CANONICAL EMBEDDINGS¹

BY

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ABSTRACT. In this paper the authors compare the embedding of a compact Riemann surface in its tangent bundle to the embedding as the diagonal in the product. These embeddings are proved to be first, but not second, order equivalent. The embedding of a hyperelliptic curve in its tangent bundle is described in an explicit way. Although it is not possible to be so explicit in the other cases, it is shown that in all cases, if the Riemann surface R has genus greater than two, then the blowdown of the zero section of the tangent bundle and the blowdown of the diagonal in the product have the same Hilbert polynomial.

1. Introduction. In [4], we discussed general facts about embeddings of compact manifolds, particularly in codimension 1 with a negative normal bundle (see §5). In this paper we shall be concerned with a very special case: that of a compact Riemann surface of genus $g \geq 2$ into its tangent bundle (as the zero section). In this case the tangent bundle is negative and the zero section can be blown down (for terminology and notation, see [2]). We shall describe, as fully as we can, the analytic properties of the isolated singularity appearing at the blown-down zero section. There is a big difference between the hyperelliptic and nonhyperelliptic cases. For our purposes, a hyperelliptic Riemann surface is one which can be realized as a 2-sheeted cover of $\mathbf{P}^1$ ([3, p. 247], see Definition 3.1 below). The function theory on such a surface can be made explicit, and as a result we can write down the equations defining the blown-down variety. In the nonhyperelliptic case this is not possible; however, Noether's theorem (Theorem 1.7) makes the situation much easier to deal with in a descriptive way (see Theorem 5.6 of [4]).

In this section we shall collect the necessary preliminary results, then in §2 we discuss the nonhyperelliptic case. In §§3–5 we deal with the hyperelliptic case in full detail. Finally in §6 we compare this embedding with the diagonal embedding in the Cartesian product; these embeddings are equivalent to first order (Theorem 4.2 of [4]).

First, we summarize some general remarks about the Grauert blowdown. Let X be an analytic space and Q a compact subvariety of X . Let $\pi: X \rightarrow \tilde{X}$ be the quotient map identifying Q to a point, denoted q . We make $\tilde{X}$ into a ringed space

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via the presheaf

$$\tilde{x}^\emptyset(U) = x^\emptyset(\pi^{-1}(U)), \quad U \text{ open in } X. \quad (1.1)$$

This makes $\pi: X \rightarrow \tilde{X}$ a morphism.

1.2. DEFINITION. Q is *exceptional* in X if $(\tilde{X}, \tilde{x}^\emptyset)$ is an analytic space.

If X is normal, $\tilde{x}^\emptyset$ is a sheaf of germs of functions on $\tilde{X}$, and $\tilde{X}$ is also normal. Note that, by definition

$$\tilde{x}^\emptyset_q = H^0(Q, x^\emptyset), \quad \tilde{x}m_q = H^0(Q, \mathcal{G})$$

where m_q is the maximal ideal at q and $\mathcal{G}$ is the ideal sheaf of Q in X . We should not expect that

$$H^0(Q, \mathcal{G}^k) = \tilde{x}m_q^k, \quad k > 1$$

(see Proposition 6.5), so we cannot study the filtration of $\tilde{x}Q_q$ by powers of the maximal ideal via the corresponding filtration on X along Q . Instead we have to use a result in [1]: Let $\hat{m}^k$ be the ideal sheaf on X generated by $\pi^{-1}m_q^k$ ($\hat{m}^k = x^\emptyset \cdot \pi^{-1}m_q$). For $\mathcal{F}$ a sheaf on X , $\pi_*\mathcal{F}$ is the direct image sheaf.

1.3. THEOREM [1, p. 113]. $\pi_*(\hat{m}^k)_q \subset m_q^{F(k)}$ where $F(k) \rightarrow \infty$ as $k \rightarrow \infty$.

REMARK. In general $\mathcal{G}^k$ and $\hat{m}^k$ are different. Let p_1 and p_2 be two points on a compact Riemann surface R such that $H^0(R, (p_1 + p_2)) \cong \mathbb{C}$ (if R has genus 2 this is generically the case). Let σ generate $H^0(R, (p_1 + p_2))$ and let $r \in R$ be such that $\sigma(r) \neq 0$. Then $H^0(R, (p_1 + p_2 - r)) = 0$. For if τ is the canonical section of (r) ,

$$H^0(R, (p_1 + p_2 - r)) \xrightarrow{\tau} H^0(R, (p_1 + p_2))$$

is injective. If ξ were a nonzero section of $(p_1 + p_2 - r)$, $\tau \cdot \xi$ would be a section of $(p_1 + p_2)$ vanishing at r and thus not a multiple of σ .

Now $(p_1 + p_2 - r)$ is a positive divisor, so if L is the line bundle of this divisor, L^{-1} is negative, and the zero section Q can be blown down to a point q . We have the exact sequence

$$0 \rightarrow H^0(L^{-1}, \mathcal{G}^2) \rightarrow H^0(L^{-1}, \mathcal{G}) \rightarrow H^0(R^1, \mathcal{G}/\mathcal{G}^2).$$

But $\mathcal{G}/\mathcal{G}^2 = (p_1 + p_2 - r)$, thus $H^0(L^{-1}, \mathcal{G}) = m_q \cong H^0(L^{-1}, \mathcal{G}^2)$, so $\hat{m}_q \subset \mathcal{G}^2$ is distinct from $\mathcal{G}$.

