q -EXTENSION OF THE p -ADIC GAMMA FUNCTION

BY

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ABSTRACT. p -adic functions depending on a parameter q , $0 < |q - 1|_p < 1$, are defined which extend Y. Morita's p -adic gamma function and the derivative of J. Diamond's p -adic log-gamma function in the same way as the classical q -gamma function $\Gamma_q(x)$ extends $\Gamma(x)$. Properties of these functions which are analogous to the basic identities satisfied by $\Gamma_q(x)$ are developed.

1. Introduction. A generalized gamma function $\Gamma_q(x)$, depending on a parameter $0 < q < 1$, was introduced and studied by F. H. Jackson, R. Askey, G. E. Andrews and others. Defined as

$$\Gamma_q(x) = (1 - q)^{1-x} \frac{(1 - q)(1 - q^2)(1 - q^3) \cdots}{(1 - q^x)(1 - q^{x+1})(1 - q^{x+2}) \cdots},$$

it satisfies relations which generalize the well-known identities for the gamma function, and in the limit as $q \rightarrow 1^-$ it becomes $\Gamma(x)$.

The purpose of this article is to construct and study the properties of a natural q -extension $\Gamma_{p,q}$ of Morita's p -adic gamma function Γ_p [13] and a q -extension $\psi_{p,q}$ of the derivative of Diamond's p -adic log-gamma function G_p [4]. Recall that Γ_p is a function from the p -adic integers $\mathbb{Z}_p$ to the p -adic units $\mathbb{Z}_p^*$ defined by

$$\Gamma_p(x) = \lim_{n \rightarrow x} (-1)^n \prod'_{j < n} j,$$

where n runs over positive integers and $\prod'$ means that indices j divisible by p are omitted. On the other hand, G_p is a function on the complement of $\mathbb{Z}_p$ in Ω_p (where Ω_p is the p -adic completion of the algebraic closure of the field $\mathbb{Q}_p$ of p -adic numbers, with norm $|\cdot|_p$ for which $|p|_p = p^{-1}$). It is defined as

$$G_p(x) = \lim_{N \rightarrow \infty} p^{-N} \sum_{0 < j < p^N} (x + j)(\log_p(x + j) - 1),$$

where $\log_p$ is the Iwasawa p -adic logarithm [7]. Although G_p is not $\log_p \Gamma_p$, it has the following two connections with $\log_p \Gamma_p$ (see [4] and [6]). (1) If we let $G_p^*(x) = G_p(x) - G_p(x/p)$, which is defined on the complement of $\mathbb{Z}_p^*$ by the same formula as G_p with Σ replaced by Σ' , then $G_p^*(x) = \log_p \Gamma_p(x)$ for $x \in p\mathbb{Z}_p$. (2) $\log_p \Gamma_p(x) = \sum_{0 < j < p, x+j \notin p\mathbb{Z}_p} G_p((x + j)/p)$ for $x \in \mathbb{Z}_p$.

For simplicity suppose $p > 2$.

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2. q -extension of Γ_p .

THEOREM 1. Let $t \in \Omega_p$, $|t|_p < 1$, $t \neq 0$. Set $q = 1 + t$, and for any n define

$$\Gamma_{p,q}(n) = (-1)^n \prod'_{j < n} \frac{1 - q^j}{1 - q}.$$

Then $\Gamma_{p,q}$ extends to a continuous function on $\mathbb{Z}_p$, and $\lim_{q \rightarrow 1} \Gamma_{p,q} = \Gamma_p$.

PROOF. We first prove two general lemmas.

