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REALIZATION OF SQUARE-INTEGRABLE REPRESENTATIONS OF UNIMODULAR LIE GROUPS ON L^2-COHOMOLOGY SPACES

BY

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ABSTRACT. An analogue of the "Langlands conjecture" is proved for a large class of connected unimodular Lie groups having square-integrable representations (modulo their centers). For nilpotent groups, it is shown (without restrictions on the group or the polarization) that the L^2-cohomology spaces of a homogeneous holomorphic line bundle, associated with a totally complex polarization for a flat orbit, vanish except in one degree given by the "deviation from positivity" of the polarization. In this degree the group acts irreducibly by a square-integrable representation, confirming a conjecture of Moscovici and Verona. Analogous results which improve on theorems of Satake are proved for extensions of a nilpotent group having square-integrable representations by a reductive group, by combining the theorem for the nilpotent case with Schmid's proof of the Langlands conjecture. Some related results on Lie algebra cohomology and the "Harish-Chandra homomorphism" for Lie algebras with a triangular decomposition are also given.

0. Introduction. Since the appearance of Kirillov's thesis **[21]** on nilpotent Lie groups, the key unifying idea in the study of unitary representations of more or less arbitrary connected Lie groups has been the association of irreducible or at least primary representations with coadjoint orbits or "generalized orbits." This one principle is the basis for what one may call the Kirillov-Kostant "orbit method," which encompasses many of the deepest results of the representation theory of both solvable and semisimple Lie groups. (See, for example, **[3]**, **[5]**, **[14]**, **[21]**, **[22]**, **[25]**–**[28]**, **[35]**, **[36]**, **[40]**, **[43]**, **[46]**, **[49]**. This is not by any means a complete list–it is only a small sample of the literature to suggest the scope of the subject.) At the very least, it seems that for any connected Lie group G, all the representations needed to decompose the regular representation should be obtainable from some sort of "quantization process" involving polarizations for elements of the (real) dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G that satisfy some sort of integrality condition.

The most familiar instance of this construction is the one used by Kirillov to construct all the irreducible unitary representations of a nilpotent Lie group, and which one can also use to construct the (unitary) principal series of a complex (or more generally, quasi-split) semisimple group. In this situation, one starts with an element $\lambda \in \mathfrak{g}^*$ and a real polarization $\mathfrak{h}$ for λ, that is a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ that is

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a maximal totally isotropic subspace of $\mathfrak{g}$ with respect to the alternating bilinear form $B_\lambda(\cdot,\cdot) = \lambda([\cdot,\cdot])$. If one assumes that the connected subgroup H of G corresponding to $\mathfrak{h}$ is closed and has a unitary character χ_λ with differential $i\lambda$, then inducing χ_λ to G will give a unitary representation of G. Under favorable circumstances, this representation will be irreducible and independent of the choice of $\mathfrak{h}$. (Of course this is a vast oversimplification since one usually has to worry about nonconnectedness of the stabilizer G_λ of λ, "admissibility" of $\mathfrak{h}$, and the "Pukanszky condition." These problems disappear in the nilpotent case, though.)

However, even to deal with the Borel-Weil theorem for compact groups and the representations of the four-dimensional "oscillator group," one needs to consider complex polarizations, which are subalgebras not of $\mathfrak{g}$ but of its complexification. When such a polarization is positive (see 1.1 below), one can often construct an irreducible representation of G by an analogue of the inducing process called "holomorphic induction." Roughly speaking, this involves considering L^2-sections of a homogeneous line bundle that are holomorphic in certain directions given by the polarization. This process can be used to construct the Borel-Weil realization of the irreducible representations of a compact group, the Harish-Chandra holomorphic discrete series for a noncompact semisimple group corresponding to a hermitian symmetric space **[17]**, the irreducible representations of a type I solvable group **[5]** or of a group with cocompact nilradical **[36]**, **[27]**, and even the traceable factor representations of a group with cocompact radical **[35]**, **[36]** that need not be type I.

But to treat general Lie groups, even positive complex polarizations are not sufficient. The orbits corresponding to the nonholomorphic discrete series of semisimple groups do not admit positive polarizations, and in fact even some of the square-integrable representations of the semidirect product of $SL(2, \mathbf{R})$ and the 3-dimensional Heisenberg group cannot be obtained by holomorphic induction. Nevertheless, Schmid's proof **[42]**, **[43]** of the "Langlands conjecture" has given the orbit method yet another success–a geometrical realization of the discrete series on the L^2-cohomology of line bundles over coadjoint orbits. (An alternative geometrical realization using harmonic spinors **[4]** is also related to the orbit picture for general Lie groups, and in fact may ultimately make it possible to dispense with polarizations entirely in the case of nilpotent groups–see **[47]**, **[48]**, **[30]**.)

The purpose of this paper is therefore to consider the representations of a fairly broad class of unimodular Lie groups on L^2-cohomology spaces of line bundles with respect to nonpositive polarizations. Our primary concern is with totally complex polarizations, which seem inextricably liked to square-integrable representations whenever they occur. In the nilpotent case, the most general polarizations for which harmonically induced representations can be defined can be reduced to this case anyway (see §3). For semisimple groups, the tempered representations needed for the Plancherel theorem **[18]**, **[46]** can all be constructed by unitary induction from cuspidal parabolics once one knows how to construct the discrete series. (Alternatively, they may be constructed directly from orbits by a method that essentially comes down to this **[46**, §8].) For more general groups (including almost all solvable ones), the place of the square-integrable representations in the general pattern is very obscure; nevertheless, the geometrical realization of the discrete series is an interesting subject in itself and is of course a prerequisite to understanding how to construct more general representations from coadjoint orbits or "generalized orbits."

It should be pointed out that the study of totally complex polarizations in the nonunimodular case appears considerably more complicated, as is evidenced by the breakdown of the Connes-Moscovici index theorem **[10]** and by the very complicated conditions of Rossi-Vergne **[39]**, Fujiwara **[15]**, and Zaitsev **[50]** for nontriviality of holomorphically induced representations even from a positive polarization. We are apparently a long way from understanding this situation.

The plan of this paper is as follows. §1 contains the basic definitions concerning (not necessarily positive) polarizations and some facts about their behavior in the nilpotent case. These are used in §2 in the proof of Theorem 2.4, which is a sort of a nilpotent analogue of the Borel-Weil-Bott-Kostant Theorem **[24]** on Lie algebra cohomology.[2] This result is then used in §3 to prove our first main objective, a conjecture of Moscovici and Verona. (This is Theorem 2 of **[31]** with all the technical conditions on the polarization removed.) We also briefly discuss the correct interpretation of "harmonic induction" for polarizations that are not totally complex, again for nilpotent groups with square-integrable representations. §4 is concerned with certain nonnilpotent groups with square-integrable representations, including in particular the U-groups studied by Anh **[1]**, plus some solvable unimodular groups with a particularly simple structure. The primary objective here is to combine the results of §3 with the (very deep) results of Schmid on the semisimple case to give an improved version of some results of Satake **[41]**. The main results are Theorem 4.3, which relates square-integrable representations to coadjoint orbits, and Theorem 4.8, which proves a generalized form of the "Langlands conjecture." §5 is independent of most of the rest of the paper but is included because it helps to "explain" the results of §4. The purpose of this section is to show that the universal enveloping algebras for the groups considered in §4 are sufficiently like those of semisimple groups so that one would expect similarities in the representation theory.

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[2]After this paper was completed, the author received a copy of *Lie cohomology of representations of nilpotent Lie groups and holomorphically induced representations* by R. Penney, which was written almost simultaneously with the present work. Penney gives another proof of Theorem 2.4 and then deduces Theorem 3.1 in the same way. Both proofs use the same inductive framework and case-by-case analysis, based on the equivalents of Lemmas 1.5 and 1.6 of this paper. However, the proofs differ in their analysis of the various cases. The author prefers the present proof because of the relative simplicity of Lemmas 2.2 and 2.3, which have some independent interest. Penney's proof, on the other hand, has the advantage of not requiring spectral sequences and only using a simple algebraic lemma (his Lemma 2) instead.

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He also wishes to thank Michel Duflo, first of all for suggesting many of the ideas of §5 in lectures given at the University of Maryland in December, 1978, and secondly for suggesting several improvements in the original version of the manuscript (including a generalization of Theorem 2.4, deferred to a future paper). Finally, but perhaps most importantly, thanks are due to Richard Penney for detecting several errors in the original proof of Theorem 2.4 and for suggesting the use of weak polarizations as a means of avoiding them.

The following terminological and notational conventions are in effect throughout the paper. The word "representation" when applied to groups always means "strongly continuous unitary representation." We usually shall not distinguish between a representation and its unitary equivalence class. The only fields used are the real and complex numbers $\mathbf{R}$ and $\mathbf{C}$, except in §5 where most remarks are valid over any algebraically closed field of characteristic zero. Lie algebras are denoted by lower-case Gothic letters. For Lie algebras over $\mathbf{R}$, the corresponding connected Lie group (simply connected unless constrained by context to be a subgroup of a larger group) is denoted by the corresponding capital Roman letter. If $\mathfrak{a}$ is a real Lie algebra, $\mathfrak{a}^*$ denotes its real dual space, $\mathfrak{a}_{\mathbf{C}}$ the complexification of $\mathfrak{a}$, and $U(\mathfrak{a}_{\mathbf{C}})$ its complexified enveloping algebra. The symbols $\mathcal{S}$ and $\hat{\otimes}$ denote the Schwartz space of rapidly decreasing smooth functions and the projective tensor product of topological vector spaces, respectively. (The only topological vector spaces needed are Fréchet spaces of the familiar sort.)

1. Preliminaries on polarizations.

1.1. Let $\mathfrak{g}$ be a Lie algebra over $\mathbf{R}$, $\mathfrak{g}^*$ its dual vector space over $\mathbf{R}$. An element λ of $\mathfrak{g}^*$ defines an alternating bilinear form B_λ on $\mathfrak{g}$ (which we extend by $\mathbf{C}$-bilinearity to the complexification $\mathfrak{g}_{\mathbf{C}}$) by $B_\lambda(\cdot,\cdot) = \lambda([\cdot,\cdot])$. By a *polarization* for λ, we mean a (complex) Lie subalgebra of $\mathfrak{g}_{\mathbf{C}}$ such that

(i) $\mathfrak{h}$ is a maximal totally isotropic subspace for $\mathfrak{g}_{\mathbf{C}}$ with respect to B_λ and

(ii) $\mathfrak{h} + \bar{\mathfrak{h}}$ (often called $\mathfrak{e}_{\mathbf{C}}$) is a Lie subalgebra of $\mathfrak{g}_{\mathbf{C}}$. Here "$^-$" denotes complex conjugation of $\mathfrak{g}_{\mathbf{C}}$ relative to the real form $\mathfrak{g}$. If $\mathfrak{h}$ satisfies (i) but not necessarily (ii), we call $\mathfrak{h}$ a *weak polarization*. Everything to be said in this section is valid for weak polarizations. From B_λ we can also manufacture a hermitian sesquilinear form H_λ on $\mathfrak{h}$ by

$$H_\lambda(u, v) = iB_\lambda(u, \bar{v}) = i\lambda([u, \bar{v}]) \quad \text{for } u, v \in \mathfrak{h}.$$

$\mathfrak{h}$ is called *positive* if H_λ is nonnegative, *negative* if H_λ is nonpositive, *totally real* if $H_\lambda \equiv 0$ (equivalently, if $\mathfrak{h} = \bar{\mathfrak{h}}$), *totally complex* if H_λ is nondegenerate modulo the radical $\mathfrak{g}_{\lambda,\mathbf{C}}$ of B_λ (equivalently, if $\mathfrak{h} + \bar{\mathfrak{h}} = \mathfrak{g}_{\mathbf{C}}$).

1.2. A certain nonnegative integer $q(\mathfrak{h}, \lambda)$ will turn out to be an important invariant of the pair $(\mathfrak{h}, \lambda)$. We will call it the *negativity index* of the polarization since it measures the deviation of $\mathfrak{h}$ from being totally complex and positive. To define it, let $\mathfrak{g}_{\lambda,\mathbf{C}}$ be the radical of B_λ as before and let $\mathfrak{d}_{\mathbf{C}} = \mathfrak{h} \cap \bar{\mathfrak{h}}$ be the radical of H_λ. Then H_λ induces a nondegenerate (although not necessarily definite) hermitian inner product on $\mathfrak{h}/\mathfrak{d}_{\mathbf{C}}$. We get

$$q(\mathfrak{h}, \lambda) = \dim_{\mathbf{C}}(\mathfrak{d}_{\mathbf{C}}/\mathfrak{g}_{\lambda,\mathbf{C}}) + (\text{no. of minus signs in the signature of } H_\lambda \text{ on } \mathfrak{h}/\mathfrak{d}_{\mathbf{C}})$$
$$= \dim_{\mathbf{C}}(\mathfrak{h}/\mathfrak{g}_{\lambda,\mathbf{C}}) - (\text{no. of plus signs in the signature of } H_\lambda \text{ on } \mathfrak{h}/\mathfrak{d}_{\mathbf{C}}).$$

1.3. In the rest of this section we will be interested only in the case when $\mathfrak{g}$ is nilpotent. Via the Kirillov correspondence [**21**], λ defines an irreducible representation π_λ of the connected, simply connected group G having Lie algebra $\mathfrak{g}$ (or more precisely, a unitary equivalence class of such representations). The number $n = \dim_{\mathbf{C}}(\mathfrak{h}/\mathfrak{g}_{\lambda,\mathbf{C}})$ which appears in the second expression for $q(\mathfrak{h}, \lambda)$ is independent of the choice of $\mathfrak{h}$ and is the Gelfand-Kirillov dimension of π_λ; it is the unique n such that the image of $U(\mathfrak{g}_{\mathbf{C}})$ under π_λ is isomorphic to the Weyl algebra A_n (over $\mathbf{C}$). Furthermore, one may take the space of C^∞-vectors for π_λ, which we denote by $(\pi_\lambda)_\infty$, to be the Schwartz space $\mathcal{S}(\mathbf{R}^n)$, in such a way that the action of $U(\mathfrak{g}_{\mathbf{C}})$ on $(\pi_\lambda)_\infty$ just becomes the standard representation of A_n by polynomial differential operators in n variables. (See [**21**, Theorem 7.1] and [**20**, Proposition 3.3].)

1.4. For the rest of this section, we make the further assumption that G has one-dimensional center Z and has square-integrable representations modulo Z (see [**28**]). This forces the dimension of G to be odd. We fix $\lambda \in \mathfrak{g}^*$ with nontrivial restriction to the center $\mathfrak{z}$ of $\mathfrak{g}$; then π_λ is square-integrable modulo Z and its restriction to Z has differential $i\lambda|\mathfrak{z}$ [**28**, Theorem 1].

In this situation, let $\mathfrak{h}$ be a weak polarization for λ. Then $\mathfrak{g}_\lambda = \mathfrak{z}$ and $n = (\dim_{\mathbf{C}} \mathfrak{h}) - 1 = ((\dim G) - 1)/2$. We will need to compare $q(\mathfrak{h}, \lambda)$ with the negativity index of a weak polarization for some functional μ gotten by restricting λ to an ideal $\mathfrak{g}'$ of $\mathfrak{g}$. Therefore assume $n > 0$, so that $\mathfrak{z}^{(2)}(\mathfrak{g}) \supsetneq \mathfrak{z}$, and choose $y \in \mathfrak{z}^{(2)}(\mathfrak{g}) \setminus \mathfrak{z}$. Let $\mathfrak{g}'$ be the centralizer of y in $\mathfrak{g}$, which is an ideal of codimension 1 in $\mathfrak{g}$. For convenience, put $\mathfrak{n} = \mathfrak{z}_{\mathbf{C}} + \mathbf{C}y$, which is a 2-dimensional central ideal in $\mathfrak{g}'_{\mathbf{C}}$.

Case I. $\mathfrak{h} \subseteq \mathfrak{g}'_{\mathbf{C}}$. Since $\mathfrak{n}$ is central in $\mathfrak{g}'_{\mathbf{C}}$, $[\mathfrak{n}, \mathfrak{h}] = 0$ and $\mathfrak{n} + \mathfrak{h}$ is subordinate to λ; hence $\mathfrak{n} \subseteq \mathfrak{h}$ since $\mathfrak{h}$ is maximal subordinate. We may assume $\lambda(y) = 0$, in which case the restriction of λ to $\mathfrak{g}'$ induces a linear functional ν on $\mathfrak{g}'/\mathbf{R}y$.

LEMMA 1.5. *In this situation, the group L corresponding to $\mathfrak{g}'/\mathbf{R}y$ has square-integrable representations, and the image $\tilde{\mathfrak{h}}$ of $\mathfrak{h}$ in $\mathfrak{g}'_{\mathbf{C}}/\mathbf{C}y$ is a weak polarization for ν. Also, $q(\mathfrak{h}, \lambda) = q(\tilde{\mathfrak{h}}, \nu) + 1$.*

PROOF. To show ν gives rise to square-integrable representations of L, it is enough by [**28**, Theorem 1] to show that the stabilizer L_ν of ν in $\mathfrak{l}^*$ is just the image $\tilde{Z}$ of Z. But if $t \in \mathfrak{g}'$ and the image $\tilde{t}$ of t in $\mathfrak{l}$ is in $\mathfrak{l}_\nu$, then for all s in $\mathfrak{g}'$, $\nu([\tilde{t}, \tilde{s}]) = 0$, or $\lambda([t, s]) = 0$ (since $\lambda(y) = 0$); hence $t \in \mathfrak{n}$ and $\tilde{t} \in \tilde{\mathfrak{z}}$. That $\tilde{\mathfrak{h}}$ is a weak polarization for ν is obvious by dimension counting.

Finally, note that $\mathfrak{n} \subseteq \mathfrak{h} \cap \bar{\mathfrak{h}}$, so that the map from $\mathfrak{g}'_{\mathbf{C}}$ to $\mathfrak{g}'_{\mathbf{C}}/\mathbf{C}y$ has one-dimensional kernel on $\mathfrak{h} \cap \bar{\mathfrak{h}}$ and induces an isomorphism from the nondegenerate quotient of H_λ to the nondegenerate quotient of H_ν. Thus H_λ and H_ν have the same number of minus signs in their signatures, and $\dim_{\mathbf{C}}(\mathfrak{d}_{\mathbf{C}}/\mathfrak{z}_{\mathbf{C}}) = \dim_{\mathbf{C}}(\tilde{\mathfrak{d}}_{\mathbf{C}}/\tilde{\mathfrak{z}}_{\mathbf{C}}) + 1$. The assertion about negativity indices follows.

