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THE ESSENTIAL NORM OF AN OPERATOR AND ITS ADJOINT

BY

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ABSTRACT. We consider the relationship between the essential norm of an operator T on a Banach space X and the essential norm of its adjoint T^*. We show that these two quantities are not necessarily equal but that they are equivalent if X^* has the bounded approximation property. For an operator into the sequence space c_0, we give a formula for the distance to the compact operators and show that this distance is attained. We introduce a property of a Banach space which is useful in showing that operators have closest compact approximants and investigate which Banach spaces have this property.

Let X denote a Banach space, let $\mathcal{L}(X)$ denote the set of all operators (bounded linear transformations) on X, and let $\mathcal{K}(X)$ denote the set of all compact operators on X (recall that an operator is said to be compact if the image of the unit ball has compact closure). The essential norm $\|T\|_e$ of an operator T is the distance to the compact operators:

$$\|T\|_e = \inf\{\|T - K\|: K \in \mathcal{K}(X)\}.$$

In §1 we show that if X is any Banach space and $T \in \mathcal{L}(X)$, then $\|T^*\|_e = \|T^{**}\|_e$. We show that for many Banach spaces one has $\|T\|_e = \|T^*\|_e$, but we also give an example where $\|T\|_e = 1$ and $\|T^*\|_e = \frac{1}{2}$. We prove that if X^* has the metric approximation property then $\|T^*\|_e \geqslant \frac{1}{2}\|T\|_e$ for all $T \in \mathcal{L}(X)$ (see Theorem 3). We have recently been informed that A. M. Davie has proved that $\operatorname{dist}(T^*, \mathcal{F}(X^*)) \geqslant \frac{1}{3}\operatorname{dist}(T, \mathcal{F}(X))$ for any Banach space X, where $\mathcal{F}(X)$ denotes the finite rank operators on X.

In Proposition 4 we consider operators from an arbitrary Banach space into the sequence space c_0. We give a formula for the distance of such an operator from the compact operators. Furthermore we show that this distance is always attained.

In §2 we introduce a property of a Banach space which is useful in showing that operators have closest compact approximants. We investigate which Banach spaces have this property and we discuss the relationship to M-ideals.

The strong operator topology (SOT) will be useful in both sections of the paper. Recall that a net $\{T_\alpha\} \subset \mathcal{L}(X)$ is said to converge to $T \in \mathcal{L}(X)$ in the strong operator topology (written $T_\alpha \to T$ (SOT)) if $\|T_\alpha x - Tx\| \to 0$ for each vector $x \in X$.

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1. The essential norm of an adjoint operator. For $T \in \mathcal{L}(X)$ we have $\|T\| = \|T^*\|$. Since an operator is compact if and only if its adjoint is compact, it is easy to see that

$$\|T\|_e \geqslant \|T^*\|_e. \tag{1}$$

In this section we study the relationship between $\|T\|_e$ and $\|T^*\|_e$.

We regard X as being a subspace of X^{**} under the canonical embedding. If X is reflexive, then $\|T\|_e = \|T^*\|_e$, as is seen by applying (1) to T^*. More generally, suppose there is a projection P of norm one of X^{**} onto X. If $T \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X^{**})$, then

$$\|T^{**} - K\| \geqslant \|P(T^{**} - K)|X\| = \|T - (PK|X)\| \geqslant \|T\|_e$$

and therefore $\|T^{**}\|_e \geqslant \|T\|_e$, which by (1) implies that $\|T\|_e = \|T^*\|_e$.

It is well known (and easy to see) that if Y is any Banach space, then there is a projection of norm one of Y^{***} onto Y. Thus the above comments imply that if X is the dual of some Banach space, then $\|T\|_e = \|T^*\|_e$ for every $T \in \mathcal{L}(X)$. If u is a positive σ-finite measure, then there is a norm-one projection of $L^1(u)^{**}$ onto $L^1(u)$ [4, Chapter III, Theorem 8, pp. 163–164]. Thus $\|T\|_e = \|T^*\|_e$ for every $T \in \mathcal{L}(L^1(u))$. These results might lead one to believe that $\|T\|_e$ always equals $\|T^*\|_e$; however we now give a counterexample.

