TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 261, Number 1, September 1980

BINARY SEQUENCES WHICH CONTAIN NO *BBb*

BY

EARL D. FIFE

ABSTRACT. A (one-sided) sequence or (two-sided) bisequence is irreducible provided it contains no block of the form BBb, where b is the initial symbol of the block B. Gottschalk and Hedlund [Proc. Amer. Math. Soc. **15** (1964), 70–74] proved that the set of irreducible binary bisequences is the Morse minimal set M. Let M^+ denote the one-sided Morse minimal set, i.e. $M^+ = \{x_0x_1x_2 \ldots : \ldots x_{-1}x_0x_1 \ldots \in M\}$. Let P^+ denote the set of all irreducible binary sequences. We establish a method for generating all $x \in P^+$. We also determine $P^+ - M^+$.

Considering P^+ as a one-sided symbolic flow, P^+ is not the countable union of transitive flows, thus P^+ is considerably larger than M^+. However M^+ is the ω-limit set of each $x \in P^+$, and in particular M^+ is the nonwandering set of P^+.

0. Introduction. The Morse minimal set has been characterized [3] as the set of all doubly-infinite sequences on two symbols which have the property that they contain no block of the form BBb, where b is the initial symbol of the block B. Let us call (two-sided) bisequences, (one-sided) sequences and blocks (finite strings) which satisfy this property *irreducible*. What is the set of all irreducible sequences on two symbols? Is it the same as the one-sided version of the Morse minimal set? We shall develop a procedure to construct every irreducible sequence. We then use this to show that, not only does the set of all of them properly contain the one-sided Morse minimal set, but that there are uncountably many irreducible sequences for which no "tail" is in the one-sided Morse minimal set.

This problem of irreducibility was first considered by Axel Thue [8] in 1912. He was concerned with constructing bisequences which repeated in a uniformly minimal fashion. It is evident that every binary (i.e. using two symbols) sequence and bisequence must contain a block of the form BB, where each occurrence of B represents the same block. Thus there is no stronger nonrepetitive condition for binary sequences or bisequences which holds for blocks of all lengths than that they be irreducible. Thue constructed what is now known as the Morse-Thue bisequence and established its rôle in determining all irreducible binary bisequences.

Independently, Marston Morse constructed the Morse-Thue bisequence in 1917 while working on his dissertation. He first published it in 1921 in [6]. Later in 1944, Morse and Hedlund [7] proved the bisequence to be irreducible. Finally in 1964, Gottschalk and Hedlund [3] proved the aforementioned characterization of the Morse minimal set. It was not until after [3] had appeared in print that Thue's

Received by the editors August 2, 1978 and, in revised form, September 18, 1979.
AMS (MOS) subject classifications (1970). Primary 05B30, 54H20.
Key words and phrases. Morse minimal set, nonrepetitive sequences.

work, having appeared in a relatively obscure journal, became well known. For a more complete history, see Hedlund [**4**].

Although the problem of determining all irreducible binary bisequences has been of sufficient interest to have been solved at least twice, the analogous problem for binary sequences (one-sided) has remained unsolved. We solve it here.

It is not always easy to decide whether or not a binary block is irreducible. To illustrate, the 200-block

0010011010010110011010011001011001101001
0110100110010110100101100110100110010110
0110100101101001100101100110100110010110
1001011001101001011010011001011010010110
0110100110010110011010010110100110010110

is not irreducible, but no block of the form BBb is readily recognizable.

In the solution to the two-sided problem, this recognition difficulty can be avoided. Any block which appears in some irreducible bisequence must appear in a special type of block called a Morse block (see §1), and Morse blocks are relatively easy to recognize. Unfortunately, blocks which appear in irreducible sequences need not appear in any Morse block. Thus a different approach is needed.

We establish a method for generating all irreducible binary sequences by associating with each binary sequence a sequence on three symbols which we call an algorithm sequence. (The algorithm sequence is actually used to generate the binary sequence.) The existence of a block of the form BBb in the binary sequence is reflected in the existence of an easily recognizable block in the algorithm sequence. From the algorithm sequence we can also easily determine whether or not an irreducible binary sequence can be extended to an irreducible binary bisequence. There are, in fact, uncountably many irreducible binary sequences which cannot be so extended.

The results of this paper are contained in the author's doctoral dissertation written at Wesleyan University. The author is grateful to Professor Ethan M. Coven for his valuable suggestions.

1. Preliminaries. A *flow* (X, T) consists of a nonempty compact, metrizable space X and a continuous map T of X into itself. A subset E of X is *invariant* provided $T(E) \subseteq E$. If X' is a nonempty, closed, invariant subset of X, then (X', T) is a *subflow* of (X, T). A flow (X, T) is *minimal* provided it contains no subflows other than itself.

If E is a nonempty subset of X, then $\{T^nE: n \geqslant 0\}$ is called the *orbit* of E and is denoted $\mathcal{O}(E)$. The closure of $\mathcal{O}(E)$ is denoted $\mathrm{Cl}\,\mathcal{O}(E)$ and is called the *orbit-closure* of E. If E is a nonempty subset of X, then $(\mathrm{Cl}\,\mathcal{O}(E), T)$ is a subflow of (X, T).

Let S^+ denote the set of all binary sequences and S the set of all binary bisequences. That is

$$S^+ = \{x = x_0x_1x_2 \ldots : x_i = 0 \text{ or } 1 \text{ for } i = 0, 1, 2, \ldots\}$$

and

$$S = \{x = \ldots x_{-1}x_0x_1 \ldots : x_i = 0 \text{ or } 1 \text{ for } i = \ldots, -1, 0, 1, \ldots\}.$$

(As a reminder to the reader, throughout this paper we shall superscript sets of binary sequences with a + .) Given the product topology, S and S^+ are compact, metrizable spaces homeomorphic to the Cantor set. A compatible metric for S (for S^+) is given by $d(x, y) = 1/(k + 1)$ where k is the largest nonnegative integer such that $x_i = y_i$ for $|i| \leqslant k$ ($i \leqslant k$). Let σ: $S \to S$ (σ: $S^+ \to S^+$) be defined by $[\sigma(x)]_i = x_{i+1}$ for $i = \ldots, -1, 0, 1, \ldots$ ($i = 0, 1, 2, \ldots$). The map σ, called the *shift transformation*, is continuous. Any subflow of (S, σ) or (S^+, σ) is called a *symbolic flow*.

We shall have occasion to pass from bisequences to sequences. Thus if $x \in S$ we shall denote $x_0x_1x_2 \ldots$ by x^+.

An *n-block* is a string of n consecutive 0's and 1's. Blocks are an essential tool in the study of symbolic flows in that they represent the cylinder sets which form a basis of open and closed sets of the topology, e.g. in place of the set $U_B = \{x \in S^+: x_i \ldots x_{i+n-1} = b_1 \ldots b_n\}$, we consider the block $B = b_1 \ldots b_n$. Thus an arbitrary open set about a point x can be taken to be a block which appears in x starting at a specified place. The n-block $x_k \ldots x_{k+n-1}$ will often be denoted by $x[k; n]$. The length of a block B will be denoted by $l(B)$, and the set of all blocks will be denoted by $\mathfrak{B}$.

The *dual block* of $B = b_1 \ldots b_n$ is the block $\bar{B} = \bar{b}_1 \ldots \bar{b}_n$ where $\bar{b}_i = 0$ if $b_i = 1$, and $\bar{b}_i = 1$ if $b_i = 0$. For example, if $B = 011011$, then $\bar{B} = 100100$.

We define a sequence of blocks $A_0, A_1, A_2, \ldots$ inductively by letting $A_0 = 0$ and $A_{n+1} = A_n\bar{A}_n$ for $n \geqslant 0$. Thus $A_1 = 01$, $A_2 = 0110$, and $A_3 = 01101001$. Notice that A_n is a 2^n-block. A block B is a *Morse block* provided $B = A_n$ or $\bar{A}_n$ for some $n \geqslant 0$.

A block C is *reducible* provided there is a block B with initial symbol b such that BBb is a subblock of C, i.e. for some integer $i \geqslant 0$ and some integer $n \geqslant 1$, $c_{i+k} = c_{i+k+n}$ for all $0 \leqslant k \leqslant n$. A block which is not reducible is *irreducible*. Let $\mathfrak{P} = \{B \in \mathfrak{B} : B \text{ is irreducible}\}$. It is evident that if $B \in \mathfrak{P}$ and C is a subblock of B, then $C \in \mathfrak{P}$. Morse and Hedlund showed in [7] that every Morse block is an element of $\mathfrak{P}$.

Let $P^+ = \{x \in S^+: \text{no reducible block appears in } x\}$, equivalently $P^+ = \{x \in S^+: x_i \ldots x_{i+n} \in \mathfrak{P} \text{ for all } i, n \geqslant 0\}$. Observe that P^+ is a nonempty, closed, invariant (under σ) subset of S^+; thus (P^+, σ) is a subflow of (S^+, σ).

We define $\mu \in S$ by $\mu_0 \ldots \mu_{2^n-1} = A_n$ for each $n \geqslant 0$, and $\mu_{-i} = \mu_{i-1}$ for each $i \geqslant 1$. μ is the *Morse-Thue bisequence*, and $\mu^+ = \mu_0\mu_1\mu_2 \ldots$ is the *Morse-Thue sequence*. Morse and Hedlund proved in [7] that no reducible block appears in μ. Thus $\mu^+ \in P^+$.

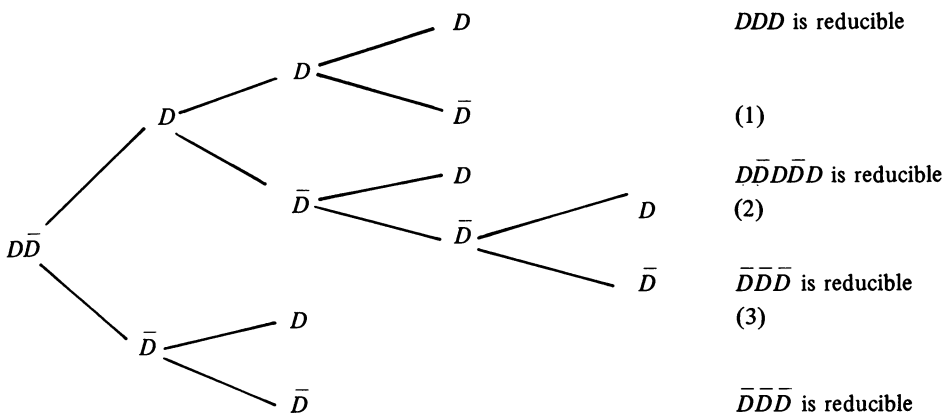
Let $M^+ = \text{Cl}\, \mathcal{O}(\mu^+)$ and $M = \text{Cl}\, \mathcal{O}(\mu)$. Observe that M^+ is a collection of sequences, and M is a collection of bisequences. The flow (M, σ) is the (two-sided) *Morse minimal set*. (The Morse minimal set is usually defined as $\text{Cl}\{\sigma^n\mu: n = 0, \pm 1, \pm 2, \ldots\}$, however since (M, σ) is minimal, nonnegative powers of σ suffice.) The flow (M^+, σ) is the *one-sided Morse minimal set*.