Case II. $\mathfrak{h} \not\subseteq \mathfrak{g}'_{\mathbf{C}}$. In this case, $\mathfrak{h} \cap \mathfrak{g}'_{\mathbf{C}}$ is an ideal of codimension 1 in $\mathfrak{h}$, and $\mathfrak{m} = \mathfrak{h} \cap \ker \lambda \cap \mathfrak{g}'_{\mathbf{C}}$ is an ideal of codimension 1 in $\mathfrak{h} \cap \ker \lambda$. As before, if we assume $\lambda(y) = 0$, λ induces a functional ν on $\mathfrak{l} = \mathfrak{g}'/\mathbf{R}y$, and ν determines a square-integrable representation of L.

LEMMA 1.6. *In this situation, the image* $\tilde{\mathfrak{h}}$ *of* $\mathfrak{h} \cap \mathfrak{g}'_{\mathbf{C}}$ *in* $\mathfrak{l}_{\mathbf{C}} = \mathfrak{g}'_{\mathbf{C}}/\mathbf{C}y$ *is again a weak polarization for* ν. *Then* $q(\mathfrak{h}, \lambda)$ *and* $q(\tilde{\mathfrak{h}}, \nu)$ *are related as follows*:

(a) *if* $\mathfrak{h} \cap \bar{\mathfrak{h}} \underset{\neq}{\supset} \mathfrak{h} \cap \bar{\mathfrak{h}} \cap \mathfrak{g}'_{\mathbf{C}}$, *then* $q(\mathfrak{h}, \lambda) = q(\tilde{\mathfrak{h}}, \nu) + 1$;

(b) *if* $\mathfrak{h} \cap \bar{\mathfrak{h}} \subseteq \mathfrak{g}'_{\mathbf{C}}$ *and* $\tilde{\mathfrak{h}} \cap \bar{\tilde{\mathfrak{h}}} =$ *image of* $(\mathfrak{h} \cap \bar{\mathfrak{h}})$ *in* $\mathfrak{l}_{\mathbf{C}}$, *then the orthogonal complement of* $\mathfrak{h} \cap \mathfrak{g}'_{\mathbf{C}}$ *in* $\mathfrak{h}$ (*relative to* H_λ) *is* 1-*dimensional modulo* $\mathfrak{h} \cap \bar{\mathfrak{h}}$, *and* H_λ *is definite on this* 1-*dimensional space. If it is positive, then* $q(\mathfrak{h}, \lambda) = q(\tilde{\mathfrak{h}}, \nu)$; *if it is negative*, $q(\mathfrak{h}, \lambda) = q(\tilde{\mathfrak{h}}, \nu) + 1$;

(c) *if* $\mathfrak{h} \cap \bar{\mathfrak{h}} \subseteq \mathfrak{g}'_{\mathbf{C}}$ *and* $\tilde{\mathfrak{h}} \cap \bar{\tilde{\mathfrak{h}}}$ *is strictly bigger than the image of* $\mathfrak{h} \cap \bar{\mathfrak{h}}$ [*i.e.*, $((\mathfrak{h} \cap \mathfrak{g}'_{\mathbf{C}}) + \mathfrak{n}) \cap ((\bar{\mathfrak{h}} \cap \mathfrak{g}'_{\mathbf{C}}) + \mathfrak{n}) \underset{\neq}{\supset} (\mathfrak{h} \cap \bar{\mathfrak{h}}) + \mathfrak{n}$], *then* $q(\mathfrak{h}, \lambda) = q(\tilde{\mathfrak{h}}, \nu)$.

Cases (a), (b), *and* (c) *are mutually exclusive, and all can occur.*

PROOF. It is clear that the three cases are mutually exclusive and exhaustive. Examples of all three will be given later, along with an example of Case I (the situation of Lemma 1.5).

(a) Suppose $\mathfrak{h} \cap \bar{\mathfrak{h}} \underset{\neq}{\supset} \mathfrak{h} \cap \bar{\mathfrak{h}} \cap \mathfrak{g}'_{\mathbf{C}}$. Then $\dim_{\mathbf{C}}(\mathfrak{h} \cap \bar{\mathfrak{h}}) = \dim_{\mathbf{C}}(\mathfrak{h} \cap \bar{\mathfrak{h}} \cap \mathfrak{g}'_{\mathbf{C}}) + 1$. Choose basis elements $t_1, \ldots, t_n$ for $\mathfrak{h} \cap \mathfrak{g}'_{\mathbf{C}}$ whose images $\tilde{t}_1, \ldots, \tilde{t}_n$ diagonalize H_ν on $\tilde{\mathfrak{h}}$; more precisely, assume $H_\nu(\tilde{t}_i, \tilde{t}_j) = 0$ for $i \neq j$ and

$$H_\nu(\tilde{t}_i, \tilde{t}_i) = \begin{Bmatrix} 0 & \text{for } 1 \leqslant i \leqslant p \\ -1 & \text{for } p + 1 \leqslant i \leqslant p + r \\ +1 & \text{for } p + r + 1 \leqslant i \leqslant n \end{Bmatrix}.$$

By definition of the negativity index, $q(\tilde{\mathfrak{h}}, \nu) = p + r - 1$, and by construction of ν from λ, $H_\lambda(t_i, t_j) = H_\nu(\tilde{t}_i, \tilde{t}_j)$. Since $\mathfrak{h} \cap \bar{\mathfrak{h}} \not\subseteq \mathfrak{g}'_{\mathbf{C}}$, we may choose $t_0 \in \mathfrak{h} \cap \bar{\mathfrak{h}}$ not in $\mathfrak{g}'_{\mathbf{C}}$. Then t_0 lies in the radical of H_λ, so we have diagonalized H_λ by means of $t_0, \ldots, t_n$. It is clear that the signature of H_λ has $p + 1$ zeroes and r minus signs, so $q(\mathfrak{h}, \lambda) = p + r = q(\tilde{\mathfrak{h}}, \nu) + 1$.

(b) Suppose $\mathfrak{h} \cap \bar{\mathfrak{h}} \subseteq \mathfrak{g}'_{\mathbf{C}}$. Then if $t \in \mathfrak{h} \setminus (\mathfrak{h} \cap \mathfrak{g}'_{\mathbf{C}})$, $t \notin \mathfrak{h} \cap \bar{\mathfrak{h}} = \operatorname{rad}(H_\lambda)$, so there exists $s \in \mathfrak{h}$ with $H_\lambda(t, s) \neq 0$. If $\tilde{\mathfrak{h}} \cap \bar{\tilde{\mathfrak{h}}} = \operatorname{rad} H_\nu$ is equal to the image of $\mathfrak{h} \cap \bar{\mathfrak{h}}$, then if we construct a basis diagonalizing H_ν as in (a), the elements $t_1, \ldots, t_p$ will be a basis for the radical of H_λ. The orthogonal complement of $\mathfrak{h} \cap \mathfrak{g}'_{\mathbf{C}}$ will be one-dimensional modulo this radical, and if $t_0 \in \mathfrak{h} \setminus (\mathfrak{h} \cap \mathfrak{g}'_{\mathbf{C}})$ is orthogonal to $\mathfrak{h} \cap \mathfrak{g}'_{\mathbf{C}}$, then since it is not in the radical of H_λ, we must have $H_\lambda(t_0, t_0) \neq 0$. Thus the signature of H_λ has the same number of zeroes as that of H_ν, and will have one extra plus or minus sign. The rest is clear.

(c) Again suppose $\mathfrak{h} \cap \bar{\mathfrak{h}} \subseteq \mathfrak{g}'_{\mathbf{C}}$, but suppose $\tilde{\mathfrak{h}} \cap \bar{\tilde{\mathfrak{h}}}$ is bigger than the image of $\mathfrak{h} \cap \bar{\mathfrak{h}}$. Since $\dim_{\mathbf{C}}(\mathfrak{n}/\mathfrak{z}_{\mathbf{C}}) = 1$, the radical of H_ν then has dimension one larger than that of H_λ. Choose a basis as above; we may assume that $t_2, \ldots, t_p \in \mathfrak{h} \cap \bar{\mathfrak{h}}$ and that $t_1 \notin \operatorname{rad} H_\lambda$. Choose $t_0 \in \mathfrak{h} \setminus (\mathfrak{h} \cap \mathfrak{g}'_{\mathbf{C}})$ orthogonal to $t_2, \ldots, t_n$ as in (b). Since $H_\lambda(t_1, t_j) = 0$ for $j \geqslant 1$ and $t_1 \notin \operatorname{rad} H_\lambda$, necessarily $H_\lambda(t_1, t_0) \neq 0$. The matrix of

H_λ in the basis $t_0, \ldots, t_n$ then has the form

$$\left[\begin{array}{cc|ccc} \beta & \alpha & & 0 & \\ \bar{\alpha} & 0 & & & \\ \hline & & 0_{p-1} & 0 & 0 \\ & 0 & 0 & -I_r & 0 \\ & & 0 & 0 & I_{n-p-r} \end{array}\right], \qquad \alpha \in \mathbf{C} \setminus \{0\},$$

and since the determinant of $\left(\begin{smallmatrix} \beta & \alpha \\ \bar{\alpha} & 0 \end{smallmatrix}\right)$ is negative, the signature of H_λ has $(p - 1)$ zeroes, $(r + 1)$ minus signs, and $(n - p - r + 1)$ plus signs. In particular, we have $q(\mathfrak{h}, \lambda) = (p - 1) - 1 + (r + 1) = p + r - 1 = q(\tilde{\mathfrak{h}}, \nu)$.

1.7. Examples. We give examples to illustrate the four cases above. It will suffice to take for $\mathfrak{g}$ the 5-dimensional Heisenberg Lie algebra with basis x_1, x_2, y_1, y_2, z, where $[x_1, y_1] = [x_2, y_2] = z$, $[x_1, x_2] = [x_1, y_2] = [y_1, y_2] = [y_1, x_2] = 0$. Let $y = y_2$, so $\mathfrak{g}'$ is spanned by x_1, y_1, y_2, and z. Finally, let $\lambda(z) = 1$, $\lambda(x_j) = \lambda(y_j) = 0$ for $j = 1, 2$.

(1) Take $\mathfrak{h}$ to be spanned by y_1, y_2, and z. This is an example of Case I, which happens to be totally real (although there are more general examples).

(2) Take $\mathfrak{h}$ to be spanned by x_1, x_2, and z. This is an example of Case II(a), which happens to be totally real (although there are more general examples).

(3) Let $\mathfrak{h} = \mathbf{C}(x_2 \pm iy_2) + \mathbf{C}x_1 + \mathbf{C}z$. Then $\mathfrak{h} \cap \bar{\mathfrak{h}} = \mathbf{C}x_1 + \mathbf{C}z \subseteq \mathfrak{g}'_{\mathbf{C}}$, and $\tilde{\mathfrak{h}} = \bar{\tilde{\mathfrak{h}}} = \mathbf{C}\tilde{x}_1 + \mathbf{C}\tilde{z}$. This is an example of Case II(b) in which the signature depends on the $\pm$ sign in $x_2 \pm iy_2$.

(4) Let $\mathfrak{h} = \mathbf{C}(x_1 + iy_2) + \mathbf{C}(x_2 + iy_1) + \mathbf{C}z$. Then $\mathfrak{h}$ is totally complex, and $\mathfrak{h} \cap \mathfrak{g}'_{\mathbf{C}} = \mathbf{C}(x_1 + iy_2) + \mathbf{C}z$. In the mapping $\mathfrak{g}' \to \mathfrak{l}$, y_2 goes to 0; hence $\tilde{\mathfrak{h}} = \mathbf{C}\tilde{x}_1 + \mathbf{C}\tilde{z}$ is totally real (even though $\mathfrak{h}$ is totally complex). This is an example of Case II(c).

2. Cohomology of nilpotent Lie algebras of differential operators acting on the Schwartz space. This section contains applications of §1 to the computation of some Lie algebra cohomology groups. We use standard facts about the Hochschild-Serre spectral sequence for Lie algebra cohomology relative to an ideal that may be found, for instance, in [7, Chapter XVI, §6]. Some of these results were previously obtained in [33] by similar means; however, the present methods offer an improvement in several respects:

(a) We do not require our polarizations to be totally complex. This has an interesting geometrical consequence which will be discussed in §3.4. Anyway, the generalization seems more or less unavoidable since the natural inductive process starting from a totally complex polarization often leads to consideration of polarizations that are not totally complex for a smaller group.

(b) We do not require our polarizations to be commutative.

(c) Penney's inductive step in the non-Heisenberg case [33, p. 32] appears to break down for 5-dimensional groups, since then his $\mathfrak{N}_0$ and $\mathfrak{M}_0$ coincide. This could be remedied by treating this case separately or appealing to Example 1 in [31, §5], but this seems aesthetically unappetizing.

The following special fact, essentially noted by Penney, is crucial to what follows:

LEMMA 2.1. *Let* $\mathfrak{a}$ *be the* 1-*dimensional abelian Lie algebra over* $\mathbf{C}$, *acting on* $\mathcal{S}(\mathbf{R})$ *by multiples of the differential operator* $d/dx + \alpha x$, *where* $\alpha \in \mathbf{C}$ *and* x *is the coordinate on* $\mathbf{R}$. *Then*

$$\begin{cases} H^0(\mathfrak{a}, \mathcal{S}(\mathbf{R})) = 0 & \textit{if } \operatorname{Re}\alpha < 0, \mathbf{C} \textit{ if } \operatorname{Re}\alpha > 0, \\ H^1(\mathfrak{a}, \mathcal{S}(\mathbf{R})) = \mathbf{C} & \textit{if } \operatorname{Re}\alpha < 0, 0 \textit{ if } \operatorname{Re}\alpha > 0. \end{cases}$$

PROOF. The solution space of the differential equation $(d/dx + \alpha x)f(x) = 0$ is spanned by $f(x) = e^{-\alpha x^2/2}$, which lies in $\mathcal{S}$ exactly when $\operatorname{Re}\alpha > 0$. This proves the result about H^0. Similarly, given $g \in \mathcal{S}$, the solution of $(d/dx + \alpha x)f(x) = g(x)$ with initial condition $f(x) \to 0$ as $x \to -\infty$ is given by

$$f(x) = e^{-\alpha x^2/2}\int_{-\infty}^{x} e^{\alpha t^2/2} g(t)\, dt,$$

which lies in $\mathcal{S}$ exactly when either $\operatorname{Re}\alpha > 0$ or else $\operatorname{Re}\alpha < 0$ and $\int_{-\infty}^{\infty} e^{\alpha t^2/2} g(t)\, dt = 0$ (see proofs of [**33**, Sublemmas 21 and 22]). This proves the assertion about H^1.

We also need one other cohomology computation, based on properties of the Cauchy-Riemann operator:

LEMMA 2.2. *Let* $\mathfrak{a}$ *be the two-dimensional complex abelian Lie algebra, acting on* $\mathcal{S}(\mathbf{R}^2)$ *by linear combinations of the polynomial differential operators*

$$\zeta = x_1 + ix_2 \quad \textit{and} \quad \partial/\partial\bar{\zeta} + P = \tfrac{1}{2}(\partial/\partial x_1 + i\partial/\partial x_2) + P(x_1, x_2),$$

where x_1 *and* x_2 *are the coordinate functions on* $\mathbf{R}^2$, ζ *is the complex coordinate on* $\mathbf{C}$ (*identified with* $\mathbf{R}^2$), *and* P *is a polynomial function in two variables. Then*

$$H^k(\mathfrak{a}, \mathcal{S}(\mathbf{R}^2)) = \begin{cases} 0 & \textit{if } k \neq 1, \\ \mathbf{C} & \textit{if } k = 1, \end{cases}$$

and the spectral sequence with E_2 *terms* $H^p(\mathbf{C}(\partial/\partial\bar{\zeta} + P), H^q(\mathbf{C}\zeta, \mathcal{S}(\mathbf{R}^2)))$ *converging to* $H^k(\mathfrak{a}, \mathcal{S}(\mathbf{R}^2))$ *collapses so as to have*

$$E_2^{p,q} = \begin{cases} 0 & \textit{unless } p = 0, q = 1, \\ \mathbf{C} & \textit{if } p = 0, q = 1. \end{cases}$$

PROOF. Obviously the fact about the spectral sequence is stronger than that about the cohomology alone. And since ζ acts injectively on $\mathcal{S}(\mathbf{R}^2)$, $E_2^{p,q} = 0$ if $q = 0$. To reduce the calculation of $E_2^{1,1}$ and $E_2^{0,1}$ to standard facts in complex analysis, it is helpful to apply [**19**, Theorem 1.4.4] to construct a function $\varphi \in C^\infty(\mathbf{R}^2)$ such that $\partial\varphi/\partial\bar{\zeta} = P$. Then $\partial/\partial\bar{\zeta} + P$ can be rewritten as $D = e^{-\varphi}(\partial/\partial\bar{\zeta}) \circ e^{\varphi}$.

First we show that $E_2^{1,1} = 0$. This amounts to showing that if $f \in \mathcal{S}(\mathbf{R}^2)$, then f can be written in the form

$$f(\zeta) = \zeta g(\zeta) + e^{-\varphi(\zeta)}\frac{\partial}{\partial\bar{\zeta}}(e^{\varphi(\zeta)}h(\zeta))$$

for some g, $h \in \mathcal{S}(\mathbf{R}^2)$. First choose $\psi \in C_c^\infty(\mathbf{R}^2)$ such that $\psi \equiv 1$ on some neighborhood of $(0, 0)$. By [**19**, Theorem 1.4.4] again, there exists $\xi \in C^\infty(\mathbf{R}^2)$ such that $\partial\xi/\partial\bar{\zeta} = e^\varphi f$. Then if $h = \psi e^{-\varphi}\xi$, $h \in C_c^\infty(\mathbf{R}^2)$ and $k = f - Dh$ vanishes identically in some neighborhood of $(0, 0)$. Hence $k(x_1, x_2)/(x_1 + ix_2)$ makes sense and defines a function $g \in \mathcal{S}(\mathbf{R}^2)$, and $f = \zeta g + Dh$.

