ and $E_{n}=E_{n-1}$. Otherwise choose $\left(x_{n}, \xi_{n}\right) \in \Gamma\left[E_{n-1}\right]$ with $\xi_{n} \leqslant$ $2 \alpha_{E_{n-1}}$ and let

$$
E_{n}=E_{n-1} \cap\left(\left|T x_{n}\right| \leqslant A \xi_{n}\right)
$$

This completes the induction. Now note that $\lambda\left(E_{n}\right) \leqslant \lambda\left(E_{n-1}\right)-\xi_{n}^{-p}, n=$ $1,2, \ldots$, and hence $\Sigma \xi_{n}^{-p} \leqslant 1$. In particular, $\xi_{n} \rightarrow \infty$ and hence $\alpha_{E_{n}} \uparrow \infty$. Let

$$
E_{\infty}=\bigcap_{n=1}^{\infty} E_{n} .
$$

Then $\alpha_{E_{\infty}} \geqslant \alpha_{E_{n}}$ for all $n$ and hence $\alpha_{E_{\infty}}=\infty$. We deduce $\lambda\left(E_{\infty}\right)<1-\varepsilon$ and so for some $N, \lambda\left(E_{N}\right)<1-\varepsilon$.

If $s \in K \backslash E_{N}$, then there exists $1 \leqslant n \leqslant N$ such that $s \in E_{n-1} \backslash E_{n}$ so that $\xi_{n}{ }^{-1}\left|T x_{n}(s)\right|>A$. Hence

$$
\left(\sum_{n=1}^{N} \xi_{n}^{-2}\left|T x_{n}(s)\right|^{2}\right)^{1 / 2}>A, \quad s \in K \backslash E_{N}
$$

However there is a set $F \in \mathscr{B}$ with $\lambda(F) \geqslant 1-\varepsilon$, by (3.1.2) such that

$$
\left(\sum_{n=1}^{N} \xi_{n}^{-2}\left|T x_{n}(s)\right|^{2}\right)^{1 / 2} \leqslant A\left(\sum_{n=1}^{N} \xi_{n}^{-p}\right)^{1 / p} \leqslant A, \quad s \in F
$$

Thus $F$ and $K \backslash E_{N}$ are disjoint but $\lambda(F)+\lambda\left(K \backslash E_{N}\right)>1$. This contradiction establishes the theorem.

Corollary 3.2. Under the assumptions of Theorem 3.1, there exists $\varphi \in L_{0}$ with $\varphi>0$ a.e. such that $M_{\varphi} T \in \mathcal{E}(X, L(p, \infty))$ where $M_{\varphi} f=\varphi \cdot f, f \in L_{0}$.

Proof. For each $n \in \mathbf{N}$ pick $E_{n} \in \mathscr{B}$ with $\lambda\left(E_{N}\right) \geqslant 1-1 / n$ and such that $P_{E_{n}} T \in \mathcal{L}(X, L(p, \infty))$ with $\left\|P_{E_{n}} T\right\|=\beta_{n}$ say. Let $F_{n}=E_{n} \backslash\left(E_{1} \cup \cdots \cup E_{n-1}\right)$
and define

$$
\varphi(s)=2^{-n} \beta_{n}^{-1}, \quad s \in F_{n} .
$$

Then $M_{\varphi} P_{F_{n}} T \in \mathcal{L}(X, L(p, \infty))$ and $\left\|M_{\varphi} P_{F_{n}} T\right\| \leqslant 2^{-n}$. Hence

$$
M_{\varphi} T=\sum M_{\varphi} P_{F_{N}} T \in \mathcal{L}(X, L(p, \infty))
$$

4. Local convergence of operators on $L_{p}(0<p<1)$. Suppose $X$ is a $p$-Banach space. We define on $\mathcal{L}\left(L_{p}, X\right)$ a vector topology which we call the topology of local convergence as follows: a base of neighborhoods of 0 consists of sets of the form

$$
\mathscr{2}(\varepsilon)=\left\{T: \exists B \in \mathscr{B}, \lambda(B) \geqslant 1-\varepsilon,\left\|T P_{B}\right\|<\varepsilon\right\},
$$

for $\varepsilon>0$. These sets form a base for a metrizable vector topology on $\mathcal{E}\left(L_{p}, X\right)$ weaker than the usual quasinorm topology.

Proposition 4.1. The topology of local convergence is stronger than the topology of pointwise convergence on bounded sets. The unit ball of $\mathcal{L}\left(L_{p}, X\right)$ is complete for local convergence.

Proof. Suppose $T_{n} \rightarrow 0$ in local convergence with $\left\|T_{n}\right\| \leqslant M<\infty$ and $f \in L_{p}$. There exists a sequence $\varepsilon_{n} \rightarrow 0$ and $B_{n} \in \mathscr{B}$ with $\lambda\left(B_{n}\right) \geqslant 1-\varepsilon_{n}$ such that $\left\|T_{n} P_{B_{n}}\right\|$ $\leqslant \varepsilon_{n}$. Hence

$$
\left\|T_{n} f\right\| \leqslant\left(\varepsilon_{n}^{p}\|f\|^{p}+\left\|T_{n}\right\|^{p}\left\|f-P_{B_{n}} f\right\|^{p}\right)^{1 / p} \rightarrow 0 \quad \text { as } n \rightarrow \infty,
$$

since

$$
\int_{K-B_{n}}|f(t)|^{p} d \lambda(t) \rightarrow 0, \quad \text { as } \lambda\left(K-B_{n}\right)<\varepsilon_{n} \rightarrow 0 .
$$

The second half of the proposition is an immediate consequence of the first. If $\left\|T_{n}\right\| \leqslant 1$ and $T_{n}$ is a Cauchy sequence for local convergence then $T_{n}$ converges pointwise to a limit $T$ and it is easy to see that $T_{n} \rightarrow T$ in local convergence.

Proposition 4.2. For each $T \in \mathcal{L}\left(L_{p}, X\right)$ there is an essentially unique Borel function $t \mapsto \eta(t ; T)(t \in K)$ such that

$$
\left\|T P_{B}\right\|=\underset{t \in B}{\operatorname{ess} \sup } \eta(t ; T)
$$

whenever $B \in \mathscr{B}$ and $\lambda(B)>0$.
Thus $T_{n} \rightarrow T$ in local convergence if and only if $\eta\left(t ; T-T_{n}\right) \rightarrow 0$ in $\lambda$-measure.
Remark. We shall call the function $\eta(\cdot, T)$ the local quasinorm of $T$.
Proof. For each $n \in \mathbf{N}$, let $E(n, j)\left(1 \leqslant j \leqslant 2^{n}\right)$ be a partitioning of $K$ into $2^{n}$ disjoint Borel sets of measure $2^{-n}$, such that

$$
E(n, j)=E(n+1,2 j-1) \cup E(n+1,2 j), \quad 1 \leqslant j<2^{n}, n=1,2, \ldots,
$$

and such that the algebra generated by the sets $\left\{E(n, j), 1 \leqslant j<2^{n}, n=1,2, \ldots\right\}$ is $\lambda$-dense in $\mathscr{B}$.

For each $n \in \mathbf{N}$, define $\varphi_{n}$ a simple Borel function on $K$ by

$$
\varphi_{n}(t)=2^{n}\left\|T 1_{E(n, j)}\right\|^{p}, \quad t \in E(n, j)
$$

Let $\mathscr{F}_{n}$ denote the (finite) algebra generated by $\left(E(n, j): 1<j<2^{n}\right)$. Then

$$
\begin{aligned}
\mathcal{E}\left(\varphi_{n+1} \mid \mathscr{F}_{n}\right)(t) & =2^{n}\left(\left\|T 1_{E(n+1,2 j-1)}\right\|^{p}+\left\|T 1_{E(n+1,2 j)}\right\|^{p}\right) \\
& \geqslant \varphi_{n}(t), \quad t \in E(n, j) .
\end{aligned}
$$

Hence $\varphi_{n}$ is an $\mathscr{F}_{n}$-submartingale and of course $0<\varphi_{n} \leqslant\|T\|^{p}$. Thus $\varphi_{n}$ converges $\lambda$-almost everywhere and in $L^{1}$-norm to a limit $\varphi$. We define

$$
\eta(t ; T)=\varphi(t)^{1 / p}, \quad t \in K .
$$

If $B \in \mathscr{B}$ and $\lambda(B)>0$, then

$$
\left\|T P_{B}\right\|=\sup \left\{\left\|T 1_{C}\right\| / \lambda(C)^{1 / p}: C \subset B, \lambda(C)>0\right\}
$$

If $C \subset B$ and $\lambda(C)>0$, then for $\varepsilon>0$ there exists $n \in N$ and $E \in \mathscr{F}_{n}$ such that $\lambda(E \Delta C)<\varepsilon$. Then

$$
\begin{aligned}
\left\|T 1_{C}\right\|^{p} & <\|T\|^{p} \varepsilon+\left\|T 1_{E}\right\|^{p} \\
& <\|T\|^{p} \varepsilon+\int_{E} \varphi_{n}(t) d \lambda(t) \\
& <\|T\|^{p} \varepsilon+\int_{E} \varphi(t) d \lambda(t)
\end{aligned}
$$

(since $\varphi_{n}$ is a submartingale),

$$
\begin{aligned}
& <2\|T\|^{p} \varepsilon+\int_{C} \varphi(t) d \lambda(t) \\
& <2\|T\|^{p} \varepsilon+\lambda(C) \underset{t \in B}{\operatorname{ess} \sup } \varphi(t) .
\end{aligned}
$$

Thus,

$$
\left\|T P_{B}\right\| \leqslant \underset{t \in B}{\text { ess } \sup _{t}} \eta(t ; T) .
$$

Conversely suppose, $\lambda(C)>0$ and $\eta(t ; T) \geqslant \alpha, t \in C$. For $\varepsilon>0$, choose $n \in \mathbf{N}$ and $E \in \mathscr{F}_{n}$ as before. Thus for some $m \geqslant n$,

$$
\int_{C} \varphi_{m} d \lambda \geqslant \alpha^{p} \lambda(C)-\varepsilon
$$

and so

$$
\int_{E} \varphi_{m} d \lambda \geqslant \alpha^{p} \lambda(C)-\varepsilon\left(1+\|T\|^{p}\right)
$$

Let $E_{1}, \ldots, E_{l}$ be the atoms of $\mathscr{F}_{m}$ contained in $E$.

$$
\begin{aligned}
\left\|T 1_{E_{,}}\right\|^{p} & \leqslant\left\|T 1_{E_{, n c}}\right\|^{p}+\left\|T 1_{E_{,} \backslash C}\right\|^{p} \quad(1<j<l) \\
& \leqslant\left\|T P_{B}\right\|^{p} \lambda\left(E_{j} \cap C\right)+\|T\|^{p} \lambda\left(E_{j} \backslash C\right) .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\int_{E} \varphi_{m} d \lambda & <\left\|T P_{B}\right\|^{p} \lambda(E \cap C)+\|T\|^{p} \lambda(E \backslash C) \\
& <\left\|T P_{B}\right\|^{p} \lambda(C)+\varepsilon\|T\|^{p}
\end{aligned}
$$

Hence

$$
\left\|T P_{B}\right\|^{p} \lambda(C) \geqslant \alpha^{p} \lambda(C)-\varepsilon\left(1+2\|T\|^{p}\right)
$$

As $\varepsilon>0$ is arbitrary, $\left\|T P_{B}\right\| \geqslant \alpha$, and this implies

$$
\left\|T P_{B}\right\| \geqslant \underset{t \in B}{\operatorname{ess} \sup } \eta(t ; T)
$$

Essential uniqueness of $\eta$ is obvious and the final part follows easily from the definition of local convergence.

The next result is a technical lemma whose usefulness will emerge later.
Lemma 4.3. Suppose $G$ is a subset of $\mathfrak{E}\left(L_{p}, X\right)$ which is closed in $\mathcal{L}\left(L_{p}, X\right)$ in its usual topology. Suppose $\mathcal{G}$ has the properties:
(4.3.1) If $T \in \mathscr{G}$ and $B \in \mathscr{B}$ then $T P_{B} \in \mathscr{G}$.
(4.3.2) If $T \in \mathcal{E}\left(L_{p}, X\right)$ and for every $\varepsilon>0$ there exists $B \in \mathscr{B}$ with $\lambda(B)>1$ $\varepsilon$ and $T P_{B} \in \mathscr{G}$ then we have $T \in \mathscr{G}$.
Then $\mathscr{G}$ is closed under local convergence.
Proof. Suppose $T_{n} \in \mathscr{G}$ and $T_{n} \rightarrow T$ locally. Then by Egoroff's theorem, for any $\varepsilon>0$ there exists $B \in \mathscr{B}$ with $\lambda(B)>1-\varepsilon$ and

$$
\underset{t \in B}{\operatorname{ess} \sup } \eta\left(t ; T-T_{n}\right) \rightarrow 0
$$

i.e.,

$$
\left\|\left(T-T_{n}\right) P_{B}\right\| \rightarrow 0
$$

Hence by (4.3.1), $T P_{B} \in \mathscr{G}$ and so by (4.3.2), $T \in \mathscr{G}$.
Proposition 4.4. Suppose $X$ is a $\sigma$-complete $p$-Banach lattice with the Fatou property, $\left\|\sup _{n} x_{n}\right\|=\sup _{n}\left\|x_{n}\right\|$, whenever $x_{n}$ is a bounded increasing sequence of positive elements (e.g. a $\sigma$-complete $p$-Banach function space, see §2). Then $\mathcal{L}\left(L_{p}, X\right)$ is a $p$-Banach lattice, and if $T \in \mathcal{L}\left(L_{p}, X\right)$ then $\||T|\|=\|T\|$ and $\eta(t ;|T|)=$ $\eta(t ; T)$ a.e.

Proof. Clearly $\mathcal{L}\left(L_{p}, X\right)$ is an ordered $p$-Banach space. To show that it is a lattice, consider the sets $E(n, j)$ defined in 4.1. We define

$$
|T| 1_{E(n, j)}=\sup _{m>n} \sum_{k=1}^{2^{m-n}}\left|T 1_{E\left(m, 2^{\left.m-n_{j}-k+1\right)}\right.}\right|,
$$

where the supremum is taken in the $\sigma$-complete lattice $X$. The sequence on the right-hand side is increasing and bounded and we have

$$
\left\||T| 1_{E(n, j)}\right\|^{p}=\sup _{m>n}\left\|\sum_{k=1}^{2^{m-n}} \mid T 1_{E\left(m, 2^{m-n_{j}-k+1}\right)}\right\|^{p} \leqslant\|T\|^{p} \lambda(E(n, j)) .
$$

It is easy to see that $|T|$ may be extended to a positive linear operator in $\mathcal{E}\left(L_{p}, X\right)$ and that $\||T|\|=\|T\|$. Clearly, $|T| \geqslant \pm T$ and is the least operator with this property. Thus, $\mathcal{L}\left(L_{p}, X\right)$ is a $p$-Banach lattice.

If $B \in \mathscr{B}$ then clearly $|T| P_{B} \geqslant \pm T P_{B}$ so that $|T| P_{B} \geqslant\left|T P_{B}\right|$ and $|T| P_{K \backslash B} \geqslant$ $\left|T P_{K \backslash B}\right|$. Conversely, $\left|T P_{B}\right|+\left|T P_{K \backslash B}\right| \geqslant \pm T$ so that

$$
\left|T P_{B}\right|+\left|T P_{K \backslash B}\right| \geqslant|T| P_{B}+|T| P_{K \backslash B}
$$

Hence, $|T| P_{B}=\left|T P_{B}\right|$ and so $\left\||T| P_{B}\right\|=\left\|T P_{B}\right\|$. From this it follows that $\eta(t ;|T|)=\eta(t ; T)$ а.е.
5. Dominated operators. Let $X$ be a quasi-Banach function space and let $T$ : $X \rightarrow L_{0}$ be a linear operator. We shall say that $T$ is dominated if there is a positive linear operator $P: X \rightarrow L_{0}$ such that $P f \geqslant|T f|$ a.e., $f \in X$.