REMARK 1.1. (i) $M^+ = \{x^+ \in S^+ : x \in M\}$.
(ii) (M^+, σ) is a subflow of (P^+, σ).

2. Algorithm sequences. We wish to establish a method for generating irreducible sequences. We shall do this by considering the problem of how to extend an irreducible block to the right to obtain an irreducible block of greater length. The following lemma, which is the one-sided analog of Lemma 4 of [3], states a minimal condition any extension must satisfy.

LEMMA 2.1. *Let $n \geqslant 0$ and $p \geqslant 0$. If $x \in P^+$ and if $x[p; 2^{n+1}] = A_nA_n, A_n\bar{A}_n, \bar{A}_nA_n$ or $\bar{A}_n\bar{A}_n$, then for each $m \geqslant 0$, $x[p + m \cdot 2^n, 2^n] = A_n$ or $\bar{A}_n$.*

In order to repeatedly apply this lemma to increasing values of n, we wish to extend a Morse block $D\bar{D}$ to an irreducible block which ends in a Morse block twice as long. From the following tree, it is evident that there are only three such extensions.



Thus the only ways to extend $D\bar{D}$ to an irreducible block which ends in a Morse block twice as long are

(1) $D\bar{D}DD\bar{D}$,

(2) $D\bar{D}D\bar{D}\bar{D}DD$ and

(3) $D\bar{D}\bar{D}D$.

To make this process rigorous we introduce the following definitions.

Let $\mathfrak{B}' = \{B \in \mathfrak{B} : B$ ends in 01 or 10$\}$.

The *canonical decomposition* of a block $B \in \mathfrak{B}'$ is $B = CD\bar{D}$ where $D\bar{D}$ is the maximal terminal Morse block of B. It should be noted that C might be the empty block. However D is never empty since $B \in \mathfrak{B}'$ if and only if the maximal terminal Morse block of B has length at least 2. To illustrate, we list the canonical decomposition of three blocks (where the dots separate the blocks C, D and $\bar{D}$).

$$01101001 = 0110 \cdot 1001, \qquad 010110 = 01 \cdot 01 \cdot 10, \qquad 0100110 = 010 \cdot 01 \cdot 10.$$

Notice that in the first case, C is the empty block.

Define three maps a_1^*, a_2^*, and a_3^*, each mapping $\mathfrak{B}'$ into $\mathfrak{B}'$, by

$$a_1^*(CD\bar{D}) = CD\bar{D}DD\bar{D}, \qquad a_2^*(CD\bar{D}) = CD\bar{D}D\bar{D}\bar{D}D$$

and

$$a_3^*(CD\bar{D}) = CD\bar{D}\bar{D}D.$$

For example, recalling that $010 \cdot 01 \cdot 10$ is the canonical decompostion of 0100110, we have

$$a_1^*(0100110) = 010 \cdot 01 \cdot 10 \cdot 01 \cdot 01 \cdot 10 = 0100110010110,$$
$$a_2^*(0100110) = 010 \cdot 01 \cdot 10 \cdot 01 \cdot 10 \cdot 10 \cdot 01 = 010011001101001$$

and

$$a_3^*(0100110) = 010 \cdot 01 \cdot 10 \cdot 10 \cdot 01 = 01001101001.$$

To aid in remembering the evaluations of each of these maps, observe the following.

(i) If C is the empty block, then the evaluation of each a_i^* is precisely one of the irreducible extensions of $D\bar{D}$ observed from the tree.

(ii) The a_i^*'s are subscripted in lexicographical order where D proceeds $\bar{D}$.

(iii) The evaluation of each a_i^* ends in a block obtained from a Morse 4-block by substituting D and $\bar{D}$ for 0 and 1.

It is convenient to think of these maps as algorithms for extending blocks in $\mathfrak{B}'$, hence we call them *algorithms*. Composition of algorithms is read from right to left and is denoted by juxtaposition. Thus $a_i^* a_j^*(B)$ means first apply a_j^* to B and then apply a_i^* to $a_j^*(B)$.

The composition of n algorithms is called an *algorithm n-block*. The algorithm n-block B^* is subscripted with positive integers increasing from right to left, i.e. $B^* = b_n^* \ldots b_1^*$. (Of course since a_1^*, a_2^* and a_3^* have been designated as specific algorithms, an algorithm block such as $a_1^* a_2^* a_3^*$ is not at variance with this subscripting convention.)

Let $\mathfrak{B}^*$ denote the set of all nonempty algorithm blocks, and let $\mathfrak{P}^* = \{B^* \in \mathfrak{B}^*: B^*(01) \text{ is irreducible}\}$.

Since our objective is to generate sequences in S^+ as well as blocks in $\mathfrak{B}$, we shall also consider left sequences of algorithms called *algorithm sequences*. They will be denoted x^*, y^* or z^*, subscripted, as in the case of algorithm blocks, from right to left with positive integers, e.g. $x^* = \ldots x_3^* x_2^* x_1^*$. The set of all algorithms sequences is denoted S^*.

Let $\mathfrak{B}'' = \{B \in \mathfrak{B}': l(C) < 2l(D)$ where $CD\bar{D}$ is the canonical decomposition of $B\}$.

The members of S^* may be thought of as maps from $\mathfrak{B}''$ into S^+ as follows. $x^*(B) = y = y_0 y_1 y_2 \ldots$, where for each $n \geqslant 1$, $y[0; n]$ is the initial n-block of $x_n^* \ldots x_1^*(B)$.

Remark. If $B \in \mathfrak{B}''$ and $n \geqslant 1$, then $x_n^* \ldots x_1^*(B)$ is an initial subblock of $x_{n+1}^* \ldots x_1^*(B)$. It follows that $x^*(B)$ is well defined.

It is an easy exercise to show that $\mu^+ = \ldots a_3^* a_3^* a_3^*(01)$.

Let $P^* = \{x^* \in S^*: x^*(01) \in P^+\}$. We shall show that in order to study P^+, it is sufficient (in some sense) to consider P^* (Theorems 2.3 and 2.11).

The following lemma is a direct consequence of Lemma 2.1, and its proof is illustrated by the tree following Lemma 2.1.

LEMMA 2.2. *Let $x \in P^+$ and let $x[p, 2^{k+1}] = D\bar{D}$ where $D = A_k$ or $\bar{A}_k$. Then there exists an integer $r \geqslant 2^{k+2}$ and an algorithm b^* such that $x[p; r] = b^*(D\bar{D})$.*

THEOREM 2.3. *Let $x \in S^+$ where $x_1 = \bar{x}_0$. Then $x \in P^+$ if and only if there exists an algorithm sequence $x^* \in P^*$ such that $x^*(01) = x$ or $\bar{x}$. Furthermore, if such an x^* exists, it is unique.*

PROOF. It suffices to prove $\Rightarrow$. Let $x \in P^+$. Since $\bar{x}$ is also in P^+, we may suppose that $x_0x_1 = 01$. By the definition of P^*, it suffices to find an $x^* \in S^*$ such that $x = x^*(01)$.

Let k be the smallest positive integer such that $x_0 \ldots x_k$ is not an initial subblock of $B^*(01)$ for any $B^* \in \mathfrak{B}^*$.

If $x_2x_3 = 00$, then since $x \in P^+$, $x_4 = 1$. Thus $x_0 \ldots x_4 = 01001 = a_1^*(01)$, so $k \geqslant 5$.

If $x_2x_3 = 01$, then since $x \in P^+$, $x_4x_5 = 10$. Thus $x_0 \ldots x_5 = 010110 = a_2^*(01)$, so $k \geqslant 6$.

If $x_2 = 1$, then since $x \in P^+$, $x_3 = 0$. Thus $x_0 \ldots x_3 = 0110 = a_3^*(01)$, so $k > 4$.

Therefore there exists an algorithm b^* and an integer r, $3 \leqslant r < k$, such that $b^*(01) = x_0 \ldots x_r$.

Let B^* be the algorithm block of greatest length such that $B^*(01)$ is an initial subblock of $x_0 \ldots x_k$. Let $CD\bar{D}$ be the canonical decomposition of $B^*(01)$, and let $C = x_0 \ldots x_{p-1}$. By Lemma 2.2, there exists an integer $t > 2l(D\bar{D})$ and an algorithm c^* such that $x[p; t] = c^*(D\bar{D})$. Thus $x[0; t+p] = c^*B^*(01)$. If $t + p > k + 1$, then $x_0 \ldots x_k$ is an initial subblock of $c^*B^*(01)$, contrary to the choice of k. If $t + p < k + 1$, then $c^*B^*(01)$ is an initial subblock of $x_0 \ldots x_k$, contrary to the choice of B^*. Therefore $x = x^*(01)$ for some $x^* \in S^*$.

It is readily verified that x^* is unique. $\square$

The reader is invited to use the procedure indicated in the proof of Theorem 2.3 to verify that

$$A_0\bar{A}_1A_2\bar{A}_3 \ldots = \ldots x_1^*x_1^*x_1^*(01).$$

It might also be instructive to express the 200-block in the introduction in terms of algorithms.

We may consider the elements of P^+ to be of two basic types–those which begin 01 or 10, and those which begin 001 or 110. Theorem 2.3 related those elements in P^+ of the former type to P^*. Before considering the theorem relating those elements in P^+ of the latter type to P^* (Theorem 2.11), we shall need several facts concerning binary blocks and algorithm blocks.

Let θ be the substitution θ: $0 \to 01$, $1 \to 10$. Extend θ to a map of $\mathfrak{B}$ into $\mathfrak{B}$ by $\theta(b_1 \ldots b_n) = \theta(b_1) \ldots \theta(b_n)$. We shall use the following three properties of the substitution θ without comment.

(i) If $C, D \in \mathfrak{B}$, then $\theta(CD) = \theta(C)\theta(D)$.

(ii) If $B \in \mathfrak{B}$, then $\theta(\bar{B}) = \overline{\theta(B)}$.

(iii) $\theta(A_n) = A_{n+1}$ for each $n \geqslant 0$.

We denote successive applications of θ by exponentiation. Thus $\theta^n(B)$ denotes $\theta(\theta^{n-1}(B))$ for all $n \geqslant 2$.

REMARK 2.4. Let $B^* \in \mathfrak{B}^*$ and let $B \in \mathfrak{B}''$. Then

(i) $\theta[B^*(B)] = B^*(\theta(B))$,

(ii) $\theta[B^*(01)] = B^*a_3^*(01)$.