Finally, we must show that $H^1(\mathfrak{a}, \mathcal{S}(\mathbf{R}^2)) = E_2^{0,1}$ is one-dimensional i.e., that D has 1-dimensional kernel on $\mathcal{S}(\mathbf{R}^2)/(x_1 + ix_2)\mathcal{S}(\mathbf{R}^2)$. Suppose that $f \in \mathcal{S}(\mathbf{R}^2)$ is a representative for a coset of $\zeta\mathcal{S}$ killed by D. Then there exists a function $h \in \mathcal{S}(\mathbf{R}^2)$ such that $(Df)(\zeta) = \zeta h(\zeta)$, or $\partial(fe^\varphi)/\partial\bar{\zeta} = e^\varphi\zeta h$. Let $k(\zeta) = f(\zeta)e^{\varphi(\zeta)}\zeta^{-1}$ for $\zeta \neq 0$. Then k is C^∞ in the punctured plane and $\partial k/\partial\bar{\zeta} = e^\varphi h$. By [**19**, Theorem 1.4.4] again, there exists $\psi \in C^\infty(\mathbf{R}^2)$ with $\partial\psi/\partial\bar{\zeta} = e^\varphi h$ everywhere. Hence $\partial(k - \psi)/\partial\bar{\zeta} = 0$ in $\mathbf{C} \setminus \{0\}$, i.e., $k - \psi$ is holomorphic. Since f, φ, and ψ are C^∞, the formula for k indicates that $k - \psi$ can have at worst a simple pole at 0, and will have a removable singularity if $f(0) = 0$. Thus k extends to a C^∞ function in the whole plane when $f(0) = 0$, and in this case, $f = \zeta e^{-\varphi}k$. Since f and its derivatives die rapidly at infinity, so do $e^{-\varphi}k$ and its derivatives; hence $f \in \zeta\mathcal{S}$ when $f(0) = 0$. On the other hand, if $f = e^{-\varphi}$ in some neighborhood of 0, then clearly $f \notin \zeta\mathcal{S}$, but yet Df vanishes in a neighborhood of 0; hence $Df \in \zeta\mathcal{S}$. So the kernel of D on $\mathcal{S}/\zeta\mathcal{S}$ is exactly one-dimensional.

The following technical lemma, showing that under suitable hypotheses, Lie algebra cohomology "commutes with direct integrals," will also be needed:

LEMMA 2.3. *Suppose a (complex) Lie algebra* $\mathfrak{m}$ *acts on* $\mathcal{S}(\mathbf{R}^n) \cong \mathcal{S}(\mathbf{R}) \hat{\otimes} \mathcal{S}(\mathbf{R}^{n-1})$ *by a "continuous direct integral"*

$$\sigma = \int^{\oplus} \pi \circ \alpha(s)\, ds,$$

where π *is a fixed representation of* $\mathfrak{m}$ *on* $\mathcal{S}(\mathbf{R}^{n-1})$ *by polynomial differential operators and* $\{\alpha(s)\}$ *is a one-parameter group of automorphisms of* $\mathfrak{m}$. (*By this we mean that for* $x \in \mathfrak{m}$, $f \in \mathcal{S}(\mathbf{R})$, $g \in \mathcal{S}(\mathbf{R}^{n-1})$, *and* $s \in \mathbf{R}$,

$$(\sigma(x)(f \otimes g))(s, \cdot) = f(s)(\pi(\alpha(s)x)g)(\cdot).)$$

Assume that for each q, $H^q(\pi(\mathfrak{m}), \mathcal{S}(\mathbf{R}^{n-1}))$ *may be identified with a Fréchet space* V_q. (*By this we mean that if we view* $H^q(\pi(\mathfrak{m}), \mathcal{S}(\mathbf{R}^{n-1}))$ *as the qth cohomology group* Z^q/B^q *of the standard complex* $\mathrm{Hom}_{\mathbf{C}}(\Lambda^*\mathfrak{m}, \mathcal{S}(\mathbf{R}^{n-1}))$, *and if we give the cochain spaces their natural Fréchet topologies, then we assume* B^q *is closed in* Z^q *and* Z^q/B^q *is topologically isomorphic with* V_q. *See* [**33**, *Section* II] *for further discussion of the topologies here.*) *Then for each* q,

$$H^q(\sigma(\mathfrak{m}), \mathcal{S}(\mathbf{R}^n)) \cong \mathcal{S}(\mathbf{R}) \hat{\otimes} V_q.$$

Note. In most applications of this lemma, V_q will be either zero- or one-dimensional, so that it is not necessary to complete the algebraic tensor product.

PROOF. Let C_π^q, C_σ^q denote the cochain groups $\mathrm{Hom}_{\mathbf{C}}(\Lambda^q\mathfrak{m}, \mathcal{S}(\mathbf{R}^{n-1}))$ and $\mathrm{Hom}_{\mathbf{C}}(\Lambda^q\mathfrak{m}, \mathcal{S}(\mathbf{R}^n))$, respectively, and let $\delta_\pi^q\colon C_\pi^q \to C_\pi^{q+1}$, $\delta_\sigma^q\colon C_\sigma^q \to C_\sigma^{q+1}$ denote the corresponding coboundary operators (as in [7, Chapter XIII, §8]). For $s \in \mathbf{R}$, let e_s: $\mathbb{S}(\mathbf{R}^n) \to \mathbb{S}(\mathbf{R}^{n-1})$ denote "evaluation at s," defined by $e_s(f)(x_1, \ldots, x_{n-1}) = f(s, x_1, \ldots, x_{n-1})$ for $f \in \mathbb{S}(\mathbf{R}^n)$. Then e_s induces maps $C^q_\sigma \to C^q_\pi$ in the obvious way, which we denote by the same symbol, and $e_s \circ \delta_\sigma = \delta_{\pi \circ \alpha(s)} \circ e_s$, so that e_s maps cocycles to cocycles, coboundaries to coboundaries, and induces maps

$$H^q(\sigma(\mathfrak{m}), \mathbb{S}(\mathbf{R}^n)) \to H^q(\pi \circ \alpha(s)(\mathfrak{m}), \mathbb{S}(\mathbf{R}^{n-1})).$$

Composing with the map on cohomology induced by $\alpha(-s)$ and putting these together for all s, we get maps

$$\theta^q\colon H^q(\sigma(\mathfrak{m}), \mathbb{S}(\mathbf{R}^n)) \to \mathbb{S}(\mathbf{R}) \hat{\otimes} H^q(\pi(\mathfrak{m}), \mathbb{S}(\mathbf{R}^{n-1})) = \mathbb{S}(\mathbf{R}) \hat{\otimes} V_q.$$

Our problem is to show θ^q is an isomorphism.

For this we need the fact (a consequence of the nuclearity of $\mathbb{S}$) that if E is a closed subspace of a Fréchet space F, then $\mathbb{S} \hat{\otimes} E$ may be identified with a closed subspace of $\mathbb{S} \hat{\otimes} F$ [**16**, Chapitre II, §3, no. 1, Corollaire à Proposition 10], and $\mathbb{S} \hat{\otimes} F/\mathbb{S} \hat{\otimes} E$ is topologically isomorphic with $\mathbb{S} \hat{\otimes} (F/E)$. (There is clearly a continuous injection from $\mathbb{S} \hat{\otimes} F/\mathbb{S} \hat{\otimes} E$ into $\mathbb{S} \hat{\otimes} (F/E)$. By the open mapping theorem, it will be a topological isomorphism if it is surjective. For the surjectivity, see [**16**, Chapitre I, §1, no. 2, Proposition 3].)

Now let us complete the proof. To show θ^q is surjective, we must show that any Schwartz function φ: $\mathbf{R} \to V_q$ comes from an element of (ker δ^q_σ). Since $V_q =$ (ker δ^q_π)/(im δ^{q-1}_π), by the results on Fréchet spaces just quoted, φ can be lifted to a Schwartz function $\tilde{\varphi}$: $\mathbf{R} \to$ (ker δ^q_π). "Twisting" by α, we can view $\tilde{\varphi}$ as an element ψ of C^q_σ such that $e_s(\psi) = \alpha(s)\tilde{\varphi}(s)$. But then $e_s \circ \delta^q_\sigma(\psi) = \delta^q_{\pi \circ \alpha(s)}(\alpha(s)\tilde{\varphi}(s)) = 0$ for all s; hence $\delta^q_\sigma(\psi) = 0$ and the cohomology class of ψ maps to φ under θ^q. To prove θ^q is injective, suppose $\psi \in C^q_\sigma$ is a cocycle whose cohomology class maps to 0 under θ^q. This means $e_s(\psi)$ is a $\pi \circ \alpha(s)$-coboundary for each s. Now the space of q-coboundaries for π may be identified with $C^{q-1}_\pi/(\ker \delta^{q-1}_\pi)$. Again, by the above result on Fréchet spaces, the Schwartz function defined by ψ from $\mathbf{R}$ to this space can be lifted to a Schwartz function φ: $\mathbf{R} \to C^{q-1}_\pi$, and $e_s(\psi) = \delta^{q-1}_{\pi \circ \alpha(s)}(\sigma(s)\varphi(s))$. Thus φ defines an element of C^{q-1}_σ such that $e_s(\delta_\sigma(\varphi)) = e_s(\psi)$ for each s, so that $\delta_\sigma(\varphi) = \psi$ and the cohomology class of ψ is trivial.

We are now ready for the statement and proof of the main theorem of this section. For this we adopt the notation of §1 above. The idea of the proof is to use the Hochschild-Serre spectral sequence and Lemma 2.3 to reduce all cohomology calculations down to the situations of Lemmas 2.1 and 2.2. Note that Lemma 2.1 alone will not suffice; Lemma 2.2 is needed as soon as one deals with the (unique up to isomorphism) 5-dimensional non-Heisenberg group with one-dimensional center, square-integrable representations, and a totally complex polarization. (This is [**31**, §5, Example 1] or the group $\Gamma_{5,3}$ in the notation of [**11**]. The group $\Gamma_{5,6}$ in Dixmier's list has square-integrable representations but no totally complex polarizations.)

THEOREM 2.4. *Let* $\mathfrak{g}$ *be a nilpotent Lie algebra with one-dimensional center* $\mathfrak{z}$ *such that the corresponding group* G *has square-integrable representations modulo* Z. *Let* $\lambda \in \mathfrak{z}^* - \{0\}$, *extended as before to a complex linear functional on* $\mathfrak{g}_{\mathbf{C}}$, *let* $\mathfrak{h} \subseteq \mathfrak{g}_{\mathbf{C}}$ *be a weak polarization for* λ, *and let* $\pi_{-\lambda}$ *denote the irreducible representation of* G, *the* *differential of whose central character is $-i\lambda$. Let $(\pi_{-\lambda})_\infty$ denote the C^∞-vectors of $\pi_{-\lambda}$ viewed as a $\mathfrak{g}_{\mathbf{C}}$-module, and let $q(\mathfrak{h}, \lambda)$ be the negativity index of $\mathfrak{h}$, as in* §1. *Then*

$$H^k(\mathfrak{h} \cap \ker \lambda, (\pi_{-\lambda})_\infty) = \begin{cases} 0 & \textit{for } k \neq q(\mathfrak{h}, \lambda), \\ \mathbf{C} & \textit{for } k = q(\mathfrak{h}, \lambda).^3 \end{cases}$$

PROOF. Let $\dim_{\mathbf{R}} \mathfrak{g} = 2n + 1$, so that $\dim_{\mathbf{C}} \mathfrak{h} = n + 1$ and $\dim_{\mathbf{C}}(\mathfrak{h} \cap \ker \lambda) = n$. The proof is by induction on n. When $n = 0$, there is nothing to prove. When $n = 1$, $\mathfrak{g}$ is Heisenberg, $(\pi_{-\lambda})_\infty$ is $\mathcal{S}(\mathbf{R})$, and $\mathfrak{h}$ acts by homogeneous first-order polynomial differential operators. This case is just a restatement of Lemma 2.1, for if x, y, z are a basis for $\mathfrak{g}$ with $[x, y] = z$, we may (without loss of generality, since x and y play symmetrical roles) assume $\mathfrak{h}$ contains $x + \beta y$ for some $\beta \in \mathbf{C}$. If we represent x as usual by d/dt and y by $-i\lambda(z)t$, then $\mathfrak{h}$ acts by multiples of $d/dt + \alpha t$, where $\alpha = -i\lambda(z)\beta$. Also, we have

$$H_\lambda(x + \beta y, x + \beta y) = i\lambda\big(\big[x + \beta y, x + \bar{\beta} y\big]\big) = i\big(\bar{\beta} - \beta\big)\lambda(z) = 2\,\mathrm{Im}(\beta)\lambda(z),$$

so that when $\mathrm{Im}(\beta)\lambda(z) > 0$, $q(\mathfrak{h}, \lambda) = 0$ and $\mathrm{Re}\,\alpha > 0$, and when $\mathrm{Im}(\beta)\lambda(z) \leqslant 0$, $q(\mathfrak{h}, \lambda) = 1$ and $\mathrm{Re}\,\alpha \leqslant 0$.

Now assume $n > 1$, and suppose that the theorem is known for algebras of smaller dimension. We choose $y \in \mathfrak{z}^{(2)}(\mathfrak{g}) \setminus \mathfrak{z}$, let $\mathfrak{g}' = \mathfrak{z}_{\mathfrak{g}}(y)$, $\mathfrak{n} = \mathfrak{z}_{\mathbf{C}} + \mathbf{C}y$, and consider the various cases of §1.

Case I. $\mathfrak{h} \subseteq \mathfrak{g}'_{\mathbf{C}}$ and $\mathfrak{n} \subseteq \mathfrak{h}$. The Hochschild-Serre spectral sequence for the ideal $\mathbf{C}y$ of $\mathfrak{h}$ gives

$$H^p((\mathfrak{h} \cap \ker \lambda)/\mathbf{C}y, H^q(\mathbf{C}y, (\pi_{-\lambda})_\infty)) \Rightarrow H^k(\mathfrak{h} \cap \ker \lambda, (\pi_{-\lambda})_\infty).$$

As a $\mathbf{C}y$-module, $(\pi_{-\lambda})_\infty$ is just $\mathcal{S}(\mathbf{R}) \hat{\otimes} \mathcal{S}(\mathbf{R}^{n-1})$, where $\mathbf{C}y$ acts by multiples of $x_1 \otimes 1$ (x_1 the coordinate in the first factor). So $H^0(\mathbf{C}y, (\pi_{-\lambda})_\infty) = 0$ and $H^1(\mathbf{C}y, (\pi_{-\lambda})_\infty) \cong \mathcal{S}(\mathbf{R}^{n-1})$. Thus the spectral sequence collapses and

$$H^k(\mathfrak{h} \cap \ker \lambda, (\pi_{-\lambda})_\infty) \cong H^{k-1}((\mathfrak{h} \cap \ker \lambda)/\mathbf{C}y, \mathcal{S}(\mathbf{R}^{n-1})).$$

Here we may identify $(\mathfrak{h} \cap \ker \lambda)/\mathbf{C}y$ with $\tilde{\mathfrak{h}} \cap \ker \nu$, where $\tilde{\mathfrak{h}}$ is (as in §1) a weak polarization for the functional ν of §1 on $\mathfrak{l} = \mathfrak{g}'/\mathbf{R}y$, and $\mathcal{S}(\mathbf{R}^{n-1})$ may be identified with $(\pi_{-\nu})_\infty$. By inductive hypothesis, $H^{k-1}(\tilde{\mathfrak{h}} \cap \ker \nu, (\pi_{-\nu})_\infty) = 0$ for $k - 1 \neq q(\tilde{\mathfrak{h}}, \nu)$, $\mathbf{C}$ for $k - 1 = q(\tilde{\mathfrak{h}}, \nu)$. But by Lemma 1.5 $q(\mathfrak{h}, \lambda) = q(\tilde{\mathfrak{h}}, \nu) + 1$. Thus $H^k(\mathfrak{h} \cap \ker \lambda, (\pi_{-\lambda})_\infty) = 0$ if $k \neq q(\mathfrak{h}, \lambda)$, $\mathbf{C}$ if $k = q(\mathfrak{h}, \lambda)$.

Case II. $\mathfrak{h} \not\subseteq \mathfrak{g}'_{\mathbf{C}}$ and $\mathfrak{h} \cap \mathfrak{g}'_{\mathbf{C}}$ is an ideal of codimension 1 in $\mathfrak{h}$.

Note that $\pi_{-\lambda}$ decomposes, when restricted to G', into a direct integral $\int^{\oplus}(\pi_{-\mu})\,d\mu$, where μ ranges over the affine space of elements of $(\mathbf{R}y + \mathfrak{z})^*$ restricting to λ on $\mathfrak{z}$, and where $\pi_{-\mu}$ denotes the square-integrable representation of G' whose central character has differential $-i\mu$. If μ_0 denotes the restriction of our original λ to $\mathfrak{n}$, then each μ is conjugate to μ_0 via an element s of G (not in G'). If we assume as we may that $\lambda(y) = 0$, then the kernel of $\pi_{-s\cdot\mu_0}$ has Lie algebra spanned by $(\mathrm{Ad}\, s)y$, and $\pi_{-s\cdot\mu_0}$ is lifted from a representation of the quotient of G'

[3]The reason for relating cohomology of $\pi_{-\lambda}$ to the negativity index of $(\mathfrak{h}, \lambda)$ (instead of relating cohomology of π_λ to the "positivity index" of $(\mathfrak{h}, \lambda)$) is that π_λ will appear in L^2-cohomology of the line bundle defined by λ exactly when $\pi_{-\lambda}$ has nontrivial Lie algebra cohomology.

by this group. Furthermore, the element s effects an isomorphism from L (the group with Lie algebra $\mathfrak{g}'/\mathbf{R}y$) to $G'/\ker \pi_{-s\cdot\mu_0}$ carrying $\pi_{-\mu_0}$ to $\pi_{-s\cdot\mu_0}$. Corresponding to the direct integral decomposition of $\pi_{-\lambda}$, we also have a "smooth direct integral" decomposition of $(\pi_{-\lambda})_\infty$ as $\mathcal{S}(\mathbf{R}^n) \cong \mathcal{S}(\mathbf{R}) \hat{\otimes} \mathcal{S}(\mathbf{R}^{n-1})$, where y acts only on the first factor and where the second factor in the tensor product may be identified with $(\pi_{-\mu_0})_\infty$.