LEMMA 1. Let $P_1(X), \dots, P_h(X) \in \mathbb{Q}[X]$. Then there exists $Q(X) \in \mathbb{Q}[X]$ such that for all n

$$\sum_{i_1 < i_2 < \dots < i_h < n} P_1(i_1) \cdots P_h(i_h) = Q(n).$$

PROOF OF LEMMA. Use induction on h . For $h = 1$ the lemma follows because $\sum_{i_1=1}^{n-1} i_1^l$ is a polynomial in n for any l . Suppose the lemma holds for $h - 1$. We have

$$\begin{aligned} \sum_{i_1 < i_2 < \dots < i_h < n} P_1(i_1) \cdots P_h(i_h) &= \sum_{i_2 < \dots < i_h < n} \left(\sum_{i_1 < i_2} P_1(i_1) \right) P_2(i_2) \cdots P_h(i_h) \\ &= \sum_{i_2 < \dots < i_h < n} (Q_1(i_2) P_2(i_2)) P_3(i_3) \cdots P_h(i_h) \end{aligned}$$

by the lemma for $h = 1$. But this is of the form $Q(n)$ by the lemma for $h - 1$ applied to the polynomials $Q_1 P_2, P_3, \dots, P_h$.

LEMMA 2. Let $P_k(X) \in \mathbb{Q}[X]$, $k = 1, 2, \dots$; $A_j(t) = 1 + \sum_{k=1}^{\infty} P_k(j)t^k \in \mathbb{Q}[[t]]$, $j = 1, 2, \dots$. Then there exist $Q_k(X) \in \mathbb{Q}[X]$ such that for all n ,

$$\prod_{j < n} A_j(t) = 1 + \sum_{k=1}^{\infty} Q_k(n)t^k.$$

PROOF OF LEMMA. For each k let $s = \{s_1, \dots, s_h\}$ run through the set S of partitions of ordered positive integers s_i whose sum is k . The coefficient of t^k in $\prod_{j < n} A_j(t)$ is clearly

$$\sum_{s \in S} \sum_{i_1 < i_2 < \dots < i_h < n} P_{s_1}(i_1) \cdots P_{s_h}(i_h).$$

Since the first sum is finite, this is a polynomial in n by Lemma 1.

PROOF OF THEOREM 1. Let $P_k(X) = (X - 1)(X - 2) \cdots (X - k)/(k + 1)!$. Then

$$\frac{1 - q^j}{1 - q} = \frac{(1 + t)^j - 1}{t} = j \left(1 + \sum_{k=1}^{\infty} P_k(j)t^k \right).$$

Further let $\tilde{n} = [(n - 1)/p] + 1$, and let $\tilde{P}_k(X) = P_k(pX)$. Finally, let

$$A_j(t) = 1 + \sum_{k=1}^{\infty} P_k(j)t^k, \quad \tilde{A}_j(t) = 1 + \sum_{k=1}^{\infty} \tilde{P}_k(j)t^k.$$

Then

$$\Gamma_{p,q}(n) = \Gamma_p(n) \frac{\prod_{j < n} A_j(t)}{\prod_{j < \tilde{n}} \tilde{A}_j(t)}.$$

By Lemma 2, the quotient can be written $(1 + \sum Q_k(n)t^k)/(1 + \sum \tilde{Q}_k(\tilde{n})t^k)$.

Note that $\Gamma_{p,q}(n) = (-1)^n \prod_{j < n, p \nmid j} (1 + (1+t) + \cdots + (1+t)^{j-1}) \in \mathbb{Z}[t]$, and hence $\Gamma_{p,q}(n)/\Gamma_p(n) = 1 + \sum_{k=1}^{\infty} R_k(n)t^k \in \mathbb{Z}_p[[t]]$ for each n (where the R_k are some functions of n), i.e., $|R_k(n)|_p < 1$ for all k, n . Since each R_k is a finite expression in Q_{k_1} and $\tilde{Q}_{k_2}$, $k_1, k_2 \leq k$, and since each $\tilde{Q}_{k_2}$ depends continuously on $\tilde{n}$ and hence on n , and Q_{k_1} is just a polynomial in n , it follows that R_k is a bounded (by 1) continuous function of n . Since $|t|_p < 1$, the theorem follows. Q.E.D.

THEOREM 2.