EXAMPLE 1. There exists a Banach space X and an operator $T \in \mathcal{L}(X)$ such that $\|T\|_e \neq \|T^*\|_e$.

Let $X = l^1 \oplus c_0$ with the norm $\|(x, y)\| = \|x\|_1 + \|y\|_\infty$. Thus $X^* = l^\infty \oplus l^1$ with norm $\|(a, b)\| = \max(\|a\|_\infty, \|b\|_1)$. Define $T \in \mathcal{L}(X)$ by $T(x, y) = (0, x)$. Then $\|T\| = 1$ and $T^* \in \mathcal{L}(X^*)$ is given by $T^*(a, b) = (b, 0)$. We now show that $\|T\|_e = 1$. Let e_n denote the vector that is zero except for a one in the nth place. Let K be any compact operator on X. Then

$$\|T - K\| = \|T^* - K^*\| \geqslant \|(T^* - K^*)(0, e_n)\| \geqslant \|(e_n, 0)\| - \|K^*(0, e_n)\|. \tag{2}$$

Since $(0, e_n) \to (0, 0)$ weak-* in X^* and K^* is weak-* continuous, $K^*(0, e_n) \to (0, 0)$ weak-* in X^*. Since K^* is compact this implies that $\|K^*(0, e_n)\| \to 0$. Thus (2) shows that $\|T\|_e \geqslant 1$, as asserted.

We now show that $\|T^*\|_e \leqslant \frac{1}{2}$. Let $d = (1, 1, 1, \ldots) \in l^\infty$. Define $l \in \mathcal{L}(X^*)$ by

$$L(a, b) = \left(\frac{1}{2}\left(\sum b_n\right)d, 0\right).$$

Since L has a one-dimensional range, it is compact. Also

$$\begin{aligned}
\|(T^* - L)(a, b)\| &= \left\|\left(b - \frac{1}{2}\left(\sum b_n\right)d, 0\right)\right\| \\
&= \frac{1}{2}\|(b_1 - b_2 - b_3 - \cdots, -b_1 + b_2 - b_3 - b_4 \cdots, \cdots)\|_\infty \\
&\leqslant \frac{1}{2}\sum |b_n| \leqslant \frac{1}{2}\|(a, b)\|.
\end{aligned}$$

Therefore $\|T^*\|_e \leqslant \|T^* - L\| \leqslant \frac{1}{2}$ completing the example.

Note that the remarks preceding the example show that if $T \in \mathcal{L}(l^1)$, then $\|T\|_e = \|T^*\|_e$. Also, if $T \in \mathcal{L}(c_0)$, then $\|T\|_e = \|T^*\|_e$. (Proposition 4(iii) gives a formula for $\|T\|_e$. Properties of l^1 can then be used to verify that $\|T\|_e = \|T^*\|_e$; we omit the details.) Example 1 thus shows that the direct sum of two spaces for which equality holds may fail to have this property.

Example 1 raises the question of whether for each Banach space X there is a constant $c > 0$ such that $\|T\|_e \geqslant \|T^*\|_e \geqslant c\|T\|_e$ for all $T \in \mathcal{L}(X)$. The example presented shows that it may be necessary to choose $c \leqslant \frac{1}{2}$. In fact for the above example it can be shown that $\|T^*\|_e = \frac{1}{2}\|T\|_e$. Theorem 3 will show that the constant $c = \frac{1}{2}$ will work for almost all the common Banach spaces (see the Remark following Theorem 3). First we require a lemma.

Recall that a Banach space X is said to have the λ-metric approximation property if there is a net $\{T_\alpha\}$ of finite rank operators on X such that $\|T_\alpha\| \leqslant \lambda$ for all α, and $\|T_\alpha x - x\| \to 0$ for each $x \in X$; i.e., $T_\alpha \to 1$ (SOT).

The following lemma shows that the operators on X^* that arise in the definition of the λ-metric approximation property for X^* can actually be taken to be adjoints of operators on X. This result has been independently obtained in the separable case by M. Feder [**6**, Proposition 4]. Lemma 2 will also be used in §2.