By Lemma 1.6 we have three subcases, (a), (b), and (c), to consider. First assume we are in case (a) or case (b), and let t_0 be the basis element of $\mathfrak{h}$ not in $\mathfrak{g}'_{\mathbf{C}}$ constructed in the proof of Lemma 1.6. Note that $\mathfrak{m} = \mathfrak{g}'_{\mathbf{C}} \cap \ker \lambda \cap \mathfrak{h}$ is an ideal of codimension 1 in $\mathfrak{h} \cap \ker \lambda$. In case (a), we may assume $t_0 = w$ is real, i.e., belongs to $\mathfrak{g}$ and not just to $\mathfrak{g}_{\mathbf{C}}$. In case (b), we may assume $t_0 = w + iy$, where $w \in \mathfrak{g} \setminus \mathfrak{g}'$. (Indeed, we may clearly choose t_0 so that $\lambda([t_0, y]) = 1$. Let w and v be the real and imaginary parts of t_0, so that $B_\lambda(w, y) = 1$ and $B_\lambda(v, y) = 0$, i.e., $v \in \mathfrak{g}'$. Depending on the sign of $H_\lambda(t_0, t_0)$, we may assume that $B_\lambda(w, v)$ is either -1 or 1. Now $B_\lambda(\bar{t}_0, \mathfrak{h} \cap \mathfrak{g}'_{\mathbf{C}}) = 0$ since t_0 is orthogonal to $\mathfrak{h} \cap \mathfrak{g}'_{\mathbf{C}}$ with respect to H_λ, and $B_\lambda(t_0, \mathfrak{h} \cap \mathfrak{g}'_{\mathbf{C}}) = 0$ since $t_0 \in \mathfrak{h}$ and $\mathfrak{h}$ is B_λ-totally isotropic. So both w and v are B_λ-orthogonal to $\mathfrak{h} \cap \mathfrak{g}'_{\mathbf{C}}$. Now $w - iy$ (if $B_\lambda(w, v) = -1$) or $w + iy$ (if $B_\lambda(w, v) = 1$) is B_λ-orthogonal to both t_0 and to $\mathfrak{h} \cap \mathfrak{g}'_{\mathbf{C}}$, hence lies in $\mathfrak{h}$ since $\mathfrak{h}$ is maximal B_λ-totally isotropic.) In either case, the one-parameter group generated by w will conjugate $\pi_{-\mu_0}$ to the other $\pi_{-\mu}$'s and will map $\mathfrak{m}$ into itself. The action of $\mathfrak{m}$ in the various $\pi_{-\mu}$'s can thus be viewed (via this one-parameter group) as coming from the action of $\tilde{\mathfrak{h}}$ on $(\pi_{-\mu_0})_\infty$, so that $(\pi_{-\lambda})_\infty$, as an $\mathfrak{m}$-module, may be identified with a continuous direct integral of conjugates of $(\pi_{-\mu_0})_\infty$, as in 2.3. Now $\pi_{-\mu_0}$, viewed as a representation of L, is associated with the linear functional ν (the image of μ_0 on $\mathfrak{l}$), and so $H^q(\mathfrak{m}, (\pi_{-\mu_0})_\infty) \cong H^q(\tilde{\mathfrak{h}} \cap \ker \nu, (\pi_{-\nu})_\infty)$, which is given by the induction hypothesis to be one-dimensional for $q = q(\tilde{\mathfrak{h}}, \nu)$, zero otherwise. Hence by Lemma 2.3,

$$H^q(\mathfrak{m}, (\pi_{-\lambda})_\infty) \cong \mathcal{S}(\mathbf{R}) \otimes H^q(\tilde{\mathfrak{h}} \cap \ker \nu, (\pi_{-\nu})_\infty) = \begin{cases} 0 & \text{if } q \neq q(\tilde{\mathfrak{h}}, \nu), \\ \mathcal{S}(\mathbf{R}) & \text{if } q = q(\tilde{\mathfrak{h}}, \nu). \end{cases}$$

Furthermore, the action of $(\mathfrak{h} \cap \ker \lambda)/\mathfrak{m}$ on $H^{q(\tilde{\mathfrak{h}},\nu)}(\mathfrak{m}, (\pi_{-\lambda})_\infty)$ will be given by multiples of a first-order differential operator acting on $\mathcal{S}(\mathbf{R})$, namely, the image of t_0. This operator on $\mathcal{S}(\mathbf{R})$ will be of the form considered in Lemma 2.1, with the sign of the real part of α depending on the sign of $H_\lambda(t_0, t_0)$. Thus

$$H^p(\mathbf{C}t_0, \mathcal{S}(\mathbf{R})) = \begin{cases} 0 & \text{if } p = 0 \text{ and } H_\lambda(t_0, t_0) < 0 \\ & \text{or } p = 1 \text{ and } H_\lambda(t_0, t_0) > 0, \\ \mathbf{C} & \text{if } p = 0 \text{ and } H_\lambda(t_0, t_0) > 0 \\ & \text{or } p = 1 \text{ and } H_\lambda(t_0, t_0) < 0. \end{cases}$$

Since we have a spectral sequence

$$H^p((\mathfrak{h} \cap \ker \lambda)/\mathfrak{m}, H^q(\mathfrak{m}, (\pi_{-\lambda})_\infty)) \Rightarrow H^k(\mathfrak{h} \cap \ker \lambda, (\pi_{-\lambda})_\infty)$$

with only one nonzero $E_2^{p,q}$ term, we finally conclude that $H^k(\mathfrak{h} \cap \ker \lambda, (\pi_{-\lambda})_\infty)$ is nonzero for only one value of k, and for this value of k, it is one-dimensional. The degree of the nontrivial cohomology will be

$$\begin{cases} q(\tilde{\mathfrak{h}}, \nu) & \text{if } H_\lambda(t_0, t_0) > 0, \\ q(\tilde{\mathfrak{h}}, \nu) + 1 & \text{if } H_\lambda(t_0, t_0) < 0. \end{cases}$$

But by Lemma 1.6(a)(b), this is just $q(\mathfrak{h}, \lambda)$.

We are still left with Case II(c), in which $y \in (\mathfrak{h} \cap \mathfrak{g}'_{\mathbf{C}}) + (\bar{\mathfrak{h}} \cap \mathfrak{g}'_{\mathbf{C}})$, hence $\mathfrak{h} \cap \mathfrak{g}'_{\mathbf{C}}$ contains an element of the form $y + iw$, $w \in \mathfrak{g}'$ (this is the element t_1 in the proof of Lemma 1.6). Things are slightly more complicated in this case because we will have $\mu(y + iw) \neq 0$ for $\mu \neq \mu_0$; hence $(\pi_{-\mu})_\infty$ and $(\pi_{-\mu_0})_\infty$ are not conjugate as $\mathfrak{m}$-modules and we cannot identify $H^q(\mathfrak{m}, (\pi_{-\lambda})_\infty)$ with $\mathcal{S}(\mathbf{R}) \otimes H^q(\mathfrak{m}, (\pi_{-\mu_0})_\infty)$.

In the simplest examples of this subcase, $\mathfrak{m}$ splits as a Lie algebra semidirect sum $\mathfrak{m} = \mathbf{C}(y + iw) + \mathfrak{k}$, where $\mathfrak{k}$ is some ideal in $\mathfrak{m}$ of codimension one. Under these circumstances, one can show using the induction hypothesis, Lemma 1.6(c), and Lemma 2.3 that

$$H^q(\mathfrak{k}, (\pi_{-\lambda})_\infty) = \begin{cases} 0 & \text{if } q \neq q(\mathfrak{h}, \lambda) - 1, \\ \mathcal{S}(\mathbf{R}) \hat{\otimes} \mathcal{S}(\mathbf{R}) = \mathcal{S}(\mathbf{R}^2) & \text{if } q = q(\mathfrak{h}, \lambda) - 1, \end{cases}$$

where $\mathbf{C}w$ acts by multiples of $x_1 \otimes 1$ and $\mathbf{C}y$ acts by multiples of $1 \otimes x_2$. The cohomology of $\mathfrak{h} \cap \ker \lambda$ can then be computed by first using the spectral sequence for the ideal $\mathfrak{k}$ of $\mathfrak{m}$ to compute the $\mathfrak{m}$-cohomology from the $\mathfrak{k}$-cohomology, then using the spectral sequence with $E_2^{p,q}$ terms

$$H^p((\mathfrak{h} \cap \ker \lambda)/\mathfrak{m}, H^q(\mathfrak{m}, (\pi_{-\lambda})_\infty)).$$

Because we can choose an element of $\mathfrak{h} \cap \ker \lambda$, not in $\mathfrak{m}$, that acts essentially by $(\partial/\partial x_1 + i\partial/\partial x_2)/2 + P$, P a polynomial in x_1 and x_2, the situation reduces to that of Lemma 2.2. This lemma then completes the proof.

However, as Richard Penney has kindly pointed out to the author (correcting an error in a preliminary version of this paper), $y + iw$ may instead lie in the derived algebra of $\mathfrak{m}$, in which case we cannot find an ideal of $\mathfrak{m}$ supplementary to $\mathbf{C}(y + iw)$. This necessitates a somewhat more elaborate calculation, although the ultimate result is the same. What follows is essentially Penney's argument, somewhat recast in the present language of spectral sequences.

As mentioned above, we may realize $(\pi_{-\lambda})_\infty$ as $\mathcal{S}(\mathbf{R}) \otimes (\pi_{-\mu_0})_\infty$ so that y acts by multiplication by $ix_1 \otimes 1$ (x_1 the coordinate on the first factor). If t_0 is as in the proof of Lemma 1.6(c), then t_0 may be assumed to act by $\partial/\partial x_1 \otimes 1$ + (terms commuting with $x_1 \otimes 1$). Let W be the subspace of $(\pi_{-\lambda})_\infty$ consisting of elements which, when viewed as Schwartz functions $f\colon \mathbf{R} \to (\pi_{-\mu_0})_\infty$, satisfy

$$(t_0)^m \cdot f|_{x_1 = 0} = 0 \quad \text{for all } m > 0,$$

or in other words, which vanish to infinite order at $x_1 = 0$ (relative to the differential operator given by t_0). Since t_0 normalizes $\mathfrak{m}$, W is an $(\mathfrak{h} \cap \ker \lambda)$-submodule of $(\pi_{-\lambda})_\infty$.

LEMMA 2.5 (PENNEY).[4] *The element* $y + iw$ *of* $\mathfrak{m}$ *acts bijectively on* W. *The natural map* $f \mapsto (f|_{x_1=0}, t_0 \cdot f|_{x_1=0}, \ldots)$ *identifies* $(\pi_{-\lambda})_\infty/W$ *with the* $\mathfrak{h}$-*module of formal power series* $(\pi_{-\mu_0})_\infty[[x]]$ (*on which* t_0 *acts by* $\partial/\partial x$).

We do not include the proof of this lemma here, since it is somewhat long and computational. It should be remarked, however, that it is clear that $y + iw$ acts bijectively on the submodule of M consisting of functions vanishing in a neighborhood of $x_1 = 0$. (Away from $x_1 = 0$, $y + iw$ has spectrum bounded away from the real axis, since w is skew-adjoint and $[y, w] = 0$.) The difficulty is thus to check that $y + iw$ acts similarly on functions only vanishing to infinite order at 0.

Next we need an algebraic lemma, also present in Penney's paper, but in a rather disguised form.

LEMMA 2.6. *Let* $\mathfrak{a}$ *be a finite-dimensional Lie algebra* (*say over* $\mathbf{C}$), *let* $\mathfrak{k}$ *be the one-dimensional subalgebra of* $\mathfrak{a}$ *spanned by a nilpotent element* v, *and let* M *be a* $\mathfrak{a}$-*module. The* E_1 *terms of the Hochschild-Serre spectral sequence for* $H^*(\mathfrak{a}, M)$ *associated to the subalgebra* $\mathfrak{k}$ *of* $\mathfrak{a}$ *are*

$$E_1^{p,q} = H^q(\mathfrak{k}, \operatorname{Hom}(\Lambda^p(\mathfrak{a}/\mathfrak{k}), M)),$$

nonzero for at most $q = 0$ *and* $q = 1$.

(a) *If* v *acts injectively on* M, *then* $E_1^{p,0} = 0$ *for all* p. *If* v *acts surjectively on* M, *then* $E_1^{p,1} = 0$ *for all* p. *Thus, if* v *acts bijectively on* M, $H^*(\mathfrak{a}, M) = 0$.

(b) *Suppose furthermore that* $\mathfrak{k}$ *is complemented by an ideal* $\mathfrak{m}$ *of* $\mathfrak{a}$, *that* v *acts surjectively on* M, *and that there is an exact sequence of* $\mathfrak{m}$-*modules* $M \xrightarrow{\phi} N \to 0$ *with* $M = M^{\mathfrak{k}} \oplus (\ker \phi)$ (*as vector spaces–in general,* $M^{\mathfrak{k}}$ *is not a submodule*). *Then* ϕ *induces isomorphisms*

$$\begin{aligned} H^p(\mathfrak{a}, M) &\cong H^p(\mathfrak{a}, \mathfrak{k}; M) \ (\textit{relative Lie algebra cohomology}) \\ &\xrightarrow{\cong} H^p(\mathfrak{m}, N). \end{aligned}$$

PROOF. (a) If $f \in \operatorname{Hom}(\Lambda^j(\mathfrak{a}/\mathfrak{k}), M)^{\mathfrak{k}}$, then $v \cdot (f(x)) = f(v \cdot x)$ for $x \in \Lambda^j(\mathfrak{a}/\mathfrak{k})$; hence for large n, $v^n \cdot (f(x)) = f(v^n \cdot x) = 0$ and $f = 0$ provided v acts injectively on M. Similarly, if v acts surjectively on M and $g \in \operatorname{Hom}(\Lambda^j(\mathfrak{a}/\mathfrak{k}), M)$, we can choose h_0 such that $v \cdot (h_0(x)) = g(x)$ for all x. Then $(g - v \cdot h_0)(x) = h_0(v \cdot x)$, and we can choose $\tilde{h}_1$ with $v \cdot (\tilde{h}_1(x)) = h_0(x)$, then set $h_1(x) = \tilde{h}_1(v \cdot x)$. This gives $(g - v \cdot h_0 - v \cdot h_1)(x) = \tilde{h}_1(v^2 \cdot x)$, etc. Since v acts nilpotently on $\Lambda^j(\mathfrak{a}/\mathfrak{k})$, we see that $\mathfrak{k}$ acts surjectively on $\operatorname{Hom}(\Lambda^j(\mathfrak{a}/\mathfrak{k}), M)$ and $E_1^{j,1} = 0$.

(b) As for the second part of the lemma, recall that $H^p(\mathfrak{a}, \mathfrak{k}; M) = E_2^{p,0}$ is the cohomology of the complex $\operatorname{Hom}(\Lambda^*(\mathfrak{a}/\mathfrak{k}), M)^{\mathfrak{k}}$. The fact that this coincides with the cohomology of $\mathfrak{a}$ follows from the collapsing of the spectral sequence in (a) above. Since $\mathfrak{a} = \mathfrak{k} + \mathfrak{m}$, any cochain determines (by composition with ϕ) an element of $\operatorname{Hom}(\Lambda^*\mathfrak{m}, N)$. Furthermore, the mapping ϕ_*: $\operatorname{Hom}(\Lambda^*(\mathfrak{a}/\mathfrak{k}), M)^{\mathfrak{k}} \to \operatorname{Hom}(\Lambda^*\mathfrak{m}, N)$ obtained in this way is a bijection because of the nilpotence of the action of v on $\mathfrak{m}$. (If f is a cochain sent to 0, this means first that f vanishes on

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[4] See R. Penney, *Lie cohomology of representations of nilpotent Lie groups* ... in this issue of these TRANSACTIONS, Lemmas 7 and 8.

$(\Lambda^*\mathfrak{m})^{\mathfrak{r}}$, then on $(\Lambda^*\mathfrak{m})^{v^2}$, etc., hence $f = 0$. Surjectivity follows from surjectivity of ϕ and of the action of v.) One checks readily that ϕ_* commutes with the differentials of the two cochain complexes, hence defines isomorphisms of cohomology groups.

PROOF OF THEOREM (CONTINUED). With the aid of the lemmas, we rapidly conclude the proof of 2.4. First note that by the first statement of 2.5 and by the last assertion of Lemma 2.6(a) (with $v = y + iw$), $H^k(\mathfrak{h} \cap \ker \lambda, W) = 0$ for all k. Thus if $M = (\pi_{-\mu_0})[[x]]$, the second statement of 2.5 and the long exact cohomology associated to $0 \to W \to (\pi_{-\lambda})_\infty \to (\pi_{-\lambda})_\infty / W \to 0$ imply that $H^k(\mathfrak{h} \cap \ker \lambda, (\pi_{-\lambda})_\infty) \cong H^k(\mathfrak{h} \cap \ker \lambda, M)$ for all k. Now apply (b) of Lemma 2.6 with $v = t_0$, $N = (\pi_{-\mu_0})_\infty$. All of the hypotheses are satisfied since $\partial/\partial x$ acts surjectively on power series in x, with kernel the "constant functions." Thus by the lemma, $H^k(\mathfrak{h} \cap \ker \lambda, M) \cong H^k(\mathfrak{m}, (\pi_{-\mu_0})_\infty)$ for all k. By Lemma 1.6(c), $q(\tilde{\mathfrak{h}}, \nu) = q(\mathfrak{h}, \lambda)$; thus by inductive hypothesis, $H^k(\mathfrak{m}, (\pi_{-\mu_0})_\infty)$ vanishes except when $k = q(\mathfrak{h}, \lambda)$ and is one-dimensional in this case. Since we have seen that $H^k(\mathfrak{h} \cap \ker \lambda, (\pi_{-\lambda})_\infty) \cong H^k(\mathfrak{m}, (\pi_{-\mu_0})_\infty)$ for all k, we are done.

3. Applications of Lie algebra cohomology to the study of harmonically induced representations of nilpotent groups. The results of §2 easily imply an analogue of the "Langlands conjecture" for nilpotent Lie groups, via an argument that has more or less become standard since it was introduced by Schmid **[42]**, **[43]** in the semisimple case. However, the technical details in the nilpotent case are somewhat different **[31]**, **[33]**, because K-finiteness is no longer a useful condition and because there is no Casimir element in the enveloping algebra to work with.

As formulated by Moscovici and Verona **[31]**, the basic problem is the following. Suppose G is a connected, simply connected nilpotent Lie group (for the moment not assumed to have one-dimensional center or square-integrable representations). By the basic result of Kirillov theory **[21]**, there is a natural bijection from the coadjoint orbit space $\mathfrak{g}^*/G$ to $\hat{G}$. Given $\lambda \in \mathfrak{g}^*$, one can realize the irreducible representation π_λ associated with the G-orbit of λ by choosing a totally real polarization $\mathfrak{h}$ for λ and inducing up to G the character χ_λ of the corresponding subgroup H of G, with differential $i\lambda$. This is fine for treating nilpotent groups by themselves, but if G occurs as a normal subgroup of a larger Lie group S, it is desirable to choose $\mathfrak{h}$ to be invariant under the stabilizer of λ in S, which is usually impossible if we require $\mathfrak{h}$ to be totally real, but often possible if we allow $\mathfrak{h}$ to be complex **[5]**, **[13]**. When $\mathfrak{h}$ is positive, "holomorphic induction" using $\mathfrak{h}$ still produces an irreducible representation of G unitarily equivalent to the one gotten from a totally real polarization **[5**, Lemma III.1.1**]**. However, it is reasonable to ask what happens when $\mathfrak{h}$ is not positive, in which case the Hilbert space of the holomorphically induced representation degenerates to $\{0\}$. Bott's refinement **[6]** of the Borel-Weil theorem suggests that when $\mathfrak{h}$ is not positive, one should still be able to produce from $\mathfrak{h}$ an irreducible representation, but only on a higher cohomology space of the associated line bundle, not on partially holomorphic sections.