(1)

$$\Gamma_{p,q}(x+1)/\Gamma_{p,q}(x) = \begin{cases} -(1-q^x)/(1-q) & \text{if } x \in \mathbb{Z}_p^*, \\ -1 & \text{if } x \in p\mathbb{Z}_p. \end{cases}$$

(2)

$$\begin{aligned} \Gamma_{p,q}(x) \Gamma_{p,q^m}\left(\frac{1}{m}\right) \Gamma_{p,q^m}\left(\frac{2}{m}\right) \cdots \Gamma_{p,q^m}\left(\frac{m-1}{m}\right) \\ = \Gamma_{p,q^m}\left(\frac{x}{m}\right) \Gamma_{p,q^m}\left(\frac{x+1}{m}\right) \cdots \Gamma_{p,q^m}\left(\frac{x+m-1}{m}\right) \left(\frac{1-q^m}{1-q}\right)^{x-\tilde{x}} \end{aligned}$$

for any $x \in \mathbb{Z}_p$ and any positive integer m prime to p , where $\tilde{\cdot}$ is the unique continuous extension to $\mathbb{Z}_p$ of the map $n \mapsto \tilde{n} = [(n-1)/p] + 1$ used in the proof of Theorem 1. (If $x = a_0 + a_1p + a_2p^2 + \cdots$, then $\tilde{x} = (a_1 + 1) + a_2p + a_3p^2 + \cdots$ if $a_0 \neq 0$ and $a_1 + a_2p + a_3p^2 + \cdots$ if $a_0 = 0$. Note that $(1-q^m)/(1-q)$ is a p -adic unit, and the exponent can be written $a_0 - p + (p-1)\tilde{x}$ if $a_0 \neq 0$ and $(p-1)\tilde{x}$ if $a_0 = 0$.)

(3)

$$\Gamma_{p,q}(x) \Gamma_{p,1/q}(1-x) = -(-q)^{\tilde{x}-x}.$$

PROOF. Since both sides of (1), (2) and (3) are continuous in x , it suffices to prove them for $x = n$. In that case (1) follows immediately from the definition.

(2) For fixed m let A_n denote the left side of (3) for $x = n$, and let B_n denote the right side. We prove that $A_n = B_n$ by induction on n . Trivially $A_1 = B_1$. Suppose $A_n = B_n$. We have

$$\frac{A_{n+1}}{A_n} = \frac{\Gamma_{p,q}(n+1)}{\Gamma_{p,q}(n)} = \begin{cases} -(1-q^n)/(1-q) & \text{if } p \nmid n, \\ -1 & \text{if } p \mid n, \end{cases}$$

and

$$\begin{aligned} \frac{B_{n+1}}{B_n} &= \frac{\Gamma_{p,q^m}(n/m+1)}{\Gamma_{p,q^m}(n/m)} \left(\frac{1-q^m}{1-q}\right)^{1+\tilde{n}-(\widetilde{n+1})} \\ &= \begin{cases} (-(1-q^n)/(1-q^m)) \cdot ((1-q^m)/(1-q)) & \text{if } p \nmid n, \\ -1 \cdot 1 & \text{if } p \mid n. \end{cases} \end{aligned}$$

Hence $A_{n+1}/A_n = B_{n+1}/B_n$, and so $A_{n+1} = B_{n+1}$. This completes the induction.

(3) is easily proved by induction in the same way as (2). Q.E.D.

REMARKS. 1. Comparing parts (2) and (3) of Theorem 2 gives support for a comment once made by B. H. Gross that the Euler reflection formula for the gamma function should be thought of as the (-1) -case of the multiplication formula. The right side of (3) can be written as

$$\Gamma_{p,q}(1) \left(\frac{1 - q^{-1}}{1 - q} \right)^{x - \bar{x}}$$

(recall $\Gamma_{p,q}(1) = -1$). This similarity between the multiplication and the reflection formulas for Γ_p was not clear until we looked at its q -extension $\Gamma_{p,q}$.

