FACTORIZATION OF CURVATURE OPERATORS

BY

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ABSTRACT. Let V be a real finite-dimensional vector space with inner product and let R be a curvature operator, i.e., a symmetric linear map of the bivector space $\Lambda^2 V$ into itself. Necessary and sufficient conditions are given for R to admit factorization as $R = \Lambda^2 L$, with L a symmetric linear map of V into itself. This yields a new characterization of Riemannian manifolds that admit local isometric embedding as hypersurfaces of Euclidean space.

1. Introduction. A curvature operator is a symmetric linear map $R: \Lambda^2 V \rightarrow \Lambda^2 V$, where V is a real n -dimensional vector space with inner product and $\Lambda^2 V$ is the associated $\binom{n}{2}$ -dimensional bivector space, supplied with the inner product induced from V . The problem solved in this paper is the following: to characterize those curvature operators $R: \Lambda^2 V \rightarrow \Lambda^2 V$ that can be factored as $R = \Lambda^2 L$ for suitable symmetric linear maps $L: V \rightarrow V$.

This problem is closely related to the local embedding problem for Riemannian manifolds. Namely, if M^n is an abstract Riemannian manifold, one asks for the least integer p for which M^n is locally isometrically embeddable into Euclidean space E^{n+p} . It is known that $p \leq \frac{1}{2}n(n+1)$ in the C^∞ case, and that p depends on the curvature tensor of M^n . At each point of M^n , this tensor can be represented as a symmetric linear map $R: \Lambda^2 V \rightarrow \Lambda^2 V$, where V is the tangent space, endowed with the inner product given by the metric tensor. If M^n is isometrically embeddable into E^{n+p} , then R must at each point admit factorization as a sum $R = \Lambda^2 L_1 + \cdots + \Lambda^2 L_p$ for some symmetric linear maps $L_i: V \rightarrow V$. In general, it is not known how to determine the minimum-length factorization of R . The purpose of this paper is to characterize those R which admit the shortest possible factorization, namely, $p = 1$.

For this case, the factorization is in fact equivalent to local embeddability into E^{n+1} . Namely, T. Y. Thomas has shown [8] that an abstract Riemannian manifold M^n with rank $R \geq 6$ admits local isometric embedding into E^{n+1} if and only if R can be factored as $R = \Lambda^2 L$ at each point. *Hence, the main result of the present paper provides necessary and sufficient conditions for local isometric embedding into E^{n+1} when curvature has rank ≥ 6 .*

The $p = 1$ local embedding problem has been considered before by several authors, each working with quite different methods. Namely, T. Y. Thomas [8] obtained a solution by algebraic means, involving certain 3×3 determinants in the

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matrix components of R . Rozenson [7] noted that the determinants in Thomas' solution were not coordinate-invariant; she obtained some rather complicated tensor expressions involving R , and gave a solution in terms of these tensors. Yanenko [11] gave essentially the same solution as Thomas, although he worked with differential 2-forms rather than just the matrix components of R .

The methods of this paper are different. The solution given here involves the properties of R in relation both to the vector space structure and to the exterior product structure of the space $\Lambda^2 V$. In fact, the principal condition on R is that it must preserve the decomposability of bivectors. It is hoped that the methods used here will be useful also for studying the $p \geq 2$ case.

The main result of the paper is Theorem 7.8, contained in §7 below.

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2. Plan of work. Let $R: \Lambda^2 V \rightarrow \Lambda^2 V$ be a curvature operator. Our problem is to find conditions on R that permit R to be factored as $R = \Lambda^2 L$, for a suitable linear map $L: V \rightarrow V$. This problem was solved in [9] for the case of nonsingular R , by means of a two-stage procedure. Namely, R was first factored as $R = \pm \Lambda^2 L$ and then the minus sign was removed. The results obtained in [9] can be applied to the case of an R with nontrivial kernel as follows.

Since R is symmetric, the vector space $\Lambda^2 V$ admits the orthogonal decomposition $\Lambda^2 V = \ker R + \operatorname{im} R$, and R splits into the sum $R = R_0 + R_1$, where $R_1: \operatorname{im} R \rightarrow \operatorname{im} R$ is a nonsingular linear map, and $R_0: \ker R \rightarrow \operatorname{im} R$ is identically zero. If it happens that the subspace $\operatorname{im} R$ of $\Lambda^2 V$ is of the form $\Lambda^2 U$ for some subspace U of V , then the first result of [9] (cf. Proposition 7.1 below) can be used to factor the map R_1 as $R_1 = \pm \Lambda^2 L_1$, for a suitable linear map $L_1: U \rightarrow U$. The full factorization $R = \pm \Lambda^2 L$ is then obtained by setting $L = L_0 + L_1$, where $L_0: V \rightarrow U$ is the zero map. Finally, the minus sign can be removed by the same technique as in [9].

Therefore, the first task in this paper is to find conditions on R that allow its image to be factored as $\operatorname{im} R = \Lambda^2 U$, for a suitable subspace U of V .

We assume the elements of multilinear algebra (cf. [6], [10], and [4] especially for inner products).

3. Decomposability preserving maps and the subspace U . A linear map $R: \Lambda^2 V \rightarrow \Lambda^2 V$ is said to *preserve decomposability* if each nonzero decomposable bivector either has a nonzero decomposable image or gets mapped to zero. Since we include zero in the set G of decomposable bivectors, this happens precisely when $R(G) \subset G$.

Preservation of decomposability is the main condition on R which is needed to factor the image of R as $\operatorname{im} R = \Lambda^2 U$. In this section we give some equivalent formulations and general consequences of this condition, and we also define the subspace $U \subset V$. The proofs are omitted, since they are straightforward. We remind the reader that decomposable bivectors α, β are called *adjacent* if $\alpha\beta = 0$.