One is therefore naturally led to study the L^2-cohomology of the "partially holomorphic" line bundle defined by λ and $\mathfrak{h}$. As pointed out by Moscovici and

Verona [31, p. 63], this only makes sense when E/D carries an E-invariant hermitian structure, or when $\mathfrak{h}$ is *relatively ideal*, i.e., $\mathfrak{h} \cap \bar{\mathfrak{h}}$ is an ideal of $\mathfrak{h}$. (Here E and D are the groups with Lie algebras $\mathfrak{e} = (\mathfrak{h} + \bar{\mathfrak{h}}) \cap \mathfrak{g}$, and $\mathfrak{d} = \mathfrak{h} \cap \mathfrak{g}$, respectively. By [5, Theorem I.4.10], every positive polarization for a nilpotent group is relatively ideal.) In this case, if one divides out by $\mathfrak{d} \cap \ker \lambda$ (which is also easily seen to be the Lie algebra of the kernel of the Kirillov representation of E associated with $\lambda|_{\mathfrak{e}}$), one may as well assume $\mathfrak{d}$ is one-dimensional, is the center of $\mathfrak{e}$, and is the Lie algebra of the stabilizer of $\lambda|_{\mathfrak{e}}$ in E. In particular, E has square-integrable representations and satisfies all the hypotheses of §§1 and 2. Then $\mathfrak{h}$ defines an invariant complex structure on E/D, χ_λ defines a homogeneous holomorphic line bundle $\mathfrak{L}_\lambda$ on E/D, and one can define the L^2-harmonic spaces (or L^2-cohomology spaces) of $\mathfrak{L}_\lambda$ [31, §2]. By inducing these representations of E up to G, one gets unitary representations of G which will be degenerate or irreducible exactly when the harmonic representation of E is so. Thus to study harmonically induced representations of G it is enough to replace G by E and assume G has one-dimensional center and square-integrable representations.

Another motivation for the study just of groups with square-integrable representations comes from Penney's notion of "canonical objects" in the Kirillov theory [32]. Given λ as above, one can associate to λ certain Lie subalgebras $\mathfrak{h}_\infty(\lambda)$ and $\mathfrak{k}_\infty(\lambda)$ of $\mathfrak{g}$, with the properties that $\mathfrak{h}_\infty(\lambda)$ is an ideal in $\mathfrak{k}_\infty(\lambda)$ and that each algebra is the orthogonal complement of the other for the bilinear form associated with λ. In particular, $\mathfrak{h}_\infty(\lambda)$ contains the Lie algebra $\mathfrak{g}_\lambda$ of the stabilizer of λ in G and is subordinate to λ. There exist polarizations $\mathfrak{h}$ for λ with $\mathfrak{h}_\infty(\lambda)_{\mathbf{C}} \subseteq \mathfrak{h} \subseteq \mathfrak{k}_\infty(\lambda)_{\mathbf{C}}$; in fact, very often (but not always) all polarizations have this property. Thus (by induction in stages) the Kirillov representation of G associated with λ can be constructed by first constructing the Kirillov representation of $K_\infty(\lambda)$ associated with the restriction of λ and then by inducing. However, the representation of $K_\infty(\lambda)$ defined by λ is square-integrable modulo $H_\infty(\lambda)$, so that if we divide out by $\mathfrak{h}_\infty(\lambda) \cap \ker \lambda$, consider a polarization $\mathfrak{h}$ contained in $\mathfrak{k}_\infty(\lambda)_{\mathbf{C}}$, and replace $\mathfrak{g}$ by $\mathfrak{k}_\infty(\lambda)$, we are back in the situation of §§1 and 2.

We are now ready for our main result on nilpotent groups, which was conjectured by Moscovici and Verona in [31].

THEOREM 3.1. *Let G be a connected, simply connected Lie group with one-dimensional center Z and square-integrable representations modulo Z. Let $\lambda \in \mathfrak{g}^*$ and assume $\lambda|_{\mathfrak{z}}$ is nontrivial (so π_λ is irreducible). Let $\mathfrak{h}$ be a totally complex polarization for λ, so that the line bundle $\mathfrak{L}_\lambda$ over G/Z is holomorphic with respect to the associated complex structure. Let $\mathcal{H}^q(\mathfrak{L}_\lambda, \mathfrak{h})$ denote the qth L^2-cohomology space of $\mathfrak{L}_\lambda$, i.e., the Hilbert space of L^2 $\mathfrak{L}_\lambda$-valued harmonic* $(0, q)$*-forms. Then $\mathcal{H}^q(\mathfrak{L}_\lambda, \mathfrak{h})$ vanishes for $q \neq q(\mathfrak{h}, \lambda)$ (as defined in* 1.2) *and G acts irreducibly, via a representation equivalent to π_λ, on $\mathcal{H}^{q(\mathfrak{h},\lambda)}(\mathfrak{L}_\lambda, \mathfrak{h})$.*

PROOF. Since $\mathfrak{z}$ acts by $i\lambda$ on all spaces of $\mathfrak{L}_\lambda$-valued forms, $\mathcal{H}^q(\mathfrak{L}_\lambda, \mathfrak{h})$ will be a multiple of π_λ for each q. By [31, Lemma 4] or [33, Theorem 10], the multiplicity of π_λ in $\mathcal{H}^q(\mathfrak{L}_\lambda, \mathfrak{h})$ may be identified with the dimension of the "formal harmonic space"

$$\mathcal{H}^q(\pi_{-\lambda}, \mathfrak{h}) = \{\Phi \in \mathcal{H}_{-\lambda} \otimes \Lambda^q(\mathfrak{h} \cap \ker \lambda)^*: \Phi \perp \mathrm{range}(\delta + \delta^*)\},$$

where $\mathcal{H}_{-\lambda}$ is the Hilbert space of $\pi_{-\lambda}$, δ is the "formal coboundary operator" for $(\mathfrak{h} \cap \ker \lambda)$-cohomology, and δ^* is the formal adjoint of δ. But as a consequence of the ellipticity of the $\bar{\partial}$-Laplacian $\square = (\delta + \delta^*)^2 = \delta\delta^* + \delta^*\delta$, the formal harmonic space (which may be identified with the kernel of the closure of $\square$) is actually contained in $(\pi_{-\lambda})_\infty \otimes \Lambda^q(\mathfrak{h} \cap \ker \lambda)^*$ [33, Proposition 3] and so defines a subspace of the (algebraic!) Lie algebra cohomology space $H^q(\mathfrak{h} \cap \ker \lambda, (\pi_{-\lambda})_\infty)$. In fact, when the coboundary map for the standard complex defining $H^q(\mathfrak{h} \cap \ker \lambda, (\pi_{-\lambda})_\infty)$ has closed image in the Fréchet space of q-cochains, the formal harmonic space and Lie algebra cohomology space coincide [33, Theorem 2]. By Theorem 2.4, the Lie algebra cohomology space vanishes except when $q = q(\mathfrak{h}, \lambda)$, when it is one-dimensional. Thus $\mathcal{H}^q(\pi_{-\lambda}, \mathfrak{h}) = 0$ except when $q = q(\mathfrak{h}, \lambda)$, and in this case, the coboundary map has a range which is of finite codimension in the space of cocycles, hence is closed by the closed graph theorem. So this proves the result, along with a "Hodge theorem" for the Lie algebra cohomology (every cohomology class has a unique representative in the formal harmonic space).

COROLLARY 3.2. *Let G be any connected, simply connected nilpotent Lie group, let $\lambda \in \mathfrak{g}^*$, and let $\mathfrak{h}$ be a relatively ideal polarization for λ. Let $\pi^q(\lambda, \mathfrak{h}, G)$ be the harmonically induced representation associated to $\mathfrak{h}$ and λ, in the sense of* [31]. *Then $\pi^q(\lambda, \mathfrak{h}, G)$ vanishes for $q \neq q(\mathfrak{h}, \lambda)$, and is irreducible and equivalent to π_λ when $q = q(\mathfrak{h}, \lambda)$.*

PROOF. As remarked above, this reduces immediately to the case of square-integrable representations.

REMARK 3.3. Theorem 3.1 gives an interesting example of the index theorem of Connes and Moscovici [10]. In fact, let $E = \sum_{q \text{ even}} \Lambda^q(\mathfrak{h} \cap \ker \lambda)^*$, $F = \sum_{q \text{ odd}} \Lambda^q(\mathfrak{h} \cap \ker \lambda)^*$, and let $\mathcal{E}$, $\mathcal{F}$ be the induced vector bundles over G/Z (when we view E and F as trivial Z-modules). Then $\bar{\partial} + \bar{\partial}^*$ is a G-invariant elliptic operator from sections of $\mathcal{L}_\lambda \otimes \mathcal{E}$ to sections of $\mathcal{L}_\lambda \otimes \mathcal{F}$; hence by the index theorem, the L^2 kernel and cokernel of $\bar{\partial}$ have finite formal degree, i.e., are finite multiples of π_λ. However the kernel and cokernel of $\bar{\partial} + \bar{\partial}^*$ are just the sums of the even and the odd L^2-cohomology spaces, respectively. Thus the formal harmonic spaces $\mathcal{H}^q(\pi_{-\lambda}, \mathfrak{h})$ of the proof of 3.1 are all finite-dimensional. Furthermore, the index theorem gives

$$\sum_q (-1)^q (\text{mult. of } \pi_\lambda \text{ in } \mathcal{H}^q(\mathcal{L}_\lambda, \mathfrak{h})) = \pm 1$$

(the term on the right is evaluated exactly as in [31, p. 70]). This is of course what one gets from 3.1; conversely, we could prove 3.1 from the index theorem if we had a sufficiently powerful vanishing theorem for L^2-cohomology. ([29] applies in the case of "sufficiently generic" λ.) The reader will of course note the extreme similarity with the arguments of [4, §3].

3.4. Since Theorem 2.4 was proved for arbitrary polarizations, not just for totally complex or even relatively ideal ones, it is of interest to see if Theorem 3.1 has an analogue for polarizations that are not totally complex. To fix ideas, suppose G and λ are as in 3.1 but $\mathfrak{h}$ is totally real. Let $\mathfrak{m} = \mathfrak{h} \cap \ker \lambda \cap \mathfrak{g}$ and let M be the corresponding subgroup of G. By analogy with the proof of 3.1, we expect the Lie algebra cohomology of $\mathfrak{h} \cap \ker \lambda$ (which we computed) to correspond to the formal harmonic spaces

$$\mathcal{H}^q(\pi_{-\lambda}, \mathfrak{h}) = \{\Phi \in \mathcal{H}_{-\lambda} \otimes \Lambda^q \mathfrak{m}^*: \Phi \perp \operatorname{im}(d + d^*)\},$$

where d now corresponds to exterior differentiation in the direction of $\mathfrak{m}$. In other words, we are led to consider $\mathcal{H}^q(\mathcal{L}_\lambda, \mathfrak{h}) = \{L^2\ \mathcal{L}_\lambda$-valued q-forms on G/Z, harmonic in the direction of $\mathfrak{m}\}$. Unfortunately, because of the global L^2 condition (with respect to the invariant measure on G/Z, not on G/MZ), these spaces vanish for all q. In fact, this is not surprising since when $\mathfrak{h}$ is not totally complex, the "Laplacian" of the $\bar{\partial}$ complex is not elliptic, only partially elliptic in the direction of E. However, in the totally real case, Theorem 2.4 says that the space spanned by the ranges of the operators $\pi_{-\lambda}(x)$, $x \in \mathfrak{m}$, has codimension 1 in $(\pi_{-\lambda})_\infty$ (and as mentioned above, is necessarily closed by the closed graph theorem). Hence, up to scalars, there is exactly one element of $(\pi_{-\lambda})_{-\infty}$ (the dual of $(\pi_{-\lambda})_\infty$) annihilating this space. This statement is exactly Corollary 3.4.3 of **[20]**. The general case of Theorem 2.4 may be viewed as a generalization of Howe's observation.

4. A large class of Lie groups with square-integrable representations. In this section, we combine the above results on nilpotent groups with W. Schmid's proof of the "Langlands conjecture" for semisimple groups **[42]**, **[43]** to obtain geometric realizations of the square-integrable representations of a large class of Lie groups in terms of the Kirillov-Kostant "orbit picture". Throughout this section we make the following standing assumption:

Hypothesis 4.1. G is a connected Lie group with a closed connected normal nilpotent subgroup N such that

(i) G/N is reductive,

(ii) the center Z of N is central in G, and

(iii) N has square-integrable representations modulo Z.

PROPOSITION 4.2. *If G satisfies* 4.1, *then G is unimodular.*

PROOF. Since N and G/N are unimodular, it is enough to show that the conjugation action of G on N preserves Haar measure. Since Z is central in G, this action fixes Z pointwise, hence will be unimodular if and only if the action of G on $\mathfrak{n}/\mathfrak{z}$ (the Lie algebra of N/Z) is unimodular. Let $\lambda \in \mathfrak{n}^*$ have nonzero Pfaffian (see **[28**, §3] for the definition)–the set of such λ's is Zariski dense in $\mathfrak{n}^*$ by assumption (iii). Then λ defines a nondegenerate alternating bilinear form B_λ on $\mathfrak{n}/\mathfrak{z}$. If $g \in G$, then $g \cdot \lambda$ has the same restriction to $\mathfrak{z}$ as λ (since $\mathfrak{z}$ is central), hence lies in $N \cdot \lambda$ by **[28**, Theorem 1]. Therefore, changing g by an element of N if necessary, we may assume g stabilizes λ. Then g acts on $\mathfrak{n}/\mathfrak{z}$ by a symplectic transformation for B_λ, hence acts unimodularly.

The importance of groups satisfying 4.1 is indicated by several results of N. Anh. By [**1**, Theorem 4.4], any connected Lie group with nilpotent radical must satisfy 4.1 (with $N = \mathrm{rad}(G)$) if it has square-integrable representations modulo its center. (The restrictions on the center of G/N are easy to remove using [**46**, §3].) On the other hand, a group satisfying 4.1 with $S = G/N$ semisimple will have square-integrable representations modulo its center if and only if S has square-integrable representations, i.e., $\mathrm{rk}\, S = \mathrm{rk}\, K$, where K is the inverse image in S of a maximal compact subgroup of the adjoint group of S. Furthermore, according to [**2**], the class of all connected unimodular Lie groups having square-integrable irreducible representations modulo their centers is not much more general than the set of those satisfying 4.1. (Essentially, to get the general case one need only replace the nilpotent group N by a solvable "H-group". On the other hand, many (perhaps all?) H-groups, for instance [**38**, Example 4.13], satisfy 4.1 with respect to some normal nilpotent subgroup, although not necessarily with respect to the nilradical.)

It is convenient to have a characterization of those groups, among those satisfying 4.1, which have square-integrable representations modulo their centers, in terms of their coadjoint orbit spaces. The following theorem does this and more, in that it provides a parametrization of the square-integrable representations whenever they exist. The result is not really new; it is basically an explication of [**2**, Théorème 1, Corollaire à Théorème 5] in our situation, but we include a detailed proof for later reference.

THEOREM 4.3. *Let G be a Lie group with center $Z(G)$ satisfying* 4.1 *with respect to some simply connected normal nilpotent subgroup N. Then the regular representation of G is type* I. *Furthermore, G has square-integrable representations modulo $Z(G)$ if and only if there exist G-orbits O in $\mathfrak{g}^*$ for which*

(i) *for any $\mu \in O$, $G_\mu/Z(G)$ is compact (here G_μ = stabilizer of μ in G), and*

(ii) *O is "integral", i.e., for any $\mu \in O$, there is a character χ_μ of G_μ with differential $i\mu|_{\mathfrak{g}_\mu}$.*

Finally, if G has square-integrable representations, then the set $\hat{G}_d$ of equivalence classes of such is in natural one-to-one correspondence with the set of regular (cf. [**12**, §1.11]) *orbits satisfying* (i) *and* (ii)′ (*"admissibility"–see below*).

PROOF. By [**28**, Theorem 6], the regular representation of N is a direct integral of square-integrable representations π_λ, each of which extends to a projective representation $\tilde{\pi}_\lambda$ of G by the proof of 4.2. Also, by the proof of 4.2, $G = N \cdot G_\lambda$ (G_λ = stabilizer in G of an element $\lambda \in \mathfrak{n}^*$ in the Kirillov orbit associated with π_λ), and $N \cap G_\lambda = Z$. Let $\alpha_\lambda \in H^2(G/N, \mathbf{T})$ be the Mackey obstruction to extending π_λ to G. Then α_λ^{-1} determines a central extension of G/N by $\mathbf{T}$, whose universal covering will be of the form (semisimple) $\times$ (vector group) $\times$ (Heisenberg), hence type I. (This is because G/N is reductive; hence its universal covering is of the form $S \times$ (vector group) with S semisimple, and α_λ^{-1} must be trivial on S.) This proves the first statement.

As far as the second statement is concerned, we see by a trivial case of [**23**, Theorem 2.3] that G will have a square-integrable representation with central character $e^{i\lambda}$ on Z if and only if G/N has a square-integrable α_λ^{-1}-representation. If we assume (as we may without loss of generality) that G/N is simply connected, we may write the central extension of G/N by $\mathbf{T}$ defined by α_λ^{-1} as $S \times V_\lambda \times H_\lambda$, where S is semisimple, V_λ is a vector group, and H_λ is either $\mathbf{T}$ itself or else a Heisenberg group with compact center. Since H_λ always has square-integrable representations, we conclude that G has a square-integrable representation with central character $e^{i\lambda}$ if and only if the following two conditions are satisfied:

(4.4)(iii) S (which is isomorphic with $G/\mathrm{rad}(G)$) satisfies the Harish-Chandra condition $\mathrm{rk}\, K = \mathrm{rk}\, S$, where K is the inverse image in S of a maximal compact subgroup of the adjoint group of S, and

(iv) the image of V_λ in G is compact modulo $Z(G)$.

Note that if (ii) is satisfied, then the cocycle defining the group extension

$$1 \to N \to G \to (G/N) \cong S \times V_\lambda \times (H_\lambda/\mathbf{T}) \to 1$$

must be trivial on V_λ. Since it is always trivial on S, we see that G is an extension

$$1 \to (S \times V) \cdot N \to G \to W \to 1, \tag{4.5}$$

where $(S \times V) \cdot N$ is a semidirect product with $\mathrm{Ad}_{\mathfrak{n}}(V)$ a torus, W is a vector group of even dimension, and α_λ^{-1} is totally skew on W. (The last condition will be true for almost all λ if it is true for one λ.)