Let R be a compact Riemann surface. We shall let $T \rightarrow R$ denote the tangent bundle to R and $K \rightarrow R$ its dual (the *canonical bundle* of R). In general, if $L \rightarrow R$ is any line bundle we shall let Z (or Z_L if the context requires that L be specified) represent the zero section of L . If the genus g of R is at least 2, then T is a negative bundle and its zero section can be blown down to a point in a normal analytic space V [2]. K^n can be viewed as the sheaf of germs of holomorphic functions on T which are homogeneous of degree n on the fibers; these are the *n-fold differentials* on R . Thus $H^0(R, K^n)$ ($n \geq 0$) can be identified with a subspace of $\mathcal{O}_{v,0}$, the local ring of V at the blown-down zero section. Conversely, if $f \in \mathcal{O}_{v,0}$, we can lift f back to T and expand it in a Taylor series along Z . The terms in this Taylor series are in $H^0(R, K^n)$, $n \geq 0$. Thus the ring $\mathcal{K}(R) = \bigoplus_{n \geq 0} H^0(R, K^n)$ is contained in $\mathcal{O}_{v,0}$ and every element of $\mathcal{O}_{v,0}$ can be approximated (in any reasonable topology) by

$\mathcal{H}(R)$. Thus $\mathcal{H}(R)$ serves to determine the variety V (in fact, in the algebraic sense it is just the function ring of V).

Let $L \rightarrow R$ be any line bundle on R , and $\sigma_0, \dots, \sigma_{n-1} \in H^0(R, L^*)$. Since these are holomorphic functions on L which are linear on the fibers, the map $F = (\sigma_0, \dots, \sigma_{n-1}): L \rightarrow \mathbb{C}^n$ maps L onto a set $F(L)$ which is homogeneous (if $z \in F(L)$, $t \in \mathbb{C}$, then $tz \in F(L)$). If the σ 's have no base point (for each $p \in R$, there is a j such that $\sigma_j(p) \neq 0$), then F is proper (since $F^{-1}(0) = Z$ is compact) and $F(L)$ is an analytic subvariety of $\mathbb{C}^n$. For $z_0, \dots, z_{n-1}$, coordinates for $\mathbb{C}^n$, let $[z_0, \dots, z_{n-1}]$ represent the corresponding homogeneous coordinates for $\mathbb{P}^{n-1}$, and $\pi: \mathbb{C}^n - \{0\} \rightarrow \mathbb{P}^{n-1}$ the projection map $(z_0, \dots, z_{n-1}) \rightarrow [z_0, \dots, z_{n-1}]$. Since $\pi \circ F$ is constant on the fibers of L it defines a map $[F]: R \rightarrow \mathbb{P}^{n-1}$ such that $F(L) - \{0\} = \pi^{-1}([F](R))$ and $F(L)$ is the cone over the projective variety $[F](R)$.

1.4. PROPOSITION. $F: L - Z \rightarrow \mathbb{C}^n - \{0\}$ is an embedding if and only if $[F]: R \rightarrow \mathbb{P}^{n-1}$ is an embedding.

PROOF. (We may replace R by any compact manifold.) $L - Z$ and $\mathbb{C}^n - \{0\}$ are fibered by $\mathbb{C}^*$, F preserves the fibers, and $[F]$ is the induced fiber map. Since F is linear and thus automatically biholomorphic on the fibers we easily see that F is injective if and only if $[F]$ is injective.

Let $p \in R$ and suppose $\sigma_0(p) \neq 0$ (this is true for some σ). Using inhomogeneous coordinates on $\mathbb{P}^{n-1}$: $u_j = z_j z_0^{-1}$ ($1 \leq j \leq n-1$), then the equations $u_j = \sigma_j / \sigma_0$, $1 \leq j \leq n-1$, define the map $[F]$, and we can take $z_0, u_1, \dots, u_{n-1}$ as coordinates for $\mathbb{C}^n$; so F is defined by $z_0 = \sigma_0$, $u_j = \sigma_j \sigma_0^{-1}$. Since $d\sigma_0$ is not zero on the direction $\{u_j = \text{constant}, 1 \leq j \leq n-1\}$, $\text{rank}_\xi dF = \text{rank}_p d[F] + 1$ for $\xi \neq 0$ in the fiber over p . Thus F has maximal rank at ξ if and only if $[F]$ has maximal rank at p .

1.5. PROPOSITION. Suppose $[F]: R \rightarrow \mathbb{P}^{n-1}$ is an embedding. If R is not a projective space and F linear, then V has a singularity at the vertex.

PROOF. If 0 is not a singular point of V , we find $f_1, \dots, f_{n-d}$ ($d = \dim V$) vanishing on V with $df_1(0), \dots, df_{n-d}(0)$ independent. But since V is homogeneous these differentials also vanish on V . But $\dim V = d$, so V must coincide with the linear space on which these differentials vanish. Thus $[F](R)$ is a linearly embedded projective space.

In this article we shall make a close study of the Grauert blowdown $\tilde{T}$ of the tangent bundle T to a compact Riemann surface R of genus $g > 2$. Let $w_0, \dots, w_{g-1}$ be a basis for $H^0(R, K)$, and consider the associated map $F: T \rightarrow \mathbb{C}^g$. Since there are no base points, $[F]$ is well defined. We have to distinguish the cases where $[F]$ is or is not an embedding. The following results are classical, and essential to our work. (See Proposition 4.2.)