PROOF. (i): Observe that if $CD\bar{D}$ is the canonical decomposition of B, then $\theta(C) \cdot \theta(D) \cdot \theta(\bar{D})$ is the canonical decomposition of $\theta(B)$. The result then follows via induction on $l(B^*)$.

(ii): Note that $a_3^*(01) = \theta(01)$. $\square$

LEMMA 2.5. *Let $x^* \in S^*$ and let $B \in \mathfrak{B}''$. Then $\theta[x^*(B)] = x^*(\theta(B))$.*

PROOF. Use Remark 2.4. $\square$

The following lemma and its proof are contained in the proof of Theorem 3.1 of [7].

LEMMA 2.6. *Let $B = b_0 \dots b_t \in \mathfrak{B}$ have the property that for at least one of $k = 0$ or $k = 1$, $b_{k+2n+1} = \bar{b}_{k+2n}$ for all nonnegative integers n such that $k + 2n + 1 < t$. Let C be a block with initial symbol c such that CCc is a subblock of B. Then $l(C)$ is even.*

LEMMA 2.7. *Let B be a binary n-block and let $\theta(B) = b_0 \dots b_{2n-1}$. Then $b_{2j+1} = \bar{b}_{2j}$ for all j, $0 \leqslant j < n$.*

PROOF. See §12.29 of [2]. $\square$

LEMMA 2.8. *Let $B^* \in \mathfrak{B}^*$ and let $B^*(01) = c_0 \dots c_t$. Then*

(i) *There exists $k = 0$ or 1 such that*

$$c_{k+2n+1} = \bar{c}_{k+2n} \quad \text{for } 0 \leqslant n \leqslant \tfrac{1}{2}(t - k - 1).$$

(ii) *If B is a block with initial symbol b such that BBb is a subblock of $B^*(01)$, then $l(B)$ is even.*

PROOF. (i): Let $B^* = b_q^* \dots b_1^*$ and let $C^* = b_q^* \dots b_2^*$. If $b_1^* = a_1^*$, then

$$B^*(01) = C^*(01001) = 0 \cdot [C^*(1001)] = 0 \cdot [C^*(\theta(10))] = 0 \cdot \theta[C^*(10)].$$

Thus by Lemma 2.7, (i) holds with $k = 1$.

The proofs for $b_1^* = a_2^*$ and $b_1^* = a_3^*$ are analogous.

(ii): Apply Lemma 2.6 to part (i). $\square$

LEMMA 2.9. *Let $B \in \mathfrak{B}$. Then*

(i) *$B \in \mathfrak{P}$ if and only if $\theta(B) \in \mathfrak{P}$,*

(ii) *$B^* \in \mathfrak{P}^*$ if and only if $B^*a_3^* \in \mathfrak{P}^*$.*

PROOF. (i): ($\Rightarrow$) Let $\theta(B) = b_0 \dots b_{2p-1}$ and let $B = b_0' \dots b_{p-1}'$. Then

$$b_{2i} = b_i' \quad \text{for all } 0 \leqslant i \leqslant p - 1. \tag{1}$$

Suppose $\theta(B) \notin \mathcal{P}$. Then some subblock of $\theta(B)$ has the form CCc, where C is a block with initial symbol c. By Lemma 2.6, $l(C)$ is even, call it $2n$. Let

$$b_k \ldots b_{k+4n} = CCc \text{ be the first subblock of } \theta(B) \text{ of this type.} \tag{2}$$

Thus

$$b_{k+i} = b_{k+2n+i} \quad \text{for } 0 \leqslant i \leqslant 2n. \tag{3}$$

By Lemma 2.7 we have

$$b_{2i+1} = \bar{b}_{2i} \quad \text{for each } i, 0 \leqslant i \leqslant p-1. \tag{4}$$

We show that k is even. Suppose k is odd. Then by (4) and (3), $\bar{b}_{k-1} = b_k = b_{k+2n} = \bar{b}_{k+2n-1}$, that is $b_{k-1} = b_{k+2n-1}$. Similarly $b_{k+2n-1} = b_{k+4n-1}$. Thus

$$b_{k-1} = b_{k+2n-1} = b_{k+4n-1}. \tag{5}$$

Combining (3) and (5), we have $b_{k-1+i} = b_{k-1+2n+i}$ for $0 \leqslant i \leqslant 2n$, i.e. for $C' = b_{k-1} \ldots b_{k+2n-2}$, $C'C'c'$ is a reducible subblock of $\theta(B)$, contradicting (2). Thus k is even; say $k = 2m$.

Define $D = d_0 \ldots d_{n-1}$ by $d_i = b_{k+2i}$ for $0 \leqslant i \leqslant n-1$. Then by (1), $D = b_k b_{k+2} b_{k+4} \ldots b_{k+2n-2} = b_{2m} b_{2m+2} \ldots b_{2m+2n-2} = b'_m b'_{m+1} \ldots b'_{m+n-1}$. Thus DDd_0 is a subblock of B. Therefore $B \notin \mathcal{P}$.

($\Leftarrow$) Suppose $B \notin \mathcal{P}$. Then there exists a block C with initial symbol c such that CCc is a subblock of B. Since c is the initial symbol of both $\theta(C)$ and $\theta(c)$, it follows that $\theta(C)\theta(C)c$ is a subblock of $\theta(B)$. Thus $\theta(B) \notin \mathcal{P}$.

(ii): Combine (i) with Remark 2.4. $\square$

LEMMA 2.10. *$x \in P^+$ if and only if $\theta(x) \in P^+$.*

THEOREM 2.11. *Let $x \in S^+$ with $x_0 = x_1 = \bar{x}_2$. Then $x \in P^+$ if and only if there exists an $x^* \in P^*$ such that*

(i) $x^*(001) = x$ or $\bar{x}$ and

(ii) $x^* a_2^* \in P^*$.

Furthermore if such an x^* exists, it is unique.

PROOF. Without loss of generality we may suppose that $x_0 = 0$. Thus $x_0x_1x_2 = 001$.

($\Leftarrow$) Suppose there exists an $x^* \in S^*$ such that $x^*(001) = x$ or $\bar{x}$ and $x^*a_2^* \in P^*$. By our assumption, $x^*(001) = x$ rather than $\bar{x}$. From Lemma 2.5,

$$\theta(x) = \theta[x^*(001)] = x^*(\theta(001)) = x^*(010110) = x^*a_2^*(01).$$

Since $x^*a_2^* \in P^*$, $\theta(x) \in P^+$ and therefore by (2.10), $x \in P^+$.

($\Rightarrow$) Suppose $x \in P^+$. Then $\sigma x \in P^+$. Furthermore $[\sigma x]_0[\sigma x]_1 = 01$. Thus by Theorem 2.3, there exists a unique $x^* \in P^*$ such that $x^*(01) = \sigma x$. It then follows that

$$x = 0 \cdot [\sigma x] = 0 \cdot [x^*(01)] = x^*(001),$$

thus proving (i). By Lemma 2.10, $\theta(x) \in P^+$. Now applying θ to both sides of (i) and using Lemma 2.5, we have

$$\theta(x) = \theta[x^*(001)] = x^*(\theta(001)) = x^*(010110) = x^*a_2^*(01).$$

Hence $x^*a_2^*(01) \in P^+$, that is $x^*a_2^* \in P^*$, thus proving (ii). $\square$

3. Determination of all irreducible sequences. From Theorems 2.3 and 2.11, in order to determine whether a binary sequence is irreducible, we only need to consider its algorithm sequence. Thus we now wish to determine which algorithm sequences are in $P^* = \{x^* \in S^*: x^*(01) \text{ is irreducible}\}$. We do this by first determining which algorithm blocks are in $\mathcal{P}^*$ (Theorem 3.14), and then concluding that P^* consists of all sequences in S^* which have the property that every block lies in $\mathcal{P}^*$ (Corollary 3.15).

We begin with a remark and a series of lemmas which provide useful information about algorithm blocks in $\mathcal{P}^*$.

LEMMA 3.1. *If $B^* \in \mathcal{P}^*$ and C^* is a subblock of B^*, then $C^* \in \mathcal{P}^*$.*

PROOF. Let $C^* = b_n^* \dots b_k^*$ and let $CD\bar{D}$ be the canonical decomposition of $b_{k-1}^* \dots b_1^*(01)$. (If $k = 1$, then C is the empty block and $D\bar{D} = 01$.) Now use Lemma 2.9 and the fact that $D\bar{D} = A_{t-1}\bar{A}_{t-1} = \theta^{t-1}(01)$ or $D\bar{D} = \bar{A}_{t-1}A_{t-1} = \theta^{t-1}(10)$. $\square$

REMARK 3.2. Let $B^* \in \mathcal{B}^*$. Then

(i) $B^*a_1^*(01) = 0 \cdot [B^*a_3^*(10)]$.

(ii) $B^*a_2^*(01) = 01 \cdot [B^*a_3^*(01)]$.

(iii) $l(B^*a_2^*(01)) = l(B^*a_1^*(01)) + 1$.

(iv) If $x_0 \dots x_p = B^*a_2^*(01)$, then p is odd and $x_{2n+1} = \bar{x}_{2n}$ for $n = 0, 1, \dots, \frac{1}{2}(p-1)$.

Proof of (iv): From (ii) and Remark 2.4,

$$x_0 \dots x_p = 01 \cdot [B^*a_3^*(01)] = \theta(0) \cdot \theta[B^*(01)] = \theta[0 \cdot [B^*(01)]].$$

Therefore $l(x_0 \dots x_p)$ is even, i.e. p is odd. Now apply Lemma 2.7. $\square$

LEMMA 3.3. *Let $B^* \in \mathcal{P}^*$. Then*

(i) If $B^*a_1^* \notin \mathcal{P}^*$, then some initial subblock of $B^*a_1^*(01)$ is of the form BBb.

(ii) If $B^*a_2^* \notin \mathcal{P}^*$, then some initial subblock of $B^*a_2^*(01)$ is of the form BBb.

PROOF. (i): Suppose $B^*a_1^* \notin \mathcal{P}^*$, i.e. $B^*a_1^*(01) = x_0 \dots x_p$ is reducible. If no initial subblock of $x_0 \dots x_p$ is of the form *BBb*, then $x_1 \dots x_p = B^*a_3^*(10)$ is reducible, i.e. $B^*a_3^* \notin \mathcal{P}^*$. But by Lemma 2.9 $B^* \notin \mathcal{P}^*$, contrary to the hypothesis.

(ii): Let $B^*a_2^* \notin \mathcal{P}^*$ and let $B^*a_2^*(01) = x_0 \dots x_p$. Suppose that $x_i \dots x_{i+2k}$ is of the form *BBb*, that is $x_{i+j} = x_{i+j+k}$ for $0 \leq j \leq k$. By an argument similar to (i) we can show that $i < 2$.