LEMMA 2. *Suppose that X^* has the λ-metric approximation property. Then there is a net $\{S_\alpha\} \subset \mathcal{L}(X)$ such that S_α^* has finite rank, $\|S_\alpha^*\| \leqslant \lambda$ and $S_\alpha^* \to 1$ (SOT).*

PROOF. Let $\{T_\alpha\} \subset \mathcal{L}(X^*)$ be a net of finite rank operators that satisfies the conditions of the definition of the λ-metric approximation property. Let $\{\varepsilon_\alpha\}$ be a bounded net of positive numbers such that $\varepsilon_\alpha \to 0$ (for example, take $\varepsilon_\alpha = 1/(1 + \dim T_\alpha X^*)$). By a result based on the principle of local reflexivity (see [**9**, Corollary 3.2]), for each α there exists a finite-dimensional operator $A_\alpha \in \mathcal{L}(X)$ such that $A_\alpha^* | T_\alpha X^* = T_\alpha | T_\alpha X^*$ and $\|A_\alpha\| \leqslant \|T_\alpha\|(1 + \varepsilon_\alpha)$.

Now for any $\theta \in X^*$ and any α we have

$$\begin{aligned} A_\alpha^*\theta &= A_\alpha^*(\theta - T_\alpha\theta) + A_\alpha^*(T_\alpha\theta) = A_\alpha^*(\theta - T_\alpha\theta) + T_\alpha(T_\alpha\theta) \\ &= A_\alpha^*(\theta - T_\alpha\theta) + T_\alpha(T_\alpha\theta - \theta) + T_\alpha\theta. \end{aligned}$$

Since $\{\|A_\alpha^*\|\}$ and $\{\|T_\alpha\|\}$ are bounded and $T_\alpha\theta \to \theta$, the above equation shows that $A_\alpha^*\theta \to \theta$. Let $S_\alpha = A_\alpha/(1 + \varepsilon_\alpha)$. Then the net $\{S_\alpha^*\}$ satisfies all the conditions necessary for the λ-metric approximation property for X^*. Q.E.D.

THEOREM 3. *Let X be a Banach space such that X^* has the λ-metric approximation property. Then*

$$\|T\|_e \geqslant \|T^*\|_e \geqslant (1/(1 + \lambda))\|T\|_e$$

for every $T \in \mathcal{L}(X)$.

PROOF. Let $T \in \mathcal{L}(X)$. Let $\{S_\alpha\} \subset \mathcal{L}(X)$ be the net of operators whose existence is guaranteed by the lemma. Let $R_\alpha = 1 - S_\alpha$. Then $\|R_\alpha\| \leqslant 1 + \lambda$ and $R_\alpha^* \to 0$ (SOT). We have $T = TR_\alpha + TS_\alpha$. Since TS_α is compact we have $\|T\|_e \leqslant \underline{\lim}\|TR_\alpha\|$.

Let $K \in \mathcal{K}(X^*)$. Then

$$(1 + \lambda)\|T^* - K\| \geq \|R_\alpha^*(T^* - K)\| \geq (\|R_\alpha^* T^*\| - \|R_\alpha^* K\|).$$

Since $R_\alpha^* \to 0$ (SOT) and $\{\|R_\alpha^*\|\}$ is bounded and K is compact, we have $\|R_\alpha^* K\| \to 0$. Hence

$$(1 + \lambda)\|T^*\|_e \geq \overline{\lim}\, \|R_\alpha^* T^*\| \geq \underline{\lim}\, \|TR_\alpha\| \geq \|T\|_e.$$

This completes the proof.

REMARK. Many common Banach spaces have the 1-metric approximation property, which is usually called the metric approximation property. To see that l^∞ has the metric approximation property, let $\alpha = A_1 \cup A_2 \cup \cdots \cup A_n$ denote a partition of the positive integers into disjoint sets $A_1, \ldots, A_n$. Define $T_\alpha: l^\infty \to l^\infty$ by $(T_\alpha x)(j) = x(m)$, where m is the smallest integer such that $m \in A_k$ and where k is such that $j \in A_k$. We say that $\alpha \geq \alpha'$ if the partition corresponding to α is finer than the partition corresponding to α'. It is now easy to verify that the net $\{T_\alpha\}$ has the properties required by the definition of the metric approximation property. Finally, the space $X^* = l^\infty \oplus l^1$ occurring in Example 1 has the metric approximation property because it is the (sup-norm) direct sum of two spaces with this property.