1.6. THEOREM [3, pp. 247, 255]. If R is hyperelliptic, $[F](R)$ is the normal rational curve parametrized by $t \rightarrow [1, t, \dots, t^{g-1}]$ and $[F]$ is a two-sheeted branched covering. If R is not hyperelliptic $[F]$ is an embedding.

Part (1) of the following result is a theorem of Max Noether and (2) is due to Petri.

1.7. THEOREM [3, p. 253], [5], [6], [7]. *Let R be a nonhyperelliptic Riemann surface of genus $g \geq 3$, and let K be the canonical bundle of R . Let $S^*H^0(K)$ be the symmetric algebra of $H^0(K)$, i.e., $S^*H^0(K) = \bigoplus_{n \geq 0} S^n H^0(K)$ with its ring structure.*

(1) *The canonical map $\phi: S^*H^0(K) \rightarrow \bigoplus_{n \geq 0} H^0(K^n)$ is surjective.*

(2) *The kernel of ϕ is generated by its elements of degree 2 and 3, except for $g = 3$, when it is generated by an element of degree four.*

2. Nonhyperelliptic singularities. Let R be a nonhyperelliptic Riemann surface of genus $g \geq 3$. Let $\{w_0, \dots, w_{g-1}\}$ be a basis for $H^0(R, K)$, the space of holomorphic differentials of order 1. Then, as we have observed, (Theorem 1.2), the mapping $[F] = [w_0, \dots, w_{g-1}]$ of R into $\mathbf{P}^{g-1}$ is an embedding. Considered as a map $F = (w_0, \dots, w_{g-1})$ of the tangent bundle T into $\mathbf{C}^g$, F maps T onto the cone $V = \pi^{-1}([F](R))$, blowing the zero section down to the vertex (by Proposition 1.4). (Here $\pi: \mathbf{C}^g - \{0\} \rightarrow \mathbf{P}^{g-1}$ is the defining map.) $T \rightarrow R$ is the pull-back via $[F]$ of the tautological bundle on $\mathbf{P}^{g-1}$ and therefore $F: T \rightarrow V$ is the quadratic transform of the origin of V . By Noether's theorem the hypotheses of Proposition 5.6 of [4] are satisfied, so V is normal. Summarizing we have

2.1. LEMMA. *Let R be a nonhyperelliptic Riemann surface. The image V of the tangent bundle T via a basis of $H^0(K)$ is an analytic subvariety of $\mathbf{C}^g$ with a normal isolated singularity at the origin. T is the quadratic transform of V .*

Furthermore, Theorem 1.7 applies immediately to give

2.2. THEOREM. (i) $g = 3$. V is a hypersurface in $\mathbf{C}^3$ defined by a homogeneous polynomial of degree 4.

(ii) $g > 3$. V is the zero locus in $\mathbf{C}^g$ of a set of homogeneous polynomials of degree 2 and 3.

In either case, V is the cone over the canonical curve $[F](R)$.

2.3. PROPOSITION. *The multiplicity $\mu_0(V)$ of V at the origin is $2g - 2$.*

PROOF. Since V is the cone over the curve $C = [F](R)$ in $\mathbf{P}^{g-1}$, $\mu_0(V) = \deg C$. Let

$$H = \left\{ \sum_{j=0}^{g-1} a_j z_j = 0 \right\}$$

be a hyperplane in $\mathbf{P}^{g-1}$. By Bezout's theorem [3, p. 670] $\deg C = (C, H)$ where (C, H) is the intersection multiplicity. But (C, H) is the degree of the divisor $\sum_{j=0}^{g-1} a_j w_j$ on R . Since $\sum a_j w_j$ is a holomorphic one-form on R it has degree $2g - 2$, unless it is identically zero. But, since the w_j are independent, $\sum a_j w_j = 0$ only when all the a_j are zero.

2.4. PROPOSITION. *The minimal embedding dimension of V at 0 is g .*

PROOF. Suppose f is holomorphic in a neighborhood of 0, and vanishes identically on V . Let $df(0) = \sum_{j=0}^{g-1} a_j z_j$. Since V is a cone, $df(0)$ vanishes identically on V also. But $df(0) \circ F = \sum a_j w_j \equiv 0$ implies $a_0 = \cdots = a_{g-1} = 0$, so $df(0) = 0$. Thus V is already minimally embedded in $\mathbb{C}^g$.

Finally, our last observation holds for the Grauert blow down of T in both the hyperelliptic and nonhyperelliptic cases.

2.5. PROPOSITION. V is Gorenstein.

PROOF. By definition, we have to produce a global nowhere-vanishing 2-form on $V - \{0\} = T - Z$. Let $\{U_\alpha\}$ be a system of local trivializations of T with coordinates z_α, t_α (t_α is the fiber coordinate). The transition functions are

$$z_\alpha = f_{\alpha\beta}(z_\beta), \quad t_\alpha = f'_{\alpha\beta}(z_\beta)t_\beta.$$

We easily compute that

$$\frac{dz_\alpha \wedge dt_\alpha}{t_\alpha^2} = \frac{dz_\beta \wedge dt_\beta}{t_\beta^2}$$

so the global form is given by

$$w = \frac{dz_\alpha \wedge dt_\alpha}{t_\alpha^2} \text{ in } U_\alpha.$$

3. The hyperelliptic case: the algebra of holomorphic differentials. We continue to study the tangent bundle T of a (nonsingular) curve R of genus $g \geq 2$. However, for the next few sections we shall deal only with the case where R is hyperelliptic.