Suppose $i = 1$. Then $x_{1+j} = x_{1+j+k}$ for $0 \leq j \leq k$. By Remark 3.2, p is odd and $x_{2n+1} = \bar{x}_{2n}$ for $0 \leq n \leq \frac{1}{2}(p-1)$. In particular, $x_0 = \bar{x}_1$ and $x_{2k} = \bar{x}_{2k+1}$. Furthermore by Lemma 2.6, k is even, so $x_k = \bar{x}_{k+1}$. Thus we have $x_0 = \bar{x}_1 = \bar{x}_{k+1} = x_k$ and $x_0 = \bar{x}_{k+1} = \bar{x}_{2k+1} = x_{2k}$, that is $x_0 = x_k = x_{2k}$. Therefore $x_j = x_{j+k}$ for $0 \leq j \leq k$; equivalently $x_0 \dots x_{2k}$ is an initial block of the form *BBb* as desired. $\square$

LEMMA 3.4. *$B^*a_1^* \in \mathcal{P}^*$ if and only if $B^*a_2^* \in \mathcal{P}^*$.*

PROOF. By Lemma 3.1, if $B^* \notin \mathcal{P}^*$, and $B^*a_1^* \notin \mathcal{P}^*$ and $B^*a_2^* \notin \mathcal{P}^*$. So suppose $B^* \in \mathcal{P}^*$.

Let $x_0 \ldots x_p = B^*a_2^*(01)$ and $y_0 \ldots y_q = B^*a_1^*(01)$. By Remark 3.2, p is odd, $q = p - 1$ and

$$x_1 \ldots x_p = 1 \cdot [B^*a_3^*(01)] = \overline{0 \cdot [B^*a_3^*(10)]} = \overline{y_0 \ldots y_{p-1}}.$$

Equivalently,

$$y_j = \bar{x}_{j+1} \quad \text{for } 0 \leqslant j \leqslant p - 1. \tag{1}$$

($\Leftarrow$) If $B^*a_1^* \notin \mathcal{P}^*$, it follows directly from (1) that $B^*a_2^* \notin \mathcal{P}^*$.

($\Rightarrow$) Suppose $B^*a_2^* \notin \mathcal{P}^*$. Then by Lemma 3.3 an initial subblock of $B^*a_2^*(01)$ is of the form BBb, i.e. there exists an integer $k \geqslant 1$ such that $x_i = x_{i+k}$ for $0 \leqslant i \leqslant k$. From (1), it follows that $y_i = \bar{x}_{i+1} = \bar{x}_{i+1+k} = y_{i+k}$ for $0 \leqslant i \leqslant k - 1$. Furthermore by Remark 3.2, $x_{2k} = \bar{x}_{2k+1}$. Thus $y_k = \bar{x}_{k+1} = \bar{x}_1 = 0$ and $y_{2k} = \bar{x}_{2k+1} = x_{2k} = x_0 = 0$. Since $y_0 = 0$, we have $y_0 = y_k = y_{2k}$. Therefore $y_i = y_{i+k}$ for $0 \leqslant i \leqslant k$, i.e. $y_0 \ldots y_{2k}$ is of the form BBb. Hence $B^*a_1^* \notin \mathcal{P}^*$. $\square$

LEMMA 3.5. *Let $B^* \in \mathcal{B}^*$. Then*

(i) *$B^*a_1^* \in \mathcal{P}^*$ if and only if $B^*a_3^*a_3^*a_1^* \in \mathcal{P}^*$,*

(ii) *$B^*a_2^* \in \mathcal{P}^*$ if and only if $B^*a_3^*a_3^*a_2^* \in \mathcal{P}^*$.*

PROOF. By Lemma 3.4, it suffices to prove (i), and by Lemma 3.1, (i) is valid if $B^* \notin \mathcal{P}^*$. So suppose $B^* \in \mathcal{P}^*$.

Let $x_0 \ldots x_p = B^*a_1^*(01)$ and $y_0 \ldots y_q = B^*a_3^*a_3^*a_1^*(01)$. Then $x_0 \ldots x_p = 0 \cdot [B^*(1001)]$ and

$$y_0 \ldots y_q = 0 \cdot [B^*a_3^*a_3^*(1001)] = 0 \cdot \theta^2[B^*(1001)]. \tag{1}$$

Hence

$$\theta^2(x_t) = y_{4t-3}y_{4t-2}y_{4t-1}y_{4t} \quad \text{for } 1 \leqslant t \leqslant p, \tag{2}$$

and therefore

$$x_t = y_{4t-3} \quad \text{for } 1 \leqslant t \leqslant p. \tag{3}$$

Furthermore, since $\theta^2(b) = b\bar{b}\bar{b}b$, (2) gives us

$$y_{4t-3} = y_{4t} \quad \text{for } 1 \leqslant t \leqslant p. \tag{4}$$

($\Leftarrow$) Suppose $B^*a_1^* \notin \mathcal{P}^*$. By Lemma 3.3 there exists an integer $k \geqslant 1$ such that $x_0 \ldots x_{2k}$ is of the form BBb; equivalently

$$x_i = x_{i+k} \quad \text{for } 0 \leqslant i \leqslant k. \tag{5}$$

By (2) and (5) we have

$$y_{4j-3}y_{4j-2}y_{4j-1}y_{4j} = \theta^2(x_j) = \theta^2(x_{j+k}) = y_{4j+4k-3}y_{4j+4k-2}y_{4j+4k-1}y_{4j+4k}$$

for $1 \leqslant j \leqslant k$, i.e. $y_i = y_{i+4k}$ for $1 \leqslant i \leqslant 4k$.

However by (3), (4) and (5), $y_0 = 0 = x_0 = x_k = y_{4k-3} = y_{4k}$. Thus $y_i = y_{i+4k}$ for $0 \leqslant i \leqslant 4k$. Therefore $y_0 \ldots y_{8k}$ is of the form BBb, and hence $B^*a_3^*a_3^*a_1^* \notin \mathcal{P}^*$.

($\Rightarrow$) Suppose $B^*a_3^*a_3^*a_1^* \notin \mathcal{P}^*$. By Lemma 3.3 there exists an integer $k \geqslant 1$ such that $y_0 \ldots y_{2k}$ is of the form BBb, that is

$$y_i = y_{i+k} \quad \text{for } 0 \leqslant i \leqslant k. \tag{6}$$

We show that k is a multiple of 4. Observe that

$$y_0 \ldots y_8 = 010010110 \tag{7}$$

is the initial 9-block of $B^* a_3^* a_3^* a_1^*(01)$. By (1), $B^* a_3^* a_3^* a_1^*(01) = 0 \cdot \theta^2[B^*(1001)]$ and, since $\theta^2(b) = b\bar{b}\bar{b}b$,

$$y_{4i+1} y_{4i+2} y_{4i+3} y_{4i+4} = 0110 \text{ or } 1001 \quad \text{for } 0 \leqslant i \leqslant \tfrac{1}{4}(q-4). \tag{8}$$

Now by Lemma 2.8, k is even, so if $k = 4n + 2$, then $k + 3 = 4n + 5 = 4(n + 1) + 1$. Consequently by (8),

$$y_{k+3} y_{k+4} y_{k+5} y_{k+6} = 0110 \text{ or } 1001.$$

But by (6) and (7),

$$y_{k+3} y_{k+4} y_{k+5} y_{k+6} = y_3 y_4 y_5 y_6 = 0101.$$

Hence $k = 4n$ for some n. Furthermore from (7), $n \neq 1$ because $y_0 \ldots y_8$ is not of the form BBb. Thus $n \geqslant 2$.

From (6) we have $y_i = y_{i+4n}$ for $0 \leqslant i \leqslant 4n$, and in particular $y_{4j-3} = y_{4j+4n-3}$ for $0 < j \leqslant n$. Thus by (3), $x_j = y_{4j-3} = y_{4(j+n)-3} = x_{n+j}$ for $1 \leqslant j \leqslant n$. Furthermore from (6), (4) and (3), $x_0 = 0 = y_0 = y_{4n} = y_{4n-3} = x_n$. Thus $x_j = x_{j+n}$ for $0 \leqslant j \leqslant n$, i.e. $x_0 \ldots x_{2n}$ is of the form BBb. Therefore $B^* a_1^* \notin \mathcal{P}^*$. $\square$

An *inadmissible block* is an algorithm block of the form $b_{2n+3}^* \ldots b_1^*$ for $n \geqslant 0$ where

(i) $b_1^* = a_1^*$ or a_2^*,

(ii) $b_{2n+3}^* b_{2n+2}^* = a_1^* a_2^*,\ a_2^* a_3^*$ or $a_3^* a_1^*$

and if $n \geqslant 1$

(iii) $b_i^* = a_3^*$ for each i, $2 \leqslant i \leqslant 2n + 1$.

Observe that $b_q^* \ldots b_1^*$ is inadmissible if and only if $b_q^* b_{q-1}^* b_1^*$ is inadmissible and b_{q-1}^* and b_1^* are separated by an even number of a_3^*'s. Thus each of the following is an inadmissible block:

$$a_1^* a_2^* a_1^*, \quad a_2^* a_3^* a_3^* a_3^* a_1^* \quad \text{and} \quad a_3^* a_1^* a_3^* a_3^* a_3^* a_3^* a_2^*.$$

We shall show that $\mathcal{P}^*$ consists of all algorithm blocks which contain no inadmissible subblocks (Theorem 3.14).

LEMMA 3.6. *If $B^* \in \mathcal{P}^*$, then no subblock of B^* is an inadmissible block.*

PROOF. By Lemma 3.1, it suffices to show that if $B^* = b_{2n+3}^* \ldots b_1^*$ is inadmissible, then $B^* \notin \mathcal{P}^*$. Observe that

$$a_1^* a_2^* a_1^*(01) = \underline{010011001011} \cdot \underline{010011001011} \cdot \underline{0},$$

$$a_2^* a_3^* a_1^*(01) = \underline{01001011} \cdot \underline{01001011} \cdot \underline{0} \cdot 01101001$$

and

$$a_3^* a_1^* a_1^*(01) = \underline{010011} \cdot \underline{010011} \cdot \underline{0} \cdot 010110$$

each contain a block of the form BBb as indicated. The result now follows from Lemmas 3.4, 3.5 and induction on n. $\square$

LEMMA 3.7. *Let* $B^* = b_n^* \dots b_1^* \in \mathfrak{B}^*$ *be such that*

(i) $n > 3$,

(ii) *if* C^* *is a proper subblock of* B^*, *then* $C^* \in \mathcal{P}^*$ *and*

(iii) *there exists a block* B *with initial symbol* b *such that* BBb *is an initial subblock of* $B^*(01)$.

Then $b_{n-3}^* \dots b_1^*(01)$ *is a subblock of* B.