2. In the classical case the type of argument in Theorem 2 above will quickly reveal that $\Gamma_q(x)\Gamma_{1/q}(1-x)(-q)^x$ is periodic of period 1 (where we use D. Moak's Γ_q for $q > 1$ [11] to define $\Gamma_{1/q}(1-x)$). But this periodic function is not a constant (as any continuous function on $\mathbf{Z}_p$ with period 1 must be); it turns out to equal a constant divided by the theta-function

$$\sum_{n=-\infty}^{\infty} (-1)^{-(n+x)} q^{(n+x-1/2)^2/2}.$$

3. Table of properties.

gamma function	Classical Case its q -extension ($0 < q < 1$)
(1) $\Gamma(n+1) = n!$	$\Gamma_q(n+1) = \prod_{j=1}^n (1 + q + q^2 + \cdots + q^{j-1})$
(2) $\frac{\Gamma(x+1)}{\Gamma(x)} = x$, $\Gamma(1) = 1$	$\frac{\Gamma_q(x+1)}{\Gamma_q(x)} = \frac{(1-q^x)}{(1-q)}$, $\Gamma(1) = 1$.
Γ and Γ_q are uniquely characterized by (2) together with convexity of their logarithm.	
(3) $\Gamma(x)\Gamma\left(\frac{1}{m}\right) \cdots \Gamma\left(\frac{m-1}{m}\right)$ $= \Gamma\left(\frac{x}{m}\right)\Gamma\left(\frac{x+1}{m}\right) \cdots$ $\Gamma\left(\frac{x+m-1}{m}\right) m^{x-1}$ for any positive integer m	$\Gamma_q(x)\Gamma_{q^m}\left(\frac{1}{m}\right) \cdots \Gamma_{q^m}\left(\frac{m-1}{m}\right)$ $= \Gamma_{q^m}\left(\frac{x}{m}\right)\Gamma_{q^m}\left(\frac{x+1}{m}\right) \cdots$ $\Gamma_{q^m}\left(\frac{x+m-1}{m}\right) \left(\frac{(1-q^m)}{(1-q)}\right)^{x-1}$ for any positive integer m
(4) $\Gamma(x) \cdot \Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$	$\Gamma_q(x)\Gamma_q(1-x)$ $= \sum_{n=-\infty}^{\infty} \frac{(1-q)(-1)^n q^{n(n+1)/2}}{(1-q^{n+x})}$
(5) $\text{dlog } \Gamma(1) = -\gamma$	$\text{dlog } \Gamma_q(1) = -\gamma_q$.

p -adic Case

gamma function

its q -extension ($0 < |q - 1|_p < 1$)

$$\begin{aligned}
 (1) \quad \Gamma_p(n+1) &= (-1)^{n+1} \prod_{j=1}^n j & \Gamma_{p,q}(n+1) &= (-1)^{n+1} \prod_{j=1}^n \\
 & & & \cdot (1 + q + q^2 + \cdots + q^{j-1}) \\
 (2) \quad \frac{\Gamma_p(x+1)}{\Gamma_p(x)} &= \begin{cases} -x & \text{if } x \in \mathbf{Z}_p^*, \\ -1 & \text{if } x \in p\mathbf{Z}_p, \end{cases} & \frac{\Gamma_{p,q}(x+1)}{\Gamma_{p,q}(x)} &= \begin{cases} -\frac{(1-q^x)}{(1-q)} & \text{if } x \in \mathbf{Z}_p^*, \\ -1 & \text{if } x \in p\mathbf{Z}_p, \end{cases} \\
 \Gamma_p(1) &= -1 & \Gamma_{p,q}(1) &= -1.
 \end{aligned}$$

Γ_p and $\Gamma_{p,q}$ are uniquely characterized by (2) together with continuity.