Denote by $\mathfrak{N}(K)$ the space of regular Borel measures on $K$.
Theorem 5.1. Suppose $X$ is a separable quasi-Banach function space. Then if $T \in \mathcal{L}\left(X, L_{0}\right), T$ is dominated if and only if there is a weak*-Borel map $t \mapsto \nu_{t}$ ( $K \mapsto \mathfrak{T}(K)$ ) such that
(5.1.1) If $B \in \mathscr{B}$ and $\lambda(B)=0$ then $\left|\nu_{t}\right|(B)=0$ a.e.
(5.1.2) If $f \in X$, then $f$ is $\left|\nu_{t}\right|$-integrable a.e.
(5.1.3) $T f(t)=\int_{K} f(s) d \nu_{t}(s)$ a.e.

Proof. First observe that if (5.1.1), (5.1.2) and (5.1.3) hold then (observing that $t \rightarrow\left|\nu_{t}\right|$ is weak*-Borel, see [11]),

$$
P f(t)=\int_{K} f(s) d\left|\nu_{t}\right|(s)
$$

defines a positive linear operator from $X$ into $L_{0}$. In fact, (5.1.2) shows that $P$ defines a linear map. To show continuity suppose $f_{n} \in X$ and $\left\|f_{n}\right\|<2^{-n}$; then $g=\Sigma\left|f_{n}\right| \in X$ and $P f_{n} \rightarrow 0$ a.e. by applying the Dominated Convergence Theorem.

Conversely let us suppose $T$ is dominated. As in [11, Theorem 3.1], we need only consider the case when $K$ is totally disconnected. Suppose $\Sigma$ is the countable algebra of clopen subsets of $K$. By modifying each on a set of measure zero we may suppose $A \mapsto T 1_{A}(t)(A \in \Sigma), A \rightarrow P 1_{A}(t)(A \in \Sigma)$ additive for all $t \in K$ and that $\left|T 1_{A}(t)\right| \leqslant P 1_{A}(t), A \in \Sigma, t \in K$. Extend by linearity to $f \in S(\Sigma)$, the linear space of all simple continuous functions on $K$. Then

$$
|T f(t)| \leqslant|P f(t)| \leqslant\|f\|_{\infty} P 1_{K}(t), \quad f \in S(\Sigma) .
$$

Hence there exist measures $\nu_{t}, \mu_{t} \in \mu(K)$ such that $T f(t)=\int_{K} f d \nu_{t}, f \in S(\Sigma)$, $P f(t)=\int_{K} f d \mu_{t}, f \in S(\Sigma)$, and $\left|\nu_{t}\right| \leqslant \mu_{t}$ and $\left\|\mu_{t}\right\|=P 1_{K}(t)$.

For each $f \in S(\Sigma)$, the maps $t \mapsto \int f d \nu_{t}, t \mapsto \int f d \mu_{t}$ are Borel and hence as $S(\Sigma)$ is dense in $C(K)$, the maps $t \mapsto \nu_{t}, t \mapsto \mu_{t}$ are weak*-Borel. Note that (5.1.1) and (5.1.2) must hold.

Now suppose $f \in X$ and $f \geqslant 0$. Then there is a sequence $f_{n} \in S(\Sigma)$ such that $0<f_{n} \uparrow f$ a.e. Since $X$ is separable, it follows that $f_{n} \rightarrow f$ in $X$ (cf. [5]). By passing to a subsequence we may therefore suppose $T f_{n} \rightarrow T f$ a.e. Now $P f_{n} \leqslant P f$ a.e. for all $n$ and hence by the Monotone Convergence Theorem, $f$ is $\mu_{r}$-integrable for a.e. $t$. Hence $f$ is $\left|\nu_{t}\right|$-integrable a.e. and so by the Dominated Convergence Theorem,

$$
\begin{aligned}
T f(t) & =\lim _{n \rightarrow \infty} T f_{n}(t) \quad \text { a.e. } \\
& =\lim _{n \rightarrow \infty} \int f_{n} d \nu_{t} \quad \text { a.e. } \\
& =\int f d \nu_{t} \quad \text { a.e. }
\end{aligned}
$$

The theorem now follows by linearity.
We shall call the map $t \mapsto \nu_{t}$ of 5.1 the kernel of $T$. Note that it is essentially unique.

Corollary 5.2. If $T$ is dominated there exists an operator $|T|=\sup (+T,-T)$ in $\mathcal{L}\left(X, L_{0}\right)$ and $|T|$ has kernel, $t \rightarrow\left|\nu_{t}\right|$.

Remark. Since $\mathcal{L}\left(L_{1}, L_{1}\right)$ is a lattice (Chacon-Krengel [1]), every $T \in \mathcal{L}\left(L_{1}, L_{1}\right)$ is dominated, and the same is true for $\mathcal{E}\left(L_{p}, L_{p}\right)(0<p<1)$ [11]. That $\mathcal{L}\left(L_{1}, L_{0}\right)$ is not a lattice and hence that there exist nondominated operators $T: L_{1} \rightarrow L_{0}$ is due to Pryce [27] (Pryce only observes that $\mathcal{E}\left(l_{1}, L_{0}\right)$ is not a lattice, but the same argument holds for $L_{1}$ ). See also Nikisin [22].

We shall say that the kernel $t \mapsto \nu_{t}$ is atomic if $\nu_{t} \in \mathscr{R}_{a}(K)$ a.e. where $\mathscr{R}_{a}(K)$ is the space of purely atomic measures (of the form $\Sigma a_{n} \delta\left(t_{n}\right)$ where $\Sigma\left|a_{n}\right|<\infty$ ). Hence if $t \mapsto \nu_{t}$ is atomic it may be redefined in a Borel set of measure zero so that $\nu_{t} \in \mathscr{R}_{a}(K)$ everywhere.

The following theorem is (essentially) proved in Theorem 3.2 of [11].
Theorem 5.3. If $t \mapsto \nu_{t}$ is an atomic kernel then there are Borel maps $a_{n}: K \rightarrow \mathbf{R}$, $\sigma_{n}: K \rightarrow K(n=1,2, \ldots)$ so that

$$
\begin{gather*}
\left|a_{n}(t)\right| \geqslant\left|a_{n+1}(t)\right|, \quad n=1,2, \ldots  \tag{5.3.1}\\
\sigma_{n}(t) \neq \sigma_{m}(t), \quad m \neq n \in \mathbf{N}  \tag{5.3.2}\\
\nu_{t}=\sum_{n=1}^{\infty} a_{n}(t) \delta\left(\sigma_{n} t\right), \quad t \in K \tag{5.3.3}
\end{gather*}
$$

Definitions. A dominated operator $T$ will be called elementary if its kernel $t \mapsto \nu_{t}$ is atomic and the support of $\nu_{t}$ is (almost everywhere) at most one point; thus $T$ is of the form

$$
T f(t)=a(t) f(\sigma t) .
$$

$T$ is locally elementary if there exist Borel sets ( $B_{n}: n \in \mathbf{N}$ ) which are disjoint and satisfy $\lambda\left(\cup B_{n}\right)=1$ and such that $T P_{B_{n}}$ is elementary for each $n$.
$T$ is of finite type if $t \mapsto \nu_{t}$ is atomic and the support of $\nu_{t}$ is almost everywhere finite.

Remark. $T$ is of finite type if and only if $T$ is the restriction of an endomorphism of $L_{0}$ (Kwapien [17], see also [11]).

Theorem 5.4. If $T$ is a dominated operator of finite type and $\varepsilon>0$ then there exists $A \in \mathscr{B}$ with $\lambda(A) \geqslant 1-\varepsilon$ and such that $P_{A} T$ is locally elementary.

Proof. Let $\nu_{t}=\sum_{n=1}^{\infty} a_{n}(t) \delta\left(\sigma_{n} t\right)$ as in Theorem 5.3. Then there exists a closed set $A$ with $\lambda(A) \geqslant 1-\varepsilon$ such that for some $N, a_{n}(t)=0$ if $t \in A$ and $n>N$ and $\sigma_{1}, \ldots, \sigma_{N}$ are continuous on $A$. Now each $t \in K$ has a neighborhood $V_{t}$ such that the sets $A \cap \sigma_{i}^{-1}\left(V_{t}\right)(1 \leqslant i \leqslant N)$ are pairwise disjoint and by a compactness argument we may cover $K$ with finitely many disjoint Borel sets $B_{1}, \ldots, B_{m}$ so that the sets $A \cap \sigma_{i}^{-1}\left(B_{j}\right)(1 \leqslant i \leqslant N)$ are pairwise disjoint for each $j$. Then $P_{A} T P_{B_{j}}$ is elementary.

Now for $\mu \in \mathscr{T}(K)$ and $x \geqslant 0$ define

$$
\Delta_{x}(\mu)=\sum\{\delta(t):|\mu|\{t\}>x\}
$$

Lemma 5.5. If $t \mapsto \nu_{t}$ is an atomic kernel then $t \mapsto \Delta_{x}\left(\nu_{t}\right)$ is weak*-Borel.
Proof. Write $\nu_{t}=\sum_{n=1}^{\infty} a_{n}(t) \delta\left(\sigma_{n} t\right)$. Then

$$
\Delta_{x}\left(\nu_{t}\right)=\sum_{\left|a_{n}(t)\right|>x} \delta\left(\sigma_{n} t\right)
$$

is easily seen to be weak*-Borel.
For $x \geqslant 0$ define

$$
\alpha_{x}(B)=\int_{K} \Delta_{x}\left(\nu_{t}\right)(B) d \lambda(t) .
$$

Then for each $x>0, \alpha_{x}$ is a positive (possibly infinite) measure, satisfying $\lambda(B)=0 \Rightarrow \alpha_{x}(B)=0$. It follows that $\alpha_{x}$ has a (possibly infinite) Radon-Nikodym derivative. For each rational $x$ let $w(\cdot, x)$ be a Borel derivative of $\alpha_{x}$. Since $\alpha_{x} \geqslant \alpha_{y}$ if $x<y$, we may suppose $w(t, x) \geqslant w(t, y), t \in K, x \geqslant y$. Now define for any $(t, x) \in K \times[0, \infty)$,

$$
w(t, x)=\sup _{\substack{y>x \\ y \in Q}} w(t, y)
$$

(5.5.1) $w$ is monotone-decreasing and lower-semicontinuous in $x$ for each fixed $t \in K$.
(5.5.2) $w(t, x)$ is a derivative of $\alpha_{x}$ for each $x \geqslant 0$.
(5.5.3) $w$ is Borel on $K \times[0, \infty)$.
(5.5.1) is clear. To prove (5.5.2) suppose $y_{n} \in Q$ and $y_{n} \downarrow x$. Then $\alpha_{y_{n}}(B) \uparrow \alpha_{x}(B)$ for each $B \in \mathscr{B}$ and the Monotone Convergence Theorem gives the result. For (5.5.3), observe

$$
w(t, x)=\sup _{m} w\left(t,-\frac{1}{m}[-m x]\right)
$$

where $[x]$ is the largest integer $\leqslant x$, and the functions

$$
(t, x) \mapsto w\left(t,-\frac{1}{m}[-m x]\right)
$$

are clearly Borel.
By an application of Fubini's theorem it can be seen that up to sets of product measure zero, the function $w$ satisfying (5.5.2) and (5.5.3) is unique.

Definition. The function $w$ is called the distribution of the operator $T$ (or kernel $\left.t \mapsto \nu_{t}\right)$.
6. Operators from $L_{p}$ into $L(p, \infty)$ and $L_{0}$ where $0<p<1$. The space $L(1, \infty)$ is not locally convex, although Cwikel and Saghar [3] show that its dual is nontrivial. For $p \leqslant 1$, Saghar [29] and Hunt [8] show that $L(p, q)$ is $r$-convex whenever $r<p$ and $r \leqslant q$; in particular, $L(p, \infty)$ is $r$-convex for any $r<p$. The next result improves this; we note here that an inequality equivalent to Theorem 6.1 has been obtained by Pisier and Zinn (unpublished).

Theorem 6.1. If $0<p<1, L(p, \infty)$ is $p$-convex.
Proof. Let $M_{p}=2^{1 / p} p(1-p)^{-1}$. We shall show that if $f_{1}, \ldots, f_{n} \in L(p, \infty)$,

$$
\left\|f_{1}+\cdots+f_{n}\right\| \leqslant M_{p}\left(\left\|f_{1}\right\|^{p}+\cdots+\left\|f_{n}\right\|^{p}\right)^{1 / p}
$$

Let $h=f_{1}+\cdots+f_{n}$ and let $h^{*}$ be its decreasing rearrangement. For $0<\tau<1$ let $A=A(\tau)=\left(|h| \geqslant h^{*}(\tau)\right)$. Then $\lambda(A) \geqslant \tau$. For $1 \leqslant k \leqslant n$, let

$$
\tau_{k}=\frac{1}{2} \tau \frac{\left\|f_{k}\right\|^{p}}{\left\|f_{1}\right\|^{p}+\cdots+\left\|f_{n}\right\|^{p}}
$$

Choose Borel sets $E_{k}, 1 \leqslant k \leqslant n$, in $K$ such that $\lambda\left(E_{k}\right)=\tau_{k}$ and $\left|f_{k}(t)\right| \geqslant f_{k}^{*}\left(\tau_{k}\right)$, $t \in E_{k}$. Let $E=E_{1} \cup \cdots \cup E_{n}$; then $\lambda(E) \leqslant(1 / 2) \tau$.

Now,

$$
\begin{aligned}
h^{*}(\tau) & \leqslant \inf _{t \in A}|h(t)| \leqslant \inf _{t \in A \backslash E}|h(t)| \\
& \leqslant \frac{1}{\lambda(A \backslash E)} \int_{A \backslash E}|h(t)| d \lambda(t) \leqslant \frac{2}{\tau} \int_{K \backslash E}|h(t)| d \lambda(t) \\
& \leqslant \frac{2}{\tau} \sum_{k=1}^{n} \int_{K \backslash E_{k}}\left|f_{k}(t)\right| d \lambda(t) \leqslant \frac{2}{\tau} \sum_{k=1}^{n} \int_{\tau_{k}}^{\infty}\left\|f_{k}\right\| x^{-1 / p} d x \\
& =\frac{2 p}{\tau(1-p)} \sum_{k=1}^{n}\left\|f_{k}\right\| \tau_{k}^{1-1 / p}=\left(\frac{2}{\tau}\right)^{1 / p} \frac{p}{1-p}\left(\sum_{k=1}^{n}\left\|f_{k}\right\|^{p}\right)^{1 / p}
\end{aligned}
$$

and the result follows.
Now it is possible to requasinorm $L(p, \infty)$ if $0<p<1$ as a $p$-Banach function space, as in §2.

Corollary 6.2. $\mathcal{L}\left(L_{p}, L(p, \infty)\right)$ and $\mathcal{L}\left(L_{p}, L_{0}\right)$ are lattices.
Proof. By Proposition 4.4 and Corollary 3.2.
It follows that every $T \in \mathcal{L}\left(L_{\rho}, L_{0}\right)$ is dominated. We now show that the kernel is atomic.

Theorem 6.3. Suppose $T \in \mathcal{E}\left(L_{p}, L_{0}\right)$; then if $p<r<1$ the kernel $\nu_{t}$ of $T$ satisfies $\left\|\nu_{t}\right\|_{r}<\infty$ a.e. In particular, $\nu_{t}$ is atomic.