We conclude this section with some results concerning operators from an arbitrary Banach space X into c_0. Recall that the dual of c_0 can be identified with l^1. Let $\{e_n\}$ $(n = 1, 2, \ldots)$ denote the usual coordinate vectors in l^1. Let $\mathcal{K}(X, c_0)$ denote the space of all compact operators from X into c_0.

PROPOSITION 4. *Let X be a Banach space and let* $T \in \mathcal{L}(X, c_0)$. *Then*

(i) $\|T\| = \|T^*\| = \sup\|T^* e_n\|$.

(ii) *T is compact if and only if* $\|T^* e_n\| \to 0$.

(iii) $\operatorname{dist}(T, \mathcal{K}(X, c_0)) = \overline{\lim}\|T^* e_n\|$.

(iv) *There exists a closest compact operator to T.*

PROOF. (i) This is easy.

(ii) One easily shows that if $\|T^* e_n\| \to 0$, then T^* is compact. Conversely, suppose that T^* is compact, but that $\|T^* e_n\| > \varepsilon > 0$ for infinitely many values of n. Choose a subsequence (still denoted $\{e_n\}$) for which $T^* e_n$ is norm convergent. Since $e_n \to 0$ weak-$*$ and T^* is weak-$*$ continuous, $T^* e_n \to 0$ weak-$*$. Hence $T^* e_n$ converges to zero in norm, which is a contradiction.

(iii) and (iv) We may assume that T is not compact and thus $\overline{\lim}\|T^* e_n\| > 0$. Let $K: X \to c_0$ be compact. Then

$$\|T - K\| = \|T^* - K^*\| \geq \overline{\lim}\, \|(T^* - K^*)e_n\| \geq \overline{\lim}\, \|T^* e_n\|. \tag{3}$$

To complete the proof we must show that there is a compact operator K whose distance to T is equal to the right-hand side of (3). Let

$$r_k = \max\left(0, \frac{\|T^* e_k\| - \overline{\lim}\, \|T^* e_n\|}{\|T^* e_k\|}\right).$$

Thus $r_k \to 0$ as $k \to \infty$. Define $K: X \to c_0$ by

$$Kx = (r_1(Tx, e_1), r_2(Tx, e_2), \ldots).$$

Since $Tx \in c_0$ we see that $Kx \in c_0$ and K is a bounded linear transformation from X into c_0. One verifies that $K^* e_k = r_k T^* e_k$; thus $\|K^* e_k\| \to 0$ and therefore K is compact by (ii). Finally, by (i)

$$\|T - K\| = \sup\|(T^* - K^*)e_n\| = \overline{\lim}\,\|T^* e_n\|.$$

This completes the proof of Proposition 4.

2. The Basic Inequality. Part (iv) of Proposition 4 shows that every operator into c_0 has a closest compact approximant. In this section we will study a general property which ensures the existence of closest compact operators.

DEFINITION. A Banach space X is said to satisfy the Basic Inequality if for each $T \in \mathcal{L}(X)$ and each bounded net $\{A_\alpha\} \subset \mathcal{L}(X)$ such that $A_\alpha \to 0$ (SOT) and $A_\alpha^* \to 0$ (SOT) the following is true. For each $\varepsilon > 0$ there exists an index β such that

$$\|T + A_\beta\| \leqslant \varepsilon + \max(\|T\|, \|T\|_e + \|A_\beta\|).$$

The Basic Inequality was originally defined in [2] where, however, sequences were used instead of nets. Theorem 2 of that paper states that l^p $(1 < p < \infty)$ satisfies the Basic Inequality. This theorem remains true with the new definition of the Basic Inequality and with the same proof (but replacing sequences by nets). It was stated in [2] (see the end of §2) that l^1 satisfies the Basic Inequality; however we will see in Theorem 7 that with the new definition this is no longer true. We now restate Theorem 1 of [2] using nets rather than sequences.