Now let us rephrase (4.4) in terms of coadjoint orbits. Suppose (4.4) holds with respect to some $\lambda \in \mathfrak{n}^*$ with nonzero Pfaffian and (4.5) is the corresponding decomposition of G. Without loss of generality, we may suppose $S \times V \subseteq G_\lambda$. First choose $\alpha \in \mathfrak{s}^*$ with stabilizer a Cartan subgroup T of S contained in K. (This is possible since $\mathrm{rk}\, K = \mathrm{rk}\, S$; furthermore, note that T is connected by [**45**, Proposition 1.4.1.4].) Also choose $\beta \in \mathfrak{v}^*$ and regard the triple (α, β, λ) as a functional on the Lie algebra $\mathfrak{m}$ of $M = (S \times V) \cdot N$. Extend this functional to obtain an element μ of $\mathfrak{g}^*$. We claim $G_\mu/Z(G)$ is compact. Indeed, if $g \in G_\mu$, then in particular $g \in G_\lambda = S \times V \times \tilde{W}$, where $\tilde{W} \supseteq Z$ is two step nilpotent and $\tilde{W}/Z = W$, W as in (4.5). If we write $g = svw$ with $s \in S$, $v \in V$, and $w \in \tilde{W}$, then since g stabilizes the restriction of μ to G_λ, we have $s \in T$ and $w \in Z$. Thus $G_\mu = T \times V \times Z$, which is compact modulo $Z(G)$. Furthermore it is clear that we can choose α and β so that $e^{i\mu}$ exponentiates to $T \times V \times Z$, hence to G_μ. Thus if G has square-integrable representations, it has orbits satisfying (i) and (ii). Also note that G_μ is always connected in this case.

Conversely, suppose (i) and (ii) are satisfied for $O = G \cdot \mu$, and let $\lambda = \mu|_{\mathfrak{n}}$. By (ii), $N_\lambda \subseteq G_\mu \cap N = Z$; hence λ has nonzero Pfaffian. Put $G_\lambda = S \times V \times \tilde{W}$ with S semisimple, V vector, $\tilde{W}$ nilpotent with $\tilde{W} \cap Z(G) = Z$. The stabilizer in G_λ of the restriction of μ to $\mathfrak{s} + \mathfrak{v} + \tilde{\mathfrak{w}}$ is compact modulo $Z(G)$; hence S has a Cartan subgroup contained in K, V is compact modulo $Z(G)$, $\tilde{W}$ has square-integrable representations modulo Z, and (iii) and (ii) of (4.4) hold.

Finally, let us consider the parametrization of the square-integrable representations of G when they exist. We may as well assume G is simply connected and we are interested in representations of G extending the representation π_λ of N, where $\lambda \in \mathfrak{n}^*$ is fixed and has nonzero Pfaffian. Furthermore, we may divide out by the kernel of π_λ and assume Z is one-dimensional, $\lambda|_{\mathfrak{z}} \neq 0$. The classification of the square-integrable representations of G now reduces to the problem of finding all the square-integrable α_λ^{-1} representations of G/N, or equivalently, of finding all square-integrable (ordinary) representations of $G_\lambda = S \times V \times \tilde{W}$, where $\tilde{W}$ is Heisenberg with center Z and we have fixed a central character on Z. By the Harish-Chandra parametrization of $\hat{S}_d$ (as extended to groups with possibly infinite center in [46, §3]), the fiber $\hat{G}_{d,\lambda}$ of $\hat{G}_d$ over π_λ is parametrized by $(\hat{T}'/\mathcal{W}) \times \hat{V}$, where $\hat{T}'$ is the "regular" part of the dual of T and $\mathcal{W}$ is the Weyl group of (G, T). This parameter space in turn may be identified with the set of regular coadjoint orbits of $S \times V$ satisfying (i) and (ii)–see [46, §7]. (There is an unavoidable confusion at this point due to the fact that there are basically two ways to associate representations to orbits, that differ by half the sum of the positive roots. In the case of a compact group, this amounts to parametrizing representations by highest weights or else parametrizing them by the coadjoint orbits that support the Fourier transform of the character. If the latter parametrization is used, only regular orbits are associated to representations. For our purposes, the second parametrization is more suitable, since this becomes the Harish-Chandra parametrization in the noncompact semisimple case.)[5] Finally, given a coadjoint orbit for $S \times V$, we choose the corresponding orbit for $G_\lambda = S \times V \times \tilde{W}$ with restriction $\lambda|_{\mathfrak{z}}$ on $\mathfrak{z}$ and extend to a G-orbit in $\mathfrak{g}^*$. Since $G = NG_\lambda$, this orbit restricts only to the one we started with on $\mathfrak{g}_\lambda$, so we have our desired parametrization of $\hat{G}_d$.

We will be interested in studying realizations on L^2-cohomology spaces of the representations classified in Theorem 4.3. Since our tool for decomposing L^2-cohomology will be, as in §3, to relate formal harmonic spaces to Lie algebra cohomology of the C^∞-vectors, it is useful to have a description of the latter. The following is probably a special case of some (as yet unproven) general theorem about C^∞-vectors for tensor products:

THEOREM 4.6. *Let G be a connected Lie group with a connected normal nilpotent subgroup N, let π be a representation of N whose equivalence class is fixed by G, let α be the "Mackey obstruction" of π, let $\tilde{\pi}$ be the projective extension of π to G, and let σ be a unitary α^{-1}-representation of G/N. (Thus $\tilde{\pi} \otimes \sigma$ is a well-defined (ordinary) unitary representation of G.) Then the space of C^∞-vectors of $\tilde{\pi} \otimes \sigma$ may be identified (topologically) with $\mathcal{S}(\mathbf{R}^n) \hat{\otimes} \sigma_\infty$, where n is the Gelfand-Kirillov dimension of π and σ_∞ is the space of C^∞-vectors of σ with the C^∞-topology (see [45, §4.4.1]).*

[5]For a very readable discussion of the relative merits of the two ways to associate representations to orbits, see §8 of [27]. Strictly speaking, what we have said is valid only if G is simply connected. Otherwise, α_λ may not be trivial on the maximal semisimple subgroup S_λ of G (it may be of order 2) and ρ, "half the sum of the positive roots," may not be integral (it is always half-integral at worst). The correct parameterization of $\hat{G}_d$ in the general case is by "admissible" rather than by integral orbits, which are orbits that define representations of the universal covering group that descend to ordinary (rather than projective) representations of G. For example, if G is the semidirect product of $SL(2, \mathbf{R})$ and the 3-dimensional Heisenberg group, $\hat{G}_d$ corresponds to discrete series representations of the twofold covering of $SL(2, \mathbf{R})$ that do *not* descend to the linear group. The admissible orbits in this case are exactly the half-integral ones.

In what follows, we are primarily interested in *constructing* (rather than parameterizing) representations, so it will suffice to deal with the simply connected case.

PROOF. Any C^∞-vector for $\tilde{\pi} \otimes \sigma$ must in particular be a C^∞-vector for the restriction of $\tilde{\pi} \otimes \sigma$ to N, which is $\pi \otimes 1$. We may identify the Hilbert space of $\tilde{\pi} \otimes \sigma$ with $L^2(\mathbf{R}^n) \otimes \mathcal{H}_\sigma$ (Hilbert space tensor product), where $\mathcal{H}_\sigma$ is the Hilbert space on which σ acts, or with $L^2(\mathbf{R}^n, \mathcal{H}_\sigma)$ (L^2-space of vector-valued functions). In the identification of the Hilbert space of π_λ with $L^2(\mathbf{R}^n)$, π_∞ becomes $\mathcal{S}(\mathbf{R}^n)$. Similarly, the space of C^∞-vectors for $\pi \otimes 1$ must consist of vector-valued Schwartz functions (C^∞ functions on $\mathbf{R}^n$ with values in $\mathcal{H}_\sigma$ all of whose derivatives have norms rapidly vanishing at infinity). Consider such a function f: $\mathbf{R} \to \mathcal{H}_\sigma$. Going up to a covering group or to a central extension of G/N by a torus, we may assume σ is actually an ordinary representation (instead of a cocycle representation). By Goodman's Theorem [**45**, Theorem 4.4.4.10], f will be a C^∞-vector for $\tilde{\pi} \otimes \sigma$ if and only if f lies in the domain of the closure of $((\tilde{\pi} \otimes \sigma)_\infty(x))^m$ for all $x \in \mathfrak{g}$ and for all $m \geqslant 1$.

Now the representation π_∞ of $U(\mathfrak{n}_\mathbf{C})$ may be viewed as the composition of the isomorphism $U(\mathfrak{n}_\mathbf{C})/\ker(\pi_\infty) \to A_n(\mathbf{C})$ with the standard representation of the Weyl algebra A_n on $\mathcal{S}(\mathbf{R}^n)$. Since the adjoint action of G on $U(\mathfrak{n}_\mathbf{C})$ preserves $\ker(\pi_\infty)$, it induces an action of G on $U(\mathfrak{n}_\mathbf{C})/\ker(\pi_\infty) \cong A_n(\mathbf{C})$. This action will be derived from a Lie algebra homomorphism $\mathfrak{g} \to \operatorname{Der}(A_n)$, where $\operatorname{Der}(A_n)$, the Lie algebra of derivations of A_n, is just $A_n(\mathbf{C})/\mathbf{C}$ since every derivation of A_n is inner [**12**, Lemma 4.6.8]. By a theorem of Duflo [**12**, Proposition 10.1.4], this map lifts to a Lie algebra map θ: $\mathfrak{g} \to A_n$, and $\tilde{\pi}_\infty$ is just $\pi_\infty \circ \theta$. Hence for $x \in \mathfrak{g}$, $\tilde{\pi}_\infty(x)$ is a polynomial differential operator on $\mathbf{R}^n$, and the space of C^∞-vectors of $\tilde{\pi}$ coincides with that for π_∞ (even topologically), which is $\mathcal{S}(\mathbf{R}^n)$. Let $x \in \mathfrak{g}$, put $\tilde{\pi}_\infty(x) = D_1$ and $\sigma_\infty(x) = D_2$, and suppose f: $\mathbf{R}^n \to \mathcal{H}_\sigma$ is a C^∞-vector for $\tilde{\pi} \otimes \sigma$. We want to show that for each $t \in \mathbf{R}^n$, $f(t)$ lies in the domain of the adjoint of D_2^m, all $m > 1$.

First note that $\mathcal{S}(\mathbf{R}^n) \hat{\otimes} \sigma_\infty$ may be identified with a space of rapidly decreasing functions $\mathbf{R}^n \to \sigma_\infty \subseteq \mathcal{H}_\sigma$, hence with a subspace of $L^2(\mathbf{R}^n, \mathcal{H}_\sigma)$. Furthermore, $\mathcal{S}(\mathrm{R}^n) \hat{\otimes} \sigma_\infty$ is contained in the space of C^∞-vectors for $\tilde{\pi} \otimes \sigma$, by [**45**, Proposition 4.4.1.10], and so by [**34**, Corollary 1.2], $(\tilde{\pi} \otimes \sigma)^m_\infty(x)$ is the closure of its restriction $(D_1 \hat{\otimes} 1 + 1 \hat{\otimes} D_2)^m$ to $\mathcal{S}(\mathbf{R}^n) \hat{\otimes} \sigma_\infty$. We have seen that f lies in the domain of the closure of $(D_1 \hat{\otimes} 1)^m$ for all m; hence by an easy induction on k, f lies in the domain of the closure of $D_1^j \hat{\otimes} D_2^k$ for all j and k. Furthermore, for each k, $(1 \hat{\otimes} D_2^k)^-(f)$ must again be a C^∞-vector for the action of N, hence is again represented by a smooth function $\mathbf{R}^n \to \mathcal{H}_\sigma$. For $\xi \in \sigma_\infty$ and $\varphi \in \mathcal{S}(\mathbf{R}^n)$,

$$\begin{aligned}\left\langle (1 \hat{\otimes} D_2^k)^*(f), \varphi \otimes \xi \right\rangle &= \int_{\mathbf{R}^n} \left\langle (1 \hat{\otimes} D_2^k)^*(f)(s), \xi \right\rangle \overline{\varphi(s)}\, ds \\ &= \left\langle f, \varphi \otimes D_2^k \xi \right\rangle = \int_{\mathbf{R}^n} \left\langle f(s), D_2^k \xi \right\rangle \overline{\varphi(s)}\, ds.\end{aligned}$$

Letting $\bar{\varphi} \to \delta_t$ (the Dirac measure at t) in the weak topology of $\mathcal{S}'$, we get $\langle((1 \otimes D_2^k)^*(f))(t), \xi\rangle = \langle f(t), D_2^k\xi\rangle$; hence $f(t) \in \operatorname{dom}((D_2^k)^*)$ and $(D_2^k)^*(f(t)) = ((1 \otimes D_2^k)^*(f))(t)$. Since $x \in \mathfrak{g}$ was arbitrary, applying Goodman's theorem again gives $f(t) \in \sigma_\infty$, so f: $\mathbf{R}^n \to \sigma_\infty$. The proof also shows that $\|D_2^k(f(t))\|$ is again a Schwartz function of t; since the seminorms $\xi \mapsto \|D_2^k\xi\|$ generate the topology of

σ_∞, $f \in \mathcal{S}(\mathbf{R}^n, \sigma_\infty) = \mathcal{S}(\mathbf{R}^n) \hat{\otimes} \sigma_\infty$. Thus $\mathcal{S}(\mathbf{R}^n) \hat{\otimes} \sigma_\infty$ and the space of C^∞-vectors for $\tilde{\pi} \otimes \sigma$ coincide as spaces. Since the topology on the former is stronger than that of the latter, they coincide topologically by the open mapping theorem.

Now we are ready to prove the analogues of Theorems 2.4 and 3.1 in this context.

THEOREM 4.7. *Let G be as in* 4.1, *let Z be one-dimensional, let $\mu \in \mathfrak{g}^*$ be such that $\mu|_{\mathfrak{z}} \neq 0$, and let $\mathfrak{h}$ be a polarization for μ that is "admissible for $\mathfrak{n}$", i.e., such that $\mathfrak{h} \cap \mathfrak{n}_{\mathbf{C}}$ is a polarization for $\lambda = \mu|_{\mathfrak{n}}$. Let $\tilde{\pi}_{-\lambda} \otimes \sigma$ be, as in* 4.6, *a unitary representation of G extending a multiple of the Kirillov representation of N associated to $-\lambda$. Let $\mathfrak{m}$ be a Lie subalgebra of $\mathfrak{h}$ such that $\mathfrak{m} \cap \mathfrak{n}_{\mathbf{C}} = \mathfrak{h} \cap \mathfrak{n}_{\mathbf{C}} \cap \ker \lambda$. Then for all $q \geq q(\mathfrak{h} \cap \mathfrak{n}_{\mathbf{C}}, \lambda)$,*

$$H^q(\mathfrak{m}, (\tilde{\pi}_{-\lambda} \otimes \sigma)_\infty) \cong H^{q - q(\mathfrak{h} \cap \mathfrak{n}_{\mathbf{C}}, \lambda)}(\mathfrak{m}/(\mathfrak{m} \cap \mathfrak{n}_{\mathbf{C}}), \sigma_\infty),$$

and for $q < q(\mathfrak{h} \cap \mathfrak{n}_{\mathbf{C}}, \lambda)$,

$$H^q(\mathfrak{m}, (\tilde{\pi}_{-\lambda} \otimes \sigma)_\infty) = 0.$$

Furthermore, $\mathfrak{m}$ acts on the one-dimensional cohomology space

$$H^{q(\mathfrak{h} \cap \mathfrak{n}_{\mathbf{C}}, \lambda)}(\mathfrak{m} \cap \mathfrak{n}_{\mathbf{C}}, (\tilde{\pi}_{-\lambda})_\infty)$$

(*as explained in* [**9**, *Proposition* 2.2]) *according to the linear functional* $-\frac{1}{2}\operatorname{tr} \operatorname{ad}|_{\mathfrak{m} \cap \mathfrak{n}_{\mathbf{C}}}$.

PROOF. By 4.6, $(\tilde{\pi}_{-\lambda} \otimes \sigma)_\infty$ may be identified with $(\pi_{-\lambda})_\infty \hat{\otimes} \sigma_\infty$. Furthermore, σ may be (by the discussion preceding (4.4)) identified with an irreducible representation of a direct product of a reductive and a Heisenberg group, hence has a distribution character. (By this we mean that σ maps C^∞ functions of compact support to trace class operators, so that $f \mapsto \operatorname{Tr} \sigma(f)$ defines a distribution. This follows easily from the corresponding fact for semisimple groups.) By [**8**, Théorème 2.6], σ_∞ is a nuclear Fréchet space. Replacing $\mathcal{S}(\mathbf{R})$ by σ_∞ in the proof of Lemma 2.3 (in the case of "trivial twisting"), we see that for any q,

$$H^q(\mathfrak{m} \cap \mathfrak{n}_{\mathbf{C}}, (\tilde{\pi}_{-\lambda} \otimes \sigma)_\infty) \cong H^q(\mathfrak{m} \cap \mathfrak{n}_{\mathbf{C}}, (\pi_{-\lambda})_\infty) \hat{\otimes} \sigma_\infty.$$

By Theorem 2.4, we therefore have

$$H^q(\mathfrak{m} \cap \mathfrak{n}_{\mathbf{C}}, (\tilde{\pi}_{-\lambda} \otimes \sigma)_\infty) = \begin{cases} 0 & \text{if } q \neq q(\mathfrak{h} \cap \mathfrak{n}_{\mathbf{C}}, \lambda), \\ \sigma_\infty & \text{if } q = q(\mathfrak{h} \cap \mathfrak{n}_{\mathbf{C}}, \lambda). \end{cases}$$

Moreover, these identifications respect the action of $\mathfrak{m}/\mathfrak{m} \cap \mathfrak{n}_{\mathbf{C}}$. Since we have a spectral sequence converging to $H^*(\mathfrak{m}, (\tilde{\pi}_{-\lambda} \otimes \sigma)_\infty)$ with E_2^{pq} terms

$$E_2^{pq} = H^q(\mathfrak{m}/(\mathfrak{m} \cap \mathfrak{n}_{\mathbf{C}}), H^q(\mathfrak{m} \cap \mathfrak{n}_{\mathbf{C}}, (\tilde{\pi}_{-\lambda} \otimes \sigma)_\infty)),$$

the first part of the theorem follows.