3.1. DEFINITION [3, pp. 247, 255]. R is *hyperelliptic* if R can be realized as the desingularization of the curve in $\mathbb{C}^2$ with equation

$$\rho(u, z) = u^2 - \prod_{j=1}^{2g+1} (z - e_j) = 0 \quad (3.2)$$

(with points over infinity adjoined).

In (3.2) the e_j , $1 \leq j \leq 2g+1$, are distinct nonzero complex numbers (the exclusion of 0 is a convenience and imposes no restrictions).

We consider u and z to be meromorphic functions on R satisfying the relation (3.2) and generating the field of rational functions on R . We shall now compute the algebra $K(R) = \bigoplus_{n \geq 0} H^0(K^n)$ of holomorphic differentials in terms of z and u . By the Riemann-Roch theorem

$$\dim H^0(K^n) = \begin{cases} g & \text{if } n = 1, \\ (2n-1)(g-1) & \text{if } n \geq 2. \end{cases} \quad (3.3)$$

For each j

$$w_j = z^j dz / u \quad (3.4)$$

is a meromorphic differential on R .

3.5. PROPOSITION. $w_0, \dots, w_{g-1}$ are holomorphic differentials and give a basis for $H^0(K)$.

PROOF. At finite points where $u \neq 0$, clearly the w_j are holomorphic. Let $(0, e_p) \in R$. At such a point u is a local coordinate since

$$\frac{\partial p}{\partial z}(0, e_p) = - \prod_{k \neq p} (e_p - e_k) \neq 0.$$

Call this number d_p . Then, in terms of u

$$z = e_p - d_p^{-1}u^2 + \cdots, \quad dz = -2d_p^{-1}udu + \cdots, \quad (3.6)$$

where the ellipsis refers to higher powers of u . Thus $u^{-1}dz$ (and with it all the w_j) is holomorphic at $(0, e_p)$.

At ∞ we take a local parameter t so that $z = t^{-2}$. Then, in terms of t , u has a zero of order $2g + 1$, so

$$u(t) = v(t)t^{-(2g+1)} \quad (3.7)$$

with $v(0) \neq 0$. Then

$$u^{-1}dz = -2v^{-1}t^{2g-2}dt, \quad w_j = -2v^{-1}t^{2(g-1-j)}dt, \quad (3.8)$$

so w_j is holomorphic as long as $0 \leq j \leq g-1$. Clearly these w_j are independent (over $\mathbb{C}$) so form a basis for $H^0(K)$.

We turn now to $H^0(K^n)$ for $n \geq 2$. The differential

$$\theta = z^i u^j (dz/u)^n, \quad 0 \leq i, j,$$

is holomorphic over finite points since $u^{-1}dz$ is. At ∞ , computing as above, we find

$$\theta = (-2)^n v^{j-n} t^e dt,$$

where $e = 2n(g-1) - (2i + 2gj + j)$. Since v is invertible, the differential θ is holomorphic if and only if

$$2i + 2gj + j \leq 2n(g-1). \quad (3.9)$$

We can select i and j so as to give basis for $H^0(K^n)$; to do that carefully we consider different cases.

$$g = 2, \quad n = 2. \quad \{z^i(dz/u)^2, 0 \leq i \leq 2\} \text{ are a basis.} \quad (3.10)$$

In this case (3.9) becomes $2i + 5j \leq 4$, so these are the only allowable values of i and j . A computation at ∞ shows these to be independent.

$$g = 2, \quad n > 2. \quad \begin{cases} z^i(dz/u)^n, & 0 \leq i \leq n, \\ z^i u(dz/u)^n, & 0 \leq i \leq n-3, \end{cases} \quad (3.11)$$

are a basis.

Here (3.9) becomes $2i + 5j \leq 2n$. Taking $j = 0, 1$ we obtain the indicated ranges of i . Since this gives us $2n-1$ differentials, we need only show that they are independent. Computing at ∞ , the expression (3.11) becomes

$$\begin{aligned} k_i(t)t^{2(n-i)}dt, & \quad 0 \leq i \leq n, \\ h_i(t)t^{2(n-i)-5}dt, & \quad 0 \leq i \leq n-3, \end{aligned}$$

with the k_i, h_i nonvanishing. Since the powers of t are all distinct, these are independent.

$$g > 2, \quad n \geq 2. \quad \begin{cases} z^i(dz/u)^n, & 0 \leq i \leq n(g-1), \\ z^i u(dz/u)^n, & 0 \leq i \leq (n-2)(g-1) + (g-3), \end{cases} \quad (3.12)$$

are a basis.

These are $(2n-1)(g-1)$ forms satisfying the condition (3.9), so are in $H^0(K^n)$. Again, we can demonstrate their independence by calculating at ∞ and comparing the exponents of t .

Thus (3.9)–(3.12) gives us a basis for each $H^0(K^n)$. We consider now the algebraic structure of the ring $K(R) = \bigoplus_{n \geq 0} H^0(K^n)$. We shall describe a system of (algebraic) generators and then compute the relations.

For $g = 2$, let

$$\sigma_0 = u(dz/u)^3. \quad (3.13)$$

For $g > 2$, let

$$\tau_j = z^j u(dz/u)^2, \quad 0 \leq j \leq g-3. \quad (3.14)$$

Recall also the first-order forms $w_0, \dots, w_{g-1}$ defined by (3.4).