Equivalently there exist Borel maps $a_{n}: K \rightarrow \mathbf{R}, \sigma_{n}: K \rightarrow K$ such that

$$
T f(t)=\sum_{n=1}^{\infty} a_{n}(t) f\left(\sigma_{n} t\right) \quad \text { a.e., } f \in L_{p}
$$

and

$$
\sum_{n=1}^{\infty}\left|a_{n}(t)\right|^{r}<\infty \quad \text { a.e. }
$$

whenever $p<r<1$.
Proof. By Corollary 3.2 it suffices to prove the same result for $T \in$ $\mathcal{L}\left(L_{p}, L(p, \infty)\right)$. As before, we may assume $K$ totally disconnected. Let $\mathbb{Q}_{n}=\left\{A_{n, k}\right.$ : $1 \leqslant k \leqslant l(n)\}$ be a partitioning of $K$ into $l(n)$ nonempty clopen sets of diameter at most $n^{-1}$ and suppose $\mathbb{Q}_{n+1}$ refines $\mathbb{Q}_{n}$ for $n \geqslant 1$. Pick $\tau_{k}^{n} \in A_{n, k}$ for each $1 \leqslant k \leqslant l(n)$. Let $T 1_{A_{n, k}}=b_{k}^{n}, 1 \leqslant k \leqslant l(n)$, where each $b_{k}^{n}$ is Borel. We may further assume that

$$
b_{k}^{n}=\sum_{A_{n+1, j} \subset A_{n, k}} b_{j}^{n+1}
$$

everywhere. For $s \in K$, define

$$
\nu_{s}^{n}=\sum_{k=1}^{l(n)} b_{k}^{n}(s) \delta\left(\tau_{k}^{n}\right)
$$

If $p<r<1,\left\|\nu_{s}^{n}\right\|_{r}^{r}=\sum_{k=1}^{1(n)}\left|b_{k}^{n}(s)\right|^{r}=c_{n}(s)$, say. Then $\left|b_{k}^{n}\right|^{r} \in L\left(p r^{-1}, \infty\right)$ and

$$
\left\|c_{n}\right\|_{p r^{-1}, \infty} \leqslant M_{p r^{-1}}\left(\sum_{k=1}^{l(n)}\left\|\left|b_{k}^{n}\right|^{r}\right\|_{p r^{-1}, \infty}^{p r^{-1}}\right)^{p^{-1}}
$$

by Theorem 6.1. However,

$$
\left\|\left|b_{k}^{n}\right|^{r}\right\|_{p r^{-1}, \infty}=\left\|b_{k}^{n}\right\|_{p, \infty}^{r} \leqslant\|T\|^{r} \lambda\left(A_{n, k}\right)^{r p^{-1}} .
$$

Hence,

$$
\left\|c_{n}\right\|_{p r^{-1}, \infty} \leqslant M_{p r^{-1}}\|T\|^{r} .
$$

Now $0 \leqslant c_{n} \leqslant c_{n+1}$ everywhere and hence, as $L\left(p r^{-1}, \infty\right)$ is $\sigma$-complete, $c=\sup c_{n}$ $\in L\left(p r^{-1}, \infty\right)$ and $\|c\|_{p r^{-1}, \infty} \leqslant M_{p r^{-1}}\|T\|^{r}$. In particular, $\sup _{n}\left\|\nu_{s}^{n}\right\|_{r}<\infty$ a.e. and so almost everywhere the sequence $\left\{\nu_{s}^{n}: n \in \mathbf{N}\right\}$ is bounded. Since $T$ is dominated, it has a kernel $\nu_{t}$ and clearly $\nu_{t}\left(A_{n, k}\right)=\lim _{m \rightarrow \infty} \nu_{t}^{m}\left(A_{n, k}\right)$ a.e., for each $k, n$. Hence $\nu_{t}^{m} \rightarrow \nu_{t}$ weak* almost everywhere. In particular,

$$
\left\|\nu_{t}\right\|_{r} \leqslant \liminf _{m \rightarrow \infty}\left\|\nu_{t}^{m}\right\|_{r}<\infty \quad \text { a.e. }
$$

The result now follows from Theorem 5.1.
Example. Suppose ( $\xi_{t}: 0 \leqslant t \leqslant 1$ ) is a symmetric $p$-stable process in $L_{0}(K)$, i.e. if $0 \leqslant t_{1}<t_{2}<\cdots<t_{n} \leqslant 1$ then $\xi_{t_{j}}-\xi_{t_{j-1}}(2 \leqslant j \leqslant n)$ are mutually independent and

$$
\mathcal{E}\left(e^{i \tau\left(\xi_{1}-\xi\right)}\right)=\exp \left(-|t-s||\tau|^{p}\right)
$$

Then, $T f=\int f d \xi_{t}$ defines an embedding of $L_{p}$ into $L_{0}$. The preceding theorem guarantees that the sample paths $\xi_{t}(s)(0 \leqslant t \leqslant 1)$ almost everywhere are jump functions with countably many jumps $a_{n}(s)$ such that $\sum\left|a_{n}(s)\right|^{r}<\infty$ for any $r>p$.

Theorem 6.3 does not give necessary and sufficient conditions for $T \in$ $\mathcal{L}\left(L_{p}, L(p, \infty)\right)$. We now proceed to this problem.

Theorem 6.4. In order that the weak*-Borel map $t \mapsto \nu_{t}$ be the kernel of a linear operator $T \in \mathcal{E}\left(L_{p}, L(p, \infty)\right)$ it is necessary and sufficient that for some $C$,

$$
\begin{equation*}
x \alpha_{x}(B)^{1 / p}<C \lambda(B)^{1 / p}, \quad B \in \mathscr{B} . \tag{6.4.1}
\end{equation*}
$$

Then,

$$
\begin{equation*}
\|T\|<M_{p} C . \tag{6.4.2}
\end{equation*}
$$

Remark. Of course,

$$
\alpha_{x}(B)=\int_{K} \Delta_{x}\left(\nu_{t}\right)[B] d \lambda(t), \quad B \in \mathscr{B} .
$$

Proof. First suppose $T \in \mathbb{E}\left(L_{p}, L(p, \infty)\right)$. Then its kernel $\nu_{t}$ is atomic. Suppose as in Theorem 5.3, $\nu_{t}=\sum_{n=1}^{\infty} a_{n}(t) \delta\left(\sigma_{n} t\right)$. Then $|T|$ has kernel $\left|\nu_{t}\right|=$ $\sum_{n=1}^{\infty}\left|a_{n}(t)\right| \delta\left(\sigma_{n} t\right)$ and $\||T|\| \leqslant M_{p}\|T\|$. Let

$$
\mu_{t}^{n}=\sum_{k=1}^{n}\left|a_{k}(t)\right| \delta\left(\sigma_{k} t\right) .
$$

Then $\mu_{r}^{n}$ is the kernel of a positive linear operator $S_{n} \in \mathcal{L}\left(L_{p}, L(p, \infty)\right)$ of finite type. Pick $A_{n}$ with $\lambda\left(A_{n}\right) \geqslant 1-1 / n$ so that $P_{A_{n}} S_{n}$ is locally elementary. Suppose $P_{A_{n}} S_{n} P_{B_{j}}$ is elementary for $1 \leqslant j<\infty$ where $\cup_{j} B_{j}=K$.

Suppose $B \subset B_{j}$ for some $j$; let $P_{A_{n}} S_{n} 1_{B}=h$. Then

$$
P_{A_{n}} S_{n} 1_{B}=\sum_{k=1}^{n} a_{k} \cdot 1_{\sigma_{k}^{-1}(B)} \cdot 1_{A_{n}}
$$

and as the sets $A_{n} \cap \sigma_{k}^{-1}(B)$ are mutually disjoint,

$$
\begin{equation*}
\lambda(|h|>x)=\sum_{k=1}^{n} \lambda\left(\sigma_{k}^{-1} B \cap A_{n} \cap\left(\left|a_{k}\right|>x\right)\right) \tag{6.4.3}
\end{equation*}
$$

Hence

$$
x^{p} \sum_{k=1}^{n} \lambda\left(\sigma_{k}^{-1} B \cap A_{n} \cap\left(\left|a_{k}\right|>x\right)\right) \leqslant M_{p}^{p}\|T\|^{p} \lambda(B) .
$$

Letting $n \rightarrow \infty$ we have

$$
x^{p} \sum_{k=1}^{\infty} \lambda\left(\sigma_{k}^{-1} B \cap\left(\left|a_{k}\right|>x\right)\right)<M_{p}^{p}\|T\|^{p} \lambda(B)
$$

i.e.,

$$
x\left(\alpha_{x}(B)\right)^{1 / p}<M_{p}\|T\| \lambda(B)^{1 / p} .
$$

For the converse, define $\mu_{t}^{n}, A_{n}$ as before. We show that the formally defined operator $P_{A_{n}} S_{n} \in \mathcal{L}\left(L_{p}, L(p, \infty)\right)$ and $\left\|P_{A_{n}} S_{n}\right\| \leqslant C M_{p}$. This follows easily from (6.4.3) and the fact that $L(p, \infty)$ is $p$-convex. Then $P_{A_{n}} S_{n} f \uparrow P f$ for $f \geqslant 0$ and $P$ is a positive operator with $\|P\| \leqslant C M_{p} . P$ has kernel $\left|\nu_{t}\right|$ and it then follows that $\nu_{t}$ is the kernel of an operator $T$ with $|T|=P$. Hence, $\|T\|<C M_{p}$.

Corollary 6.5. In order that $\nu_{t}$ be the kernel of some $T \in \mathcal{L}\left(L_{p}, L(p, \infty)\right)$ it is necessary and sufficient that

$$
\begin{equation*}
\underset{t \in K}{\operatorname{ess} \sup } W^{p, \infty}(t)=C<\infty \tag{6.5.1}
\end{equation*}
$$

where

$$
\begin{equation*}
W^{p, \infty}(t)=\sup _{0<x<\infty} x w(t, x)^{1 / p} \tag{6.5.2}
\end{equation*}
$$

where $w$ is the distribution of $\nu_{t}$. Then

$$
\|T\| \leqslant C M_{p}
$$

Proof. If $\alpha_{x}(B) \leqslant C^{p} x^{-p} \lambda(B), B \in \mathscr{B}$, then $w(t, x)<C^{p} x^{-p}$, $\lambda$-a.e. for each $x>0$. By Fubini's theorem, for $\lambda$-a.e. $t, x^{p} w(t, x)<C^{p}, x$-a.e., but as $w$ is monotone in $x$ this means $x^{p} w(t, x) \leqslant C^{p}$. Hence $W^{p, \infty}(t)<C$, $\lambda$-a.e.

The converse is easy.
Corollary 6.6. Suppose $T_{n} \in \mathcal{L}\left(L_{p}, L(p, \infty)\right)$ have distributions $w_{n}$ and

$$
W_{n}^{p, \infty}(t)=\sup _{0<x<\infty} x w_{n}(t, x)^{1 / p}
$$

Then if $W_{n}^{p, \infty}(t) \rightarrow 0$ in $\lambda$-measure, $T_{n} \rightarrow 0$ in the topology of local convergence.
Proof. If $B \in \mathscr{B}$, then $T_{n} P_{B}$ has distribution

$$
\begin{aligned}
w_{n, B}(t, x) & =w_{n}(t, x), \quad t \in B, \\
& =0, \quad t \notin B,
\end{aligned}
$$

and hence,

$$
\left\|T_{n} P_{B}\right\| \leqslant M_{p} \underset{t \in B}{\text { ess sup }} W_{n}^{p, \infty}(t) .
$$

The result now follows from Egoroff's theorem.

## 7. Operators from $L_{p}$ into $L_{0}$; the main result.

Proposition 7.1. Suppose $0<p<1$ and $p<q<r \leqslant 2$. Suppose $T \in$ $\mathcal{L}\left(L_{p}, L(p, \infty)\right)$ is nonzero and has distribution $w$ satisfying

$$
\begin{equation*}
\lim _{x \rightarrow 0} x w(t, x)^{1 / p}=\lim _{x \rightarrow \infty} x w(t, x)^{1 / p}=0, \quad \lambda \text {-a.e. } \tag{7.1.1}
\end{equation*}
$$

Then there is a strongly embedded subspace $V$ of $L_{q}$ such that $V \cong L_{r}$ and $T \mid V$ is an isomorphism.
[ $V$ is strongly embedded in $L_{q}$ if the $L_{q}-$ and $L_{0}$-topologies agree on $V$.]
Proof. Consider the following property of operators $T \in \mathcal{L}\left(L_{p}, L(p, \infty)\right)$.
(7.1.2) There exists $c>0$ and $B \in \mathscr{B}$ with $\lambda(B)>0$ such that whenever $C \in \mathscr{B}$ with $C \subset B$ and $\lambda(C)>0$ then there is a strongly embedded subspace $V[C]$ of $L_{q}(C)$ such that $V_{C} \cong L_{r}$ and $\|T f\|_{p, \infty} \geqslant c\|f\|_{p}, f \in V[C]$.

Let $\mathscr{G}$ be the subset of $\mathcal{E}\left(L_{p}, L(p, \infty)\right)$ of all $T$ for which (7.1.2) fails to hold. $\mathscr{G}$ is certainly closed and satisfies the condition of Lemma 4.3. Hence $\mathscr{G}$ is closed also for local convergence.

Suppose $\mathscr{G}$ contains an elementary operator $S \neq 0$, say $S f(t)=a(t) f(\sigma t)$. Then there is a subset $A$ of $K$ with $\lambda(A)>0$ such that $0<\varepsilon \leqslant|a(t)|<2 \varepsilon, t \in A$, and then the measure $\mu(B)=\lambda\left(\sigma^{-1} B \cap A\right), B \in \mathscr{B}$, is $\lambda$-continuous, with $\lambda$-derivative $\varphi$ say. Choose $E \in \mathscr{B}$ with $\lambda(E)>0$ so that $0<\delta \leqslant \varphi(t)<2 \delta, t \in E$.

Now let $\Lambda$ be the standard embedding of $L_{r}$ into $L_{q}$ by using an $r$-stable process (cf. [16] and the Example after 6.3; also cf. [8] for an order-preserving embedding). Observe that $\Lambda$ embeds $L_{r}$ strongly into $L_{q}$ and that for some constant $\gamma$ depending only on $p,\|\Lambda f\|_{p, \infty}=\gamma\|\Lambda f\|_{p}, f \in L_{r}$. Suppose $C \in \mathscr{B}, C \subset E$ and $\lambda(C)>0$. Let $\theta_{C}: C \rightarrow K$ be any Borel map such that $\lambda\left(\theta_{C}^{-1}(B)\right)=\lambda(C) \lambda(B), B \in \mathscr{B}$. Define $R_{C}: L_{0}(K) \rightarrow L_{0}(K)$ by

$$
\begin{aligned}
R_{C} f & =f\left(\theta_{C} t\right), \quad t \in C, \\
& =0, \quad t \notin C .
\end{aligned}
$$

Let $V=V[C]=R_{C} \Lambda\left(L_{r}\right)$. Then $V[C]$ is isomorphic to $L_{r}$ and strongly embedded in $L_{q}(C)$. If $f \in V[C],\|f\|_{p, \infty}=\gamma\|f\|_{p}$. For $x>0, f \in V[C]$,

$$
\begin{aligned}
\lambda(|S f|>x) & \geqslant \lambda\left(A \cap\left(|f \circ \sigma|>x \varepsilon^{-1}\right)\right) \\
& =\mu\left(|f|>x \varepsilon^{-1}\right) \geqslant \delta \lambda\left(|f|>x \varepsilon^{-1}\right) .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
x \lambda(|S f|>x)^{1 / p} & \geqslant x \delta^{1 / p} \lambda\left(|f|>x \varepsilon^{-1}\right)^{1 / p} \\
& =\varepsilon \delta^{1 / p}\left(x \varepsilon^{-1}\right) \lambda\left(|f|>x \varepsilon^{-1}\right)^{1 / p}
\end{aligned}
$$

Hence,

$$
\|S f\|_{p, \infty} \geqslant \varepsilon \delta^{1 / p}\|f\|_{p, \infty}=\gamma \varepsilon \delta^{1 / p}\|f\|_{p} .
$$

This shows (7.1.2) does hold with $c=\gamma \varepsilon \delta^{1 / p}$ and $B=E$. We conclude $S=0$.
Now suppose $T \in \mathscr{G}$ has kernel $\nu_{t}$, where $\nu_{t}=\sum_{n=1}^{\infty} a_{n}(t) \delta\left(\sigma_{n} t\right)$ as in Theorem 5.3, and the distribution $w$ of $T$ satisfies (7.1.1).