PROOF. Use the fact that for any $D^* = d_k^* \dots d_1^* \in \mathfrak{B}^*$,

$$2^{k+1} \leqslant l(D^*(01)) \leqslant 2(2^{k+1} - 1)$$

to show that if $l(b_{n-3}^* \dots b_1^*(01)) \geqslant l(B)$, then (ii) is not satisfied. □

The main result of the section–the determination of $\mathcal{P}^*$– is proved by induction on the length of the algorithm blocks. Lemma 3.8 begins the induction and Lemmas 3.9, 3.11–3.13 are the individual cases we shall need to consider in the inductive portion of the proof. There are 363 algorithm blocks of length less than or equal to 5, thus the verification of Lemma 3.8 was done by computer.

LEMMA 3.8. *Let* $B^* \in \mathfrak{B}^*$ *with* $l(B^*) \leqslant 5$. *Then* $B^* \in \mathcal{P}^*$ *if and only if no inadmissible block appears in* B^*.

LEMMA 3.9. *Let* $B^* \in \mathcal{P}^*$ *be such that*

(i) $l(B^*) \geqslant 3$ *and*

(ii) *every proper subblock of* $B^* a_1^* a_1^*$ *is in* $\mathcal{P}^*$.

Then $B^* a_1^* a_1^* \in \mathcal{P}^*$.

PROOF. Let $B^* = b_n^* \dots b_1^*$. We first note that $B^* a_1^* a_1^*$ cannot be inadmissible, for if $C^* = B^* a_1^* a_1^*$, then $c_2^* \neq a_3^*$.

If $n = 3$, then $l(B^* a_1^* a_1^*) = 5$, thus the lemma is valid by Lemma 3.8.

Let $n > 3$ and suppose $B^* a_1^* a_1^* \notin \mathcal{P}^*$. By Lemma 3.3 there exists a block B with initial symbol b such that BBb is an initial subblock of $B^* a_1^* a_1^*(01)$. By Lemma 3.7,

$$a_1^* a_1^*(01) = A_0 \bar{A}_1 A_2 \bar{A}_2 \text{ is an initial subblock of } B. \tag{1}$$

Thus $B^* a_1^* a_1^*(01) = B^*(A_0 \bar{A}_1 A_2 \bar{A}_2) = A_0 \bar{A}_1 [B^*(A_2 \bar{A}_2)]$.

Observe that $b_1^* \neq a_3^*$, for otherwise, $b_1^* a_1^* a_1^* = a_3^* a_1^* a_1^*$, which is inadmissible.

Let $C^* = b_n^* \dots b_2^*$. If $b_1^* = a_1^*$, then

$$\begin{aligned} B_1^* a_1^* a_1^*(01) &= A_0 \bar{A}_1 \cdot \left[B^*(A_2 \bar{A}_2) \right] = A_0 \bar{A}_1 \cdot \left[C^* a_1^*(A_2 \bar{A}_2) \right] \\ &= A_0 \bar{A}_1 \cdot \left[C^*(A_2 \bar{A}_2 A_2 A_2 \bar{A}_2) \right] = A_0 \bar{A}_1 A_2 \cdot \left[C^*(\bar{A}_3 A_3) \right]. \end{aligned}$$

Similarly we have that if $b_1^* = a_2^*$ then

$$B_1^* a_1^* a_1^*(01) = A_0 \bar{A}_1 A_2 \bar{A}_2 \cdot \left[C^*(A_3 \bar{A}_3) \right].$$

Thus by (1), in each case the block B, and hence the block $a_1^* a_1^*(01) = A_0 \bar{A}_1 A_2 \bar{A}_2$, must appear in either $C^*(A_3 \bar{A}_3) = \theta^3[C^*(01)]$ or $C^*(\bar{A}_3 A_3) = \theta^3[C^*(10)]$. But $\theta^3(0) = A_3$ and $\theta^3(1) = \bar{A}_3$, so $C^*(A_3 \bar{A}_3)$ and $C^*(\bar{A}_3 A_3)$ are each concatenations of A_3's and $\bar{A}_3$'s.

Since $l(A_0\bar{A}_1A_2\bar{A}_2) = 11 < 16 = 2l(A_3)$, $A_0\bar{A}_1A_2\bar{A}_2$ must appear in some $D^{(1)}D^{(2)}D^{(3)}$ where each $D^{(i)}$ is A_3 or $\bar{A}_3$. Furthermore since $A_0\bar{A}_1A_2\bar{A}_2 =$ 01001101001, it follows by inspection that the only appearance of A_3 or $\bar{A}_3$ in $A_0A_1\bar{A}_2A_2$ is as the terminal 8-block. Consequently $A_0\bar{A}_1A_2\bar{A}_2$ must be a subblock of some $D^{(1)}D^{(2)}$ where $D^{(i)} = A_3$ or $\bar{A}_3$. It is now easily shown that $A_0\bar{A}_1A_2\bar{A}_2$ is not a subblock of A_3A_3, $A_3\bar{A}_3$, $\bar{A}_3A_3$, or $\bar{A}_3\bar{A}_3$. Therefore $B^*a_1^*a_1^*(01)$ contains no initial reducible blocks, and hence $B^*a_1^*a_1^* \in \mathcal{P}^*$. □

REMARK 3.10. (i) For all $n \geq 1$, the canonical decomposition of $A_0\bar{A}_2\bar{A}_3 \ldots \bar{A}_n$ is $CD\bar{D}$ where $C = A_0\bar{A}_2A_3 \ldots \bar{A}_{n-1}$ and $D = \bar{A}_{n-1}$.

(ii) For all $n \geq 1$,

$$\underbrace{a_2^*a_2^* \ldots a_2^*}_{n\text{-times}}a_1^*(01) = A_0\bar{A}_2\bar{A}_3 \ldots \bar{A}_{n+2}.$$

(iii) For all $n \geq 2$, $A_1\bar{A}_0A_0\bar{A}_2\bar{A}_3 \ldots \bar{A}_n = A_{n+1}$.

LEMMA 3.11. *Let $B^* \in \mathcal{P}^*$ and suppose that each proper subblock of $B^*a_2^*a_2^*a_1^*$ is in $\mathcal{P}^*$. Then $B^*a_2^*a_2^*a_1^* \in \mathcal{P}^*$.*

PROOF. Let $B^* = b_n^* \ldots b_1^*$ and let $B = A_1\bar{A}_0$. By Remark 3.10, if $b_i^* = a_2^*$ for $1 \leq i \leq n$, then $B^*a_2^*a_2^*a_1^*(01) = A_0\bar{A}_2\bar{A}_3 \ldots \bar{A}_{n+4}$. Hence again by Remark 3.10, $B \cdot [B^*a_2^*a_2^*a_1^*(01)] = A_{n+5} \in \mathcal{P}$. Thus if $B^* = a_2^* \ldots a_2^*$, then $B^*a_2^*a_2^*a_1^* \in \mathcal{P}^*$.

So suppose there exists an integer i, $1 \leq i \leq n$, such that $b_i^* \neq a_2^*$, and let r be the least such integer. Now $b_r^* = a_3^*$, for otherwise $b_r^* = a_1^*$, and then the inadmissible block $a_1^*a_2^*a_2^*$ would be a subblock of $B^*a_2^*a_2^*a_1^*$, which by Lemma 3.6 is not in $\mathcal{P}^*$.

If $r = n$, then by Remark 3.10,

$$B \cdot [B^*a_2^*a_2^*a_1^*(01)] = B \cdot [a_3^*\underbrace{a_2^* \ldots a_2^*}_{(n+1)\text{-times}}a_1^*(01)] = B \cdot \left[a_3^*\left(A_0\bar{A}_2\bar{A}_3 \ldots \bar{A}_{n+3}\right)\right]$$

$$= BA_0\bar{A}_2\bar{A}_3 \ldots \bar{A}_{n+2} \cdot \left[a_3^*\left(\bar{A}_{n+3}\right)\right] = A_{n+3}\bar{A}_{n+3}A_{n+3}.$$

Since $A_{n+3}\bar{A}_{n+3}A_{n+3}$ is a subblock of

$$A_{n+6} = A_{n+3}\bar{A}_{n+3}\bar{A}_{n+3}A_{n+3}\bar{A}_{n+3}A_{n+3}A_{n+3}\bar{A}_{n+3},$$

it follows that $B \cdot [B^*a_2^*a_2^*a_1^*(01)] \in \mathcal{P}$, so $B^*a_2^*a_2^*a_1^* \in \mathcal{P}^*$.

Suppose $r < n$. Let $C^* = b_n^* \ldots b_{r+2}^*$. (If $r = n - 1$, let C^* be the empty algorithm block.) Now $b_{r+1}^* \neq a_2^*$ for otherwise the following situations occur: if $r = 1$, then $b_2^*b_1^*a_2^* = a_2^*a_3^*a_2^*$; and if $r > 1$, then $b_{r+1}^*b_r^*b_{r-1}^* = a_2^*a_3^*a_2^*$. In either case the inadmissible block $a_2^*a_3^*a_2^*$ appears. Therefore $b_{r+1}^* = a_1^*$ or a_3^*.

Suppose $b_{r+1}^* = a_1^*$. Then by Remark 3.10,

$$\begin{aligned}
B\cdot\left[B^*a_2^*a_2^*a_1^*(01)\right] &= B\cdot\left[C^*a_1^*a_3^*\underbrace{a_2^*\ldots a_2^*}_{(r+1)\text{-times}}a_1^*(01)\right]\\
&= B\cdot\left[C^*a_1^*a_3^*\left(A_0\bar{A}_2\bar{A}_3\ldots\bar{A}_{r+3}\right)\right]\\
&= BA_0\bar{A}_2\bar{A}_3\ldots\bar{A}_{r+2}\cdot\left[C^*a_1^*a_3^*\left(\bar{A}_{r+3}\right)\right]\\
&= A_{r+3}\cdot\left[C^*\left(\bar{A}_{r+3}A_{r+3}\bar{A}_{r+3}\bar{A}_{r+3}A_{r+3}\right)\right]\\
&= C^*\left(A_{r+3}\bar{A}_{r+3}A_{r+3}\bar{A}_{r+3}\bar{A}_{r+3}A_{r+3}\right)\\
&= C^*a_2^*\left(A_{r+3}\bar{A}_{r+3}\right) = C^*a_2^*\underbrace{a_3^*\ldots a_3^*}_{(r+3)\text{-times}}(01).
\end{aligned}$$

Thus if $B^*a_2^*a_2^*a_1^* \notin \mathcal{P}^*$, then

$$C^*a_2^*\underbrace{a_3^*\ldots a_3^*}_{(r+3)\text{-times}} \notin \mathcal{P}^*.$$

But by Lemma 2.9, we would then have $C^*a_2^* \notin \mathcal{P}^*$, and hence by Lemma 3.4, $C^*a_1^* \notin \mathcal{P}^*$. But $C^*a_1^* = C^*b_{r+1}^* = b_n^*\ldots b_{r+1}^*$ which by the hypothesis is in $\mathcal{P}^*$. Therefore if $b_{r+1}^* = a_1^*$, then $B^*a_2^*a_2^*a_1^* \in \mathcal{P}^*$.