3.15. THEOREM. (I) $g = 2$.

(i) Every element of $H^0(K^2)$ is a quadratic polynomial in w_0, w_1 , hence $S^2 H^0(K) \rightarrow H^0(K^2)$ is surjective;

(ii) $S^3 H^0(K) \rightarrow H^0(K^3)$ is not surjective; however

(iii) for $n \geq 3$, every element of $H^0(K^n)$ is a polynomial in w_0, w_1, σ_0 ; this polynomial is a sum of homogeneous terms of degree 1 or 0 in σ_0 and correspondingly of degree $n-3$ or n in w_0, w_1 . Thus

(iv) for $n \geq 3$, $S^{n-3} H^0(K) \otimes H^0(K^3) \rightarrow H^0(K^n)$ is surjective.

(II) $g > 2$.

(i) $S^2 H^0(K) \rightarrow H^0(K^2)$ is not surjective; however

(ii) for $n \geq 2$, every element of $H^0(K^n)$ is a polynomial in $w_0, \dots, w_{g-1}, \tau_0, \dots, \tau_{g-3}$; this polynomial is a sum of homogeneous terms of degree 1 or 0 in the τ_j and correspondingly of degree n or $n-2$ in the w_i . Thus

(iii) for $n \geq 2$, $S^{n-2} H^0(K) \otimes H^0(K^2) \rightarrow H^0(K^n)$ is surjective.

PROOF. (I)(i) The basis (3.10) can be written as $\{w_0^2, w_0 w_1, w_1^2\}$. Thus (i) is proven.

(ii) σ_0 is not a polynomial in w_0, w_1 since dz and u^{-1} do not appear to the same power in σ_0 .

(iii) To show the surjectivity in orders $n > 2$, it suffices to write the basis elements (3.11) as monomials in w_0, w_1, σ_0 of degree 1 or 0 in σ_0 . This clearly can be done

$$z^i(dz/u)^n = w_0^{n-i} w_1^i \quad (i \leq n); \quad z^i u(dz/u)^n = w_0^{n-i-3} w_1^i \sigma_0 \quad (1 \leq n-3).$$

Thus (I) is proven.

(II)(i) No τ_j is a polynomial in the w_i since dz and u^{-1} do not appear to the same power in τ_j .

(ii) We show that the basis elements (3.12) are monomials in the w_i, τ_j of degree 1 or 0 in τ_j . Given i and n with $0 \leq i \leq n(g-1)$, write $i = k(g-1) + r$ with $k < n$ and $0 < r \leq g-1$. Then

$$z^i (dz/u)^n = (z^{g-1} dz/u)^k (z^r dz/u) (dz/u)^{n-k-1} = w_{g-1}^k w_r w_0^{n-k-1}. \quad (3.16)$$

Now, if $i \leq (n-2)(g-1) + (g-3)$ we can write

$$z^i u (dz/u)^n = [z^j u (dz/u)^2] [z^k (dz/u)^{n-2}]$$

with $j \leq g-3$ and $k \leq g-1$. The first factor is just τ_j and the second factor can be handled as above. Thus the theorem is proven.

This theorem shows that Noether's theorem (1.7) is not true in the hyperelliptic case, and gives the appropriate version. We will describe the analogue of Petri's theorem in §5.

4. The embedding in Euclidean space. Let T be the tangent bundle to R . In this section we shall consider the n th order differentials on R explicitly as holomorphic functions on T which are homogeneous of order n on the fibers. Let

$$\begin{cases} F_0: T \rightarrow \mathbb{C}^g: F_0 = (w_0, \dots, w_{g-1}), \\ F: T \rightarrow \mathbb{C}^3: F = (w_0, w_1, \sigma_0) \quad \text{if } g = 2, \\ F: T \rightarrow \mathbb{C}^{2g-2}: F = (w_0, \dots, w_g, \tau_0, \dots, \tau_{g-3}) \quad \text{if } g > 2. \end{cases} \quad (4.1)$$

4.2. PROPOSITION. *The canonical map F_0 takes T onto the cone V_0 over the rational curve in P^{g-1} parametrized by $t \rightarrow [1, t, \dots, t^{g-1}]$. F_0 maps Z down to the vertex, and is a 2-sheeted branched covering map on $T - Z$.*

PROOF. At finite points (z, u, ξ) , where ξ is the fiber coordinate,

$$F_0(z, u, \xi) = \left(\frac{dz(\xi)}{u}, \frac{zdz(\xi)}{u}, \dots, \frac{z^{g-1} dz(\xi)}{u} \right).$$

Obviously, the image is dense in the cone V_0 , so F_0 maps all of T onto V_0 . $F_0(z, u, \xi) = F_0(z', u', \xi')$ if and only if $z = z'$ and $(u, \xi) = \pm (u', \xi')$, so the map is strictly 2 : 1 where $\xi \neq 0$ and $u \neq 0$. This calculation holds at ∞ as well.

We now observe that the image V of F is precisely the Grauert blowdown of T . Then we shall compute the equations defining V , and in the next section conclude that the image V_0 of F_0 is (essentially) the tangent cone to V .

4.3. THEOREM. *$F: T \rightarrow V$ is the Grauert blowdown of T . T is obtained from V by monoidal transform of $\mathfrak{m}^3$ ($\mathfrak{m}^2$) if $g = 2$ ($g > 2$) ($\mathfrak{m}$ is the maximal ideal of V at 0).*

PROOF. The results of §3 tell us that the coordinates of F , as functions on T , generate the algebra $\bigoplus_{n \geq 0} H^0(R, T^{*n})$. Thus the first statement follows from Theorem 5.6 of [4]. The second statement follows from Theorem 5.5 of [4].