THEOREM. *Let X be a Banach space that satisfies the Basic Inequality and let $T \in \mathcal{L}(X) \sim \mathcal{K}(X)$. Let $\{T_\alpha\} \subset \mathcal{K}(X)$ be a bounded net of compact operators such that $T_\alpha \to T$ (SOT) and $T_\alpha^* \to T^*$ (SOT). Then there exists a sequence of indices $\{\alpha(k)\}$ and a sequence of nonnegative real numbers $\{a_k\}$ such that $\Sigma a_k = 1$ and $\|T - K\| = \|T\|_e$, where $K = \Sigma a_k T_{\alpha(k)}$.*

The proof is the same as the proof of Theorem 1 of [2], changing sequences to nets where appropriate. With the new definition of the Basic Inequality, the Corollary to Theorem 1 [2] which showed the nonuniqueness of K is still valid (with the same proof).

Theorem 5 gives conditions on a Banach space which ensure that every operator has a closest compact operator. It is an improvement of the second corollary following Theorem 2 of [2]. A Banach space is said to have the bounded approximation property if it has the λ-metric approximation property (which was defined in the previous section) for some λ.

THEOREM. 5. *If X is a Banach space that satisfies the Basic Inequality, and if X^* has the bounded approximation property, then each operator on X has a closest compact approximant.*

PROOF. By hypothesis, X^* has the λ-metric approximation property for some λ. Thus by Lemma 2 there exists a net $\{S_\alpha\} \subset \mathcal{L}(X)$ such that S_α^* has finite rank, $\|S_\alpha^*\| \leqslant \lambda$ and $S_\alpha^* \to 1$ (SOT). It follows that S_α has finite rank and $S_\alpha \to 1$ (WOT); here WOT denotes the weak operator topology. We now use the net $\{S_\alpha\}$ to construct a new net of finite rank operators, bounded by λ, converging to 1 (SOT) and such that the adjoints also converge to 1 (SOT).

For each fixed index β,

$$1 \in \overline{\{ S_\alpha : \alpha \geqslant \beta \}}^{\mathrm{WOT}} \subset \overline{\operatorname{conv}\{S_\alpha : \alpha \geqslant \beta\}}^{\mathrm{WOT}} = \overline{\operatorname{conv}\{S_\alpha : \alpha \geqslant \beta\}}^{\mathrm{SOT}}.$$

Here conv denotes the convex hull and the bar denotes closure in the indicated topology; the last equality follows from Corollary VI.1.5, p. 477 of [4]. Thus for each β and each SOT-open subset $\mathcal{O}$ of $\mathcal{L}(X)$ containing 1, there exists $V_{\beta,\mathcal{O}} \in \mathcal{O} \cap \operatorname{conv}\{S_\alpha : \alpha \geqslant \beta\}$. Note that $V_{\beta,\mathcal{O}}$ has finite rank and $\|V_{\beta,\mathcal{O}}\| \leqslant \lambda$. The SOT-open subsets of $\mathcal{L}(X)$ containing 1 are directed by reverse inclusion and the pairs $(\beta, \mathcal{O})$ are directed by the usual product ordering. It immediately follows that $V_{\beta,\mathcal{O}} \to 1$ (SOT). We now show that $V_{\beta,\mathcal{O}}^* \to 1$ (SOT) also. Indeed, let $\mathfrak{p}$ be an SOT-open convex subset of $\mathcal{L}(X^*)$ containing 1. For β sufficiently large, $S_\alpha^* \in \mathfrak{p}$ for all $\alpha \geqslant \beta$. Thus $V_{\beta,\mathcal{O}}^* \in \mathfrak{p}$ for all such β. Thus $V_{\beta,\mathcal{O}}^* \to 1$ (SOT) as claimed.

Let $T \in \mathcal{L}(X)$. To show that T has a closest compact approximant, let $T_{\beta,\mathcal{O}} = TV_{\beta,\mathcal{O}}$. Then $\{T_{\beta,\mathcal{O}}\}$ is a uniformly bounded net of finite rank operators such that $T_{\beta,\mathcal{O}} \to T$ (SOT) and $T_{\beta,\mathcal{O}}^* \to T^*$ (SOT). As noted above, Theorem 1 of [2] applies in these circumstances, and thus T has a closest compact approximant. Q.E.D.

We note that the proof of Theorem 5 shows that if X^* has the λ-metric approximation property, then X also has the λ-metric approximation property (see [12, p. 34]). Our proof further shows that there is a net of operators in $\mathcal{L}(X)$ ($V_{\beta,\mathcal{O}}$ in the proof) such that these operators and their adjoints both satisfy the conditions required for the λ-metric approximation property (on X and X^*, respectively).