PROPOSITION 3.1. (i) *If R preserves decomposability, then it also preserves adjacency.*

(ii) *R preserves decomposability if and only if $R(x_1x_2)R(x_3x_4) = -R(x_1x_3)R(x_2x_4)$ for all $x_1, x_2, x_3, x_4 \in V$.*

(iii) *R preserves decomposability if and only if there exists a well-defined linear map $\Lambda^2R: \Lambda^4V \rightarrow \Lambda^4V$ satisfying the equation $\Lambda^2R(\alpha\beta) = R\alpha R\beta$ for all $\alpha, \beta \in \Lambda^2V$.*

(iv) *If R preserves decomposability and is symmetric, then for all $\alpha, \beta \in \text{im } R$, $\alpha\beta = 0$ if and only if $R\alpha R\beta = 0$.*

DEFINITION 3.2. Let $U \subset V$ be the subspace spanned by all planes $\{\omega\}$ belonging to decomposable bivectors $\omega \in \text{im } R$.

PROPOSITION 3.3. (i) *If $\text{im } R$ is spanned by decomposable bivectors $\omega_1, \dots, \omega_N$, then U is spanned by the planes $\{\omega_1\}, \dots, \{\omega_N\}$ of V , and $\text{im } R \subset \Lambda^2U$.*

(ii) *If R preserves decomposability, then $\text{im } R \subset \Lambda^2U$.*

Definition 3.2 is suitable for defining U only when R preserves decomposability. We shall indicate some more general definitions for U , since these permit $\dim U$ to be related to other known numerical invariants of R . Define the absolute nullspace of a curvature operator R to be the subspace N in V consisting of all vectors x such that $Rxy = 0$ for all y in V ; define its "rank" subspace to be the smallest subspace W in V such that $\text{im } R \subset \Lambda^2W$. Then $\dim N$ is called the nullity index of R (cf. [5, p. 347]), and $\dim W$ is called the "rank" of R (cf. [11, §9 on p. 34 and §4 on p. 76]). The codimension of N is called the conullity or type of R . Clearly, it can also be described as the rank of the linear mapping $F: V \rightarrow \text{Hom}(V, \Lambda^2V)$, where $F(x)y = Rxy$ (cf. [8, p. 186]).

PROPOSITION 3.4. (i) $W = N^\perp$; hence "rank" $R = \text{type } R$.

(ii) *If R preserves decomposability, then $U = W = N^\perp$; hence $\dim U = \text{type } R$.*

(iii) *If $R = \pm \Lambda^2L$ for some linear map $L: V \rightarrow V$, then $U = \text{im } L$; hence $\dim U = \text{type } R = \text{rank } L$.*

4. Factoring $\text{im } R$ when $\Lambda^2R \not\equiv 0$. Henceforth we assume R is a decomposability-preserving curvature operator. In this section we obtain the factorization $\text{im } R = \Lambda^2U$ for the case $\Lambda^2R \not\equiv 0$.

Any pair of decomposable bivectors α_1, α_2 with $\alpha_1\alpha_2 \neq 0$ can be written as $\alpha_1 = a_1a_2, \alpha_2 = a_3a_4$, where a_1, a_2, a_3, a_4 are independent. If we suppose that $R\alpha_1R\alpha_2 \neq 0$, then we can similarly write $R\alpha_1 = b_1b_2, R\alpha_2 = b_3b_4$, where b_1, b_2, b_3, b_4 are independent.

LEMMA 4.1. *We may assume that*

$$\begin{aligned} Ra_1a_2 &= b_1b_2, & Ra_1a_3 &= b_1b_3, \\ \left. \begin{matrix} Ra_2a_3 \\ Ra_1a_4 \end{matrix} \right\} &= \pm \begin{cases} b_2b_3 \\ b_1b_4 \end{cases} \quad \text{or} \quad \pm \begin{cases} b_1b_4 \\ b_2b_3 \end{cases} & (\text{both } + \text{ or } -), \\ Ra_2a_4 &= b_2b_4, & Ra_3a_4 &= b_3b_4. \end{aligned}$$

PROOF. Consider the planes $\{Ra_i a_j\}$ in U . Define vectors b_1, b_2, b_3, b_4 by the conditions

$$\begin{aligned} \{Ra_1 a_2\} \cap \{Ra_1 a_3\} &= \{b_1\}, & \{Ra_1 a_2\} \cap \{Ra_2 a_4\} &= \{b_2\}, \\ \{Ra_1 a_3\} \cap \{Ra_3 a_4\} &= \{b_3\}, & \{Ra_2 a_4\} \cap \{Ra_3 a_4\} &= \{b_4\}. \end{aligned}$$

Then these b_i are independent and $Ra_1 a_2 = x_1 b_1 b_2$, $Ra_3 a_4 = x_2 b_3 b_4$, $Ra_1 a_3 = x_3 b_1 b_3$, $Ra_2 a_4 = x_4 b_2 b_4$, where x_i are nonzero scalars. Put $b'_1 = x_3 b_1$, $b'_2 = x_1 b_2 / x_3$, $b'_3 = b_3$, $b'_4 = x_2 b_4$; then $Ra_1 a_2 = b'_1 b'_2$, $Ra_3 a_4 = b'_3 b'_4$, and $Ra_2 a_4 = y b'_2 b'_4$. But now Proposition 3.1(ii) implies that $b'_1 b'_2 b'_3 b'_4 = -y b'_1 b'_3 b'_2 b'_4$, whence $y = 1$. Dropping the primes, we have the first two and the last two of the desired equations.