If $T \neq 0$ then $S \neq 0$ where $S$ has kernel $a_{1}(t) \delta\left(\sigma_{1} t\right)$; of course $S \in$ $\mathcal{L}\left(L_{p}, L(p, \infty)\right)$ and $|S| \leqslant|T|$.
For $n \in \mathbf{N}$, let $\mathscr{B}_{n}=\left\{B_{n, 1}, \ldots, B_{n,(n)}\right\}$ be a partition of $K$ into $l(n)$ disjoint Borel sets of diameter at most $n^{-1}$. Assume $\mathscr{B}_{n+1}$ refines $\mathscr{B}_{n}$. Let $C_{n, i}=\sigma_{1}^{-1} B_{n, i}$ and

$$
T_{n}=\sum_{i=1}^{l(n)} P_{C_{n, i}} T P_{B_{n, i}}
$$

Then $T_{n} \in \mathcal{G} . T_{n}$ has kernel $\nu_{t}^{n}$ where $\nu_{t}^{n}(B)=\nu_{t}\left(B \cap B_{n, i}\right), t \in C_{n, i}$. For fixed $t$, if $t \in C_{n, i(n)}$,

$$
\begin{aligned}
\left\|\nu_{t}^{n}-a_{1}(t) \delta\left(\sigma_{1} t\right)\right\| & =\left|\nu_{t}^{n}-a_{1}(t) \delta\left(\sigma_{1} t\right)\right|(K) \\
& =\left|\nu_{t}-a_{1}(t) \delta\left(\sigma_{1} t\right)\right|\left(B_{n, i(n)}\right)
\end{aligned}
$$

and so

$$
\begin{aligned}
\underset{n \rightarrow \infty}{\lim \sup }\left\|\nu_{t}^{n}-a_{1}(t) \delta\left(\sigma_{1} t\right)\right\| & \leqslant\left|\nu_{t}-a_{1}(t) \delta\left(\sigma_{1} t\right)\right| \lim \sup B_{n, i(n)} \\
& =\left|\nu_{t}-a_{1}(t) \delta\left(\sigma_{1} t\right)\right|\left\{\sigma_{1} t\right\}=0 .
\end{aligned}
$$

Thus, for $x>0$,

$$
\Delta_{x}\left(\nu_{t}^{n}-a_{1}(t) \delta\left(\sigma_{1} t\right)\right) \rightarrow 0
$$

and $\Delta_{x}\left(\nu_{t}^{n}-a_{1}(t) \delta\left(\sigma_{1} t\right)\right)$ is monotone decreasing.
For $x>0$, let

$$
\alpha_{x}^{n}(B)=\int_{K} \Delta_{x}\left(\nu_{t}^{n}-a_{1}(t) \delta\left(\sigma_{1} t\right)\right)(B) d \lambda(t)
$$

Since

$$
\Delta_{x}\left(\nu_{t}^{n}-a_{1}(t) \delta\left(\sigma_{1} t\right)\right) \leqslant \Delta_{x}\left(\nu_{t}\right)
$$

we may apply the Dominated Convergence Theorem to deduce $\alpha_{x}^{n}(B) \downarrow 0$ for $B \in \mathscr{B}$.

Now let $w_{n}$ be the distribution of $T_{n}-S$. We can assume $w_{n}$ monotone decreasing and since $\int_{B} w_{n}(t, x) d \lambda(t)=\alpha_{x}^{n}(B), B \in \mathscr{B}$, we have $w_{n}(t, x) \downarrow 0$ a.e. $(t, x) \in K \times[0, \infty)$ (apply Fubini's theorem). Hence, for $\lambda$-a.e. $t, w_{n}(t, x) \downarrow 0,0<x$ $<\infty$. Now fix $t$ and let

$$
\bar{w}_{n}(t, x)=\limsup _{y \rightarrow x} w_{n}(t, y)
$$

Then

$$
\bar{w}_{n}(t, x) \leqslant w_{n}\left(t, \frac{1}{2} x\right), \quad 0<x<\infty
$$

and so

$$
\bar{w}_{n}(t, x) \downarrow 0
$$

and each $\bar{w}_{n}$ is upper-semicontinuous. Hence so is $x \bar{w}_{n}(t, x)^{1 / p}$. By Dini's theorem,

$$
x w_{n}(t, x)^{1 / p} \downarrow 0
$$

uniformly on compact subsets of $(0, \infty)$. By (7.1.1), we can conclude convergence is uniform on $(0, \infty)$, i.e. $W_{n}^{p, \infty}(t) \rightarrow 0, \lambda$-a.e. Hence $T_{n} \rightarrow S$ in local convergence and so $S \in \mathscr{G}$. This contradiction proves the proposition.

Theorem 7.2. Suppose $0<p<1$ and $T \in \mathcal{L}\left(L_{p}, L_{0}\right)$ with $T \neq 0$. Then if $p<r$ $<2$, there is a subspace $V$ of $L_{p}$ isomorphic to $L_{r}$ and such that $T$ is an isomorphism on $V$.

Proof. We may suppose $r<1$. By Nikisin's theorem there is a subset $B$ of $K$ such that $S=P_{B} T \neq 0, S \in \mathcal{E}\left(L_{p}, L(p, \infty)\right)$ and $S \in \mathcal{L}\left(L_{r}, L(r, \infty)\right.$ ). Now choose $p<q_{1}<q_{2}<r$.

Let $w$ be the distribution of $S$. Then

$$
\sup _{0<x<\infty} x^{p} w(t, x)<C_{1} \quad \text { a.e. }
$$

and

$$
\sup _{0<x<\infty} x^{r} w(t, x)<C_{2} \quad \text { a.e. }
$$

Hence if $q=q_{1}$ or $q_{2}$,

$$
\begin{aligned}
x^{q} w(t, x) & \leqslant \min \left(C_{1} x^{q-p}, C_{2} x^{q-r}\right) \\
& \leqslant C_{1} x^{q-p}, \quad 0 \leqslant x \leqslant 1 \\
& \leqslant C_{2} x^{q-r}, \quad 1 \leqslant x<\infty
\end{aligned}
$$

Hence,

$$
x^{q} w(t, x) \leqslant \max \left(C_{1}, C_{2}\right), \quad 0 \leqslant x<\infty,
$$

and

$$
\lim _{x \rightarrow \infty} x w(t, x)^{1 / q}=\lim _{x \rightarrow \infty} x w(t, x)^{1 / q}=0 .
$$

Thus $S \in \mathcal{E}\left(L_{q}, L(q, \infty)\right)$.
By Proposition 7.1, there is a strongly embedded subspace $V$ of $L_{q_{2}}$ isomorphic to $L_{r}$ such that $T$ maps $V$ isomorphically from $L_{q_{1}}$ into $L\left(q_{1}, \infty\right)$. Since $V \subset L_{q_{2}}$, $T(V) \subset L\left(q_{2}, \infty\right)$. The $L\left(q_{2}, \infty\right)$-topology on $T(V)$ is stronger than the $L\left(q_{1}, \infty\right)$ topology, but the $L_{q_{2}}{ }^{-}$and $L_{q_{1}}$-topologies agree on $V$. Hence the $L\left(q_{1}, \infty\right)$ - and $L\left(q_{2}, \infty\right)$-topologies and all intermediate $L_{s}$-topologies agree on $T(V)$, i.e., $T(V)$ is strongly embedded in $L\left(q_{1}, \infty\right)$. Hence $T$ maps $V$ isomorphically into $L_{0}$ (of course the $L_{q_{2}}{ }^{-}$and $L_{p}$-topologies agree on $V$ ).
8. Embedding $l_{p}$ and $L_{p}$ in $p$-Banach function spaces. Let $X$ be a $p$-Banach function space. Then we can apply Nikisin's theorem (Corollary 3.2) to deduce that there is a function $\varphi \in L_{0}$, with $\varphi>0$ such that $M_{\varphi}(X) \subset L(p, \infty)$. It follows that we can, without loss of generality, suppose that every $p$-Banach function space considered is contained in $L(p, \infty)$ and $\|f\|_{X} \geqslant\|f\|_{p, \infty}, f \in X$ [simply replace $X$ by $M_{\varphi}(X)$ ].

Theorem 8.1. Suppose $X$ is a $p$-Banach function space satisfying:
(8.1.1) There exists $c>0$ and $r>0$ such that whenever $f_{1}, \ldots, f_{n} \in X$ and $\left|f_{i}\right| \wedge\left|f_{j}\right|=0, i \neq j$, then

$$
\left\|f_{1}+\cdots+f_{n}\right\| \geqslant c\left(\sum\left\|f_{i}\right\|^{r}\right)^{1 / r} .
$$

Then if $X$ contains a subspace isomorphic to $l_{p}$, there is a sequence of positive elements with disjoint support equivalent to the usual basis of $l_{p}$.

Remark. Equivalently $l_{p}$ embeds in $X$ as a sublattice.
Proof. Suppose (8.1.1) holds and that $Y$ is a subspace of $X$ isomorphic to $l_{p}$. We consider two possibilities: (a) $Y$ fails to be strongly embedded in $X$, and (b) $Y$ is strongly embedded in $X$.
(a) In this case $Y$ contains a sequence $g_{n} \rightarrow 0$ in $L_{0}$ but such that $\left\|g_{n}\right\| \geqslant \varepsilon>0$ and $\left\{g_{n}\right\}$ is equivalent to the usual basis of $l_{p}$. Here $\left\{g_{n}\right\}$ can be obtained as a block basis of the original basis of $Y$, by a standard gliding hump argument.

Observe that (8.1.1) implies that whenever $0<f_{n} \uparrow f$ a.e. in $X$ then $\left\|f-f_{n}\right\| \rightarrow 0$. Hence if $f \geqslant 0$ and $f \in X$ and $A_{n} \in \mathscr{B}$ with $\lambda\left(A_{n}\right) \rightarrow 0$ then $\left\|f \cdot 1_{A_{n}}\right\| \rightarrow 0$.

Now choose an increasing sequence of integers $\{m(n)\}$, a decreasing sequence $\left\{\varepsilon_{n}\right\}$, and a sequence $A_{n} \in \mathscr{B}$ such that

$$
\begin{gather*}
\lambda\left(A_{n}\right)<\varepsilon_{n}, \quad n=1,2, \ldots,  \tag{8.1.2}\\
\left\|g_{m(n)} 1_{K-A_{n}}\right\|<\varepsilon_{n}, \quad n=1,2, \ldots,  \tag{8.1.3}\\
0<2 \varepsilon_{n+1}<\varepsilon_{n}, \quad n=1,2, \ldots,  \tag{8.1.4}\\
\text { if } \lambda(A)<2 \varepsilon_{n+1},\left\|g_{m(n)} \cdot 1_{A}\right\|<\varepsilon_{n}, \quad n=1,2, \ldots . \tag{8.1.5}
\end{gather*}
$$

Pick $\varepsilon_{1}=1$. Now suppose $\varepsilon_{1}, \ldots, \varepsilon_{n}, A_{1}, \ldots, A_{n-1}, m(1), \ldots, m(n-1)$ have been chosen where $n \geqslant 1$.

Let $h=\Sigma 2^{-n}\left|g_{n}\right| ; h \in X$ and $h \geqslant 0$. Now by Egoroff's theorem, since $g_{n} \rightarrow 0$ in measure, there exists $m(n)>m(n-1)$ and $A_{n} \in \mathscr{B}$ with $\lambda\left(A_{n}\right)<\varepsilon_{n}$ and

$$
\left|g_{m(n)}\right| 1_{K-A_{n}} \leqslant \varepsilon_{n}\|h\|^{-1} h .
$$

Then

$$
\left\|g_{m(n)} 1_{K-A_{n}}\right\|<\varepsilon_{n} .
$$

Now choose $\varepsilon_{n+1}$ so that $\varepsilon_{n+1}>0$ and (8.1.4) and (8.1.5) hold. This completes the induction.

Let $B_{n}=A_{n} \backslash \cup_{k>n} A_{k}$ and $f_{n}=g_{m(n)} \cdot 1_{B_{n}}$. Then

$$
\left\|g_{m(n)} L-f_{n}\right\|^{p}<\left\|g_{m(n)} \cdot 1_{K-A_{n}}\right\|^{p}+\left\|g_{m(n)} \cdot 1_{U_{k>n} A_{k}}\right\|^{p}<2 \varepsilon_{n}^{p}
$$

since $\lambda\left(\cup_{k>n} A_{k}\right)<2 \varepsilon_{n+1}$. Hence,

$$
\left\|g_{m(n)}-f_{n}\right\| \rightarrow 0
$$

and so, passing to a subsequence, we may assume $\left\{f_{n}\right\}$ equivalent to the usual $l_{p}$-basis.
(b) We shall show that condition (b) leads to a contradiction. Let $\left\{f_{n}\right\}$ be equivalent to the usual $l_{p}$-basis in $Y$ and hence also in $L(p, \infty)$. Let $V=\left\{f_{n}\right.$ : $n \in N\}$ and suppose

$$
M=\sup _{n}\left\|f_{n}\right\|_{X}
$$

and

$$
\left\|g_{1}+\cdots+g_{n}\right\|_{p, \infty}>a n^{1 / p}
$$

whenever $g_{1}, \ldots, g_{n}$ are distinct elements of $V$, where $a>0$.
Let $u$ be chosen so that $u>1$ and

$$
u^{1 / p-1}>\frac{2^{(1 / p)+2} M p}{a(1-p)}
$$

and let

$$
\alpha=\frac{a}{(u-1)} 2^{-(2 / p)-2}
$$

For each $k$, let $A_{k}$ be the set of integers $\theta_{n}^{k}$ such that $\theta_{1}^{k}<\theta_{2}^{k}<\cdots$ and

$$
2^{-\theta_{n}^{k} / p_{k}^{*}}\left(2^{-\theta_{n}^{k}}\right)>2^{-2 / p_{\alpha}} .
$$

We claim $\left|A_{k}\right|<\infty$ and $\left|A_{k}\right|<M^{r} 2^{3 r / p_{\alpha}}{ }^{-1} C^{-r}$. Indeed, suppose $\theta_{1}^{k}<\theta_{2}^{k}$ $<\cdots<\theta_{m}^{k} \in A_{k}$; then

$$
f_{k}=\gamma_{1}+\cdots+\gamma_{m}
$$

where the $\gamma_{i}$ 's have disjoint support and

$$
\left\|\gamma_{i}\right\|_{p, \infty} \geqslant 2^{-3 / p} \alpha
$$

Hence,

$$
\left\|f_{k}\right\|_{X} \geqslant c m^{1 / r_{2}} 2^{-3 / p_{\alpha}}
$$

and so

$$
m \leqslant M^{r} 2^{3 r / p} \alpha_{\alpha}^{-1} c^{-r}
$$

Now we may pass to a subsequence of $f_{k}$ (also called $f_{k}$ ) such that $\left|A_{k}\right|=L$ is constant and for some $l \leqslant L, \theta_{i}^{k}=\theta_{i}, i \leqslant l, k=1,2, \ldots$, and $\theta_{i}^{k+1}>\theta_{l}^{k}, i>l$, $k=1,2, \ldots$ Hence if $\rho>\theta_{l}, \rho$ belongs to at most one set $A_{k}$. Now suppose $n>2^{\theta_{1}}$, and let

$$
h_{n}=f_{1}+\cdots+f_{n}
$$

For some $\tau=\tau_{n}, 0<\tau \leqslant 1$,

$$
\tau^{1 / p} h_{n}^{*}(\tau) \geqslant \frac{1}{2} a n^{1 / p}
$$

Let $A=\left(\left|h_{n}\right| \geqslant h_{n}^{*}(\tau)\right)$ and choose $E_{1}, \ldots, E_{n} \in \mathscr{B}$ so that $\lambda\left(E_{i}\right)=\tau / 2 n(1<$ $i \leqslant n)$ and $\left|f_{i}(t)\right| \geqslant f_{i}^{*}(\tau / 2 n), t \in E_{i}, 1 \leqslant i \leqslant n$. Let $E=E_{1} \cup \cdots \cup E_{n}$ so that $\lambda(E)<\tau / 2$ and $\lambda(A) \geqslant \tau$.