By a similar argument, if $b_{r+1}^* = a_3^*$ then $B^*a_2^*a_2^*a_1^* \in \mathcal{P}^*$. □

The following two lemmas are proved in a manner similar to that of the previous lemma.

LEMMA 3.12. *Let $B^* \in \mathcal{P}^*$ and suppose that each proper subblock of $B^*a_3^*a_2^*a_1^*$ is in $\mathcal{P}^*$. Then $B^*a_3^*a_2^*a_1^* \in \mathcal{P}^*$.*

LEMMA 3.13. *Let $B^* \in \mathcal{P}^*$ and suppose that each proper subblock of $B^*a_1^*a_3^*a_1^*$ is in $\mathcal{P}^*$. Then $B^*a_1^*a_3^*a_1^* \in \mathcal{P}^*$.*

THEOREM 3.14. *Let $B^* \in \mathcal{B}^*$. Then $B^* \in \mathcal{P}^*$ if and only if no inadmissible block appears in B^*.*

PROOF. ($\Rightarrow$) See Lemma 3.6.

($\Leftarrow$) The proof is by induction on $l(B^*)$. Suppose B^* contains no inadmissible blocks. If $l(B^*) \leqslant 5$, then by Lemma 3.8, $B^* \in \mathcal{P}^*$.

Suppose that $n > 5$, and that if $C^* \in \mathcal{B}^*$, where $l(C^*) < n$, and C^* contains no inadmissible blocks, then $C^* \in \mathcal{P}^*$. Let $B^* = b_n^*\ldots b_1^*$.

Since B^* contains no inadmissible blocks, neither does $b_n^*\ldots b_2^*$. So by the inductive hypothesis, $b_n^*\ldots b_2^* \in \mathcal{P}^*$. If $b_1^* = a_3^*$, then by Lemma 2.9, $B^* \in \mathcal{P}^*$. So suppose $b_1^* \neq a_3^*$. By Lemma 3.4, we may suppose $b_1^* = a_1^*$.

Since B^* contains no inadmissible blocks, $b_3^*b_2^* \neq a_1^*a_2^*$, $a_2^*a_3^*$ or $a_3^*a_1^*$. If $b_2^* = a_1^*$, then by Lemma 3.9, $B^* \in \mathcal{P}^*$. If $b_3^*b_2^* = a_2^*a_2^*$, $a_3^*a_2^*$ or $a_1^*a_3^*$, then by Lemmas 3.11, 3.12 and 3.13 respectively, $B^* \in \mathcal{P}^*$. If $b_3^*b_2^* = a_3^*a_3^*$, then by Lemma 3.5, $B^* \in \mathcal{P}^*$ if and only if $D^* = b_n^*\ldots b_5^*b_4^*b_1^* \in \mathcal{P}^*$. It is easy to check that no inadmissible block appears in D^*. But $l(D^*) = n - 2 < n$, so by the inductive hypothesis, $D^* \in \mathcal{P}^*$. Therefore $B^* \in \mathcal{P}^*$. □

The following corollary is a direct consequence of Theorem 3.14.

COROLLARY 3.15. *Let $x^* \in S^*$. Then $x^* \in P^*$ if and only if no inadmissible blocks appear in x^*.*

REMARK 3.16. A sufficient (although certainly not necessary) condition for an algorithm sequence x^* to be an element of P^* is that none of the blocks $a_1^* a_2^*$, $a_2^* a_3^*$ or $a_3^* a_1^*$ appear in x^*. For example, $\dots a_1^* a_1^* a_1^* \in P^*$.

4. Determination of the one-sided Morse minimal set. Recall that the one-sided Morse minimal set M^+ is the set $\{x^+ \in S^+ \colon x \in M\}$. In this section we show how to determine whether or not a binary sequence $x = x^*(B)$ (where $B =$ 01, 10, 001 or 110) is an element of M^+ by considering x^*.

Let $\mathfrak{M} = \{B \in \mathfrak{B} : B$ appears in $\mu^+\}$; equivalently $\mathfrak{M} = \{B \in \mathfrak{B} : B$ appears in some Morse block$\}$. It is evident that if $B \in \mathfrak{M}$, then every subblock of B is also in $\mathfrak{M}$.

Let $\mathfrak{M}^* = \{B^* \in \mathfrak{B}^* \colon B^*(01) \in \mathfrak{M}\}$ and let $M^* = \{x^* \in S^* \colon x^*(01) \in M^+\}$. Observe that $\mathfrak{M} \subseteq \mathfrak{P}$, $\mathfrak{M}^* \subseteq \mathfrak{P}^*$ and $M^* \subseteq P^*$.

The correspondence between M^+ and M^* is analogous to that of P^+ and P^*. This is reflected in the similarity of the statements of Lemmas 4.3, 4.4 and 4.12 to Theorems 2.3, 2.11 and Corollary 3.15, respectively. As in the case of P^*, we first show that $M^+ \subseteq \{x^*(B) \colon x^* \in M^*$ and $B \in \mathfrak{B}''\}$ (Lemmas 4.3 and 4.4). We then determine $\mathfrak{M}^*$ (Theorem 4.11) and use $\mathfrak{M}^*$ to determine M^* (Corollary 4.12).

LEMMA 4.1. *Let $B \in \mathfrak{B}$. Then $B \in \mathfrak{M}$ if and only if $\theta(B) \in \mathfrak{M}$.*

PROOF. ($\Rightarrow$) Use the definition of $\mathfrak{M}$ and the fact that $\theta(A_n) = A_{n+1}$.

($\Leftarrow$) If $A_n = c_1 c_2 \dots c_{2^n}$, then $c_1 c_3 c_5 \dots c_{2^n - 1} = A_{n-1}$ and $c_2 c_4 c_6 \dots c_{2^n} = \bar{A}_{n-1}$. Thus if $\theta(B)$ is a subblock of A_n, then B is a subblock of A_{n-1} or $\bar{A}_{n-1}$. □

Recalling that for $y, z \in S^+, y \in \mathrm{Cl}\, \mathcal{O}(z)$ if and only if every block which appears in y also appears in z, the following is a direct consequence of Lemma 4.1.

LEMMA 4.2. *Let $x \in S^+$. Then $x \in M^+$ if and only if $\theta(x) \in M^+$.*

LEMMA 4.3. *Let $x \in S^+$ with $x_0 = \bar{x}_1$. Then $x \in M^+$ if and only if there exists an algorithm sequence $x^* \in M^*$ such that $x^*(01) = x$ or $\bar{x}$.*

LEMMA 4.4. *Let $x \in S^+$ with $x_0 = x_1 = \bar{x}_2$. Then $x \in M^+$ if and only if there exists an algorithm sequence $x^* \in M^*$ such that*

(i) *$x^*(001) = x$ or $\bar{x}$ and*

(ii) *$x^* a_2^* \in M^*$.*

PROOF. The proof is analogous to that of Theorem 2.11. □

We shall show that each block in $\mathfrak{M}$ has the property that it can be extended arbitrarily far to the left and still be in $\mathfrak{M}$. We then employ this concept to determine which algorithm blocks map 01 to a block which can be extended arbitrarily far to the left to a block in $\mathfrak{M}$ (Theorem 4.11).

Let $B = b_1 \dots b_n \in \mathfrak{B}$. The *reverse block* of B is the block $B^r = b_n \dots b_1$. Notice that $(BC)^r = C^r B^r$. In the case of Morse blocks, we have $A_{2n}^r = A_{2n}$ and $A_{2n+1}^r = \bar{A}_{2n+1}$.

LEMMA 4.5. *Let $B \in \mathfrak{B}$. Then*

(i) *If $B \in \mathfrak{M}$, then $B^r \in \mathfrak{M}$.*

(ii) *If $B \in \mathfrak{M}$ and $k > 0$, then there exists a k-block C such that $CB \in \mathfrak{M}$.*

(iii) *If B is a 2^n-block such that $BC \in \mathfrak{M}$ with $C = A_nA_n, A_n\overline{A}_n, \overline{A}_nA_n$ or $\overline{A}_n\overline{A}_n$, then $B = A_n$ or $\overline{A}_n$.*

PROOF. (i): Use the fact that $A_{2n}^r = A_{2n}$ for each $n \geq 0$.

(ii): Extend B^r to the right k places to a subblock B^rC^r of some A_n. Then by (i), $CB = (B^rC^r)^r \in \mathfrak{M}$.

(iii): By (i), $C^r \in \mathfrak{M}$, thus $C^r = \underline{\mu}^+[k; 2^{n+1}]$ for some $k \geq 0$. By Lemma 2.2, $B^r = \mu[k + 2^{n+1}; 2^n] = A_n$ or $\overline{A}_n$. Thus $B = A_n$ or $\overline{A}_n$. □

LEMMA 4.6. *If $B^* \in \mathfrak{M}^*$, then every subblock of B^* is in $\mathfrak{M}^*$.*

A *pathological block* is an algorithm block of the form $b^*_{2n+2} \ldots b^*_1$ for $n \geq 0$ where

(i) $b^*_1 = a^*_1$ or a^*_2,

(ii) $b^*_{2n+2} = a^*_1$

and if $n \geq 1$,

(iii) $b^*_i = a^*_3$ for $2 \leq i \leq 2n + 1$.

Observe that $B^* = b^*_q \ldots b^*_1$ is a pathological block if and only if $b^*_q b^*_1 = a^*_1 a^*_1$ or $a^*_1 a^*_2$, and b^*_q and b^*_1 are separated by an even number of a^*_3's. Thus each of the following is a pathological block: $a^*_1 a^*_1$, $a^*_1 a^*_3 a^*_3 a^*_2$ and $a^*_1 a^*_3 a^*_3 a^*_3 a^*_3 a^*_1$.

We shall show that $\mathfrak{M}^*$ consists of all algorithm blocks $B^* \in \mathfrak{P}^*$ which contain no pathological blocks (Theorem 4.11).

REMARK 4.7. Let $n \geq 1$. Then

(i) If n is even, then $\theta^n(01)$ ends in 01.

(ii) If n is odd, then $\theta^n(01)$ ends in 10.

LEMMA 4.8. *If $B^* \in \mathfrak{M}^*$, then no subblock of B^* is a pathological block.*

PROOF. By Lemma 4.6, it suffices to show that if B^* is a pathological block, then $B^* \notin \mathfrak{M}^*$.

Let $B^* = b^*_{2n+2} \ldots b^*_1$ be a pathological block and let $C^* = b^*_{2n+2} \ldots b^*_2$. Observe that

$$C^* = a^*_1 \underbrace{a^*_3 \ldots a^*_3}_{2n\text{-times}},$$

and that $b^*_1 = a^*_1$ or a^*_2.