Theorem 5 raises the question of which Banach spaces satisfy the Basic Inequality. For example, consider the question of whether every operator on $L^p[0, 1]$ has a closest compact approximant. For $p = 2$ the answer is yes and for $p = 1$ the answer is no [5]; the question is open for other values of p. Since $L^p[0, 1]$ has the bounded approximation property, Theorem 5 would give an affirmative answer to this question if $L^p[0, 1]$ satisfied the Basic Inequality. Unfortunately, the following theorem shows that this is not the case. It will be convenient to let L^p denote the usual Lebesgue space on the unit circle with normalized Lebesgue measure.

THEOREM 6. *Let $1 \leqslant p < \infty, p \neq 2$. Then L^p does not satisfy the Basic Inequality.*

PROOF. For $f \in L^p$, let $\hat{f}(n)$ denote the nth Fourier coefficient of f. Define T: $L^p \to L^p$ by $Tf = \hat{f}(0)$. Since T is a rank one operator, $\|T\|_e = 0$. Define A_n: $L^p \to L^p$ by $A_n f = \hat{f}(n)z^n$. Then $A_n^* f = \hat{f}(-n)z^{-n}$. By the Riemann-Lebesgue Lemma, $A_n \to 0$ (SOT) and $A_n^* \to 0$ (SOT).

The operator U_n: $L^p \to L^p$ defined by $(U_n f)(z) = f(z^n)$ is an isometry on L^p.

Thus for $f \in L^p$,

$$\|(T + A_1)f\|_p = \|U_n(T + A_1)f\|_p = \|(T + A_n)(U_n f)\|_p \leq \|T + A_n\| \, \|U_n f\|_p = \|T + A_n\| \, \|f\|_p.$$

Thus $\|T + A_1\| \leq \|T + A_n\|$. By Theorem 1 of **[14]** and the comment immediately following it, we see that $\|T + A_1\| > 1$ (for $p \neq 2$).

Clearly $\max\{\|T\|, \|T\|_e + \|A_n\|\} = 1$. Since $\|T + A_n\| \geq 1 + (\|T + A_1\| - 1)$, we see that the Basic Inequality fails when $\varepsilon = (\|T + A_1\| - 1)/2$. Q.E.D.

The above proof does not work for $p = \infty$ because it is not true that $A_n^* \to 0$ (SOT) when $p = \infty$.

In **[2]** it was shown that if $1 < p < \infty$, then l^p satisfies the Basic Inequality. We now show that this is false for $p = 1$ and $p = \infty$.

THEOREM 7. *The spaces l^1 and l^∞ do not satisfy the Basic Inequality.*

PROOF. Let $\{x_\alpha\}$ be a net of vectors in l^1 such that $\|x_\alpha\|_1 = 1$, $x_\alpha(1)$ (which is the first component of x_α) is zero and $x_\alpha \to 0$ weakly. It is possible to choose such a net because 0 is always in the weak closure of the unit sphere of an infinite dimensional Banach space (see **[3**, Chapter 15, p. 331, Problem I**]**). Similarly, let $\{\varphi_\beta\}$ be a net of vectors in l^∞ such that $\|\varphi_\beta\| = 1$ and $\varphi \to 0$ weakly (in particular, $\varphi_\beta \to 0$ weak-$*$).

Define $A_{\alpha,\beta}\colon l^1 \to l^1$ by $A_{\alpha,\beta}y = (x_\alpha \otimes \varphi_\beta)(y) = (y, \varphi_\beta)x_\alpha$. Clearly $\|A_{\alpha,\beta}\| = 1$. The pairs $\langle \alpha, \beta \rangle$ are ordered by the usual product ordering: $\langle \alpha_1, \beta_1 \rangle \leq \langle \alpha_2, \beta_2 \rangle$ if and only if $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$.

For each $y \in l^1$,

$$\|A_{\alpha,\beta}y\| = |(y, \varphi_\beta)| \, \|x_\alpha\| = |(y, \varphi_\beta)| \to 0$$

since $\varphi_\beta \to 0$ weak-$*$. Thus $A_{\alpha,\beta} \to 0$ (SOT). Similarly, since $x_\alpha \to 0$ weakly, we see that $A_{\alpha,\beta}^* \to 0$ (SOT). Finally, since $\varphi_\beta \to 0$ weakly, we see that $A_{\alpha,\beta}^{**} \to 0$ (SOT).