$$\begin{aligned}
 (3) \quad \Gamma_p(x) \Gamma_p\left(\frac{1}{m}\right) \cdots \Gamma_p\left(\frac{m-1}{m}\right) &= \Gamma_p\left(\frac{x}{m}\right) \Gamma_p\left(\frac{x+1}{m}\right) \cdots \\
 & \Gamma_p\left(\frac{x+m-1}{m}\right) m^{x-\bar{x}} & \Gamma_{p,q}(x) \Gamma_{p,q^m}\left(\frac{1}{m}\right) \cdots \Gamma_{p,q^m}\left(\frac{m-1}{m}\right) \\
 & = \Gamma_{p,q^m}\left(\frac{x}{m}\right) \Gamma_{p,q^m}\left(\frac{x+1}{m}\right) \\
 & \cdots \Gamma_{p,q^m}\left(\frac{x+m-1}{m}\right) \left(\frac{1-q^m}{1-q}\right)^{x-\bar{x}} \\
 & \text{for any positive integer } m & \text{for any positive integer } m \\
 & \text{prime to } p & \text{prime to } p
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad \Gamma_p(x) \cdot \Gamma_p(1-x) &= (-1)^{x-\bar{x}} \Gamma_p(1) & \Gamma_{p,q}(x) \cdot \Gamma_{p,1/q}(1-x) \\
 & = (-q)^{\bar{x}-x} \Gamma_{p,q}(1)
 \end{aligned}$$

$$\begin{aligned}
 (5) \quad \text{dlog } \Gamma_p(0) &= -(1-1/p)\gamma_p & \text{dlog } \Gamma_{p,q}(0) &= -(1-1/p)\gamma_{p,q} \\
 & & & \text{(see below)}
 \end{aligned}$$

Finally, $\Gamma_q \rightarrow \Gamma$ as $q \rightarrow 1^-$, and $\Gamma_{p,q} \rightarrow \Gamma_p$ as $q \rightarrow 1$.

4. q -extension of G'_p . For $x \in \Omega_p$ let $d(x) = \min_{z \in \mathbf{Z}_p} |x - z|_p$ be the distance from x to $\mathbf{Z}_p$. For $q = 1 + t$, $|t|_p < 1$, let $\epsilon_q = 1/(\log_p q|_p \cdot p^{1/(p-1)})$, and let $D_q = \{x \in \Omega_p \mid 0 < d(x) < \epsilon_q\}$. Thus, if $\text{ord}_p t > 1/(p-1)$, we have $\epsilon_q = 1/(|t|_p \cdot p^{1/(p-1)}) > 1$ and $D_q = \{x \in \Omega_p - \mathbf{Z}_p \mid |x|_p < \epsilon_q\}$; as $q \rightarrow 1$, D_q expands to all of $\Omega_p - \mathbf{Z}_p$.

If $x \in D_q$, say $|x - z|_p < \epsilon_q$ with $z \in \mathbf{Z}_p$, then $q^x = q^z e^{(x-z)\log_p q}$, and so q^x is a locally analytic function on D_q . Since also $q^x \neq 1$ here, it follows that $\log_p(1 - q^x)$ is locally analytic on D_q . We can then define

$$\begin{aligned}
 \psi_{p,q}(x) &= \lim_{N \rightarrow \infty} p^{-N} \sum_{0 \leq j < p^N} \log_p \frac{1 - q^{x+j}}{1 - q} \\
 &= -\log_p(1 - q) + \lim_{N \rightarrow \infty} p^{-N} \sum_{0 \leq j < p^N} \log_p(1 - q^{x+j}),
 \end{aligned}$$

which exists and is locally analytic on D_q by Theorem 2 of [4]. Clearly $\psi_{p,q} \rightarrow G'_p$ as $q \rightarrow 1$.

We can define $\psi_{p,q}^*(x)$ —a q -extension of $G_p^{*'}—by replacing $d(x)$ by $d^*(x) = \min_{z \in \mathbb{Z}_p} |x - z|_p$, D_q by $D_q^* = \{x \in \Omega_p | 0 < d^*(x) < \varepsilon_q\}$, and Σ by Σ' in the definition of $\psi_{p,q}$. Then$

$$\psi_{p,q}^*(x) = \psi_{p,q}(x) - \frac{1}{p} \psi_{p,q^p} \left(\frac{x}{p} \right) - \frac{1}{p} \log_p \frac{1 - q^p}{1 - q} \quad \text{for } x \in D_q.$$

We first show the relationship between $\psi_{p,q}(x)$ and $d \log_p \Gamma_{p,q}(x)/dx$.