We shall describe the quadratic transform (the monoidal transform of $\mathfrak{m}$) after computing the equations defining V . These are the relations among the holomorphic differentials, and (as we shall see) we need only consider fourth order.

We consider first the case $g = 2$. Then V is a subvariety of $\mathbb{C}^3$. Let x_0, x_1, y be the coordinates of $\mathbb{C}^3$ so that F is given by $x_0 = w_0, x_1 = w_1, y = \sigma_0$.

4.4. THEOREM. ($g = 2$). V is given by the equation

$$y^2 = Q(x_0, x_1) = x_0 x_1^5 + a_1 x_0^2 x_1^4 + \cdots + a_1 x_0^5 x_1 + x_0 x_0^6 \quad (4.5)$$

where the defining equation of R is

$$u^2 = \prod_{j=1}^5 (z - e_j) = z^5 + a_4 z^4 + \cdots + a_0. \quad (4.6)$$

PROOF. We can view z, u, dz as meromorphic functions on T , and x_0, x_1, y as holomorphic functions. We have the relations (4.6) and

$$x_0 = u^{-1} dz, \quad x_1 = zu^{-1} dz, \quad y = u^{-2}(dz)^3.$$

Hence

$$\begin{aligned} Q(x_0, x_1) &= (z^5 + Q_4 z^4 + \cdots + a_1 z + a_0)(u^{-1} dz)^6 \\ &= u^2(u^{-1} dz)^6 = (u^{-2}(dz)^3)^2 = y^2. \end{aligned}$$

Thus (4.5) is satisfied on V . Since the e_j are all distinct, Q is not a perfect square, so $y^2 - Q$ is irreducible. Since V is a hypersurface contained in the irreducible hypersurface $y^2 - Q = 0$, they must coincide.

REMARK. We can conclude that the relation $\sigma_0^2 = Q(w_0, w_1)$ generates all the relations among the holomorphic forms.

Now we turn to the case of higher genus. Let $x_0, \dots, x_{g-1}, y_0, \dots, y_{g-3}$ be coordinates of $\mathbb{C}^3$ so that F is given by $x_i = w_i, y_j = t_j$.

4.7. THEOREM. ($g > 2$). V is given by the equations

$$\begin{cases} x_0 x_2 = x_1^2, \\ x_1 x_3 = x_2^2, \\ \vdots \\ x_{g-3} x_{g-1} = x_{g-2}^2, \\ x_0 x_{g-1} = x_1 x_{g-2}, \end{cases} \quad (4.8)$$

$$\begin{cases} y_0 y_2 = y_1^2, \\ y_1 y_3 = y_2^2, \\ \vdots \\ y_{g-5} y_{g-3} = y_{g-4}^2, \end{cases} \quad (4.9)$$

$$y_j x_0 = y_0 x_j, \quad 0 \leq j \leq g-3, \quad (4.10)$$

$$y_j^2 = Q_j(x_0, \dots, x_{g-1}), \quad 0 \leq j \leq g-3, \quad (4.11)$$

where the Q_j are homogeneous polynomials of degree 4, not containing terms of the form cx_{g-1}^4 .

PROOF. First we describe the functions Q_j , $0 \leq j \leq g-3$. Write the defining equation of R as

$$u^2 = \prod_{j=1}^{2g+1} (z - e_j) = z^{2g+1} + a_{2g}z^{2g} + \cdots + a_1z + a_0. \quad (4.12)$$

In the following, we shall be using the standard shorthand for monomials

$$X^k = X_1^{k_1} \cdots X_n^{k_n} \quad \text{where } k = (k_1, \dots, k_n).$$

It turns out that the functions Q are defined by the relations among the monomials of degree 4. For each $i \leq 4(g-1)$, let $i = k(g-1) + r$ where $k < 4$ and $0 < r \leq g-1$ and define $\alpha(i)$ by the expression (3.16):

$$z^i(u^{-1} dz)^n = w^{\alpha(i)}. \quad (4.13)$$

Then for $j \leq g-3$ take

$$Q_j(x_0, \dots, x_{g-1}) = x^{\alpha(2j+2g+1)} + a_{2g}x^{\alpha(2j+2g)} + \cdots + a_1x^{\alpha(2j+1)} + a_0x^{\alpha(2j)},$$

where $a_0, \dots, a_{2g}$ are defined by (4.12).

Let W be the variety defined by the equations (4.8)–(4.11). First, we verify that V satisfies these equations, i.e., $V \subset W$. Recall that V is given as the image of F defined by $x_i = w_i, y_j = \tau_j$ where

$$w_i = z^i(u^{-1} dz), \quad \tau_j = z^j u(u^{-1} dz)^2.$$

It is immediate that (4.8)–(4.10) are satisfied. To verify (4.11) we use (4.12) as follows.

$$\begin{aligned} \tau_j^2 &= z^{2j} u^2 (u^{-1} dz)^4 = z^{2j} (z^{2g+1} + a_{2g}z^{2g} + \cdots + a_1z + a_0) (u^{-1} dz)^4 \\ &= w^{\alpha(2j+2g+1)} + a_{2g}w^{\alpha(2j+2g)} + \cdots + a_1w^{\alpha(2j+1)} + a_0w^{\alpha(2j)} \end{aligned}$$

using (4.13). Thus $y_j^2 = Q_j(x)$, verifying equation (4.11).