To get the remaining equations, set $\alpha = Ra_1 a_4$, $\beta = Ra_2 a_3$. Then Proposition 3.1(ii) implies $\alpha\beta = b_1 b_2 b_3 b_4$, which in turn implies $\alpha, \beta \in \Lambda^2\{b_1, b_2, b_3, b_4\}$. Moreover, since α, β are both adjacent to the four bivectors $b_1 b_2, b_1 b_3, b_2 b_4$ and $b_3 b_4$, we must have

$$\alpha = x b_2 b_3 + y b_1 b_4 \quad \text{and} \quad \beta = z b_2 b_3 + w b_1 b_4,$$

where x, y, z and w are scalars. Since $\alpha\alpha = 0$ and $\beta\beta = 0$, and $\alpha\beta = b_1 b_2 b_3 b_4$, it follows that $xy = 0$, $zw = 0$ and $xw + yz = 1$. This implies that either $x = 0$, $w = 0$, $yz = 1$, or $y = 0$, $z = 0$, $xw = 1$. Therefore, these two cases hold.

Case (i). $\alpha = z b_1 b_4$, $\beta = z^{-1} b_2 b_3$ or

Case (ii). $\alpha = z b_2 b_3$, $\beta = z^{-1} b_1 b_4$.

For Case (i), set $b'_1 = |z|^{1/2} b_1$, $b'_2 = |z|^{-1/2} b_2$, $b'_3 = |z|^{-1/2} b_3$, $b'_4 = |z|^{1/2} b_4$. Then $b'_1 b'_2 = b_1 b_2$, $b'_1 b'_3 = b_1 b_3$, $b'_2 b'_4 = b_2 b_4$, $b'_3 b'_4 = b_3 b_4$. However, $b'_2 b'_3 = |z|^{-1} b_2 b_3 = \pm z^{-1} b_2 b_3 = \pm \beta = \pm Ra_2 a_3$, and $b'_1 b'_4 = |z| b_1 b_4 = \pm \alpha = \pm Ra_1 a_4$, where both signs are $+$ when $z > 0$ and both are $-$ when $z < 0$.

For Case (ii) set $b'_1 = |z|^{-1/2} b_1$, $b'_2 = |z|^{1/2} b_2$, $b'_3 = |z|^{1/2} b_3$, $b'_4 = |z|^{-1/2} b_4$. Then $b'_1 b'_2 = b_1 b_2$, $b'_1 b'_3 = b_1 b_3$, $b'_2 b'_4 = b_2 b_4$ and $b'_3 b'_4 = b_3 b_4$. However, $b'_2 b'_3 = |z| b_2 b_3 = \pm z b_2 b_3 = \pm \alpha = \pm Ra_1 a_4$, and $b'_1 b'_4 = |z|^{-1} b_1 b_4 = \pm z^{-1} b_1 b_4 = \pm \beta = \pm Ra_2 a_3$, with the same type of sign. Q.E.D.

LEMMA 4.2. Let ω_1 and ω_2 be decomposable bivectors in $\text{im } R$, such that $\omega_1 \omega_2 \neq 0$. Then the 4-dimensional subspace $U_1 = \{\omega_1\} + \{\omega_2\}$ of V has the property that $\Lambda^2 U_1 \subset \text{im } R$.

PROOF. Let $\omega_1 = R\alpha_1$ and $\omega_2 = R\alpha_2$. By the orthogonal decomposition of $\Lambda^2 V$ given in §2, it may be assumed that α_1 and α_2 are in $\text{im } R$. But then Proposition 3.1(iv) implies that both α_1 and α_2 are decomposable and that $\alpha_1 \alpha_2 \neq 0$. Hence α_1, α_2 can be expressed as $\alpha_1 = a_1 a_2$, $\alpha_2 = a_3 a_4$. Let $\omega_1 = b_1 b_2$ and $\omega_2 = b_3 b_4$. Then $\{b_1, b_2, b_3, b_4\} + \{b_1 b_2\} = \{b_3 b_4\} = \{\omega_1\} + \{\omega_2\} = U_1$. Now Lemma 4.1 implies that each of the bivectors $b_1 b_2, \dots, b_3 b_4$ in the basis of $\Lambda^2 U_1$ is an image by R of a bivector in $\Lambda^2 V$; hence $\Lambda^2 U_1$ must be contained in $\text{im } R$. Q.E.D.

PROPOSITION 4.3. If $R: \Lambda^2 V \rightarrow \Lambda^2 V$ is a curvature operator which preserves decomposability, and if $\Lambda^2 R \neq 0$, then $\text{im } R = \Lambda^2 U$. Moreover, $\dim U \geq 4$ and $\text{rank } R \geq 6$.

PROOF. Let U be defined by Definition 3.2. Then by Proposition 3.3(ii), $\text{im } R \subset \Lambda^2 U$, and it suffices to prove that $\Lambda^2 R \neq 0$ implies $\Lambda^2 U \subset \text{im } R$. To establish this inclusion, it is enough to show that each decomposable bivector of $\Lambda^2 U$ is in $\text{im } R$, i.e., that $x, y \in U$ implies $xy \in \text{im } R$.

Let $x, y \in U$. By definition of U , there exist decomposable bivectors $\omega_1, \omega_2 \in \text{im } R$ such that $x \in \{\omega_1\}, y \in \{\omega_2\}$.

If $\omega_1 \omega_2 \neq 0$, then Lemma 4.2 implies that $\Lambda^2 U_1 \subset \text{im } R$, where $U_1 = \{\omega_1\} + \{\omega_2\}$. But both x and y are in U_1 , so that xy is in $\Lambda^2 U_1$; hence $xy \in \text{im } R$.