The second part of the theorem is really a consequence of M. Duflo's results on enveloping algebras. We may view $(\tilde{\pi}_{-\lambda})_\infty$ as a module X for the Weyl algebra $A = U(\mathfrak{n}_{\mathbf{C}})/I(\lambda)$, where the primitive ideal $I(\lambda)$ is the kernel of the action of $U(\mathfrak{n}_{\mathbf{C}})$ on X. Then the $\mathfrak{h}$-module structure on X is obtained via a canonical Lie algebra homomorphism θ of $\mathfrak{h}$ into A (see [**12**, Lemma 10.1.2 and Proposition 10.1.4]). So it is enough to show that for *any* A-module X made into an $\mathfrak{h}$-module in this fashion and for any $j \geq 0$, the action of $\mathfrak{m}$ on $H^j(\mathfrak{m} \cap \mathfrak{n}_{\mathbf{C}}, X)$ is given by $-\frac{1}{2}\operatorname{tr} \operatorname{ad}|_{\mathfrak{m} \cap \mathfrak{n}_{\mathbf{C}}}$.

The proof of this last fact is almost identical to that of Theorem 5.7 below. First consider the case where $j = 0$, so that $H^j(\mathfrak{m} \cap \mathfrak{n}_{\mathbf{C}}, X) = X^{\mathfrak{m} \cap \mathfrak{n}_{\mathbf{C}}}$. Let $a \in \mathfrak{m}$ and let $b = \theta(a) - \frac{1}{2}\mathrm{tr}\, \mathrm{ad}_{\mathfrak{n}_{\mathbf{C}}/(\mathfrak{h} \cap \mathfrak{n}_{\mathbf{C}})} a$. By [12, Lemma 10.1.2(ii)], we have $b \cdot (1 \otimes 1) = 0$ in the A-module

$$\mathrm{ind}(\lambda|_{\mathfrak{h} \cap \mathfrak{n}_{\mathbf{C}}}, \mathfrak{n}_{\mathbf{C}}) = U(\mathfrak{n}_{\mathbf{C}}) \otimes_{U(\mathfrak{h} \cap \mathfrak{n}_{\mathbf{C}})} \mathbf{C}_{\lambda}.$$

This shows that b lies in the annihilator of $1 \otimes 1$, which is the left ideal $A\theta(\mathfrak{m} \cap \mathfrak{n}_{\mathbf{C}})$ of A, so that b will also annihilate $X^{\mathfrak{m} \cap \mathfrak{n}_{\mathbf{C}}}$ for any A-module X. On the other hand, a acts unimodularly on $\mathfrak{n}_{\mathbf{C}}$ (by 4.2) and trivially on $\mathfrak{z}$, so that

$$\mathrm{tr}\, \mathrm{ad}_{\mathfrak{n}_{\mathbf{C}}/(\mathfrak{h} \cap \mathfrak{n}_{\mathbf{C}})} a = -\mathrm{tr}\, \mathrm{ad}_{\mathfrak{m} \cap \mathfrak{n}_{\mathbf{C}}} a.$$

This proves the theorem in the case $j = 0$.

The general case is handled by induction on j, using the standard "dimension-shifting" argument suggested by [9] and [44] (cf. the proof of 5.7 below). We assume the theorem for $0 \leqslant j < j_0$ and embed X into $E = \mathrm{Hom}_{\mathbf{C}}(A, X)$, say with quotient B. Since $U(\mathfrak{n}_{\mathbf{C}})$ and hence A are free $U(\mathfrak{m} \cap \mathfrak{n}_{\mathbf{C}})$-modules because of the Poincaré-Birkhoff-Witt Theorem, E is injective as a $U(\mathfrak{m} \cap \mathfrak{n}_{\mathbf{C}})$-module. Thus $H^{j_0}(\mathfrak{m} \cap \mathfrak{n}_{\mathbf{C}}, E) = 0$, and we get from the long exact cohomology sequence a surjection $H^{j_0 - 1}(\mathfrak{m} \cap \mathfrak{n}_{\mathbf{C}}, B) \twoheadrightarrow H^{j_0}(\mathfrak{m} \cap \mathfrak{n}_{\mathbf{C}}, X)$. Now the desired information about $H^{j_0}(\mathfrak{m} \cap \mathfrak{n}_{\mathbf{C}}, X)$ follows immediately from the inductive hypothesis applied to $H^{j_0 - 1}(\mathfrak{m} \cap \mathfrak{n}_{\mathbf{C}}, B)$.

THEOREM 4.8. *Let G be a group satisfying* 4.1 *and also having square-integrable representations modulo its center $Z(G)$, hence satisfying condition* (i) *of* 4.3. (*The proof of that theorem showed that if* (i) *is satisfied for some orbit, then both* (i) *and* (ii) *are satisfied for certain* (*possibly other*) *orbits.*) *Let O be a G-orbit in $\mathfrak{g}^*$ satisfying* (i) *and* (ii), *let $\mu \in O$, and let $\mathfrak{h}$ be a totally complex polarization for μ which is admissible for $\mathfrak{n}$, i.e., such that $\mathfrak{h} \cap \mathfrak{n}_{\mathbf{C}}$ is a polarization for $\lambda = \mu|_{\mathfrak{n}}$.* (*In general, such an $\mathfrak{h}$ may or may not exist–see the discussion in* 4.12 *below.*) *Let $\mathfrak{L}_{\mu}$ be the line bundle on G/G_{μ} associated with μ, which is holomorphic with respect to the complex structure defined by $\mathfrak{h}$, and let $\pi^q(\mu, \mathfrak{h}, G)$ denote the representation of G on $\mathcal{H}^q(\mathfrak{L}_{\mu}, \mathfrak{h})$, the qth L^2-cohomology space of $\mathfrak{L}_{\mu}$.*

Let $\mathfrak{p} = \mathfrak{h} \cap \mathfrak{n}_{\mathbf{C}} \cap \ker \lambda$. By [12, *Proposition* 10.2.1], *$\mathfrak{h} = \mathfrak{p} + (\mathfrak{h} \cap \mathfrak{g}_{\lambda,\mathbf{C}})$* (*Lie algebra semidirect sum*), *and $\mathfrak{h} \cap \mathfrak{g}_{\lambda,\mathbf{C}}$ is a polarization for $\mu|_{\mathfrak{g}_{\lambda}}$. Since $\mathfrak{g}_{\lambda}$ is of the form $\mathfrak{s} + \mathfrak{v} + \tilde{\mathfrak{w}}$ with $\mathfrak{s}$ semisimple, $\mathfrak{v}$ abelian, and $\tilde{\mathfrak{w}}$ two-step nilpotent with center $\mathfrak{z}$, we can further split $\mathfrak{h} \cap \mathfrak{g}_{\lambda,\mathbf{C}}$ as $\mathfrak{g}_{\mu,\mathbf{C}} + \mathfrak{u}$, where $\mathfrak{g}_{\mu}$ is reductive and contains a Cartan subalgebra $\mathfrak{t}$ for $\mathfrak{s}$, the group T corresponding to which is compact modulo $Z(G)$, and where $\mathfrak{u}$ is nilpotent and is normalized by $\mathfrak{g}_{\mu}$. We assume* (*without great loss of generality*) *that $\mathfrak{u} = (\mathfrak{u} \cap \mathfrak{s}_{\mathbf{C}}) \oplus (\mathfrak{u} \cap \tilde{\mathfrak{w}}_{\mathbf{C}})$* (*so that $\mathfrak{h}$ is also admissible for $\tilde{\mathfrak{w}}$ and for $\mathfrak{s}$*). *Choose an ordering for the roots of $\mathfrak{t}_{\mathbf{C}}$ in $\mathfrak{s}_{\mathbf{C}}$ so that $\mathfrak{u} \cap \mathfrak{s}_{\mathbf{C}}$ consists of negative root spaces for $\mathfrak{t}_{\mathbf{C}}$, and define a linear functional ψ on $\mathfrak{t}_{\mathbf{C}}$ by $\psi(t) = \frac{1}{2} i\, \mathrm{tr}(\mathrm{ad}\, t)|_{\mathfrak{h}}$; note that ψ is real-valued on $\mathfrak{t}$. Extend ψ to an element of $\mathfrak{g}^*$ by requiring that ψ be identically zero on $\mathfrak{n}$, $\mathfrak{v}$, $\tilde{\mathfrak{w}}$, and the root spaces for $\mathfrak{t}_{\mathbf{C}}$ in $\mathfrak{s}_{\mathbf{C}}$. ψ is the "Duflo shift" of* [14] *and* [27, §8].

Then if $(\mu + \psi)|_{\mathfrak{s}}$ is singular, $\mathcal{H}^q(\mathfrak{L}_{\mu}, \mathfrak{h}) = 0$ for all q. Otherwise, $\mathcal{H}^q(\mathfrak{L}_{\mu}, \mathfrak{h}) = 0$ for $q \neq \tilde{q}(\mathfrak{h}, \mu)$ and $\pi^{\tilde{q}(\mathfrak{h},\mu)}(\mu, \mathfrak{h}, G)$ is irreducible, square-integrable, and associated with *the coadjoint orbit of* $\mu + \psi$. *Here* $\tilde{q}(\mathfrak{h}, \mu)$ *is just* $q(\tilde{\mathfrak{h}}, \mu + \psi)$ *as defined in* 1.2, *if we let* $\tilde{\mathfrak{h}} = \mathfrak{p} + \mathfrak{b} + \mathfrak{v}_{\mathbf{C}} + \mathfrak{z}_{\mathbf{C}} \subseteq \mathfrak{h}$, *where* $\mathfrak{b}$ *is the Borel subalgebra of* $\mathfrak{s}_{\mathbf{C}}$ *containing all the negative root spaces for* $\mathfrak{t}_{\mathbf{C}}$. (*Note that* $\tilde{\mathfrak{h}}$ *is a polarization for* $\mu + \psi$ *when the latter is regular. If* μ *is also regular, the two polarizations coincide, and if* μ *is "sufficiently regular",* $\tilde{q}(\mathfrak{h}, \mu) = q(\mathfrak{h}, \mu)$.)

PROOF. Fortunately, we have so much machinery at our disposal that the proof is not as bad as the statement! To begin with, the proof of [**46**, Lemma 7.5.3] goes over without change and enables us to replace $\mathfrak{h}$ by $\tilde{\mathfrak{h}}$, $\mathfrak{g}_\mu \cap \mathfrak{s}$ by $\mathfrak{t}$. We assume in what follows that this has been done. Note that Z acts on all the $\mathcal{H}^q(\mathcal{L}_\mu, \mathfrak{h})$ by $e^{i\lambda}$, so that each π^q is a direct integral of irreducible representations of the form $\tilde{\pi}_\lambda \otimes \sigma$, where σ runs over the α_λ^{-1}-dual of G/N and the measure defining the decomposition is absolutely continuous with respect to the α_λ^{-1}-Plancherel measure β of [**23**, Theorem 2.3]. Just as in the proof of Theorem 3.1, Lemma 6 of [**42**] gives us the decomposition

$$\mathcal{H}^q(\mathcal{L}_\mu, \mathfrak{h}) \cong \mathcal{H}_\lambda \otimes \int^{\oplus} \mathcal{H}_\sigma \otimes \{\mathcal{H}^q(\tilde{\pi}_{-\lambda} \otimes \check{\sigma}, \mathfrak{h})\}_{-\mu}\, d\beta(\sigma), \tag{4.9}$$

where all the tensor products are Hilbert space tensor products, $\mathcal{H}_\lambda$ and $\mathcal{H}_\sigma$ are the Hilbert spaces of $\tilde{\pi}_\lambda$ and σ, respectively, $\check{\sigma}$ is the contragredient representation to σ, and

$$\mathcal{H}^q(\tilde{\pi}_{-\lambda} \otimes \check{\sigma}, \mathfrak{h}) = \{\Phi \in \mathcal{H}_{-\lambda} \otimes \mathcal{H}_{\check{\sigma}} \otimes \Lambda^q(\mathfrak{p} + \mathfrak{u})^*\colon \Phi \perp \operatorname{range}(\delta + \delta^*)\}. \tag{4.10}$$

Here δ is the "formal coboundary operator" for Lie algebra cohomology of the solvable Lie algebra $\mathfrak{p} + \mathfrak{u}$ (which is a complement to $\mathfrak{g}_{\mu,\mathbf{C}}$ in $\mathfrak{h}$), and the subscript $-\mu$ in (4.9) means that we further require Φ in (4.10) to be a $-i\mu$-eigenvector for $\mathfrak{g}_{\mu,\mathbf{C}}$.

Also, again as in the proof of 3.1, we may use [**33**, Theorem 2] to identify $\mathcal{H}^q(\pi_{-\lambda} \otimes \check{\sigma}, \mathfrak{h})$ with a subspace of the Lie algebra cohomology space $H^q(\mathfrak{p} + \mathfrak{u}, (\tilde{\pi}_{-\lambda} \otimes \check{\sigma})_\infty)$, which coincides with the latter if the latter is finite-dimensional. By Theorem 4.7, we have

$$H^q(\mathfrak{p} + \mathfrak{u}, (\tilde{\pi}_{-\lambda} \otimes \check{\sigma})_\infty) \cong H^{q - q(\mathfrak{h} \cap \mathfrak{n}_{\mathbf{C}}, \lambda)}(\mathfrak{u}, \check{\sigma}_\infty).$$

Assuming as usual that G/N is simply connected, we may regard $\check{\sigma}$ as an ordinary representation $\check{\sigma}_1 \otimes \sigma_{-\lambda}$ of $G_\lambda = S \times V \times \tilde{W}$, where $\check{\sigma}_1 \in (S \times V)^\wedge$ and $\sigma_{-\lambda}$ is the square-integrable representation of $\tilde{W}$ with central character $e^{-i\lambda}$. Since $\mathfrak{u}$ was assumed to split as $(\mathfrak{u} \cap \mathfrak{s}_{\mathbf{C}}) \oplus (\mathfrak{u} \cap \tilde{\mathfrak{w}}_{\mathbf{C}})$, another (relatively trivial) application of Theorem 4.7 gives

$$H^{q - q(\mathfrak{h} \cap \mathfrak{n}_{\mathbf{C}}, \lambda)}(\mathfrak{u}, \check{\sigma}_\infty) \cong H^{q - q(\mathfrak{h} \cap \mathfrak{n}_{\mathbf{C}}, \lambda) - q(\mathfrak{h} \cap \tilde{\mathfrak{w}}, \lambda)}(\mathfrak{u} \cap \mathfrak{s}_{\mathbf{C}}, (\check{\sigma}_1)_\infty).$$

By [**43**, Corollary 3.7], this latter space is finite-dimensional, so that the formal harmonic space will coincide with the Lie algebra cohomology. We have finally reduced everything down to knowing $\int_{\hat{S}}^{\oplus} H^*(\mathfrak{u} \cap \mathfrak{s}_{\mathbf{C}}, (\check{\sigma}_1)_\infty)_{-\mu - \psi_1}\, d\beta_S(\sigma_1)$, where $\psi_1(t) = \frac{1}{2}\, i \operatorname{tr}(\operatorname{ad} t)|_{\mathfrak{h} \cap \mathfrak{n}_{\mathbf{C}}}$ (this enters by the second part of Theorem 4.7 when we keep track of the $\mathfrak{t}$-eigenvalues) and where β_S is the Plancherel measure for S. This is exactly what is given by the Langlands conjecture **[42]**, **[43]**. Unfortunately, Schmid's results are only stated for linear groups, whereas we have assumed S simply connected. (At the very least, we need to deal here with the metaplectic group.) However, linearity of S is only used in **[43]** for knowing a form of Blattner's conjecture, and the author has been assured by David Vogan and Nolan Wallach that this is now known for arbitrary semisimple groups by other methods, unfortunately unpublished.[6] Since our intention here is not to go into the details of the semisimple case, we assume the full Langlands conjecture here and trust that the details will eventually be written out elsewhere.

Now let ρ be one-half the sum of the positive roots for $\mathfrak{t}_{\mathbf{C}}$ in $\mathfrak{s}_{\mathbf{C}}$, divided by i so as to be real-valued on $\mathfrak{t}$. Then we have $\psi = \psi_1 + \rho$.

By a theorem of Casselman and Osborne **[9]**, $H^*(\mathfrak{u} \cap \mathfrak{s}_{\mathbf{C}}, (\check{\sigma}_1)_\infty)_{-(\mu+\psi_1)}$ will be nonzero for only finitely many $\sigma_1 \in \hat{S}$, so that we may assume σ_1 is square-integrable. **[43]** then shows that $H^*(\mathfrak{u} \cap \mathfrak{s}_{\mathbf{C}}, (\check{\sigma}_1)_\infty)_{-(\mu+\psi_1)}$ is nonzero only if $(\mu + \psi)|_{\mathfrak{s}}$ is regular, and then only for σ_1 with Harish-Chandra parameter given by the orbit of $(\mu + \psi)|_{\mathfrak{s}}$; furthermore, the cohomology only shows up in degree

$$q_0 = \#\{\alpha \in \Delta_+^c | \langle \mu + \psi, \alpha \rangle < 0\} + \#\{\alpha \in \Delta_+^n | \langle \mu + \psi, \alpha \rangle > 0\},$$

where Δ_+^c and Δ_+^n are the compact and noncompact positive roots of $(\mathfrak{t}_{\mathbf{C}}, \mathfrak{s}_{\mathbf{C}})$, and here it is one-dimensional. (Again, we have divided by i to make everything real-valued on $\mathfrak{t}$.) However, we claim that q_0 is exactly $q(\mathfrak{b}, (\mu + \psi)|_{\mathfrak{s}})$; this is easy to check since the root spaces in $\mathfrak{b}$ diagonalize $H_{\mu+\psi}$ and since each compact (resp. noncompact) root gives rise to a copy of $\mathfrak{su}(2)$ (resp., $\mathfrak{sl}(2, \mathbf{R})$) inside $\mathfrak{s}$.