$$
\begin{aligned}
\frac{1}{2} a n^{1 / p} & \leqslant \tau^{1 / p} h_{n}^{*}(\tau) \\
& \leqslant \frac{\tau^{1 / p}}{\lambda(A \backslash E)} \int_{A \backslash E}\left|h_{n}(t)\right| d \lambda(t) \quad \text { (as in Theorem 6.1) } \\
& \leqslant 2 \tau^{1 / p-1} \sum_{k=1}^{n} \int_{K \backslash E_{k}}\left|f_{k}(t)\right| d \lambda(t) \\
& \leqslant 2 \tau^{1 / p-1} \sum_{k=1}^{n} \int_{\tau / 2 n}^{1} f_{k}^{*}(x) d x \\
& \leqslant 2 \tau^{1 / p-1} \sum_{k=1}^{n}\left\{\int_{\tau / 2 n}^{u \tau / 2 n} f_{k}^{*}(x) d x+\int_{u \tau / 2 n}^{\infty}\left\|f_{k}\right\| x^{-1 / p} d x\right\} \\
& \leqslant \frac{(u-1) \tau^{1 / p}}{n} \sum_{k=1}^{n} f_{k}^{*}\left(\frac{\tau}{2 n}\right)+2^{1 / p} \frac{n^{1 / p}}{u^{1 / p-1}} \cdot \frac{M p}{1-p} \\
& \leqslant \frac{(u-1) \tau^{1 / p}}{n} \sum_{k=1}^{n} f_{k}^{*}\left(\frac{\tau}{2 n}\right)+\frac{1}{4} a n^{1 / p} .
\end{aligned}
$$

Hence,

$$
\frac{1}{4} a n^{1 / p}<\frac{(u-1) \tau^{1 / p}}{n} \sum_{k=1}^{n} f_{k}^{*}\left(\frac{\tau}{2 n}\right) .
$$

Pick $\rho_{n}$ to be an integer such that $2^{-\rho_{n}}<\tau / 2 n<2^{1-\rho_{n}}$. Then $\tau^{1 / p}<2^{2 / p-\rho_{n} / p_{n}}{ }^{1 / p}$ and so

$$
\frac{1}{4} a n^{1 / p}<2^{2 / p-\rho_{n} / p}(u-1) n^{1 / p-1} \sum_{k=1}^{n} f_{k}^{*}\left(2^{-\rho_{n}}\right) .
$$

Hence,

$$
\alpha n \leqslant 2^{-\rho_{n} / p} \sum_{k=1}^{n} f_{k}^{*}\left(2^{-\rho_{n}}\right) .
$$

Now $\rho_{n} \geqslant \log _{2}(2 n / \tau) \geqslant \log _{2}(2 n)>\theta_{l}$. Hence $\rho_{n}$ belongs to at most one set $A_{k}$.
Thus, $2^{-\rho_{n} / p} \sum_{k=1}^{n} f_{k}^{*}\left(2^{-\rho_{n}}\right) \leqslant(n-1) \alpha 2^{-2 / p}+M$. Thus, $\alpha<2^{-2 / p} \alpha(1-1 / n)$ $+M / n$ for all $n>2^{\theta_{l}}$. This leads to a contradiction, and the theorem is proved.
We now turn to the problem of embedding $L_{p}$ in $X$. We first prove some preparatory lemmas; these will also be useful in the next section.

Lemma 8.2. Suppose $X$ is a $p$-Banach function space where $0<p<1$. Then:
(i) Any $T \in \mathcal{L}\left(L_{p}, X\right)$ has an atomic kernel.
(ii) If $S, T \in \mathcal{E}\left(L_{p}, X\right)$ have kernels $\nu_{t}$ and $\mu_{t}$ respectively, then the kernel of $S \vee T$ is $\nu_{t} \vee \mu_{t}$ almost everywhere.

Proof. (i) Since $X \subset L(p, \infty)$, this follows from Theorem 6.3.
(ii) This follows from Corollary 5.2 since $S \vee T=(1 / 2)(S+T+|S-T|)$. Of course, $|S-T|$ in $\mathcal{L}\left(L_{p}, L_{0}\right)$ is identical with $|S-T|$ in $\mathcal{L}\left(L_{p}, X\right)$.

Lemma 8.3. Suppose $X$ is a p-Banach function space with $0<p<1$. Suppose $T_{n}, T \in \mathcal{L}\left(L_{p}, X\right)$ and $0 \leqslant T_{n} \leqslant T_{n+1} \leqslant T$ for $n \geqslant 1$. Suppose $T$ has kernel $\mu_{t}$ and $T_{n}$ has kernel $\mu_{t}^{n}$. Then if $\mu_{t}^{n} \rightarrow \mu_{t}$ weak $^{*}$ a.e., $\eta(t ; T)=\lim _{n \rightarrow \infty} \eta\left(t ; T_{n}\right)$ a.e.

Proof. Clearly $\eta\left(t ; T_{n}\right)$ is monotone increasing almost everywhere. If $\lambda(B)>0$ and $\varepsilon>0, \eta\left(t ; T_{n}\right)+\varepsilon \leqslant \eta(t ; T), t \in B, n \in \mathbf{N}$ then, $\left\|T_{n} P_{B}\right\|+\varepsilon<\left\|T P_{B}\right\|, n \in$ N.

Select $B_{1} \subset B$ with $\lambda\left(B_{1}\right)>0$ such that

$$
\left\|T 1_{B_{1}}\right\| \geqslant\left(\left\|T P_{B}\right\|-\frac{1}{2} \varepsilon\right) \lambda\left(B_{1}\right)^{1 / P}
$$

Then since $\mu_{t}^{n} \leqslant \mu_{t}^{n+1} \leqslant \mu_{t}$ we have $\left\|\mu_{t}^{n}-\mu_{t}\right\| \rightarrow 0$ a.e. and hence $T_{n} 1_{B_{1}}(t) \rightarrow$ $T 1_{B_{1}}(t)$, a.e. Now by the Fatou property of the quasinorm, there exists $n$ so that

$$
\left\|T_{n} 1_{B_{1}}\right\|>\left(\left\|T P_{B}\right\|-\varepsilon\right) \lambda\left(B_{1}\right)^{1 / p}
$$

and hence

$$
\left\|T_{n} P_{B}\right\|>\left\|T P_{B}\right\|-\varepsilon
$$

This contradiction proves the result.
Lemma 8.4. Suppose $T \in \mathcal{L}\left(L_{p}, L_{0}\right)$. Then there is a sequence of operators ( $S_{n}, n \in$ N) such that

$$
\begin{equation*}
\left|S_{i}\right| \wedge\left|S_{j}\right|=0, \quad i \neq j \tag{8.4.1}
\end{equation*}
$$

$$
\begin{equation*}
S_{1}+\cdots+S_{n} \text { is locally elementary for each } n . \tag{8.4.2}
\end{equation*}
$$

If $\nu_{t}^{n}$ is the kernel of $S_{n}$ and $\mu_{t}$ is the kernel of $T$, then $\sum_{n=1}^{\infty} \nu_{t}^{n}=\nu_{t}$ weak ${ }^{*}$ a.e., and $\sum_{n=1}^{\infty}\left|\nu_{t}^{n}\right|=\left|\nu_{t}\right|$ weak* a.e.

Proof. Let

$$
\nu_{t}=\sum_{n=1}^{\infty} a_{n}(t) \delta\left(\sigma_{n} t\right)
$$

where as in Theorem 5.3, each $a_{n}: K \rightarrow \mathbf{R}$ is Borel and each $\sigma_{n}: K \rightarrow K$ is Borel, and $\sigma_{i}(t) \neq \sigma_{j}(t)$ whenever $i \neq j$. Now let

$$
\mu_{t}^{n}=\sum_{k=1}^{n} a_{k}(t) \delta\left(\sigma_{k} t\right)
$$

Then $\mu_{t}^{n}$ is the kernel of an operator $T_{n} \in \mathcal{L}\left(L_{p}, L_{0}\right)$ with $\left|T_{n}\right|<|T|$. Further, $T_{n}$ is of finite type. By Theorem 5.4 we may select $A_{n} \in \mathscr{B}$ with $\lambda\left(A_{n}\right) \geqslant 1-2^{-(n+1)}$ such that $P_{A_{n}} T_{n}$ is locally elementary. Let $B_{n}=\cap\left(A_{k}: k \geqslant n\right)$; then $\lambda\left(B_{n}\right) \geqslant 1-$ $2^{-n}$ and $P_{B_{n}} T_{n}$ is locally elementary. Finally, let $S_{1}=P_{B_{1}} T_{1}$ and $S_{n}=P_{B_{n}} T_{n}-$ $P_{B_{n-1}} T_{n-1}(n \geqslant 2)$. Then $S_{n}$ has kernel $\nu_{t}^{n}$ given by

$$
\begin{aligned}
\nu_{t}^{n} & =0, \quad t \notin B_{n}, \\
& =\mu_{t}^{n}, \quad t \in B_{n} \backslash B_{n-1}, \\
& =\mu_{t}^{n}-\mu_{t}^{n-1}, \quad t \in B_{n-1} .
\end{aligned}
$$

Clearly, $\left|\nu_{t}^{n}\right| \wedge\left|\nu_{t}^{m}\right|=0$ a.e., $m \neq n$, and (8.4.1), (8.4.2), and (8.4.3) follow easily.
Lemma 8.5. Suppose $X$ is a $p$-Banach function space satisfying (8.1.1) and that $T_{1}, \ldots, T_{n} \in \mathcal{E}\left(L_{p}, X\right)$ satisfy $\left|T_{i}\right| \wedge\left|T_{j}\right|=0, i \neq j$. Then,

$$
\begin{equation*}
\eta\left(t ; T_{1}+\cdots+T_{n}\right) \geqslant c\left\{\sum_{i=1}^{n} \eta\left(t ; T_{i}\right)^{r}\right\}^{1 / r} \quad \text { a.e. } \tag{8.5.1}
\end{equation*}
$$

Proof. Suppose first $S=T_{1}+\cdots+T_{n}$ is elementary. Then by considering kernels we see that if $\left|T_{i}\right| \wedge\left|T_{j}\right|=0(i \neq j)$ then for any $f \in L_{p}$,

$$
\left|T_{i} f\right| \wedge\left|T_{j} f\right|=0
$$

Hence, for any Borel set $B$,

$$
\left\|S 1_{B}\right\| \geqslant c\left(\sum\left\|T_{i} 1_{B}\right\|^{r}\right)^{1 / r}
$$

and (8.5.1) follows from the construction of the local quasinorm $\eta$ as in Proposition 4.2.

It now quickly follows that (8.5.1) holds for locally elementary $S$. For general $S$, let $\left(S_{n}\right)$ be chosen as in Lemma 8.4. Then if

$$
V_{m}=\left|S_{1}\right|+\cdots+\left|S_{m}\right|
$$

then

$$
V_{m}=V_{m} \wedge\left|T_{1}\right|+\cdots+V_{m} \wedge\left|T_{n}\right|
$$

and since $V_{m}$ is locally elementary,

$$
\eta\left(t ; V_{m}\right) \geqslant c\left(\sum_{i=1}^{n} \eta\left(t ; V_{m} \wedge\left|T_{i}\right|\right)^{r}\right)^{1 / r} \quad \text { a.e. }
$$

Now by Lemma 8.3,

$$
\lim _{m \rightarrow \infty} \eta\left(t ; V_{m}\right)=\eta(t ;|S|) \quad \text { a.e. }
$$

and a similar argument shows

$$
\lim _{m \rightarrow \infty} \eta\left(t ; V_{m} \wedge\left|T_{i}\right|\right)=\eta\left(t ;\left|T_{i}\right|\right) \quad \text { a.e. }
$$

The lemma now follows.
Lemma 8.6. Suppose $X$ is a $p$-Banach function space $(0<p<1)$ satisfying (8.1.1). Then the locally elementary operators are dense in $\mathcal{L}\left(L_{p}, X\right)$ in the topology of local convergence.

Proof. Suppose $S \in \mathcal{L}\left(L_{p}, X\right)$. We appeal to Lemma 8.4 to define ( $S_{n}$ ) satisfying (8.4.1), (8.4.2), and (8.4.3). Suppose $m(n)$ is an increasing sequence of integers with $m(0)=0$; then let

$$
T_{n}=\sum_{m(n-1)+1}^{m(n)} S_{i}
$$

By Lemma 8.5,

$$
\eta\left(t ; T_{1}+\cdots+T_{n}\right) \geqslant c\left(\sum_{i=1}^{n} \eta\left(t ; T_{i}\right)^{r}\right)^{1 / r} \quad \text { a.e. }
$$

Hence,

$$
c\left(\sum \eta\left(t ; T_{i}\right)^{r}\right)^{1 / r}<\eta(t ;|S|) \text { a.e. }
$$

and hence,

$$
\eta\left(t ; T_{n}\right) \rightarrow 0 \quad \text { a.e. }
$$

Thus $\sum_{i=1}^{n} S_{i}$ is a Cauchy sequence in the topology of local convergence, and as $\left|\sum_{i=1}^{n} S_{i}\right| \leqslant|T|$ by Proposition 4.1, $\sum_{i=1}^{\infty} S_{i}$ converges in this topology. Clearly, $\sum_{i=1}^{\infty} S_{i}=S$. To see this observe that given $\varepsilon>0$ there exists $B_{k} \in \mathscr{B}$ with $\lambda(B) \geqslant 1-1 / k$ and $\left\|\left(\sum_{i=1}^{\infty} S_{i}-\sum_{i=1}^{n} S_{i}\right) P_{B_{k}}\right\| \rightarrow 0$ as $n \rightarrow \infty$. Thus,

$$
\left|\sum_{i=1}^{\infty} S_{i}-\sum_{i=1}^{n} S_{i}\right| 1_{B_{k}} \rightarrow 0
$$

Hence, if $\mu_{t}$ is the kernel of $\sum_{i=1}^{\infty} S_{i},\left|\mu_{t}-\sum_{i=1}^{n} \nu_{t}^{i}\right|\left(B_{k}\right) \rightarrow 0$ as $n \rightarrow \infty$, i.e., $\left|\mu_{t}-\nu_{t}\right|\left(B_{k}\right)=0$ a.e. Thus, $\left|\mu_{t}-\nu_{t}\right|\left(\cup_{k=1}^{\infty} B_{k}\right)=0$ a.e. As $\lambda\left(K \backslash \cup B_{k}\right)=0$, $\left|\mu_{t}-\nu_{t}\right|\left(K \backslash \cup_{k} B_{k}\right)=0$ a.e. Hence $\mu_{t}=\nu_{t}$ a.e., and $S=\sum_{i=1}^{\infty} S_{i}$. The lemma follows since $S_{1}+\cdots+S_{n}$ is locally elementary for each $n$.

Theorem 8.7. Suppose $L_{p}$ is a $p$-Banach function space $(0<p<1)$ satisfying (8.1.1). Suppose $L_{p}$ embeds in $X$. Then there is an embedding $S: L_{p} \rightarrow X$ which is a lattice isomorphism, i.e., $S(f \wedge g)=S f \wedge S g, f, g \in L_{p}$ (i.e., $L_{p}$ embeds as a sublattice).