Suppose $b^*_1 = a^*_1$. Then by Remark 2.4,

$$\begin{aligned} B^*(01) &= C^* a^*_1(01) = C^*(01001) \\ &= 0 \cdot \left[C^*(1001) \right] = 0 \cdot \left[a^*_1 \underbrace{a^*_3 \ldots a^*_3}_{2n\text{-times}}(1001) \right] \\ &= 0 \cdot \left[a^*_1\left(\theta^{2n}\left(\overline{A}_1 A_1\right)\right) \right] = 0 \cdot \left[a^*_1\left(\overline{A}_{2n+1} A_{2n+1}\right) \right] \\ &= 0 \cdot \overline{A}_{2n+1} A_{2n+1} \overline{A}_{2n+1} \overline{A}_{2n+1} A_{2n+1} = 0 \cdot \overline{A}_{2n+1} A_{2n+2} \overline{A}_{2n+2}. \end{aligned}$$

If $B^*(01) \in \mathfrak{M}$, then by Lemma 4.5, there exists a $(2^{2n+1} - 1)$-block D such that $D \cdot 0 \cdot \bar{A}_{2n+1} = A_{2n+2}$ or $\bar{A}_{2n+2}$. In particular $D \cdot 0$ must be $A_{2n+1} = \theta^{2n}(01)$. But by Remark 4.7, $\theta^{2n}(01)$ ends in 01. Hence there is no such block D. Therefore if $b_1^* = a_1^*$, then $B^* \notin \mathfrak{M}^*$.

Similarly if $b_1^* = a_2^*$, then $B^* \notin \mathfrak{M}^*$. □

LEMMA 4.9. *Let $n \geq 2$, let $B^* = b_n^* \ldots b_1^* \in \mathfrak{M}^*$ and let $CD\bar{D}$ be the canonical decomposition of $B^*(01)$. Then*

(i) *If*

$$B^* = \underbrace{a_3^* \ldots a_3^*}_{n\text{-times}},$$

then C is the empty block.

(ii) *If $b_n^* = a_1^*$ or a_2^*, then there exists a block $E \in \mathfrak{M}$ such that $EC = \bar{D}D$.*

(iii) *If*

$$B^* = \underbrace{a_3^* \ldots a_3^*}_{(2k-1)\text{-times}} b_{n-2k+1}^* \ldots b_1^*$$

for some positive integer k such that $2k - 1 < n$ and $b_{n-2k+1}^* \neq a_3^*$, then there exists a block $E \in \mathfrak{M}$ such that $EC = \bar{D}$.

(iv) *If*

$$B^* = \underbrace{a_3^* \ldots a_3^*}_{2k\text{-times}} b_{n-2k}^* \ldots b_1^*$$

for some positive integer k such that $k < 2n$ and $b_{n-2k}^* \neq a_3^*$, then there exists a block $E \in \mathfrak{M}$ such that $EC = D$.

PROOF. (i): Recall from Remark 2.4,

$$\underbrace{a_3^* \ldots a_3^*}_{n\text{-times}}(01) = \theta^n(01) = A_n\bar{A}_n.$$

(ii)–(iv): The proof is by induction on n. Suppose

$$B^* \text{ is not of the form } a_3^* \ldots a_3^*. \tag{1}$$

If $n = 2$, (ii) and (iii) are readily verified, and (iv) is vacuously true.

Proceeding inductively on n, suppose that $n \geq 3$ and that (ii), (iii) and (iv) hold for all algorithm blocks of length less than n.

Let $B^* = b_n^* \ldots b_1^*$ satisfy the hypothesis of (ii), (iii) or (iv).

Let $C^* = b_{n-1}^* \ldots b_1^*$, and let $FG\bar{G}$ be the canonical decomposition of $C^*(01)$. From the inductive hypothesis, it follows that one of three situations occurs: F is the empty block, F is a nonempty terminal block of $\bar{G}G$, or F is a nonempty terminal block of $\bar{G}$. (Note that if F is a terminal block of G, then F is also a terminal block of $\bar{G}G$.) We consider each of these three cases separately.

Case 1. Suppose that F is the empty block. Observe that $G = A_{n-1}$ or $\bar{A}_{n-1}$. Furthermore, 01 is an initial block of $C^*(01)$, thus $G = A_{n-1}$. Now applying Remark 2.4,

$$C^*(01) = A_{n-1}\bar{A}_{n-1} = \theta^{n-1}(01) = \underbrace{a_3^* \ldots a_3^*}_{(n-1)\text{-times}}(01).$$

If $b_n^* = a_3^*$, then $B^* = a_3^* \dots a_3^*$, contrary to (1). So suppose $b_n^* = a_1^*$ or a_2^*.

If $b_n^* = a_1^*$, then

$$B^*(01) = b_n^* C^*(01) = a_1^*(G\bar{G}) = G\bar{G}GG\bar{G}.$$

Hence $C = G$ and $D = \bar{G}G$. Thus for $E = G\bar{G}\bar{G}$, $EC = G\bar{G}\bar{G} \cdot G = \bar{D}D$.

If $b_n^* = a_2^*$, then

$$B^*(01) = b_n^* C^*(01) = a_2^*(G\bar{G}) = G\bar{G}G\bar{G}\bar{G}G.$$

Hence $C = G\bar{G}$ and $D = G\bar{G}$. Thus for $E = \bar{G}G$, $EC = \bar{G}G \cdot G\bar{G} = \bar{D}D$.

Therefore the lemma is valid for Case 1.

Case 2. Suppose F is a nonempty terminal block of $\bar{G}G$. Let $H \in \mathfrak{M}$ such that $HF = \bar{G}G$.

We claim that $b_n^* \neq a_1^*$. For suppose $b_n^* = a_1^*$. Since $B^*(01) \in \mathfrak{M}$, by Lemma 4.5, there exists a block $K \in \mathfrak{M}$ such that $l(KF) = 2l(G)$ and $K \cdot [B^*(01)] \in \mathfrak{M}$. Now

$$K \cdot \left[B^*(01)\right] = K \cdot \left[b_n^*(FG\bar{G})\right] = K \cdot \left[a_1^*(FG\bar{G})\right] = K \cdot FG\bar{G}GG\bar{G},$$

so by Lemma 4.5 we have that $KF = GG$, $G\bar{G}$, $\bar{G}G$ or $\bar{G}\bar{G}$. But F is a terminal block of $\bar{G}G$; thus $KF \neq G\bar{G}$ or $\bar{G}\bar{G}$. Furthermore $KF \neq GG$, for otherwise the reducible block GGG is an initial block of $K \cdot [B^*(01)]$ contrary to $K \cdot [B^*(01)] \in \mathfrak{M}$. Similarly $KF \neq \bar{G}G$, for otherwise $K \cdot [B^*(01)] = \bar{G}GG\bar{G}GG\bar{G}$ which is reducible, again contrary to $K \cdot [B^*(01)] \in \mathfrak{M}$. Therefore there is no such $K \in \mathfrak{M}$. Hence by Lemma 4.5, $B^* \notin \mathfrak{M}^*$, thus proving the claim.

If $b_n^* = a_2^*$, then

$$B^*(01) = b_n^* C^*(01) = a_2^*(FG\bar{G}) = FG\bar{G}G\bar{G}\bar{G}G.$$

Hence $C = FG\bar{G}$ and $D = G\bar{G}$. Thus for $E = H$, $EC = H \cdot FG\bar{G} = \bar{G}GG\bar{G} = \bar{D}D$.

If $b_n^* = a_3^*$, let t be the greatest integer such that $b_t^* \neq a_3^*$. By (1), $t > 1$. By the inductive hypothesis, $(n - 1) - t$ cannot be odd, for then F would be a nonempty terminal block of $\bar{G}$ contrary to our supposition on F. Thus $(n - 1) - t$ is even, and so $n - t$ is odd. Furhermore

$$B^*(01) = b_n^* C^*(01) = a_3^*(FG\bar{G}) = FG\bar{G}\bar{G}G.$$

Hence $C = F$ and $D = G\bar{G}$. Thus for $E = H$, $EC = H \cdot F = \bar{G}G = \bar{D}$.

Therefore the lemma is valid for Case 2.

Case 3. If F is a nonempty terminal block of $\bar{G}$, the proof is similar to that of Case 2. □

LEMMA 4.10. *Let $B^* \in \mathfrak{M}^*$. Then*
(i) $a_2^* B^* \in \mathfrak{M}^*$ *and*
(ii) $a_3^* B^* \in \mathfrak{M}^*$.

PROOF. Use Lemma 4.9. □

THEOREM 4.11. *Let $B^* \in \mathfrak{B}^*$. Then $B^* \in \mathfrak{M}^*$ if and only if $\mathfrak{B}^* \in \mathfrak{P}^*$ and no pathological block appears in B^*.*

PROOF. ($\Rightarrow$) Use Lemma 4.8 and the fact that $\mathfrak{M} \subseteq \mathfrak{P}$.

($\Leftarrow$) Let $B^* \in \mathfrak{P}^*$ and suppose no pathological block appears in B^*. We prove that $B^* \in \mathfrak{M}^*$ by induction on $l(B^*)$.

It is readily verified that

$$B^* \in \mathfrak{M}^* \quad \text{if } l(B^*) = 1. \tag{1}$$

Suppose that $n \geqslant 2$, that $B^* = b_n^* \dots b_1^*$ and that if $C^* \in \mathfrak{P}^*$ such that $l(C^*) < n$ and no pathological block appears in C^*, then $C^* \in \mathfrak{M}^*$.

Since no pathological block appears in B^*, none appears in $b_{n-1}^* \dots b_1^*$. Furthermore $b_{n-1}^* \dots b_1^* \in \mathfrak{P}^*$. Therefore by the inductive hypothesis $b_{n-1}^* \dots b_1^* \in \mathfrak{M}^*$. Thus if $b_n^* = a_2^*$ or a_3^*, then by Lemma 4.10, $B^* \in \mathfrak{M}^*$.

So suppose $b_n^* = a_1^*$. Since no pathological block appears in B^*, $b_{n-1}^* \neq a_1^*$ or a_2^*, that is $b_{n-1}^* = a_3^*$.

By Remark 2.4, if $b_{n-1}^* \dots b_1^* = a_3^* \dots a_3^*$, then $B^*(01) = a_1^*(\theta^{n-1}(01)) = \theta^{n-1}[a_1^*(01)]$. However by (1), $a_1^*(01) \in \mathfrak{M}$, thus by Lemma 4.1, $\theta^{n-1}[a_1^*(01)] \in \mathfrak{M}$. Hence $B^* \in \mathfrak{M}^*$.