If $\omega_1 \omega_2 = 0$, then we have $\omega_1 = xz, \omega_2 = zy$ for a suitable vector z , and we may assume x, y, z independent. In this case Lemma 4.2 cannot be applied directly, so an auxiliary subspace must be used from which to get decomposable vectors of $\text{im } R$ that are not adjacent to ω_1 or ω_2 . For this purpose, let $e_1, \dots, e_n$ be a basis of V . Then Proposition 3.1(iii) implies that if $\Lambda^2 R \neq 0$ then not all $Re_i e_j Re_k e_l$ with $i < j < k < l$ can vanish. By relabeling the basis, it may be assumed that $Re_1 e_2 Re_3 e_4 \neq 0$. Then Lemma 4.2 implies that $\Lambda^2 U_0 \subset \text{im } R$, where $U_0 = \{Re_1 e_2\} + \{Re_3 e_4\}$. Let us consider the relative positions of the 4-dimensional space U_0 and the 3-dimensional space $W_0 = \{\omega_1\} + \{\omega_2\} = \{x, y, z\}$.

If $W_0 \subset U_0$, then $x, y \in U_0$, so $xy \in \Lambda^2 U_0$, whence $xy \in \text{im } R$, and we are done. On the other hand, if $W_0 \not\subset U_0$, then $U_0 \cap W_0$ has dimension ≤ 2 . Hence $(U_0 \cap W_0)^\perp \cap U_0$ has dimension ≥ 2 and therefore contains a pair of independent vectors v_1, v_2 . Setting $\omega_3 = v_1 v_2$, we have $\omega_3 \in \text{im } R, \omega_2 \omega_3 \neq 0$ and z, y, v_1, v_2 independent. Now we can apply Lemma 4.2 to ω_2, ω_3 to conclude $\Lambda^2 U_1 \subset \text{im } R$, where $U_1 = \{\omega_2\} + \{\omega_3\}$.

If $\{\omega_1\} \subset U_1$, then $x, y \in U_1$, whence $xy \in \text{im } R$, and we are done. If $\{\omega_1\} \not\subset U_1$, put $\omega_4 = yv_2$; then $\omega_1 \omega_4 \neq 0$, because $\omega_1 \omega_4 = 0$ would imply $\{\omega_1\} \subset U_1$. We again apply Lemma 4.2 to ω_1, ω_4 to conclude $\Lambda^2 U_2 \subset \text{im } R$, where $U_2 = \{\omega_1\} + \{\omega_4\} = \{x, y, z, v_2\}$. Thus $xy \in \text{im } R$. Q.E.D.

5. Correlations. The special case $\Lambda^2 R \neq 0$, $\text{rank } R = 6$ presents an anomalous feature—the possibility of R being a correlation—which must be excluded later. Therefore we include here a discussion of this case.

It follows from Proposition 4.3 that $\text{im } R = \Lambda^2 U$ and $\dim U = 4$. Hence the nonsingular map $R|_{\text{im } R}: \Lambda^2 U \rightarrow \Lambda^2 U$ satisfies the equations of Lemma 4.1, where a_1, a_2, a_3, a_4 and b_1, b_2, b_3, b_4 are now two bases of U . If $Ra_2 a_3 = \pm b_2 b_3$ and $Ra_1 a_4 = \pm b_1 b_4$, R is called a *collineation*, whereas if $Ra_2 a_3 = \pm b_1 b_4$ and $Ra_1 a_4 = \pm b_2 b_3$, R is a *correlation*.

If our R has the form $R = \pm \Lambda^2 L$, then clearly it is a collineation. Hence for the case $\Lambda^2 R \neq 0$, $\text{rank } R = 6$, the condition that R is not a correlation is necessary for R to admit factorization as $R = \pm \Lambda^2 L$.

The following three results give criteria for deciding whether R is a collineation or a correlation.

PROPOSITION 5.1. *Let $R: \Lambda^2 V \rightarrow \Lambda^2 V$ be a decomposability-preserving curvature operator having rank 6. Let $e_1, \dots, e_n$ be a basis of V such that $e_1, \dots, e_4$ is a basis of U . If $Re_1 e_2 = b_1 b_2, Re_1 e_3 = b_1 b_3, Re_2 e_3 = \pm b_2 b_3$, then R is a collineation,*

whereas if $Re_2e_3 = \pm b_1b_4$, then R is a correlation. (The vectors $b_1, \dots, b_4$ are a suitable independent set in U .)

PROOF. Clear.

PROPOSITION 5.2. R is a collineation or a correlation according to whether $\{Ra_1a_2\} + \{Ra_1a_3\} + \{Ra_2a_3\}$ is a 3- or 4-dimensional subspace of V , respectively, where a_1, a_2, a_3 is an arbitrary independent set in V .

PROOF. Consider the mapping of planes of V into planes of V defined by $\{xy\} \rightarrow \{Rxy\}$. If R is a collineation, then the set of planes containing a given line gets mapped onto a set of planes that also contain a common line, and the set of planes contained in a given 3-space gets mapped onto a set of the same type. But if R is a correlation, these types of sets of planes get reversed (cf. [3, p. 107]). Q.E.D.