Proof. Suppose $L_{p}$ does not embed as a sublattice. Let $\mathscr{G}$ be the set of all operators $T$ such that whenever $B \in \mathscr{B}$ and $\lambda(B)>0$ then $T \mid L_{p}(B)$ fails to be an isomorphism. An application of Proposition 4.2 shows that $\mathscr{G}$ is closed under local
convergence. We prove the theorem by contradiction by showing $\mathscr{G}=\mathcal{L}\left(L_{p}, X\right)$; by Lemma 8.6 we need only show every locally elementary operator is in 9 . Clearly to do this it suffices to show each elementary operator belongs to $\mathscr{G}$. Suppose $S$ is elementary and for some $B, \lambda(B)>0, S P_{B}$ is an embedding. Then $\left|S P_{B}\right|$ is a lattice isomorphism and an embedding. This contradicts our hypothesis and so $\mathcal{G}=$ $\mathcal{E}\left(L_{p}, X\right)$ and the theorem follows.
9. Symmetric function spaces. Let us denote by $\mathcal{G}$ the set of all functions $F$ : $[0, \infty) \rightarrow[0,1]$ which are monotone decreasing right-continuous and such that $\lim _{x \rightarrow \infty} F(x)=0$. Let $\Lambda_{p}(0<p<\infty)$ denote the class of maps $\Phi: \mathcal{G} \rightarrow[0, \infty]$ such that:
(9.0.1) If $F, G \in \mathcal{G}$ and $F \leqslant G$ then $\Phi(F) \leqslant \Phi(G)$.
(9.0.2) If $F_{n}, F \in \mathcal{G}$ and $F_{n}(x) \uparrow F(x), 0 \leqslant x<\infty$ then $\lim _{n \rightarrow \infty} \Phi\left(F_{n}\right)=\Phi(F)$.
(9.0.3) If $F, G \in \mathcal{G}$ and $F(0)+G(0) \leqslant 1, \Phi(F+G) \leqslant \Phi(F)+\Phi(G)$.
(9.0.4) If $\theta>0$ and $F \in \mathcal{G}$ then $\Phi\left(F_{\theta}\right)=\theta^{-p} \Phi(F)$ where $F_{\theta}(x)=F(\theta x), x \geqslant 0$.
(9.0.5) For some $F \in \mathcal{G}, 0<\Phi(F)<\infty$.

Note that by (9.0.4), $\Phi(0)=0$.
Then we define the space $L[p ; \Phi]$ to be the space of all $f \in L_{0}$ such that $\|f\|=\Phi(F)^{1 / p}<\infty$ where $F(x)=\lambda(|f|>x)$.

Theorem 9.1. If $\Phi \in \Lambda_{p}, L[p ; \Phi]$ is a $\sigma$-complete quasi-Banach function space having the Fatou property (2.0.7) and such that if $f_{1}, f_{2} \in L[p ; \Phi]$ have disjoint supports

$$
\begin{equation*}
\left\|f_{1}+f_{2}\right\|^{p} \leqslant\left\|f_{1}\right\|^{p}+\left\|f_{2}\right\|^{p} . \tag{9.1.1}
\end{equation*}
$$

Proof. First we observe that (9.0.3), (9.0.4), and (9.0.5) together imply $0<\Phi(F)$ $<\infty$ for every simple function $F$. Hence $L[p ; \Phi]$ includes all simple functions.

Next observe that (9.1.1) holds by applying (9.0.3). We use this to show that \|•\| is a quasinorm. For suppose $f, g, h \in L[p ; \Phi]$ and $h=f+g$. Choose $A \in \mathscr{B}$ with $\lambda(A)=1 / 2$. Then $h 1_{A}=f 1_{A}+g 1_{A}$ and

$$
\lambda\left(\left|h 1_{A}\right|>\frac{1}{2} x\right) \leqslant \lambda\left(\left|f 1_{A}\right|>\frac{1}{2} x\right)+\lambda\left(\left|g 1_{A}\right|>\frac{1}{2} x\right)
$$

so that

$$
\left\|h 1_{A}\right\|^{p} \leqslant 2^{p}\left(\left\|f 1_{A}\right\|^{p}+\left\|g 1_{A}\right\|^{p}\right) .
$$

Similarly,

$$
\left\|h 1_{K-A}\right\|^{p} \leqslant 2^{p}\left(\left\|f 1_{K-A}\right\|^{p}+\left\|g 1_{K-A}\right\|^{p}\right) .
$$

Hence,

$$
\|h\|^{p} \leqslant 2^{p+1}\left(\|f\|^{p}+\|g\|^{p}\right)
$$

so that

$$
\|h\| \leqslant 2^{2 / p}(\|f\|+\|g\|) .
$$

Next we show the inclusion $L[p ; \Phi] \rightarrow L_{0}$ is continuous. Indeed if $\left\|f_{n}\right\| \rightarrow 0$ then if $\varepsilon>0$ and $\delta_{n}=\lambda\left(\left|f_{n}\right|>\varepsilon\right)$, then if we denote $F_{n}(x)=\lambda\left(\left|f_{n}\right|>x\right), F_{n}>\delta_{n} 1_{[0, e)}$,
and hence, $\Phi\left(\delta_{n} 1_{[0, \varepsilon)}\right) \rightarrow 0$. If lim sup $\delta_{n}>\delta>0$, we obtain $\Phi\left(\delta 1_{[0, e)}\right)=0$, contradicting our initial remarks. Hence $f_{n} \rightarrow 0$ in measure.

Now $\|\cdot\|$ is clearly a symmetric lattice quasinorm with the Fatou property and $L[p ; \Phi]$ is a $\sigma$-complete lattice. To show $L[p ; \Phi]$ is complete it suffices to show that if $f_{n} \geqslant 0$ and $\left\|f_{n}\right\| \leqslant 2^{-n}$ then $\sum f_{n}$ converges. Clearly $\Sigma f_{n}$ converges in $L_{0}$ to $g$ say. If $G_{n}$ is the distribution of $\sum_{i=1}^{n} f_{i}$ and $G$ the distribution of $g$ then $G_{n}(x) \uparrow G(x)$ for $0 \leqslant x<\infty$ and $\sup \Phi\left(G_{n}\right)<\infty$. Hence $\Phi(G)<\infty$ and $g \in L[p ; \Phi]$. A similar argument shows that

$$
\left\|g-\sum_{i=1}^{n} f_{i}\right\| \leqslant \limsup _{m \rightarrow \infty}\left\|\sum_{i=1}^{m} f_{i}-\sum_{i=1}^{n} f_{i}\right\| \rightarrow 0 \text { and } n \rightarrow \infty
$$

Theorem 9.2. Suppose $X$ is a $p$-convex symmetric $\sigma$-complete quasi-Banach function space. Then $X=L[p ; \Phi]$ for some $\Phi \in \Lambda_{p}$.

Proof. We may suppose the lattice quasinorm on $X$ is symmetric, has the Fatou property and satisfies (9.1.1). Then define

$$
\begin{aligned}
\Phi(F) & =\|f\|^{p}, \quad f \in X \\
& =\infty, \quad f \notin X
\end{aligned}
$$

where $F \in \mathcal{G}$ and $f \in L_{0}$ is such that $\lambda(|f|>x)=F(x), x \geqslant 0$.
We do not know whether every $L[p ; \Phi]$ is $p$-convex. However we obtain a positive result with one further hypothesis.

Definition. $\Phi \in \Lambda_{p}$ is totally symmetric if

$$
\begin{equation*}
\Phi(t F)=t \Phi(F), \quad F \in \mathcal{G}, 0 \leqslant t \leqslant 1 \tag{9.2.1}
\end{equation*}
$$

A $\sigma$-complete quasi-Banach function space $X$ is totally symmetric of order $p$ if $X=L[p ; \Phi]$ for some totally symmetric $\Phi \in \Lambda_{p}$.

In terms of the quasinorm on $L[p ; \Phi]$, total symmetry implies that if $B \in \mathscr{B}$ and $\sigma: B \rightarrow K$ is any Borel map such that $\lambda\left(\sigma^{-1} A\right)=\lambda(A) \lambda(B), t \in \mathscr{B}$, then the map $R_{B}: L[p ; \Phi] \rightarrow L[p ; \Phi]$ given by

$$
\begin{aligned}
R_{B} f(t) & =f(\sigma t), \quad t \in B, \\
& =0, \quad t \notin B,
\end{aligned}
$$

satisfies $\left\|R_{B} f\right\|=\lambda(B)^{1 / p}\|f\|, f \in L[p ; \Phi]$.
Of course the spaces $L[p ; \infty]$ and $L_{p}$ are totally symmetric of order $p$; so are the intermediate Lorentz space $L(p ; q)$, where $p<q<\infty$.

Lemma 9.3. Suppose $\Phi \in \Lambda_{p}$ is totally symmetric and $0<p<1$; then there is a countable collection $\mathfrak{N}(\Phi)$ of continuous increasing functions $M:[0, \infty) \rightarrow[0, \infty)$ such that:

$$
\begin{gather*}
M(0)=0 .  \tag{9.3.1}\\
M(x+y) \leqslant M(x)+M(y), \quad x, y \geqslant 0 .  \tag{9.3.2}\\
(1-p) \Phi(F) \leqslant \sup _{M \in \mathscr{N}(\Phi)} \int_{0}^{\infty} F(x) d M(x) \leqslant \Phi(F), \quad f \in \mathcal{G} . \tag{9.3.3}
\end{gather*}
$$

Remark. $\int_{0}^{\infty} F(x) d M(x)=\int_{K} M(|f(t)|) d \lambda(t)$ if $f$ has distribution $F$.

Proof. It clearly suffices to produce $\mathfrak{N}(\Phi)$ so that (9.3.3) holds for simple $F$. Let $\mathcal{S}$ be the linear span of the simple functions in $\mathcal{G}$. For $F \in \mathcal{S}$ we define

$$
\pi(F)=\inf (t \Phi(G): F<t G, 0 \leqslant t<\infty, G \in \mathcal{G})
$$

Then $\pi$ is a sublinear functional on $\delta$; indeed it is easy to see that $\pi$ is positively homogeneous. Further, if $F_{i} \in \delta(i=1,2)$ and $\varepsilon>0$, there exist $G_{i} \in \mathcal{G}(i=1,2)$ and $t_{i} \geqslant 0(i=1,2)$ such that $t_{i} \Phi\left(G_{i}\right)<\pi\left(F_{i}\right)+\varepsilon / 2(i=1,2)$ and $F_{i}<t_{i} G_{i}$ ( $i=1,2$ ). Suppose (without loss of generality) that $t_{1} \geqslant t_{2}$ so that $t_{2}=s t_{1}$ with $0<s \leqslant 1$. Then

$$
F_{1}+F_{2}<2 t_{1}\left(\frac{1}{2} G_{1}+\frac{1}{2} s G_{2}\right)
$$

and

$$
\pi\left(F_{1}+F_{2}\right) \leqslant 2 t_{1}\left(\Phi\left(\frac{1}{2} G_{1}\right)+\Phi\left(\frac{1}{2} s G_{2}\right)\right) \leqslant \pi\left(F_{1}\right)+\pi\left(F_{2}\right)+\varepsilon
$$

Since $\varepsilon>0$ is arbitrary, $\pi$ is sublinear. If $F<0, \pi(F)=0$. If $F \in \mathcal{G}$ and $F<t G$ with $G \in \mathcal{G}$ and $t>1$ then $t^{-1} F<G$ and hence $\Phi\left(t^{-1} F\right)<\Phi(G)$. Hence $\pi(F)=$ $\Phi(F)$ for $F \in \mathcal{G}$.

Now for each $F \in \mathcal{\delta}$ with rational values and discontinuities, choose linear $\rho$ : $\delta \rightarrow \mathbf{R}$ such that $\rho(G)<\pi(G)(G \in \delta)$ and $\rho(F)=\pi(F)$, by the Hahn-Banach Theorem. The collection $\Delta$ of all such $\rho$ is countable. If $\rho \in \Delta$, then $\rho(F) \geqslant 0$ whenever $F \geqslant 0$, since $\pi(F)=0$ for $F \leqslant 0$, and $\Phi(F)=\sup _{\rho \in \Delta} \rho(F), F \in \mathcal{G}$ (since there is a sequence $F_{n} \in \delta$ with rational values and discontinuities such that $F_{n}(x) \uparrow F(x)$ for all $\left.x\right)$.

The function $\theta \mapsto \rho\left(F_{\theta}\right)$ is decreasing if $F \in \mathcal{G}$ and

$$
\begin{align*}
& (1-p) \int_{0}^{1} \rho\left(F_{\theta}\right) d \theta \leqslant(1-p) \int_{0}^{1} \theta^{-p} \Phi(F) d \theta \leqslant \Phi(F)  \tag{9.3.4}\\
& (1-p) \int_{0}^{1} \rho\left(F_{\theta}\right) d \theta \geqslant(1-p) \rho(F), \quad F \in \mathcal{G} \cap \delta \tag{9.3.5}
\end{align*}
$$

For fixed $\rho$, let

$$
N(x)=\rho\left(1_{[0, x)}\right)
$$

and

$$
M(x)=(1-p) x \int_{x}^{\infty} \frac{N(u)}{u^{2}} d u, \quad x>0
$$

We let $\mathscr{K}(\Phi)$ be the set of such functions $M$. First observe that $M(0)=0$ and that $M$ is continuous.

Now,

$$
(1-p) \int_{0}^{1} \rho\left(1_{\left[0, x \theta^{-1}\right)}\right) d \theta=(1-p) \int_{0}^{1} N\left(x \theta^{-1}\right) d \theta=M(x)
$$

upon substituting $u=x \theta^{-1}$. Since $\rho$ is a positive linear functional, $M$ is an increasing function. Further, if $F \in \mathcal{S} \cap \mathcal{G},(1-p) \int_{0}^{1} \rho\left(F_{\theta}\right) d \theta=\int_{0}^{\infty} F(x) d M(x)$ (since $M$ is continuous). Now (9.3.4) and (9.3.5) imply (9.3.3). To conclude, we show (9.3.2) holds. Indeed,

$$
\frac{M(x)}{x}=(1-p) \int_{x}^{\infty} \frac{N(u)}{u^{2}} d u
$$

is monotone decreasing and this implies (9.3.2).
Theorem 9.4. If $0<p<1$ and $\Phi \in \Lambda_{p}$ is totally symmetric, $L[p ; \Phi]$ is $p$-convex.
Proof. Suppose $f_{1}, \ldots, f_{n} \in L[p ; \Phi]$ and $h=f_{1}+\cdots+f_{n}$. For $\varepsilon>0$ select $M \in \mathfrak{R}(\Phi)$ so that

$$
\int_{K} M(|h(t)|) d \lambda(t) \geqslant(1-p)\left[\|h\|^{p}-\varepsilon\right]
$$

Then $(1-p)\left[\|h\|^{p}-\varepsilon\right] \leqslant \sum_{i=1}^{n} \int_{K} M\left(\left|f_{i}(t)\right|\right) d \lambda(t)$ by (9.3.2) and hence,

$$
(1-p)\left[\|h\|^{p}-\varepsilon\right] \leqslant \sum_{i=1}^{n}\left\|f_{i}\right\|^{p},
$$

for $\varepsilon>0$ is arbitrary,

$$
\|h\|^{p} \leqslant(1-p)^{-1} \sum_{i=1}^{n}\left\|f_{i}\right\|^{p}
$$

Now let $\mathcal{G}_{\infty}$ be the set of all monotone decreasing right-continuous functions $F$ : $[0, \infty) \rightarrow[0, \infty]$ such that $\lim _{x \rightarrow \infty} F(x)=0$. If $\Phi \in \Lambda_{p}$, we define its symmetric extension $\Psi: \mathcal{G}_{\infty} \rightarrow[0, \infty]$ by

$$
\Psi(F)=\sup (t \Phi(G): G \in \mathcal{G}, t \geqslant 0, t G \leqslant F)
$$

Then $\Psi$ satisfies the following conditions:
(9.4.1) If $F, G \in \mathcal{G}_{\infty}$, and $F \leqslant G$, then $\Psi(F) \leqslant \Psi(G)$.
(9.4.2) If $F_{n}, F \in \mathcal{G}_{\infty}$ and $F_{n}(x) \uparrow F(x), 0 \leqslant x<\infty$, then $\lim _{n \rightarrow \infty} \Psi\left(F_{n}\right)=\Psi(F)$.
(9.4.3) If $F, G \in \mathcal{G}_{\infty}, \Psi(F+G) \leqslant \Psi(F)+\Psi(G)$.
(9.4.4) If $\theta>0$ and $F \in \mathcal{G}_{\infty}, \Psi\left(F_{\theta}\right)=\theta^{-p} \Psi(F)$.
(9.4.5) If $F \in \mathcal{G}_{\infty}$ and $t \geqslant 0, \Psi(t F)=t \Psi(F)$.