THEOREM 3.

(1)

$$\psi_{p,q}^*(x) = \frac{d}{dx} \log_p \Gamma_{p,q}(x) \quad \text{for } x \in p\mathbb{Z}_p;$$

(2)

$$\frac{d}{dx} \log_p \Gamma_{p,q}(x) = \frac{1}{p} \sum_{\substack{0 \leq j < p \\ x+j \notin p\mathbb{Z}_p}} \psi_{p,q^p} \left(\frac{x+j}{p} \right) + \left(1 - \frac{1}{p} \right) \log_p \frac{1 - q^p}{1 - q} \quad \text{for } x \in \mathbb{Z}_p.$$

Note that if $\text{ord}_p(q-1) > 1/(p-1)$, then $\varepsilon_q > 1$ and $\varepsilon_{q^p} > p$, so that $x \in D_q^*$ and $(x+j)/p \in D_{q^p}$. But even if $\text{ord}_p(q-1) \leq 1/(p-1)$, in which case we may have $x \notin D_q^*$ and $(x+j)/p \notin D_{q^p}$, still $\psi_{p,q}^*(x)$ in (1) and $\psi_{p,q^p}((x+j)/p)$ in (2) are well defined because $x \in p\mathbb{Z}_p$ and $x \in \mathbb{Z}_p$, respectively.

PROOF. (1) By the definition of $\Gamma_{p,q}$, for $x \in p\mathbb{Z}_p$ we have

$$p^{-N} (\log_p \Gamma_{p,q}(x + p^N) - \log_p \Gamma_{p,q}(x)) = p^{-N} \sum'_{0 \leq j < p^N} \log_p \frac{1 - q^{x+j}}{1 - q},$$

and taking the limit as $N \rightarrow \infty$ gives (1).

(2) The sum on the right in (2) is equal to

$$\begin{aligned} & \lim_{N \rightarrow \infty} p^{-N-1} \sum_{\substack{0 \leq j < p^N \\ 0 \leq k < p \\ x+k \notin p\mathbb{Z}_p}} \log_p \frac{1 - q^{p((x+k)/p+j)}}{1 - q^p} \\ &= - \left(1 - \frac{1}{p} \right) \log_p \frac{1 - q^p}{1 - q} + \lim_{N \rightarrow \infty} p^{-N} \sum_{\substack{0 \leq j < p^N \\ x+j \notin p\mathbb{Z}_p}} \log_p \frac{1 - q^{x+j}}{1 - q} \\ &= - \left(1 - \frac{1}{p} \right) \log_p \frac{1 - q^p}{1 - q} + \lim_{N \rightarrow \infty} p^{-N} (\log_p \Gamma_{p,q}(x + p^N) - \log_p \Gamma_{p,q}(x)) \\ &= - \left(1 - \frac{1}{p} \right) \log_p \frac{1 - q^p}{1 - q} + \frac{d}{dx} \log_p \Gamma_{p,q}(x). \quad \text{Q.E.D.} \end{aligned}$$

The next theorem, giving identities for $\psi_{p,q}$, should be compared with the table in §3.

THEOREM 4. For $x \in D_q$,

(1)

$$\psi_{p,q}(x+1) - \psi_{p,q}(x) = -\frac{q^x \log_p q}{1-q^x} = \frac{d}{dx} \log_p \frac{1-q^x}{1-q},$$

(2)

$$\psi_{p,q}(x) - \frac{1}{m} \sum_{h=0}^{m-1} \psi_{p,q^m}\left(\frac{x+h}{m}\right) = \log_p \frac{1-q^m}{1-q} = \frac{d}{dx} \log_p \left(\frac{1-q^m}{1-q}\right)^{x-1}$$

for any positive integer m (not necessarily prime to p),

(3)

$$\psi_{p,q}(x) - \psi_{p,1/q}(1-x) = -\log_p q = \frac{d}{dx} \log_p (-q)^{-x}.$$