Let $\{e_n\}$ denote the standard basis vectors in l^1; so $e_n(k) = \delta_{nk}$. Define T: $l^1 \to l^1$ by $Tx = (\Sigma x(n))e_1$. Thus $\|T\| = 1$. Since T has rank one, $\|T\|_e = 0$.

For any operator $S\colon l^1 \to l^1$, it is easy to see that $\|S\| = \sup\|Se_n\|_1$. Now

$$\|(T + A_{\alpha,\beta})e_n\|_1 = \|e_1 + A_{\alpha,\beta}e_n\|_1 = \|e_1\|_1 + \|A_{\alpha,\beta}e_n\|_1$$

where the last equality holds because e_1 and $A_{\alpha,\beta}e_n = (e_n, \varphi_\beta)x_\alpha$ have disjoint supports (since $x_\alpha(1) = 0$). Thus

$$\|T + A_{\alpha,\beta}\| = \sup_n \|(T + A_{\alpha,\beta})e_n\|_1 = 1 + \sup_n \|A_{\alpha,\beta}e_n\|_1 = 2.$$

However, $\max\{\|T\|, \|T\|_e + \|A_{\alpha,\beta}\|\} = 1$. Thus the Basic Inequality fails for l^1. Since $\|T^* + A_{\alpha,\beta}^*\| = \|T + A_{\alpha,\beta}\| = 2$ while $\max\{\|T^*\|, \|T^*\|_e + \|A_{\alpha,\beta}^*\|\} = 1$ (and recalling that $A_{\alpha,\beta}^{**} \to 0$ (SOT)), we see that the Basic Inequality also fails for l^∞. Q.E.D.

Theorem 5 gives one method for showing that each operator on X has a closest compact operator. We would now like to discuss certain similarities with another method, namely, the method of M-ideals. Let F be a closed subspace of a Banach space E; the annihilator of F in E^* will be denoted by $F^\perp$. Then F is called an M-ideal if E^* has a direct sum decomposition $E^* = F^\perp \oplus G$ (for some closed subspace $G \subset E^*$) such that $\|\psi + \varphi\| = \|\psi\| + \|\varphi\|$ for all $\psi \in F^\perp$, $\varphi \in G$.

Alfsen and Effros proved that if F is an M-ideal in E, then each element of E has a closest element in F (see **[1**, Corollary 5.6]). For example, $\mathcal{K}(l^p)$ is an M-ideal in $\mathcal{L}(l^p)$ for $1 < p < \infty$ (see **[7]**), and thus each operator on l^p has a closest compact approximant. This result also follows from our Theorem 5 (since l^p satisfies the Basic Inequality; see **[2**, Theorem 2]). On the other hand it is known that $\mathcal{K}(X)$ is not an M-ideal in $\mathcal{L}(X)$ if $X = l^1$ or $X = l^\infty$ (see **[10]**, **[15]**). Note the similarity to our Theorem 7; neither of these spaces satisfies the Basic Inequality.

We turn now to L^p. Here Lima has shown **[11]** that if $1 < p < \infty, p \neq 2$, then $\mathcal{K}(L^p)$ is not an M-ideal in $\mathcal{L}(L^p)$. Our Theorem 6 shows that for these values of p, L^p does not satisfy the Basic Inequality.

The above discussion raises the question of the relation between M-ideals and the Basic Inequality. For example, consider the following two properties that a Banach space X might have: (1) $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$; (2) X satsifies the Basic Inequality. Does either of these properties imply the other?

As another piece of evidence for a possible relation between these two concepts we note that in both cases the closest compact approximant to a noncompact operator is never unique (see **[8]** and **[2**, Corollary to Theorem 1]).

We give a final example where the Basic Inequality and M-ideals have been used to prove the same result. In **[2**, Theorem 4] it was shown using the Basic Inequality that each function in L^∞ has a closest approximant in $H^\infty + C$. D. Luecking **[13]** has recently given a proof of this fact using M-ideals.

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