Now we verify that $W \subset V$. Let $q = (x_0, \dots, x_{g-1}, y_0, \dots, y_{g-3})$ be in W . We have to proceed by cases.

(A) $x_0x_1y_0 \neq 0$. By equations (4.8)–(4.10), all the coordinates are nonzero, and we can express them all in terms of x_0, x_1, y_0 :

$$q = \left(x_0, x_1, \frac{x_1^2}{x_0}, \frac{x_1^3}{x_0^2}, \dots, \frac{x_1^{g-1}}{x_0^{g-2}}, y_0, y_0 \frac{x_1}{x_0}, y_0 \left(\frac{x_1}{x_0} \right)^2, \dots, y_0 \left(\frac{x_1}{x_0} \right)^{g-3} \right). \quad (4.14)$$

Let $z = x_1x_0^{-1}$ and $u = y_0x_0^{-2}$. By (4.11)

$$\begin{aligned} y_0^2 &= Q_0(x_0, \dots, x_{g-1}) = x_0^4 Q_0(1, z, z^2, \dots, z^{g-1}) \\ &= x_0^4 (z^{2g+1} + a_{2g}z^{2g} + a_1z + a_0) = x_0^4 \prod_{j=1}^{2g+1} (z - e_j). \end{aligned}$$

Thus

$$u^2 = y_0^2 x_0^{-4} = \prod_{j=1}^{2g+1} (z - e_j),$$

so $(z, u) \in R$. Let ξ be the point in T lying above (z, u) at which $dz(\xi) = y_0 x_0^{-1}$. Then

$$F(\xi) = \left(\frac{dz(\xi)}{u}, \dots, \frac{z^{g-1} dz(\xi)}{u}, \frac{dz(\xi)^2}{u}, \dots, \frac{z^{g-3} dz(\xi)^2}{u} \right) = q$$

(by resubstituting $z = x_1 x_0^{-1}$, $u = y_0 x_0^{-2}$, $dz(\xi) = y_0 x_0^{-1}$). Thus $q \in V$.

(B) $x_0 \neq 0$, $x_1 = 0$. Then, by (4.8), $x_i = 0$ for $i \geq 2$. Thus $y_0^2 = Q_0(x_0, 0, \dots, 0) = a_0 x_0^4$, where $a_0 = -\prod e_j \neq 0$. Thus also $y_0 \neq 0$ and by (4.9), $y_j = 0$ for $j \geq 1$, so that $q = (x_0, 0, \dots, 0, y_0, 0, \dots, 0)$. Let $u = y_0 x_0^{-2}$. Then

$$u^2 = y_0^2 x_0^{-4} = a_0 = \prod_{j=0}^{2g+1} (0 - e_j),$$

so $(0, u) \in R$. Choose ξ above $(0, u)$ again so that $dz(\xi) = y_0 x_0^{-1}$ and calculate as above that $F(\xi) = q$.

(C) $x_0 \neq 0$, $x_1 \neq 0$, but $y_0 = 0$. Then, by (4.8) no x_i is zero and $x_i = x_1^i x_0^{i-1}$, $i \geq 2$. Further, by (4.10) all $y_j = 0$. By (4.11), $Q_0(x_0, \dots, x_{g-1}) = 0$. Let $z = x_1 x_0^{-1}$. Then

$$0 = x_0^{-4} Q_0(x_0, \dots, x_{g-1}) = Q_0(1, z, \dots, z^{g-1}) = \prod_{j=1}^{2g+1} (z - e_j),$$

so z is one of the roots, say e_p . Then $(z, 0) \in R$ and by (3.6), at $(z, 0)$ $dz/u = -2d_p^{-1} du$, where $d_p = -\prod_{k \neq p} (e_p - e_k)$. Choose ξ in T above $(z, 0)$ so that $du(\xi) = -\frac{1}{2} d_p x_0$. Then $F(\xi) = (x, 0) = q$.

(D) Finally, suppose $x_0 = 0$. Then all $x_i = 0$, $i < g-1$ and thus by (4.11), $y_j^2 = 0$ for all j since each monomial in Q_j involves some x_i , $i < g-1$ to a positive power. Thus $q = (0, \dots, 0, x_{g-1}, 0, \dots, 0)$ and since we know the origin is in V , we can assume $x_{g-1} \neq 0$. Recall (3.8): t is a coordinate at ∞ and $dt = -\frac{1}{2} v(0) w_{g-1}$, $v(0) \neq 0$. Take ξ above $(\infty, 0) \in R$ so that $dt(\xi) = -\frac{1}{2} v(0) x_{g-1}$. Then $F(\xi) = q$.

EXAMPLES. (4.15) $g = 2$. $V \subset \mathbb{C}^3$ is a hypersurface defined by the single equation $y^2 = Q(x_0, x_1)$. In this case $y^2 - Q$ generates the ideal of V .

(4.16) $g = 3$. $V \subset \mathbb{C}^4$ is a complete intersection defined by the equations $x_0 x_2 = x_1^2$, $y_0^2 = Q_0(x_0, x_1, x_2)$. More particularly, if R is given by $u^2 = z^7 - 1$, then the equations are $x_0 x_2 = x_1^2$, $y_0^2 = x_1 x_2^3 - x_0^4$.