Thus, provided there exists $F \in \mathcal{G}$ with $\Psi(F)<\infty$, then $\Psi \mid \mathcal{G} \in \Lambda_{p}$ and is totally symmetric.

Lemma 9.5. For $F \in \mathcal{G}$,
(9.5.1) $\Psi(F) \geqslant \Phi(F)$,
(9.5.2) $\Psi(F) \leqslant \lim \sup _{t \rightarrow 0} t^{-1} \Phi(t F)$,
(9.5.3) if $\Phi$ is totally symmetric, $\Phi(F)=\Psi(F)$.

Proof. Only (9.5.2) requires proof. If $\varepsilon>0, G \in \mathcal{G}$ and $t \geqslant 0$ such that $t G \leqslant F$ and $t \Phi(G) \geqslant \Psi(F)-\varepsilon / 2$. For $\tau=m / n \in Q$, so that $\tau<t, \tau \Phi(G) \geqslant \Psi(F)-\varepsilon$. Thus for any $N$,

$$
N n\left(\frac{G}{N n}\right)=G
$$

and hence,

$$
\Phi\left(\frac{G}{N n}\right) \geqslant \frac{1}{N n} \Phi(G)
$$

However,

$$
\frac{G}{N n} \leqslant \frac{F}{N m}
$$

and hence,

$$
\Phi\left(\frac{F}{N m}\right) \geqslant \frac{1}{N n} \Phi(G) \geqslant \frac{1}{N m}(\Psi(F)-\varepsilon) .
$$

The result now follows.
Theorem 9.6. Suppose $X$ is a symmetric $\sigma$-complete $p$-Banach function space, where $0<p<1$, so that $X=L[p ; \Phi]$ for some $\Phi \in \Lambda_{p}$. Let $\Psi$ be the symmetric extension of $\Phi$. Suppose $T \in \mathcal{L}\left(L_{p}, L_{0}\right)$ has distribution $w(t, x)$. Then a necessary and sufficient condition that $T \in \mathcal{E}\left(L_{p}, X\right)$ is that

$$
\begin{equation*}
\underset{t \in K}{\operatorname{ess} \sup } \Psi(w(t, \cdot))=C<\infty \tag{9.6.1}
\end{equation*}
$$

and then $\|T\| \leqslant C^{1 / p}(1-p)^{-1 / p}$.
Proof. Let $\nu_{t}$ be the kernel of $T$ and $\alpha_{x}(B)=\int_{K} \Delta_{x}\left(\nu_{t}\right)[B] d \lambda(t), B \in \mathscr{B}, x \geqslant 0$, so that $\alpha_{x}(B)=\int_{B} w(t, x) d \lambda(t)$. Consider the condition

$$
\begin{equation*}
\Psi\left[\alpha_{x}(B)\right] \leqslant C^{*} \lambda(B), \quad B \in \mathscr{B} . \tag{9.6.2}
\end{equation*}
$$

If $\Psi(F)=\infty$, unless $F=0$, (9.6.1) and (9.6.2) are trivially equivalent.
Otherwise, we observe that if (9.6.2) holds, then for $M \in \mathscr{T}(\Psi)$,

$$
\int_{0}^{\infty} \int_{B} w(t, x) d \lambda(t) d M(x) \leqslant C^{*} \lambda(B), \quad B \in \mathscr{B}
$$

and so by Fubini's theorem,

$$
\int_{0}^{\infty} w(t, x) d M(x) \leqslant C^{*} \quad \text { a.e. }
$$

Since $\mathscr{T}(\Psi)$ is countable, we obtain

$$
\Psi(w(t, \cdot)) \leqslant C^{*}(1-p)^{-1} \quad \text { a.e. }
$$

Thus (9.6.2) implies (9.6.1) with $C=C^{*}(1-p)^{-1}$. Reversing the reasoning (9.6.1) implies (9.6.2) with $C^{*}=C(1-p)^{-1}$.

Now suppose (9.6.2) holds. Then if $T$ is elementary, (9.6.2) implies that $T \in$ $\mathcal{L}\left(L_{p}, X\right)$ and $\|T\| \leqslant\left(C^{*}\right)^{1 / p}$, since $\alpha_{x}(B)$ is the distribution of $T 1_{B}$. Hence this statement also holds for locally elementary $T$. To obtain the result for general $T$, we appeal to Lemma 8.4. It is sufficient to show $|T| \in \mathscr{E}\left(L_{p}, X\right)$, and Lemma 8.4 allows us to define an increasing sequence $V_{n}$ of positive locally elementary operators such that $V_{n} f \uparrow T f$ a.e. for $f \geqslant 0$. Clearly the preceding argument shows $\left\|V_{n}\right\| \leqslant\left(C^{*}\right)^{1 / p}$ and so $\|T\| \leqslant\left(C^{*}\right)^{1 / p}$.

Conversely, suppose $T \in \mathcal{L}\left(L_{p}, X\right)$. Without loss of generality we may suppose $T \geqslant 0$. Suppose first $T$ is elementary and of the form $T f(t)=a(t) f(\sigma t)$ where $a \geqslant 0$ is a simple Borel function and $\sigma: K \rightarrow K$ is Borel. Let $b_{1}, \ldots, b_{l}$ be the nonzero values of $a$ and let $B_{i}=\left\{t: a(t)=b_{i}\right\}$. Since $T$ is continuous into $L_{0}$, the measures $\gamma_{i}(E)=\lambda\left(\sigma^{-1} E \cap B_{i}\right), E \in \mathscr{B}$, are absolutely continuous, and hence we may write $\gamma_{i}(C)=\int_{C} h_{i}(t) d \lambda(t), E \in \mathscr{B}$, where $h_{1}, \ldots, h_{l}$ are positive Borel functions.

Next decompose $K$ into Borel sets $A_{1}, A_{2}, \ldots$ such that $c_{i j} \leqslant h_{i}(t) \leqslant 2 c_{i j}, t \in A_{j}$, $i=1,2, \ldots, l$. Suppose $E \subset A_{j}$ (some $j$ ) and $F \subset E$ are both Borel sets of positive measure. Then the distributions of $T 1_{F}$ and $T 1_{E}$ are $x \mapsto \alpha_{x}(F)$ and $x \mapsto \alpha_{x}(E)$ and these must satisfy

$$
\frac{1}{2} \frac{\lambda(F)}{\lambda(E)} \alpha_{x}(E) \leqslant \alpha_{x}(F), \quad x \geqslant 0
$$

and hence,

$$
\Phi\left(\frac{1}{2} \frac{\lambda(F)}{\lambda(E)} \alpha_{x}(E)\right) \leqslant\|T\|^{p} \lambda(F), \quad F \subset E .
$$

Allowing $F$ to vary we obtain

$$
\begin{equation*}
\Psi\left(\alpha_{x}(E)\right) \leqslant 2\|T\|^{p} \lambda(E), \quad E \subset A_{j} \tag{9.6.3}
\end{equation*}
$$

Since $\Psi$ is subadditive, (9.6.3) holds for all $E \in \mathscr{B}$.
For general positive elementary operators we simply approximate by an increasing sequence of "simple" elementary operators as above to obtain (9.6.3). This gives (9.6.3) also for locally elementary operators and hence by Lemma 8.4 for all $T \in \mathcal{L}\left(L_{p}, X\right)$.

Corollary 9.7. $\mathfrak{E}\left(L_{p}, X\right) \neq\{0\}$ if and only if $L_{p} \subset X$.
Proof. $\mathcal{L}\left(L_{p}, X\right) \neq\{0\}$ if and only if $\Psi(F)<\infty$ for some $F \neq 0$. If $\Psi$ is totally symmetric and $\Psi\left(1_{[0,1)}\right)=1$, then it is easy to see $\Psi(F) \leqslant \int_{0}^{\infty} F(x) d\left(x^{p}\right)$. Hence $\Psi(F)<\infty$ for some $F \neq 0$ if and only if $\Psi(F) \leqslant c \int_{0}^{\infty} F(x) x^{p-1} d x$ for some $c$ and this is if and only if $\Phi(F) \leqslant c \int_{0}^{\infty} F(x) x^{p-1} d x$, i.e., $L[p ; \Phi] \supset L_{p}$.

Theorem 9.8. Let $X$ be a separable symmetric $\sigma$-complete $p$-convex function space. Then $X$ is isomorphic to a quotient of $L_{p}$ if and only if $X=L[p ; \Phi]$ for a totally symmetric $\boldsymbol{\Phi}$.

Proof. Suppose $X=L[p ; \Phi]$ where $\Phi$ is totally symmetric. Since $X$ is separable the simple functions are dense in $X$. Suppose $f$ is simple; then $f$ may be "split" so that

$$
f=f_{1}^{n}+f_{2}^{n}+\cdots+f_{2^{n}}^{n}
$$

where $f_{2 k-1}^{n+1}+f_{2 k}^{n+1}=f_{k}^{n+1}\left(1 \leqslant k \leqslant 2^{n}\right)$, then $f_{k}^{n}\left(1 \leqslant k \leqslant 2^{n}\right)$ have disjoint support and identical distributions $F_{n}$. If $F$ is the distribution of $F, F_{n}=2^{-n} F$.

Now let $E(n ; j)$ be the partitioning of $K$ described in Proposition 4.2. We define $T: L_{p} \rightarrow L[p ; \Phi]$ such that $T 1_{E(n, j)}=f_{j}^{n}$. Then, $\left\|T 1_{E(n, j)}\right\| \leqslant 2^{-n / p}\|f\|$ and hence $\|T\| \leqslant c\|f\|$, where $c$ is a constant such that

$$
\left\|f_{1}+\cdots+f_{n}\right\|^{p}<c^{p}\left(\left\|f_{1}\right\|^{p}+\cdots+\left\|f_{n}\right\|^{p}\right)
$$

for $g_{1}, \ldots, g_{n} \in L[p ; \Phi]$.
Let $\left\{h_{n}\right\}$ be a sequence of simple functions dense in the unit ball of $L[p ; \Phi]$. For each $n$, select as above $T \in \mathcal{L}\left(L_{p}, L[p, \Phi]\right)$ so that $\|T\| \leqslant c$ and $T 1_{K}=h_{n}$. Define

$$
S: l_{p}\left(L_{p}\right) \rightarrow L[p ; \Phi], \quad S\left(f_{n}\right)=\sum T_{n} f_{n}
$$

Then $\|S\| \leqslant c^{2}$ and is almost open. Hence $S$ is a surjection; since $l_{p}\left(L_{p}\right) \simeq L_{p}$, the theorem is proved.

Conversely if $X$ is a quotient of $L_{p}$, the pair ( $L_{p}, X$ ) is transitive (cf. [12]) i.e. given $f \in X$ there exists $T \in \mathcal{L}\left(L_{p}, X\right)$ such that $T 1_{K}=f$. However, Theorem 9.6 shows that if $X=L[p ; \Phi]$ then each $T \in \mathcal{L}\left(L_{p}, X\right)$ maps into $L[p, \Psi]$ where $\Psi$ is the symmetric extension of $\Phi$. Hence, $L[p, \Psi]=L[p, \Phi]=X$.

Remark. If we remove the condition of $\sigma$-completeness we must allow the possibility that $X$ is the closure of the simple functions in some $L[p ; \Phi]$ for totally symmetric $\Phi$.

Example 9.9. If $p<q<\infty, L[p, q]$ is a quotient of $L_{p}$ which contains no copy of $l_{p}$.

Proof. It is only necessary to show $l_{p}$ does not embed in $L[p ; q]$. If $F_{1}, \ldots, F_{n}$ $\in \mathcal{G}_{\infty}$,

$$
\left(\int_{0}^{\infty}\left(\sum F_{i}(x)\right)^{q / p} x^{q-1} d x\right)^{p / q} \geqslant c\left(\sum\left(\int_{0}^{\infty} F_{i}(x)^{q / p} x^{q-1} d x\right)^{p / q}\right)^{1 / r}
$$

where $r=2$ if $q \leqslant 2 p$ and $r=q / p$ if $q>2 p$, and $c$ is some constant (see [25]). Hence if $f_{1}, \ldots, f_{n} \in L[p, q]$ have disjoint supports,

$$
\left\|f_{1}+\cdots+f_{n}\right\|^{p} \geqslant c\left(\sum\left\|f_{i}\right\|^{p}\right)^{1 / r}
$$

so $L(p, q)$ satisfies (8.1.1). Hence if $l_{p}$ embeds in $L[p, q]$ there is a sequence $f_{n}$ with disjoint supports so that $\left\|f_{i}\right\| \leqslant 1$, but for all $n$ and all $l_{1}<l_{2}<\cdots<l_{n}$,

$$
\left\|f_{l_{1}}+\cdots+f_{l_{n}}\right\|>c n^{1 / p}
$$

where $c \geqslant 0$. Suppose $F_{1}, \ldots, F_{n}$ are the distributions of these functions. Since $\lambda\left(\operatorname{supp} f_{n}\right) \rightarrow 0, F_{n}(x) \rightarrow 0$ uniformly on $[0, \infty)$. By passing to a subsequence we may suppose $F_{n}$ basic in $L_{q / p}\left((0, \infty), x^{q-1} d x\right)$ and equivalent to a basic sequence with disjoint supports. Thus,

$$
\left\{\int_{0}^{\infty}\left(\sum_{i=1}^{n} F_{i}(x)\right)^{q / p} x^{q-1} d x\right\}^{p / q} \leqslant M n^{p / q}
$$

Thus $c n^{1 / p} \leqslant M n^{1 / q}$ for all $n$. This contradiction shows that $l_{p}$ does not embed in $L[p, q]$.

We conclude $\S 9$ by giving a partial result on the classification of operators on $L(p, \infty)$ for $0<p<1$.

Theorem 9.10. Suppose $T: L(p, \infty) \rightarrow L(p, \infty)$ is positive. Then $T \in \mathcal{E}\left(L_{p}\right)$ and $\|T\|_{p} \leqslant\|T\|_{p, \infty}$.