PROPOSITION 5.3. Let $R: \Lambda^2V \rightarrow \Lambda^2V$ be a decomposability-preserving curvature operator having rank 6. Let $e_1, \dots, e_n$ be a basis of V such that e_1, e_2, e_3, e_4 is a basis of U , and let R_{kl}^j be the matrix of R with respect to the basis $e_i e_j$, $i < j$, of Λ^2V . Then R is a collineation if and only if the following matrix has rank 4.

$$\begin{pmatrix} R_{12}^{23} & -R_{12}^{13} & R_{12}^{12} & 0 \\ R_{12}^{24} & -R_{12}^{14} & 0 & R_{12}^{12} \\ R_{13}^{23} & -R_{13}^{13} & R_{13}^{12} & 0 \\ R_{13}^{24} & -R_{13}^{14} & 0 & R_{13}^{12} \\ R_{23}^{23} & -R_{23}^{13} & R_{23}^{12} & 0 \\ R_{23}^{24} & -R_{23}^{14} & 0 & R_{23}^{12} \end{pmatrix}$$

PROOF. For a decomposable bivector α in Λ^2U the plane $\{\alpha\}$ spanned by it in V is given by

$$\{\alpha\} = \{x | \alpha x = 0\} = \{x | x^i \alpha^{jk} - x^j \alpha^{ik} + x^k \alpha^{ij} = 0 \text{ for } 1 \leq i < j < k \leq 4\},$$

where $\alpha = \sum \alpha^{ij} e_i e_j$ with $1 \leq i < j \leq 4$. Since α is decomposable, $\alpha\alpha = 0$, which gives $\alpha^{12}\alpha^{34} - \alpha^{13}\alpha^{24} + \alpha^{14}\alpha^{23} = 0$. From this equation it can be seen that the four equations describing $\{\alpha\}$ are dependent, and that those two with indices i, j, k equal to 1, 2, 3 and 1, 2, 4 are independent. Hence, as a subset of $\mathbf{R}^4$, $\{\alpha\}$ is the nullspace of the matrix

$$\begin{pmatrix} \alpha^{23} & -\alpha^{13} & \alpha^{12} & 0 \\ \alpha^{24} & -\alpha^{14} & 0 & \alpha^{12} \end{pmatrix}.$$

If $\alpha = Re_1e_2$, then the corresponding matrix is given by the first two rows of the 6×4 matrix in the statement of this proposition; its nullspace is precisely $\{Re_1e_2\}$, considered as a subspace of $\mathbf{R}^4$. The next two pairs of rows of that matrix correspond to $\{Re_1e_3\}$ and $\{Re_2e_3\}$. Therefore, the nullspace of the whole 6×4 matrix corresponds to the space $\{Re_1e_2\} \cap \{Re_1e_3\} \cap \{Re_2e_3\}$. Now R is a collineation if and only if this space is $\{0\}$, which occurs exactly when the 6×4 matrix has maximal rank. Q.E.D.

6. The case $\Lambda^2 R \equiv 0$. It is easy to see from Proposition 3.1 that $\Lambda^2 R \equiv 0$ if and only if $R(x_1 x_2)R(x_3 x_4) = 0$ for all x_1, x_2, x_3, x_4 in V . This is a rather severe restriction on R ; as a consequence, we can perform the factorings $\text{im } R = \Lambda^2 U$ and $R = \pm \Lambda^2 L$ both at the same time.

PROPOSITION 6.1. *Let $R: \Lambda^2 V \rightarrow \Lambda^2 V$ be a curvature operator which preserves decomposability and satisfies $\Lambda^2 R \equiv 0$. Then $R = \pm \Lambda^2 L$ for a suitable linear map $L: V \rightarrow V$ if and only if $\text{rank } R = 0$ or 1, or $\text{rank } R = \text{type } R = 3$. Moreover, in case $\text{rank } R = 0$ or 1, the $+$ sign may be assumed to hold, with L symmetric.*

PROOF. Let $a_1, \dots, a_k$ be a basis of U , with U as in Definition 3.2. Since R preserves decomposability, Proposition 3.3 states that $\text{im } R \subset \Lambda^2 U$. Hence R restricts to a linear map $\Lambda^2 U \rightarrow \Lambda^2 U$ which still has the same set as image. Therefore the bivectors $Ra_i a_j$ for $1 \leq i < j \leq k$ span $\text{im } R$, and $U = \Sigma \{Ra_i a_j\}$ accordingly. Since $\Lambda^2 R \equiv 0$, all the nonzero bivectors among these $Ra_i a_j$ must be adjacent to each other, so that all the corresponding planes $\{Ra_i a_j\}$ have 1-dimensional intersections. But then the result of [1, p. 16] implies that either (i) all of them contain one 1-dimensional subspace of V or (ii) they are all contained in one 3-dimensional subspace of V .

In case (i), $Ra_i a_j = b_1 b_{ij}$ for all $1 \leq i < j \leq k$, where b_1 is a fixed vector in U , and b_{ij} is taken equal to b_1 when $Ra_i a_j = 0$. By definition, U is spanned by the planes $\{Ra_i a_j\} = \{b_1, b_{ij}\}$ corresponding to the nonzero $Ra_i a_j$. This means that these vectors $b_1, \dots, b_{ij}, \dots$ span U , which implies that k of them are independent, including b_1 . Denote these vectors by $b_1, b_2, \dots, b_k$; they are a basis of U , and consequently the $k - 1$ bivectors $b_1 b_2, \dots, b_1 b_k$ are a basis of $\text{im } R$. Therefore R has rank $k - 1$.

In case (ii), all the planes $\{Ra_i a_j\}$ lie in one 3-dimensional subspace of V , whence U has dimension $k \leq 3$.

Now suppose $R = \pm \Lambda^2 L$ for some linear map $L: V \rightarrow V$. It follows from Proposition 3.4 that $U = \text{im } L$ and $\Lambda^2 U = \text{im } R$. This implies that $\text{rank } R = \dim \Lambda^2 U = \binom{k}{2}$. In Case (i), this means $\binom{k}{2} = k - 1$, which implies $k = 1$ or 2, so that $\text{rank } R = 0$ or 1. In Case (ii), $k \leq 3$, i.e., $k = 0, 1, 2$ or 3. Therefore, $\text{rank } R = \binom{k}{2} = 0, 1$, or 3. Hence $R = \pm \Lambda^2 L$ implies the rank and dimension conditions in the proposition, noting that by Proposition 3.4(ii), $\dim U = \text{type } R$.