If $b_{n-1}^* \dots b_1^* \neq a_3^* \dots a_3^*$, let t be the greatest integer such that $b_t^* \neq a_3^*$. Thus

$$b_n^* \dots b_t^* = a_1^* \underbrace{a_3^* \dots a_3^*}_{(n-t-1)\text{-times}} a_1^* \text{ or } a_1^* \underbrace{a_3^* \dots a_3^*}_{(n-t-1)\text{-times}} a_2^*.$$

Since no pathological block appears in B^*, $n - t - 1$ is odd. Let $CD\bar{D}$ be the canonical decomposition of $b_{n-1}^* \dots b_1^*(01)$. By Lemma 4.9 there is a block $E \in \mathfrak{M}$ such that $EC = \bar{D}$. Thus

$$\begin{aligned} D\bar{D}E \cdot [B^*(01)] &= D\bar{D}E \cdot \left[a_1^*(CD\bar{D})\right] \\ &= D\bar{D}E \cdot CD\bar{D}DD\bar{D} = D\bar{D}\bar{D}D\bar{D}DD\bar{D} \in \mathfrak{M}. \end{aligned}$$

Therefore $B^* \in \mathfrak{M}^*$. $\square$

COROLLARY 4.12. *Let $x^* \in S^*$. Then $x^* \in M^*$ if and only if no inadmissible or pathological block appears in x^*.*

Since $\dots a_1^* a_1^* a_1^* \in P^*$ (see Remark 3.16), but $\dots a_1^* a_1^* a_1^* \notin M^*$, we have that $M^* \subsetneqq P^*$. We are now able to conclude the following.

REMARK 4.13. M^+ is a proper subset of P^+.

Finally, observe that algorithm sequences give an effective method for generating all elements in the one-sided Morse minimal set. The only other known way of doing this is to use Kakutani's procedure to generate bisequences in M (see 12.47–49 of [2]), and then discard their "negative halves".

5. Comparison of P^+ and M^+. Although we know that $M^+ \neq P^+$, we have yet to compare them. In this final section we investigate the size of the set $P^+ - M^+$ and some dynamical differences between the symbolic flows (P^+, σ) and (M^+, σ).

It is a simple exercise to establish how the shift transformation of a binary sequence $x = x^*(01)$ affects the algorithm sequence x^*. If we were to do this, we would notice that, with the exception of the constantly-a_3^* algorithm sequence, the shift affects only finitely many algorithms. (This is due to the fact that if a_1^* or a_2^* appears in B^*, and if $CD\bar{D}$ is the canonical decomposition of $B^*(01)$, then C is not the empty block.)

Observe that in the sequence

$$x = \ldots a_1^* a_1^* a_1^*(01) = A_0 \bar{A}_1 A_2 \bar{A}_3 \ldots,$$

the pathological block $a_1^* a_1^*$ appears arbitrarily far to the left. Thus no matter how many times we shift, we will still have pathological blocks remaining. Therefore not only is $x \notin M^+$, but $\mathcal{O}(x) \cap M^+ = \varnothing$. The following theorem shows the abundance of such $x \in P^+$.

THEOREM 5.1. *There are uncountably many $x \in P^+$ such that $\mathcal{O}(x) \cap M^+ = \varnothing$.*

PROOF. Since $\mathcal{O}(x) \cap M^+ = \varnothing$ provided $\sigma^n x \notin M^+$ for all n, it suffices to find uncountably many algorithm sequences in P^* each of which has pathological blocks arbitrarily far to the left.

Let $E^* = \{\ldots B_2^* a_1^* a_1^* B_1^* a_1^* a_1^*\colon B_i^* = a_3^* a_2^* \text{ or } a_3^* a_3^* a_2^*\}$.

Clearly E^* is uncountable. Furthermore, since none of the blocks $a_1^* a_2^*$, $a_2^* a_3^*$ or $a_3^* a_1^*$ appear in any $z^* \in E^*$, by Remark 3.16, $E^* \subseteq P^*$. However the pathological block $a_1^* a_1^*$ appears arbitrarily far to the left in each $z \in E^*$. Thus for each $z^* \in E^*$, $\mathcal{O}(z^*(01)) \cap M^+ = \varnothing$. $\square$

We now turn to some dynamical aspects of (P^+, σ) and (M^+, σ).

LEMMA 5.2. *Let $B = 01, 10, 001$ or 110 and let $x = x^*(B)$. Then for each $n > 0$ there exists an integer k such that $\sigma^k(x)$ an infinite concatenation of A_n's and $\bar{A}_n$'s.*

PROOF. Let $CD\bar{D}$ be the canonical decomposition of $x_n^* \ldots x_1^*(B)$. It is readily verified by induction on n that $D = A_n$ or $\bar{A}_n$. Let d be the initial symbol of D, thus $D = \theta^n(d)$. Now

$$\begin{aligned} x^*(B) &= \ldots x_{n+2}^* x_{n+1}^*(CD\bar{D}) = C \cdot \left[\ldots x_{n+2}^* x_{n+1}^*(D\bar{D})\right] \\ &= C \cdot \left[\ldots x_{n+2}^* x_{n+1}^*(\theta^n(d\bar{d}))\right] = C \cdot \theta^n\left[\ldots x_{n+2}^* x_{n+1}^*(d\bar{d})\right]. \end{aligned}$$

Let $k = l(C)$, and the desired conclusion follows. $\square$

Let $x \in S^+$. The *ω-limit set* of x is the set $\omega(x) = \{y \in S^+\colon \sigma^{n_i} x \to y$ for some sequence $n_i \to +\infty\}$. Note that $y \in \omega(x)$ if and only if every block which appears in y also appears arbitrarily far to the right in x. Furthermore note that for any positive integer k, $\omega(x) = \omega(\sigma^k x)$. Since S^+ is compact, we also have that $(\omega(x), \sigma)$ is a subflow of (S^+, σ).

THEOREM 5.3. *Let $x \in P^+$. Then $\omega(x) = M^+$.*

PROOF. By Lemma 5.2, for each $n > 0$, A_n appears arbitrarily far to the right in x. Since every block which appears in M^+ must appear in some A_n, it follows that $M^+ \subseteq \omega(x)$.

To prove $\omega(x) \subseteq M^+$, we show that if $y \notin M^+$ then $y \notin \omega(x)$. Suppose $y \notin M^+$. Then there exists a block B of y such that B appears in no A_n. Let m be such that $l(B) \leqslant 2^m$. By Lemma 5.2 there exists an integer k such that $\sigma^k(x)$ is an infinite concatenation of A_m's and $\bar{A}_m$'s. By our choice of m, if B appears in $\sigma^k(x)$, then B appears in A_mA_m, $A_m\bar{A}_m$, $\bar{A}_mA_m$ or $\bar{A}_m\bar{A}_m$. But all four of these appear in $A_{m+3} = A_m\bar{A}_m\bar{A}_mA_m\bar{A}_mA_mA_m\bar{A}_m$. Thus B does not appear in $\sigma^k(x)$. Hence $y \notin \omega(\sigma^k(x)) = \omega(x)$. $\square$

COROLLARY 5.4. *(M^+, σ) is the unique minimal subflow of (P^+, σ).*

Let (X, σ) be a subflow of (S^+, σ). A point $x \in X$ is *nonwandering* provided that for every open neighborhood U of $x \in X$, $\{n \geqslant 0: \sigma^n(U) \cap U \neq \varnothing\}$ is infinite. Equivalently, we have that x is nonwandering if and only if for each initial block B of x and for each positive integer N, there exists $y \in X$ and an integer $n > N$ such that B is an initial block of both y and $\sigma^n y$.

Let $\Omega(X)$ denote the set of nonwandering points of X. We remark that $(\Omega(x), \sigma)$ is a subflow of (X, σ).

THEOREM 5.5. $\Omega(P^+) = M^+$.

PROOF. Use Lemma 5.2 and Corollary 5.4. $\square$

COROLLARY 5.6. (i) *The topological entropy of (P^+, σ) is zero.*
(ii) *(P^+, σ) is uniquely ergodic.*

PROOF. Invariant measures and topological entropy are concentrated on the nonwandering set (see pp. 35 and 138 of [1]). Klein has shown in [5] that the topological entropy of (M^+, σ) is zero and that (M^+, σ) is uniquely ergodic. The result now follows from Theorem 5.5. $\square$

We see from 5.3–5.6 that the "dynamically interesting" part of P^+ is M^+. Theorem 5.5 shows that $P^+ - M^+$ is in some sense "small". Our final result shows that it is in some sense "large".

THEOREM 5.7. *There is no countable set E such that $P^+ = \bigcup_{x \in E} \mathrm{Cl}\, \mathcal{O}(x)$.*

PROOF. By Theorem 5.3, for each $x \in P^+$, $\omega(x) = M^+$. Thus for each $x \in P^+$, $\mathrm{Cl}\, \mathcal{O}(x) = \mathcal{O}(x) \cup \omega(x) = \mathcal{O}(x) \cup M^+$.

Let $P^+ = \bigcup_{x \in E} \mathrm{Cl}\, \mathcal{O}(x)$. Then $P^+ = [\bigcup_{x \in E} \mathcal{O}(x)] \cup M^+$.

By Theorem 5.1 there is an uncountable subset F of P^+ such that $F \cap M^+ = \varnothing$. Therefore $F \subseteq \bigcup_{x \in E} \mathcal{O}(x)$. Since $\mathcal{O}(x)$ is countable for each $x \in E$, E must be uncountable. $\square$

REFERENCES

1. M. Denker, C. Grillenberger and K. Sigmund, *Ergodic theory on compact spaces*, Lecture Notes in Math., vol. 527, Springer-Verlag, Berlin and New York, 1976, pp. 35, 138.

2. W. H. Gottschalk and G. A. Hedlund, *Topological dynamics*, Amer. Math. Soc. Colloq. Publ., vol. 36, Amer. Math. Soc., Providence, R. I., 1968.

3. ______, *A characterization of the Morse minimal set*, Proc. Amer. Math. Soc. **15** (1964), 70–74.

4. G. A. Hedlund, *Remarks on the work of Axel Thue*, Nordisk Mat. Tidskr. **15** (1967), 148–150.

5. B. G. Klein, *Homomorphisms of symbolic dynamical systems*, Math. Systems Theory **6** (1972), 107–122.

6. M. Morse, *Recurrent geodesics on a surface of negative curvature*, Trans. Amer. Math. Soc. **22** (1921), 84–100.

7. M. Morse and G. A. Hedlund, *Unending chess, symbolic dynamics and a problem in semigroups*, Duke Math. J. **11** (1944), 1–7.

8. A. Thue, *Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihen*, Norske Vid. Skr. I Mat.-Natur. Kl., No. 1, 1912, 67 pp.

Department of Mathematics, Mary Washington College, Fredericksburg, Virginia 22401

Current address: Department of Mathematics, Pensacola Christian College, Pensacola, Florida 32503