PROOF. (1) This follows by Theorem 4 of [4]:

$$\begin{aligned} \psi_{p,q}(x+1) - \psi_{p,q}(x) &= \lim_{N \rightarrow \infty} p^{-N} \left(\log_p \frac{1-q^{x+p^N}}{1-q} - \log_p \frac{1-q^x}{1-q} \right) \\ &= \frac{d}{dx} \log_p \frac{1-q^x}{1-q}. \end{aligned}$$

(2) By Theorem 1(ii) of [4], $\psi_{p,q}(x)$ can also be written (for any m) as

$$\lim_{N \rightarrow \infty} \frac{1}{mp^N} \sum_{0 \leq j < mp^N} \log_p \frac{1-q^{x+j}}{1-q}.$$

Then the left side in (2) equals

$$\begin{aligned} \lim_{N \rightarrow \infty} \left(\frac{1}{mp^N} \sum_{0 \leq j < mp^N} \log_p \frac{1-q^{x+j}}{1-q} \right. \\ \left. - \frac{1}{mp^N} \sum_{0 \leq j < p^N} \sum_{0 \leq h < m} \log_p \frac{1-q^{m((x+h)/m+j)}}{1-q^m} \right) \\ = \lim_{N \rightarrow \infty} \frac{1}{mp^N} \sum_{0 \leq j < mp^N} \left(\log_p \frac{1-q^{x+j}}{1-q} - \log_p \frac{1-q^{x+j}}{1-q^m} \right) = \log_p \frac{1-q^m}{1-q}. \end{aligned}$$

(3)

$$\begin{aligned} \psi_{p,q}(x) - \psi_{p,1/q}(1-x) &= \lim_{N \rightarrow \infty} p^{-N} \sum_{0 \leq j < p^N} \left(\log_p \frac{1-q^{x+j}}{1-q} - \log_p \frac{1-q^{-(1-x+p^N-1-j)}}{1-q^{-1}} \right) \\ &= -\log_p q + \lim_{N \rightarrow \infty} p^{-N} \sum_{0 \leq j < p^N} \log_p \frac{1-q^{x+j}}{1-q^{x+j-p^N}}. \end{aligned}$$

Since

$$\begin{aligned}\log_p \frac{1 - q^{x+j}}{1 - q^{x+j-p^N}} &= -\log_p \left(1 - \frac{q^{x+j}}{1 - q^{x+j}} (q^{-p^N} - 1) \right) \\ &= \sum_{k=1}^{\infty} \frac{1}{k} \frac{q^{(x+j)k}}{(1 - q^{x+j})^k} (q^{-p^N} - 1)^k,\end{aligned}$$

and

$$q^{-p^N} - 1 = (1 + t)^{-p^N} - 1 = -p^N t + p^N(p^N + 1)t^2/2 - \dots,$$

it quickly follows that the last limit is zero. Q.E.D.

Finally, we give q -extensions of Diamond's p -adic Euler constants [4]. If $r, f \in \mathbf{Z}$, $f \geq 1$, and if $\text{ord}_p(r/f) < 0$, then define

$$\gamma_{p,q}(r, f) = -\lim_{N \rightarrow \infty} \frac{1}{fp^N} \sum_{\substack{0 \leq j < fp^N \\ j \equiv r \pmod{f}}} \log_p \frac{1 - q^j}{1 - q}.$$

Also set

$$\gamma_{p,q} = \frac{p}{p-1} \sum_{j=1}^{p-1} \gamma_{p,q}(j, p) = -\frac{p}{p-1} \lim_{N \rightarrow \infty} p^{-N} \sum'_{0 \leq j < p^N} \log_p \frac{1 - q^j}{1 - q}.$$

Then it is easy to prove the following two theorems, which generalize Theorem 14 in [4].