To prove the converse, start with $\text{rank } R = 0$. Then $R \equiv 0$, so $R = \Lambda^2 L$ for $L \equiv 0$. Next, suppose $\text{rank } R = 1$. This implies that $Re_i e_j \neq 0$ for some $i < j$, where $e_1, \dots, e_n$ is any basis of V . This nonzero $Re_i e_j$ must be decomposable, since R preserves decomposability; it also must span $\text{im } R$. Let v_1, v_2 be an orthonormal basis of the corresponding plane $\{Re_i e_j\}$. Therefore $v_1 v_2$ also spans $\text{im } R$; in fact, $\text{im } R = \Lambda^2 U$, where $U = \{v_1, v_2\}$. Since $R|_{\text{im } R}$ is nonsingular, there is a scalar $c \neq 0$ such that $Rv_1 v_2 = cv_1 v_2$. Extend v_1, v_2 to an orthonormal basis $v_1, v_2, v_3, \dots, v_n$ of V ; then $v_1 v_2, v_1 v_3, \dots, v_{n-1} v_n$ is an orthonormal basis of $\Lambda^2 V$. Since R is symmetric, $\Lambda^2 V$ splits into the orthogonal sum $\Lambda^2 V = \ker R + \text{im } R$. This implies that $v_1 v_3, \dots, v_{n-1} v_n$ is a basis of $\ker R$, so that $Rv_i v_j = 0$ when

$(i, j) \neq (1, 2)$. Now set $Lv_1 = cv_1$, $Lv_2 = v_2$, $Lv_i = 0$ for $3 \leq i$. Then $R = \Lambda^2 L$. Note that L is symmetric, and that $\text{rank } R = 1$ implies $\Lambda^2 R \equiv 0$.

Finally, suppose that $\text{rank } R = \text{type } R = 3$, i.e., $\dim U = 3$. By the first part of this proof, three of the bivectors $Ra_i a_j$ for $1 \leq i < j \leq k$, are independent and their planes span U . Now either Case (i) or Case (ii) holds.

In Case (i), it was shown above that U has a basis $b_1, \dots, b_k$ and that the bivectors $b_1 b_2, \dots, b_1 b_k$ are a basis of $\text{im } R$. Since $\text{rank } R = 3$, this basis contains exactly 3 bivectors, which implies $k = 4$. But this contradicts the hypothesis that $\dim U = 3$, which rules out this case.

In Case (ii), the planes $\{Ra_i a_j\}$ are all contained in a 3-dimensional subspace of V , which implies $\dim U \leq 3$, so that also $\dim \Lambda^2 U \leq 3$. Since by Proposition 3.3 $\text{im } R \subset \Lambda^2 U$ and, by hypothesis, $\text{rank } R = 3$, we have $\text{im } R = \Lambda^3 U$ and $\dim U = 3$. Let v_1, v_2, v_3 be an orthonormal basis of U . Then $v_1 v_2, v_1 v_3, v_2 v_3$ is an orthonormal basis of $\Lambda^2 U = \text{im } R$ and the planes $\{Rv_1 v_2\}, \{Rv_1 v_3\}, \{Rv_2 v_3\}$ span U .

Consequently, these planes intersect each other in three distinct lines $\{b_1\}, \{b_2\}, \{b_3\}$, where the vectors b_1, b_2, b_3 can be chosen so that $Rv_1 v_2 = b_1 b_2$, $Rv_1 v_3 = b_1 b_3$, $Rv_2 v_3 = cb_2 b_3$, with c a nonzero scalar. It is not hard to see that the lengths and directions of the vectors b_1, b_2, b_3 can be adjusted to get $Rv_1 v_2 = b_1 b_2$, $Rv_1 v_3 = b_1 b_3$, $Rv_2 v_3 = \pm b_2 b_3$. Now extend the basis v_1, v_2, v_3 to an orthonormal basis $v_1, v_2, v_3, v_4, \dots, v_n$ of V . Then $v_1 v_2, v_1 v_3, v_1 v_4, \dots, v_{n-1} v_n$ is an orthonormal basis of $\Lambda^2 V$. Just as in the case of $\text{rank } R = 1$ above, it follows from the symmetry of R that $Rv_i v_j = 0$ when (i, j) is not one of the pairs $(1, 2), (1, 3), (2, 3)$.

Set $Lv_1 = \pm b_1$, $Lv_2 = b_2$, $Lv_3 = b_3$ and $Lv_i = 0$ for $4 \leq i$. Then $R = \pm \Lambda^2 L$. Q.E.D.

7. The main factorization theorems. Here we bring together the results of the previous sections to obtain first $R = \pm \Lambda^2 L$ and then $R = \Lambda^2 L$. As indicated in §2, we shall use the results of [9] in a formulation extended to include the case $\dim V = 4$.

PROPOSITION 7.1. *A nonsingular curvature operator $R: \Lambda^2 V \rightarrow \Lambda^2 V$, with $\dim V \geq 4$, can be factored as $R = \pm \Lambda^2 L$, for a suitable linear map $L: V \rightarrow V$, if and only if R preserves decomposability and is not a correlation in the case $\dim V = 4$.*

PROOF. This is just [9, Theorem 1], extended to include $\dim V = 4$ (cf. [2, p. 38]). By Proposition 5.1, maps of form $R = \pm \Lambda^2 L$ are collineations, not correlations. Q.E.D.