Putting everything together, we have found that $\mathcal{H}^q(\mathcal{L}_\mu, \mathfrak{h})$ vanishes for all q when $(\mu + \psi)|_{\mathfrak{s}}$ is singular and is nonzero for exactly one value q_1 of q otherwise. Furthermore, π^{q_1} is irreducible, square-integrable, and associated with the coadjoint orbit of $\mu + \psi$ via the parametrization of 4.3. It remains to check that $q_1 = \tilde{q}(\mathfrak{h}, \mu)$. But we have shown that

$$\begin{aligned} q_1 &= q_0 + q(\mathfrak{h} \cap \mathfrak{n}_{\mathbf{C}}, \lambda) + q(\mathfrak{h} \cap \tilde{\mathfrak{w}}, \lambda) \\ &= q(\mathfrak{b}, (\mu + \psi)|_{\mathfrak{s}}) + q(\mathfrak{h} \cap \mathfrak{n}_{\mathbf{C}}, (\mu + \psi)|_{\mathfrak{n}}) + q(\mathfrak{h} \cap \tilde{\mathfrak{w}}, (\mu + \psi)|_{\tilde{\mathfrak{w}}}) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\text{(since } \psi \text{ vanishes on } \mathfrak{n} \text{ and } \tilde{\mathfrak{w}}) \\ &= q(\tilde{\mathfrak{h}}, \mu + \psi) = \tilde{q}(\mathfrak{h}, \mu) \\ &\qquad\big(\text{since } \tilde{\mathfrak{h}} \text{ is the orthogonal sum of } \mathfrak{b},\ \mathfrak{h} \cap \mathfrak{n}_{\mathbf{C}} \cap \ker \lambda, \text{ and } \mathfrak{h} \cap \tilde{\mathfrak{w}}\big). \end{aligned}$$

REMARK 4.11. Many cases of Theorem 4.8 are almost covered by **[41**, Theorems 1 and 2]. The essential difference is that we are dealing here with the full L^2-cohomology space, whereas Satake's methods only covered a certain subspace consisting of forms with a nice decomposition relative to the semidirect product decomposition of the group considered. (See **[41**, pp. 185–187] for a more precise description.) We have therefore proved that this subspace coincides with the whole space. In fact, despite the difficulty pointed out in **[41**, footnote to p. 187], it is

[6]See also the comments in G. Zuckerman, *Tensor products of finite and infinite dimensional representations of semisimple Lie groups*, Ann. of Math. (2) **106** (1977), 306.

possible to see why this should be true–in the correspondence between L^2-cohomology and Lie algebra cohomology, Satake's subspace corresponds exactly to the E_2 term in the Hochschild-Serre spectral sequence. The identity of the two spaces is due to the fact that the spectral sequence collapses. Perhaps one could show this directly by developing a spectral sequence machinery for L^2-cohomology, but the analytical details seem forbidding. Note also that, as in the nilpotent case, a weak form of 4.8 could be obtained from **[10]** and a vanishing theorem based on curvature estimates.

REMARK 4.12. It is worth discussing at this point to what extent Theorem 4.8 answers the problem of finding geometrical realizations for the square-integrable representations of the groups considered in 4.3. The difficulty here is that the author knows of no way to characterize the real Lie algebras that admit totally complex polarizations. (Among nilpotent Lie algebras, these seem to be rather rare, although among semisimple algebras they are of course common.) The most interesting class clearly included, although admittedly rather special, consists of semidirect products SN, where S is semisimple with rk K = rk S, N is Heisenberg, and S acts on N via a homomorphism into a symplectic group.

On the other hand, given a regular orbit O satisfying (i) and (ii) of 4.3 and given $\mu \in O$, it is quite plausible that there should exist a solvable polarization $\mathfrak{h}$ for μ which is admissible for $\mathfrak{n}$ and such that $\mathfrak{h} \cap \mathfrak{n}_{\mathbf{C}}$ is relatively ideal. If so, one could presumably obtain an irreducible representation of G by harmonic induction, as in the nilpotent case (cf. §3 above). However, **[11**, Théorème 2] does not seem quite strong enough to guarantee existence of such polarizations, and there are some technical complications in working out the details of the procedure, so we give here a more ad hoc method for constructing representations that still suffices for constructing all the representations of 4.3.

As usual, we reduce inductively to the case where N is Heisenberg. Let $\mu \in \mathfrak{g}^*$ be regular and satisfy (i) and (ii) of 4.3, and let $\lambda = \mu|_{\mathfrak{n}}$. We may assume $\lambda|_{\mathfrak{z}} \neq 0$, then assume $\mathfrak{z}$ is one-dimensional. (Otherwise, divide by $\mathfrak{z} \cap \ker \lambda$.) If $\mathfrak{n}$ has no proper $\mathfrak{g}$-invariant ideals other than $\mathfrak{z}$, then $\mathfrak{n}$ is Heisenberg, the image of $\mathfrak{g}_\lambda$ in Der($\mathfrak{n}$) is $\mathfrak{sp}(\mathfrak{n}/\mathfrak{z})$, and μ has a totally complex polarization. (This is the Satake situation.) Otherwise, $\mathfrak{n}$ has an abelian ideal $\mathfrak{a} \supsetneqq \mathfrak{z}$ contained in $\mathfrak{z}^{(2)}(\mathfrak{n})$ and normalized by $\mathfrak{g}$. (If $\mathfrak{n}$ is not Heisenberg, take $\mathfrak{a} = \mathfrak{z}^{(2)}(\mathfrak{n}) \cap [\mathfrak{n}, \mathfrak{n}]$. If $\mathfrak{n}$ is Heisenberg, take any $\mathfrak{g}$-invariant ideal $\mathfrak{c} \supsetneqq \mathfrak{z}$ of $\mathfrak{n}$ and let $\mathfrak{a} = \mathfrak{c} \cap \mathfrak{c}^\perp$ with respect to B_λ.) Applying the Mackey machine to the normal subgroup A of G, we see that the square-integrable representation of G associated with $\mu + \rho$ is induced from the representation of $C_N(A)G_\lambda$ associated with the restriction of $\mu + \rho$, where $C_N(A)$ is the centralizer of A in N. Now A is central in $C_N(A)$, and is the stabilizer of λ restricted to the Lie algebra of $C_N(A)$. Thus $C_N(A)$ has square-integrable representations modulo A. If $\mathfrak{a}_1 = \mathfrak{a} \cap \ker \lambda$ and A_1 is the corresponding subgroup of A, A_1 lies in the kernel of the representation of $C_N(A)$ associated with the restriction of λ. So if we replace Z by A/A_1 and N by $C_N(A)/A_1$, we are reduced to the case of a smaller nilpotent normal subgroup. Continuing inductively, we see that π is induced from a representation of a subgroup for which 4.8 applies (with N Heisenberg if we like). So 4.8

enables us to construct all square-integrable representations of groups satisfying 4.1.

5. Lie algebras with a triangular decomposition. In this final section, we show that certain properties of the enveloping algebras of semisimple Lie algebras also hold for the Lie algebras of the groups with square-integrable representations considered in 4.3. This helps "explain" why some analogue of the Langlands conjecture might be valid for such groups.

DEFINITION 5.1. Let k be an algebraically closed field of characteristic zero and let $\mathfrak{g}$ be a finite-dimensional Lie algebra over k. We say $\mathfrak{g}$ has a *triangular decomposition* $\mathfrak{h} + \mathfrak{n} + \mathfrak{n}^-$ if the following three conditions hold:

(i) $\mathfrak{h}$, $\mathfrak{n}$, $\mathfrak{n}^-$ are Lie subalgebras of $\mathfrak{g}$ such that $\mathfrak{g}$ (as a vector space) is the direct sum of $\mathfrak{h}$, $\mathfrak{n}$, and $\mathfrak{n}^-$.

(ii) $\mathfrak{h}$ is abelian, $\mathfrak{n}$ and $\mathfrak{n}^-$ are solvable and of equal dimension, $\mathfrak{h}$ normalizes $\mathfrak{n}$ and $\mathfrak{n}^-$, and $\mathfrak{n}$ (resp. $\mathfrak{n}^-$) acts unimodularly on $\mathfrak{g}/(\mathfrak{h} + \mathfrak{n})$ (resp. $\mathfrak{g}/(\mathfrak{h} + \mathfrak{n}^-)$).

(iii) There exists a regular element $f \in \mathfrak{g}^*$ such that $\mathfrak{h} = \mathfrak{g}_f$, $\mathfrak{n} + \mathfrak{n}^- \subseteq \ker f$, and $\mathfrak{h} + \mathfrak{n}$, $\mathfrak{h} + \mathfrak{n}^-$ are subordinate to f (hence are polarizations, since $\dim \mathfrak{n} = \dim(\mathfrak{g}/\mathfrak{h})/2$).

The obvious examples of such algebras $\mathfrak{g}$ are semisimple Lie algebras, where we take $\mathfrak{h}$ to be a Cartan subalgebra, $\mathfrak{n}$ (resp. $\mathfrak{n}^-$) to be the sum of the positive (resp. negative) root spaces for some ordering of the roots of $(\mathfrak{g}, \mathfrak{h})$. However, the complexifications of the Lie algebras considered earlier in this paper are also examples, if we take $\mathfrak{h} + \mathfrak{n}$ to be a totally complex polarization for some real regular linear functional f, and if $\mathfrak{n}^-$ is the complex conjugate of $\mathfrak{n}$.

What is nice about such algebras is the existence of a "Harish-Chandra homomorphism". The following proposition is basically due to M. Duflo.

PROPOSITION 5.2. *Let $\mathfrak{g}$ be a Lie algebra over k with a triangular decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{n} + \mathfrak{n}^-$. Let $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. Then there exists a unique algebra homomorphism θ: $Z(\mathfrak{g}) \to U(\mathfrak{h}) \cong S(\mathfrak{h})$ such that if $f \in \mathfrak{g}^*$ and $\mathfrak{g}_f = \mathfrak{h}$, $\mathfrak{n} \subseteq \ker f$, and if $u \in Z(\mathfrak{g})$, then $\chi_f(u) = \theta(u)(f^\sim)$. (Here we use the notation of* **[12,** 10.4.1]. *Associated to f is a primitive ideal $I(f)$ of $U(\mathfrak{g})$; $\chi_f(u)$ is the unique element of k such that $u - \chi_f(u) \in I(f)$. The precise meaning of $f^\sim$ will be explained below.)*

PROOF. It was pointed out to the author by David Vogan that the usual proof for $\mathfrak{g}$ semisimple still works. Namely, by the Poincaré-Birkhoff-Witt Theorem, we have a vector space direct sum decomposition

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus U(\mathfrak{g})\mathfrak{n} \oplus \mathfrak{n}^- U(\mathfrak{n}^-) U(\mathfrak{h}). \tag{5.3}$$

Let θ be the projection of $Z(\mathfrak{g})$ onto the first factor in the decomposition (5.3). This is a linear map, although it is not obvious that it has any other nice properties. However, if $f \in \mathfrak{g}^*$, $\mathfrak{g}_f = \mathfrak{h}$, and $\mathfrak{n} \subseteq \ker f$, then $\mathfrak{h} + \mathfrak{n}$ is a solvable polarization for f; hence by **[12,** Theorem 10.3.3], I_f, the kernel of $\operatorname{ind}^\sim(f|_{\mathfrak{h}+\mathfrak{n}}, \mathfrak{g})$, is primitive. Now the representation $\operatorname{ind}^\sim(f|_{\mathfrak{h}+\mathfrak{n}}, \mathfrak{g})$ acts on $U(\mathfrak{g}) \otimes_{U(\mathfrak{h}+\mathfrak{n})} k$, where k is viewed as a one-dimensional $(\mathfrak{h} + \mathfrak{n})$-module via $f^\sim = f + \rho$. Here ρ is another character of $\mathfrak{h} + \mathfrak{n}$, which vanishes on $\mathfrak{n}$ since $\mathfrak{n}$ acts unimodularly on $\mathfrak{g}/(\mathfrak{h} + \mathfrak{n})$ **[12,** §5.2]. Let

$u \in Z(\mathfrak{g})$, and write $u = \theta(u) + \sum b_i n_i + \sum u_j a_j$, where $n_i \in \mathfrak{n}$, $a_j \in U(\mathfrak{h})$, and $u_j \in \mathfrak{n}^- U(\mathfrak{n}^-)$. Then

$$\begin{aligned} \chi_f(u)(1 \otimes 1) = u(1 \otimes 1) &= \theta(u)(1 \otimes 1) + \sum b_i n_i (1 \otimes 1) + \sum u_j a_j (1 \otimes 1) \\ &= \theta(u)(f^\sim)(1 \otimes 1) + 0 + \sum a_j(f^\sim) u_j \otimes 1. \end{aligned}$$

Since $U(\mathfrak{g}) \otimes_{U(\mathfrak{h}+\mathfrak{n})} k \cong U(\mathfrak{n}^-)$ as $U(\mathfrak{n}^-)$-modules, this forces $\sum a_j(f^\sim) u_j \otimes 1 = 0$. Since we may vary $f^\sim$ within a Zariski open subset of $\mathfrak{h}^*$ and we may assume the a_j's are linearly independent in $U(\mathfrak{h}) \cong S(\mathfrak{h})$, it follows that $\sum u_j a_j = 0$; hence $u - \theta(u) \in U(\mathfrak{g})\mathfrak{n}$. Furthermore, we have shown that $\chi_f(u) = \theta(u)(f^\sim)$ for f as specified.

It remains to show that θ is an algebra homomorphism, uniquely determined by the above property. The uniqueness is clear from the fact that any two polynomials on $\mathfrak{h}^*$ agreeing on a Zariski open subset are equal. The homomorphism property follows from the fact that if $u_1, u_2 \in Z(\mathfrak{g})$, then

$$\theta(u_1 u_2)(f^\sim) = \chi_f(u_1 u_2) = \chi_f(u_1)\chi_f(u_2) = (\theta(u_1)\theta(u_2))(f^\sim).$$

REMARK 5.4. For nonsemisimple $\mathfrak{g}$, the proof of injectivity of θ given in [**45**, Lemma 2.3.3.5] breaks down. However, if $\mathfrak{g}$ is the complexified Lie algebra of a Lie group G for which there exist regular coadjoint orbits with compact stabilizers modulo $Z(G)$, then vanishing of 1-cohomology of compact groups with coefficients in a vector space together with Proposition 5.4.1 of [**37**] shows that these compact stabilizers in $G/Z(G)$ are conjugate within an open set. This implies existence of a Zariski open subset V of $\mathfrak{g}^*$ such that $f \in V$ implies $\mathfrak{g}_f$ is conjugate to $\mathfrak{h}$. Then by [**12**, Theorem 10.3.9], θ is injective. (The author learned this argument from a lecture of M. Duflo at the Special Year in Harmonic Analysis, University of Maryland, 1978.)[7]

5.5. EXAMPLES. When $\mathfrak{g}$ is semisimple, θ is of course the Harish-Chandra homomorphism. If $\mathfrak{g}$ is the complexified Lie algebra of a nilpotent Lie group with square-integrable representations and a totally complex polarization, then $\mathfrak{h}$ is the center of $\mathfrak{g}$ and in fact $Z(\mathfrak{g}) = U(\mathfrak{h})$ by [**28**, Theorem 3], so θ is just the identity map. A similar phenomenon occurs for certain unimodular solvable Lie algebras (the Lie algebras of the "H-groups" of [**2**])–see [**38**, Theorem 4.7].

For a nontrivial example, let $\mathfrak{g}$ be the semidirect sum of $\mathfrak{sl}(2, k)$ and the 3-dimensional Heisenberg algebra. A basis of $\mathfrak{g}$ is h, x, y, u, v, z, where $[h, x] = 2x$, $[h, y] = -2y$, $[x, y] = h$, $[u, v] = z$, $[h, u] = u$, $[h, v] = -v$, $[x, v] = u$, $[y, u] = v$. Let $\mathfrak{h} = kh + kz$, $\mathfrak{n} = kx + ku$, $\mathfrak{n}^- = ky + kv$. Then the conditions of 5.1 are satisfied, and an element in $Z(\mathfrak{g})$ not in $U(kz)$ is

$$\alpha = huv + xv^2 - yu^2 + z\left(\tfrac{1}{2}(h^2 - h) + xy + yx\right).$$

In the decomposition of (5.3), we have $\alpha = z(h^2 + 3h + 2)/2 + (2zyx - yu^2 + v^2x + hvu + 2vu)$, so $\theta(\alpha) = z(h^2 + 3h + 2)/2$.

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[7]A more refined analysis of θ in a situation generalizing this case, having the advantage that it yields a description of the image, will appear in a forthcoming paper of Duflo.

With the above preliminaries out of the way, we can now prove an analogue of [**9**, Corollary 2.7] or of [**44**, Theorem 3.3]. This in turn could be used directly (along with the link between formal harmonic spaces and Lie algebra cohomology) to prove, just as in the semisimple case, that in the situation of Theorem 4.8, the representations π occurring in the Plancherel decomposition of $\mathcal{H}^q(\mathcal{L}_\mu, \mathfrak{h})$ must all have the same infinitesimal character. (This in some sense explains why only discrete series representations appear.)

THEOREM 5.6. *Let* $\mathfrak{g}$ *be a Lie algebra over* k *with triangular decomposition* $\mathfrak{h} + \mathfrak{n} + \mathfrak{n}^-$, *and let* X *be a* $\mathfrak{g}$*-module.* (*Then, by restriction,* X *is also an* $\mathfrak{n}$*-module, and* $\mathfrak{h}$ *acts on the* $\mathfrak{n}$*-cohomology of* X *according to* [**9**, *Proposition* 2.2].) *Then for* $j \geqslant 0$, $u \in Z(\mathfrak{g})$, *and* $\xi \in H^j(\mathfrak{n}, X)$, *one has* $u \cdot \xi = \theta(u) \cdot \xi$.

PROOF. The proof (following Vogan) is by induction on j. If $j = 0$, we may identify ξ with an element of X such that $\mathfrak{n} \cdot \xi = 0$. Recall from the proof of 5.2 that $u \in \theta(u) + U(\mathfrak{g})\mathfrak{n}$; hence $u \cdot \xi = \theta(u) \cdot \xi$. Thus suppose $j_0 > 0$ and the theorem is known for $j < j_0$, and let $\xi \in H^{j_0}(\mathfrak{n}, X)$. We may embed X into an injective $U(\mathfrak{g})$-module, say into $A = \operatorname{Hom}_k(U(\mathfrak{g}), X)$, with some quotient B. Then we have a long exact cohomology sequence $\cdots \to H^{j_0-1}(\mathfrak{n}, A) \to H^{j_0-1}(\mathfrak{n}, B) \to H^{j_0}(\mathfrak{n}, X) \to H^{j_0}(\mathfrak{n}, A)$, the maps of which commute with the actions of $U(\mathfrak{h})$ and of $Z(\mathfrak{g})$. Since A is injective, $H^j(\mathfrak{n}, A) = 0$ for $j \geqslant 1$. Thus $H^{j_0}(\mathfrak{n}, X)$ is a quotient of $H^{j_0-1}(\mathfrak{n}, B)$ as a $U(\mathfrak{h})$- and $Z(\mathfrak{g})$-module (for $j_0 > 1$, we even have an isomorphism), so there exists $\eta \in H^{j_0-1}(\mathfrak{n}, B)$ mapping onto ξ. Since $u \cdot \eta = \theta(u) \cdot \eta$ by inductive hypothesis, $u \cdot \xi = \theta(u) \cdot \xi$.

COROLLARY 5.7 (CF. [**9**, COROLLARY 2.7]). *In the situation of the theorem, if* X *has an infinitesimal character* χ *and if there exists* $\xi \neq 0$ *in some* $H^j(\mathfrak{n}, X)$ *which is a weight vector for* $\mathfrak{h}$ *with weight* μ, *then* χ *is determined by* $\chi(u) = \theta(u)(\mu)$ (*for* $u \in Z(\mathfrak{g})$).

PROOF. Apply the conclusion of the theorem to ξ.

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