THEOREM 5. (1) If $d|(r, f)$, then

$$f\gamma_{p,q}(r, f) = (f/d)\gamma_{p,q^d}(r/d, f/d) - \log_p \frac{1 - q^d}{1 - q}.$$

(2) $\gamma_{p,q}(r, f) = \gamma_{p,1/q}(f - r, f) + (1/f)\log_p q$.

(3) If b is a positive integer, then $\gamma_{p,q}(r, f) = \sum_{j=0}^{b-1} \gamma_{p,q}(r + jf, bf)$.

THEOREM 6. (1) If $\text{ord}_p(r/f) < 0$ and $0 < r < f$, then

$$\psi_{p,q^f}(r/f) = -\log_p \frac{1 - q^f}{1 - q} - f\gamma_{p,q}(r, f).$$

(2) $\psi_{p,q}^*(0) = \Gamma'_{p,q}(0) = -(1 - 1/p)\gamma_{p,q}$.

REMARKS. 1. In [4] Diamond denotes a limit of the form $\lim_{N \rightarrow \infty} p^{-N} \sum_{0 \leq j < p^N} f(j, x)$ by $\int f(u, x) du$. This is the Riemann sum definition of the integral over $\mathbf{Z}_p$ with respect to the Haar distribution $\mu_{\text{Haar}}(j + p^N \mathbf{Z}_p) = p^{-N}$ if one chooses j as the representative "point" in the "interval" $j + p^N \mathbf{Z}_p$; since μ_{Haar} is not bounded, the limit of the Riemann sums does not exist independently of the choice of representative. The relationship between μ_{Haar} and the μ_z in [9] given by $\mu_z(j + p^N \mathbf{Z}_p) = z^j/(1 - z^{p^N})$ is as follows. If we choose $z = q$ with $|q - 1|_p < 1$ (but q not a p th-power root of 1), then μ_q is an unbounded distribution, and $d\mu_{\text{Haar}} = -q^{-u} \log_p q \, d\mu_q = (d/du)(q^{-u}) \, d\mu_q$ in the sense that

$$\begin{aligned} \lim_{N \rightarrow \infty} \sum_{0 \leq j < p^N} f(j, x) \mu_{\text{Haar}}(j + p^N \mathbf{Z}_p) \\ = \lim_{N \rightarrow \infty} \sum_{0 \leq j < p^N} f(j, x) (-q^{-j} \log_p q) \mu_q(j + p^N \mathbf{Z}_p). \end{aligned}$$

2. If we wanted to construct a q -extension $G_{p,q}$ of G_p having the form $G_{p,q}(x) = \lim_{N \rightarrow \infty} p^{-N} \sum_{0 \leq j < p^N} f(x + j)$, we would have to find an f whose derivative is $\log_p((1 - q^x)/(1 - q))$. It is not clear what a natural choice for such an f might be. However, one could use the technique in [9] to define a "twisted" $G_{p,q}$ by

$$G_{p,q,\xi}(x) = - \int_{\mathbf{Z}_p} \log_p \frac{1 - q^{x+u}}{1 - q} d\mu_{\xi}(u) \quad \text{for } x \in D_q,$$

where $\xi^r = 1$, $\xi \neq 1$, $p \nmid r$, and $\mu_{\xi}(j + p^N \mathbf{Z}_p) = \xi^j / (1 - \xi^{p^N})$. This function satisfies

(1)

$$\xi G_{p,q,\xi}(x + 1) - G_{p,q,\xi}(x) = \log_p \frac{1 - q^x}{1 - q},$$

(2)

$$G_{p,q,\xi}(x) - \sum_{h=0}^{m-1} \xi^h G_{p,q,\xi^m} \left(\frac{x+h}{m} \right) = \frac{1}{1-\xi} \log_p \frac{1 - q^m}{1 - q},$$

(3)

$$G_{p,q,\xi}(x) + \xi^{-1} G_{p,q^{-1},\xi^{-1}}(1-x) = \frac{1}{1-\xi} \log_p q.$$

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