Define two scalar invariants of R as follows (cf. [9, p. 201]). Set

$$\phi(R) = R_{kl}^{ij} R_{iq}^{kp} R_{jp}^{lq}, \quad \psi(R) = R_{kl}^{ij} R_{pq}^{kl} R_{ij}^{pq},$$

where summation is understood over repeated indices, and R_{kl}^{ij} is the matrix of R with respect to any basis e_i, e_j , $i < j$, corresponding to a basis $e_1, \dots, e_n$ of V .

PROPOSITION 7.2 [9, THEOREM 3]. *Suppose R is a nonsingular curvature operator such that $R = \pm \Lambda^2 L$ for a symmetric linear map $L: V \rightarrow V$, with $\dim V \geq 3$. Then $R = \Lambda^2 L$ if and only if $\phi(R) + \frac{1}{4}\psi(R) > 0$. Moreover, in case $n \equiv 3 \pmod{4}$, this inequality may be replaced by $\det R > 0$.*

THEOREM 7.3. *Let $R: \Lambda^2 V \rightarrow \Lambda^2 V$ be a curvature operator. Then $R = \pm \Lambda^2 L$ for some linear map $L: V \rightarrow V$ if and only if R preserves decomposability and satisfies one of the following additional conditions.*

- (i) $\Lambda^2 R \neq 0$ and R is not a correlation in case $\text{rank } R = 6$.
- (ii) $\Lambda^2 R \equiv 0$ and either $\text{rank } R = 0$ or 1, or $\text{rank } R = \text{type } R = 3$.

PROOF. Suppose $\Lambda^2 R \neq 0$. If R preserves decomposability, then by Proposition 4.3, $\text{rank } R \geq 6$ and $\text{im } R = \Lambda^2 U$, with $\dim U \geq 4$. If R is nonsingular, then $U = V$. If R is not a correlation, then $R = \pm \Lambda^2 L$ follows from Proposition 7.1. If R is singular, then by §2, $R_1 = R|_{\text{im } R}: \Lambda^2 U \rightarrow \Lambda^2 U$ is a nonsingular symmetric linear map. By Proposition 7.1, it factors as $R_1 = \pm \Lambda^2 L_1$, provided R_1 is not a correlation (note that R is a correlation if and only if R_1 is one). Therefore §2 gives $R = \pm \Lambda^2 L$, for a linear $L: V \rightarrow V$. Thus if R preserves decomposability and satisfies (i), then $R = \pm \Lambda^2 L$.

Suppose now that $\Lambda^2 R \equiv 0$ and let R preserve decomposability. Then Proposition 6.1 shows that (ii) implies $R = \pm \Lambda^2 L$.

Conversely, assume $R = \pm \Lambda^2 L$; clearly R preserves decomposability. If $\Lambda^2 R \equiv 0$, then Proposition 6.1 implies statement (ii). If $\Lambda^2 R \neq 0$ and $\text{rank } R = 6$, then Proposition 5.1 implies that R is not a correlation. Q.E.D.

COROLLARY 7.4. *If $R: \Lambda^2 V \rightarrow \Lambda^2 V$ is a curvature operator that preserves decomposability and has $\text{rank } R > 6$, then it admits factorization as $R = \pm \Lambda^2 L$ if and only if $\Lambda^2 R \neq 0$.*

In order to state the next theorem, we recall the Bianchi identity of Riemannian geometry. R is said to satisfy the *Bianchi identity* if

$$\langle Rx_1x_2, x_3x_4 \rangle + \langle Rx_2x_3, x_1x_4 \rangle + \langle Rx_3x_1, x_2x_4 \rangle = 0$$

for all x_1, x_2, x_3, x_4 in V .

THEOREM 7.5. *Suppose R is a curvature operator such that $R = \pm \Lambda^2 L$ for a linear map $L: V \rightarrow V$ with $\text{rank } L \geq 3$. Then L is symmetric if and only if R satisfies the Bianchi identity.*

PROOF. Substituting $R = \pm \Lambda^2 L$ into the left side of the Bianchi identity gives

$$\begin{aligned} &\langle Lx_3, x_4 \rangle (\langle Lx_2, x_1 \rangle - \langle Lx_1, x_2 \rangle) + \langle Lx_2, x_4 \rangle (\langle Lx_1, x_3 \rangle - \langle Lx_3, x_1 \rangle) \\ &\quad + \langle Lx_1, x_4 \rangle (\langle Lx_3, x_2 \rangle - \langle Lx_2, x_3 \rangle). \end{aligned}$$

If L is symmetric, this clearly reduces to zero. Conversely, if the Bianchi identity holds, this expression is identically zero. Consider any two vectors x_1, x_2 . It was observed in the proof of Proposition 6.1 that $R = \pm \Lambda^2 L$ implies $U = \text{im } L$ and $\Lambda^2 U = \text{im } R$. Hence, $\text{rank } R = \binom{k}{2}$, where $k = \dim U$. Consequently, the hypothesis, $\text{rank } L \geq 3$, implies that $\dim U \geq 3$. Therefore the vectors Lx_1, Lx_2 are in U

and do not span all of U . Hence there exists a nonzero vector $x_4 \in U$ such that $x_4 \perp$ both Lx_1 and Lx_2 . This gives the equation

$$\langle Lx_3, x_4 \rangle (\langle Lx_2, x_1 \rangle - \langle Lx_1, x_2 \rangle) = 0.$$

Now the map $L_1^{-1} = (L|U)^{-1}: U \rightarrow U$ is well defined. Set $x_3 = L_1^{-1}x_4$; then $Lx_3 = L_1x_3 = x_4$ and consequently $\langle Lx_3, x_4 \rangle = \langle x_4, x_4 \rangle \neq 0$, since x_4 was chosen nonzero. Therefore, the term in parentheses in the above equation vanishes, whence L is symmetric. Q.E.D.