Proof. We shall suppose $\|T\|_{p, \infty} \leqslant 1$. Consider the restriction $T: L_{p} \rightarrow L(p, \infty)$. We suppose first that $T$ is elementary and (as in Theorem 9.6) $T f(t)=a(t) f(\sigma t)$, where $a: K \rightarrow \mathbf{R}$ is simple and positive. Suppose $b_{1}>b_{2}>\cdots>b_{l}>0$ are the positive values assumed by $a$ and let $B_{j}=\left\{t: a(t)=b_{j}\right\}$. Let $h_{1}, \ldots, h_{l}$ be positive Borel functions so that $\lambda\left(\sigma^{-1} E \cap B_{j}\right)=\int_{E} h_{j}(t) d \lambda(t), E \in \mathscr{B}$, for $j=1,2, \ldots, l$.
Suppose $d>1$ is arbitrary. Then we can decompose $K$ into Borel sets $\left\{A_{i}\right.$ : $i=1,2, \ldots\}$ such that $c_{i j} \leqslant h_{j}(t) \leqslant d c_{i j}, t \in A_{i}, j=1,2, \ldots, l$. Fix $i$ and let
$0<\varepsilon<\lambda\left(A_{i}\right)$ be such that

$$
\varepsilon<\min \left(b_{1}^{-1}, b_{2}^{-1}, \ldots, b_{l}^{-1}, b_{1}, b_{2}, \ldots, b_{l}\right)
$$

Let $f \in L_{p}\left(A_{i}\right)$ be a function whose distribution satisfies

$$
\begin{aligned}
F(x) & =\varepsilon, \quad 0 \leqslant x \leqslant \varepsilon \\
& =\varepsilon^{1+p_{X}-p}, \quad \varepsilon \leqslant x<\varepsilon^{-1} \\
& =0, \quad x>\varepsilon^{-1}
\end{aligned}
$$

Then $\|f\|_{p, \infty}=\sup _{0<x<\infty} x F(x)^{1 / p}=\varepsilon^{1+1 / p}$.
Now $\|T f\|_{p, \infty} \leqslant \varepsilon^{1+1 / p}$, but $T f$ has distribution $G$ satisfying

$$
G(x) \geqslant \sum_{j=1}^{l} F\left(x b_{j}^{-1}\right) c_{i j}, \quad 0<x<\infty
$$

and hence

$$
G(1) \geqslant \sum_{j=1}^{l} F\left(b_{j}^{-1}\right) c_{i j}=\sum_{j=1}^{l} \varepsilon^{1+p} b_{j}^{p} c_{i j} .
$$

Thus

$$
\sum_{j=1}^{l} \varepsilon^{1+p} b_{j}^{p} c_{i j} \leqslant \varepsilon^{1+p}
$$

i.e.,

$$
\sum_{j=1}^{l} b_{j}^{p} c_{i j} \leqslant 1
$$

Now if $E \subset A_{i}$ is Borel then

$$
\begin{aligned}
\left\|T 1_{E}\right\|^{p} & =\sum_{j=1}^{l} b_{j}^{p} \lambda\left(\sigma^{-1} E \cap B_{j}\right)=\sum_{j=1}^{l} b_{j}^{p} \int_{E} h_{j}(t) d \lambda(t) \\
& \leqslant d \sum_{j=1}^{l} b_{j}^{p} c_{i j} \lambda(E) \leqslant d \lambda(E)
\end{aligned}
$$

Hence $\left\|T P_{A_{i}}\right\|_{p} \leqslant d^{1 / p}$. As this is true for all $i$,

$$
\|T\|_{p} \leqslant d^{1 / p}
$$

and as $d>1$ is arbitrary,

$$
\begin{equation*}
\|T\|_{p} \leqslant 1 \tag{9.10.1}
\end{equation*}
$$

Now it follows that (9.10.1) holds for elementary $T$ by approximation of $a(t)$. Then (9.10.1) also holds for locally elementary $T$.

Now we apply Lemma 8.4 to introduce the sequence $S_{n}$. Then for each $n$ $0 \leqslant S_{1}+\cdots+S_{n} \leqslant T$ and hence $\left\|S_{1}+\cdots+S_{n}\right\|_{p} \leqslant 1$. From this it follows easily that $\|T\|_{p} \leqslant 1$.
10. Miscellaneous results. Suppose $G$ is a compact metrizable group and $\lambda$ is Haar measure on $G$. Then the translation invariant operators $T: L_{p} \rightarrow L_{p}(0<p<$ 1) and $T: L_{p} \rightarrow L(p, \infty)$ have been classified by Sawyer [30] (for the circle group) and Oberlin [23], [24] for general locally compact groups.

Suppose $\Phi$ is a totally symmetric function in $\Lambda_{p}$. Then $l(p ; \Phi)$ denotes the sequence space of all $\left(\xi_{n}\right)$ such that $\Phi(F)<\infty$ where $F(x)$ is the number of $n$ such that $\left|\xi_{n}\right|>x$, and we also denote by $\Phi$ the symmetric extension of $\Phi$. For simplicity we shall suppose $G$ abelian in the next two theorems.

Theorem 10.1. Suppose $T: L_{p}(G) \rightarrow L[p, \Phi](G)(0<p<1)$ is translation invariant. Then

$$
\begin{equation*}
T f=\mu * f, \quad f \in L_{p} \tag{10.1.1}
\end{equation*}
$$

where $\mu=\sum_{n=1}^{\infty} \xi_{n} \delta\left(g_{n}\right)$, with $\left(g_{n}: n \in N\right)$ a sequence of points in $G$ and $\left(\xi_{n}\right) \in$ $l(p ; \Phi)$. Conversely, if $\left(\xi_{n}\right) \in l(p ; \Phi),(10.1 .1)$ defines a translation-invariant operator $T \in \mathcal{L}\left(L_{p}, L[p ; \Phi]\right)$.

Proof. Suppose $T$ has kernel $g \mapsto \nu_{g}$. Then for a dense sequence $\left\{f_{n}\right\} \subset C(G)$, we have, for fixed $g \in G$

$$
\begin{aligned}
T f_{n}(g h) & =\int_{G} f_{n}(t) d \nu_{g h} \quad \text { a.e., } h \in G \\
& =\int_{G} f_{n}(g t) d \nu_{h} \quad \text { a.e., } h \in G
\end{aligned}
$$

In particular $\left\|\nu_{g}\right\|$ is a translation-invariant function in $L^{\infty}(G)$; hence $\left\|\nu_{g}\right\|$ is constant almost everywhere and so $T \in \mathcal{L}\left(L_{1}\right)$. Thus (see [33]), $T f=\mu * f, f \in L_{1}$, where $\mu \in \mathscr{N}(K)$. By the uniqueness of the kernel, $T f=\mu * f, f \in L_{p}$, and $\mu=\sum_{n=1}^{\infty} \xi_{n} \delta\left(g_{n}\right)$ where $g_{n} \in G$ are distinct, and $\Sigma\left|\xi_{n}\right|<\infty$. Now $\nu_{h}=$ $\sum_{n=1}^{\infty} \xi_{n} \delta\left(h^{-1} g_{n}\right)$ a.e. and hence $\Delta_{x}\left(\nu_{h}\right)=\Delta_{x}(\mu)$ a.e. Thus $T$ has distribution $w$ where

$$
\begin{aligned}
w(t, x) & =\Delta_{x}(\mu) \quad \text { a.e., } 0 \leqslant x<\infty \\
& =F(x)
\end{aligned}
$$

where $F(x)$ is the number of $n$ such that $\left|\xi_{n}\right|>x$. Then $T \in \mathcal{E}\left(L_{p}, L[p ; \Phi]\right)$ provided $\Phi(F)<\infty$. The converse is easy.

Examples. (1) $T \in \mathcal{L}\left(L_{p}, L[p, \infty]\right)$ if and only if $\xi_{n} \in l(p, \infty)$, i.e., $\sup _{n} n^{1 / p} \xi_{n}^{*}$ $<\infty$, where $\left(\xi_{n}^{*}\right)$ is the decreasing rearrangment of $\left(\xi_{n}\right)$ (see Oberlin [24]).
(2) $T \in \mathcal{L}\left(L_{p} ; L[p ; q]\right)(p \leqslant q<\infty)$ if and only if $\xi_{n} \in l(p ; q)$, i.e.

$$
\begin{aligned}
& q / p \sum_{n=1}^{\infty}\left(\xi_{n}^{*}\right)^{q} \int_{n-1}^{n} t^{q / p-1} d t<\infty \\
& \sum_{n=1}^{\infty}\left(\xi_{n}^{*}\right)^{q}\left[n^{q / p}-(n-1)^{q / p}\right]<\infty
\end{aligned}
$$

which is equivalent to

$$
\sum_{n=1}^{\infty}\left(\xi_{n}^{*}\right)^{q} n^{q / p-1}<\infty
$$

(3) If $T \in \mathcal{L}\left(L_{p}, L_{0}\right)$ is translation invariant, the same proof combined with Nikisin's theorem shows that $\sup _{n} n^{1 / p} \xi_{n}^{*}<\infty$, i.e. $T \in \mathscr{L}\left(L_{p}, L(p ; \infty)\right)$.

Theorem 10.2. Suppose $\mu \in \mathscr{R}(G)$ and for each $f \in L[p ; \infty], \mu * f$ is well defined, i.e. for almost every $g \in G$,

$$
\begin{equation*}
\int_{G}|f(g h)| d|\mu|(h)<\infty . \tag{10.2.1}
\end{equation*}
$$

Then

$$
\begin{equation*}
\mu=\sum \xi_{n} \delta\left(g_{n}\right) \tag{10.2.2}
\end{equation*}
$$

where $\left(g_{n}\right)$ is a sequence of disjoint points in $G$ and

$$
\begin{equation*}
\sum\left|\xi_{n}\right|^{p}<\infty \tag{10.2.3}
\end{equation*}
$$

Conversely, if (10.2.2) and (10.2.3) hold then $\mu * f$ is defined for all $f \in L(p, \infty)$.
Proof. If $\mu * f$ is defined for each $f \in L(p, \infty)$, we define

$$
T f=|\mu| * f, \quad f \in L[p ; \infty]
$$

It is easy to see (by considering $f \in L_{1}$ first) that $T f \in L_{0}$. Also if $\left\|f_{n}\right\| \leqslant 2^{-n}$, then $\Sigma\left|f_{n}\right|$ converges in $L(p ; \infty)$. Hence $|\mu| * \Sigma\left|f_{n}\right|$ is well defined and so $|\mu| *\left|f_{n}\right| \rightarrow 0$ a.e. Then $|\mu| * f_{n} \rightarrow 0$ a.e. Thus $T: L[p ; \infty] \rightarrow L_{0}$ is a linear operator.

Now by Nikisin's theorem there is a positive function $\varphi \in L_{0}$ such that $M_{\varphi} T \in$ $\mathcal{L}(L(p ; \infty)$ ) where

$$
M_{\varphi} f(t)=\varphi(t) f(t) .
$$

Now $M_{\varphi} T \geqslant 0$ and hence by Theorem 9.10, $M_{\varphi} T \in \mathcal{E}\left(L_{p}\right)$. Now this implies that $\varphi(t)\|\mu\|_{p}<\infty$ for almost every $t$; thus (10.2.2) and (10.2.3) follow.

The converse follows easily from the $p$-convexity of $L[p ; \infty]$.
To conclude, we observe that our main theorem 7.2 implies a similar result for an $F$-space $X$ in place of $L_{0}$ provided there are enough linear operators $T \in \mathcal{E}\left(X, L_{0}\right)$ to separate points. It is easy enough to produce spaces $X$ which fail this property and for which $\mathcal{L}\left(X, L_{0}\right)=\{0\}$. An example is the space $L_{p} / H_{p}$ considered in [10] (for the same reasons as in the proof of $\mathcal{E}\left(L_{p} / H_{p}, L_{p}\right)=\{0\}$ given in that paper). We now prove a related result concerning this space. For convenience we convert it to a real space, by the process of taking real and imaginary parts.

Let $K=\Gamma_{1} \cup \Gamma_{2}$ where $\Gamma_{1}$ and $\Gamma_{2}$ denote two disjoint copies of the unit circle $\Gamma=(z:|z|=1)$. Let $\lambda=1 / 2\left(\lambda_{1}+\lambda_{2}\right)$ where $\lambda_{1}$ and $\lambda_{2}$ denote Haar measure on $\Gamma_{1}$ and $\Gamma_{2}$ respectively. Let $\hat{H}_{p}$ be the closed subspace of $L_{p}(K)$ generated by the function (where $0 \leqslant n<\infty$ )

$$
\begin{array}{rlrl}
e_{n}(z) & =\operatorname{Re} z^{n}, \quad z \in \Gamma_{1}, \\
& =\operatorname{Im} z^{n}, \quad \begin{array}{c}
z \in \Gamma_{2}, \\
e_{n}^{*}(z)
\end{array}=-\operatorname{Im} z^{n}, \quad z \quad z \in \Gamma_{1}, \\
& =\operatorname{Re} z^{n}, \quad & z \in \Gamma_{2} .
\end{array}
$$

Let $f \in \hat{H}_{p}$ if and only if $f_{1}(z)+i f_{2}(z) \in H_{p}$ where

$$
\begin{array}{ll}
f_{1}(z)=f(z), & z \in \Gamma_{1}, \\
f_{2}(z)=f(z), & z \in \Gamma_{2} .
\end{array}
$$

Thus $\hat{H}_{p}$ is a proper closed subspace of $L_{p}$.

Theorem 10.3. Let $X$ be a $\sigma$-complete $p$-Banach lattice with order-continuous quasinorm (i.e. such that for any monotone decreasing sequence $x_{n}$ with inf $x_{n}=0$ then $\left.\inf \left\|x_{n}\right\|=0\right)$. Then $\mathcal{E}\left(L_{p} / \hat{H}_{p}, X\right)=\{0\}$.

Proof. Suppose $T \in \mathcal{L}\left(L_{p} / \hat{H}_{p}, X\right)$, and let $Q: L_{p} \rightarrow L_{p} / \hat{H}_{p}$ be the quotient map. Let $S=|T Q| \in \mathcal{L}\left(L_{p}, X\right)$, and let $S 1_{K}=u \in X$. Let $Y \subset X$ be the linear span of $[-u, u]$ with $[-u, u]$ as its unit ball; then $Y$ is a Banach lattice which is an AM-space [35, p. 22]. Hence $Y$ is isometrically isomorphic to a space $C(\Omega)$ where $\Omega$ is a compact Hausdorff space; since $Y$ is order-complete, $\Omega$ is Stonian (see [35, pp. 59 and 92]). We shall identify $Y$ and $C(\Omega)$.

Now $T(C(K)) \subset C(\Omega)$ and $T: C(K) \rightarrow C(\Omega)$ is a linear map of norm one at most. Hence

$$
T f(\omega)=\int_{K} f d \nu_{\omega}, \quad f \in C(K)
$$

where $\nu_{\omega} \in \mu(K)$ and $\left\|\nu_{\omega}\right\| \leqslant 1$.
For any $\omega \in \Omega$,

$$
\int_{K} e_{n} d \nu_{\omega}=\int_{K} e_{n}^{*} d \nu_{\omega}=0, \quad n=0,1,2, \ldots
$$

Let $\mu_{1}$ and $\mu_{2}$ be the measures induced on $\Gamma$ by $\nu_{\omega} \mid \Gamma_{1}$ and $\nu_{\omega} \mid \Gamma_{2}$. Then

$$
\begin{aligned}
\int_{\Gamma} \operatorname{Re} z^{n} d \mu_{1}+\int_{\Gamma} \operatorname{Im} z^{n} d \mu_{2} & =0 \\
-\int \operatorname{Im} z^{n} d \mu_{1}+\int \operatorname{Re} z^{n} d \mu_{2} & =0
\end{aligned}
$$

so that

$$
\int_{\Gamma} z^{n} d\left(\mu_{1}-i \mu_{2}\right)=0
$$

Now by the F. and M. Riesz Theorem [6, p. 41], $\mu_{1}-i \mu_{2}$ is absolutely continuous with respect to Haar measure on the circle. Thus $\nu_{\omega}$ is $\lambda$-continuous for every $\omega \in \Omega$.

Let $U=\left\{f \in L_{p}(K):\|f\|_{\infty} \leqslant 1\right\}$. We shall show $T(U)$ is relatively compact. Suppose not; then we may find a sequence of continuous functions $f_{n}$ such that

$$
\left\|T f_{n}-T f_{m}\right\| \geqslant \varepsilon>0, \quad m \neq n
$$

and $\left\|f_{n}\right\|_{\infty} \leqslant 1$. By passing to a subsequence we may suppose ( $f_{n}$ ) converges in $\sigma\left(L_{\infty}, L_{1}\right)$ and hence that if $g_{n}=f_{n}-f_{n+1}, g_{n} \rightarrow 0, \sigma\left(L_{\infty}, L_{1}\right),\left\|T g_{n}\right\|>\varepsilon, n=$ $1,2, \ldots$ Now $\operatorname{Tg}_{n}(\omega) \rightarrow 0, \omega \in \Omega$. Let

$$
h_{n}=\sup \left(\left|T g_{n}\right|,\left|T g_{n+1}\right|, \ldots\right)
$$

in $X$. Then $h_{n} \leqslant u$ and $h_{n} \in Y$. Hence

$$
h_{n}(\omega)=\sup _{n}\left|T g_{n}(\omega)\right|
$$

except on a set of first category in $\omega$. Thus $h_{n}(\omega) \rightarrow 0$ except on a set of first category. It follows that $\inf h_{n}=0$ and so $\inf _{n}\left\|h_{n}\right\|=0$. However, $\left\|h_{n}\right\|>$ $\sup \left\|\operatorname{Tg}_{n}\right\| \geqslant \varepsilon$.

It follows that $T(U)$ is compact and hence $T(U)=0$ (see [10]).

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