THEOREM 7.6. *Suppose R is a curvature operator such that $R = \pm \Lambda^2 L$ for a symmetric linear map $L: V \rightarrow V$ having rank $L \geq 3$. Then $R = \Lambda^2 L$ if and only if $\phi(R) + \frac{1}{4}\psi(R) > 0$. Moreover, in case $n \equiv 3 \pmod{4}$, this inequality may be replaced by $\det(R|_{\text{im } R}) > 0$.*

PROOF. Since $L: V \rightarrow V$ is a symmetric linear map, V splits into the orthogonal sum $V = \ker L + \text{im } L$, and $U = \text{im } L$. Then each $x \in V$ can be written as $x = x_0 + x_1$ with $x_0 \in \ker L$ and $x_1 \in U$. Note $Lx = Lx_1 = L_1x_1$ where $L_1 = L|U: U \rightarrow U$ is a nonsingular symmetric linear map. This implies

$$Rxy = \pm LxLy = \pm L_1x_1L_1y_1 = \pm \Lambda^2 L_1(x_1y_1).$$

As noted in §2 the space $\Lambda^2 V$ can be written as $\Lambda^2 V = \ker R + \text{im } R$, with $\Lambda^2 U = \text{im } R$ here. Then $xy = (x_0 + x_1)(y_0 + y_1) = x_0y_0 + x_0y_1 + x_1y_0 + x_1y_1 = (xy)_0 + (xy)_1$ and $(xy)_1 = x_1y_1$. Hence $Rxy = R(xy)_1 = R_1x_1y_1$, where $R_1 = R|_{\Lambda^2 U}$ is a nonsingular symmetric linear map.

It follows from the above two equations for Rxy that $R = \Lambda^2 L$ if and only if $R_1 = \Lambda^2 L_1$.

Let $e_1, \dots, e_n$ be a basis of V such that $e_1, \dots, e_k$ is a basis of $U = \text{im } L$ and $e_{k+1}, \dots, e_n$ is a basis of $\ker L$. Then $Re_j e_j = R_1(e_j e_j)$ for $1 \leq i, j \leq k$ and $Re_i e_p = Re_p e_q = 0$ for $k+1 \leq p, q \leq n$. It follows that $\phi(R) = \phi(R_1)$ and $\psi(R) = \psi(R_1)$, since all terms not involving only the matrix of R_1 must vanish. Therefore $\phi(R) + \frac{1}{4}\psi(R) = \phi(R_1) + \frac{1}{4}\psi(R_1)$, and the conclusion of the theorem follows from Proposition 7.2 applied to $R_1: \Lambda^2 U \rightarrow \Lambda^2 U$ and $L_1: U \rightarrow U$. Q.E.D.

REMARK. The sign in $\pm \Lambda^2 L$ actually determines two disjoint classes of maps. Namely, if $L, M: V \rightarrow V$ are symmetric linear maps of rank ≥ 3 , then $\Lambda^2 L \neq -\Lambda^2 M$. This follows from the remark in [9, top of p. 201], via the equation $\Lambda^2 L(xy) = \Lambda^2 L_1(x_1y_1)$ obtained in the proof of Theorem 7.6 above.

The following theorem is the main result of this paper.

THEOREM 7.7. *A curvature operator $R: \Lambda^2 V \rightarrow \Lambda^2 V$ of rank > 1 can be factored as $R = \Lambda^2 L$, for a suitable symmetric linear map $L: V \rightarrow V$, if and only if it satisfies each of the following conditions.*

- (i) *R preserves decomposability.*
- (ii) *R satisfies the Bianchi identity.*
- (iii) *$\phi(R) + \frac{1}{4}\psi(R) > 0$. (If $n \equiv 3 \pmod{4}$, then $\det(R|_{\text{im } R}) > 0$ may be substituted.)*

(iv) *Either $\Lambda^2 R \not\equiv 0$ and R is not a correlation if rank $R = 6$ or $\Lambda^2 R \equiv 0$ and rank $R = \text{type } R = 3$.*

In case rank $R \leq 1$, the conditions (ii), (iii), and (iv) are deleted.

PROOF. For rank $R > 1$, see Theorems 7.3, 7.5, 7.6 and the subsequent Remark. For rank ≤ 1 , see Proposition 6.1 and its proof; in this case $\Lambda^2 R \equiv 0$. Q.E.D.

In conclusion, we make some observations about curvature operators not factorizable as $R = \Lambda^2 L$. First, if one is willing to concede condition (iii) above, then one should consider the factorization $R = \pm \Lambda^2 L$. Second, if one wants R to preserve decomposability and have rank > 3 , then Theorem 7.3 and Corollary 7.4 force $\Lambda^2 R \equiv 0$.

EXAMPLE. Let $e_1, \dots, e_n$ be an orthonormal basis of V and define (for $i < j$)

$$R e_i e_j = \begin{cases} e_1 e_j, & \text{if } i = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $\Lambda^2 R \equiv 0$ and R preserves decomposability. Since rank $R = n - 1$, this R is not factorizable as $R = \pm \Lambda^2 L$ when $n > 4$. Note that $U = V$ and that R satisfies the Bianchi identity.

The general nonfactorizable curvature operator does not preserve decomposability. If it satisfies the Bianchi identity, then it has the decomposition $R = \Lambda^2 L_1 + \dots + \Lambda^2 L_p$ mentioned in §1 (cf. [4, p. 102]).

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