(4.17) $g = 4$. $V \subset \mathbb{C}^6$ is given by the six equations

$$\begin{aligned} x_0 x_2 &= x_1^2, & x_1 x_3 &= x_2^2, & x_0 x_3 &= x_1 x_2, \\ y_1 x_0 &= y_0 x_1, & y_0^2 &= Q_0(x), & y_1^2 &= Q_1(x). \end{aligned}$$

In particular, if $u^2 = z^9 - 1$, then the last two equations become

$$y_0^2 = x_0 x_3^3 - x_0^4 = x_0(x_3^3 - x_0^3), \quad y_1^2 = x_2 x_3^3 - x_0^3 x_2 = x_2(x_3^3 - x_0^3).$$

We note that none of these equations can be eliminated.

5. Local invariants: the hyperelliptic case.

5.1. PROPOSITION. (i) ($g = 2$). The tangent cone (not counting multiplicity) to V at 0 is the linear space $y = 0$.

(ii) ($g > 2$). The tangent cone to V is given by the equations

$$\{x_0x_2 = x_1^2, x_1x_3 = x_2^2, \dots, x_{g-3}x_{g-1} = x_{g-2}^2, x_0x_{g-1} = x_1x_{g-2}, \\ y_0 = 0, \dots, y_{g-3} = 0\}. \quad (5.2)$$

PROOF. (i) is obvious. y^2 is the initial form of the function $y^2 - Q$ which generates the ideal of V .

(ii) Let W be the variety defined by the equations (5.2). Since these functions are all initial forms of functions vanishing on V (replace y_j by y_j^2), the tangent cone CV is contained in W . On the other hand $F: T \rightarrow V$ takes Z to the origin, so $dF: T|Z \rightarrow CV$. But $dF = (F_0, 0)$ and the image of F_0 is precisely the variety defined by the x -equations in (5.2). Thus $W = dF(T|Z) \subset CV$.

REMARK. Of course, if we knew that equations (4.6)–(4.9) generate the ideal of V , (ii) would also be obvious.

5.3. PROPOSITION. Let QV, QV_0 be the quadratic transforms of V, V_0 respectively. QV is a 2-sheeted branched cover of QV_0 which is locally reducible along the exceptional variety (except at branch points). The exceptional variety of QV is a $\mathbf{P}^1$.

PROOF. Let $\pi: Q \rightarrow C^{2g-2}$ be the quadratic transform of the origin in C^{2g-2} , $\pi_0: Q_0 \rightarrow C^{g-1}$ that for C^{g-1} . Then QV is the closure of $\pi^{-1}(V)$ and QV_0 is the closure of $\pi_0^{-1}V_0$. The map $F: T \rightarrow C^{2g-2}$ lifts to $\tilde{F}: T \rightarrow Q$. For we can write $\tilde{F}: T - Z \rightarrow Q$ in coordinates as

$$\tilde{F}(\xi) = (w(\xi), \tau(\xi), [w(\xi), \tau(\xi)]).$$

As $\xi \rightarrow \xi_0 \in Z$, some $w_i(\xi_0)$ is nonzero (as a section of T^*); take $i = 0$. Then, near ξ_0

$$[w_0(\xi), \dots, w_{g-1}(\xi), \tau(\xi)] = [1, \dots, w_{g-1}(\xi)/w_0(\xi), \tau(\xi)/w_0(\xi)]$$

and the expression on the right extends $\tilde{F}$. Similarly F_0 extends to $\tilde{F}_0: T \rightarrow Q_0$. Obviously, since $F, \tilde{F}$ are surjective off z , $\tilde{F}(T) = QV$, $\tilde{F}_0(T) = QV_0$. Since one of $w_0, \dots, w_{g-1}$ is nonzero at each point of Z_1 and every coordinate of τ vanishes to second order along Z , the map

$$((x, y), [x, y]) \rightarrow (x, [x]) \quad (5.4)$$

is well defined on QV and maps QV onto QV_0 taking the exceptional set of QV one-to-one onto the exceptional set of QV_0 . Otherwise (5.4) is 2 to 1, so it branches along the exceptional variety. It is easy to see this branching is reducible (generically), we verify it in the case $g = 2$.

V is given by $y^2 = Q(x_0, x_1)$. Let $[v_0, v_1, u]$ be homogeneous coordinates for $\mathbf{P}^2$; QV is the closure of the graph of the map

$$v_0 = x_0, \quad v_1 = x_1, \quad y = u \quad (\text{not all of } x_0, x_1, u \text{ are zero}).$$

Let us write this in inhomogeneous coordinates on the coordinate neighborhood $u_0 = \{v_1 \neq 0\}$. Then $\xi = v_1 v_0^{-1}$, $\zeta = uv_0^{-1}$ are coordinates for U_0 and in $\mathbb{C}^3 \times U_0$, V is given by

$$y^2 = Q(x_0, x_1), \quad y = x_0 \zeta, \quad x_1 = x_0 \xi.$$

The first equation becomes

$$x_0^2 \zeta^2 = Q(x_0, x_1) = x_0^6 Q(1, \xi) \quad \text{or} \quad \zeta^2 = x_0^4 Q(1, \xi).$$

If $Q(1, \xi) \neq 0$, it has two square roots $R_1(\xi)$, $R_2(\xi)$, and QV consists of the two nonsingular varieties coordinatized by x_0, ξ via the functions

$$\begin{aligned} \zeta &= x_0^2 R_i(\xi), \\ y &= x_0^3 R_i(\xi), \quad i = 1 \text{ or } 2. \\ x_1 &= x_0 \xi, \end{aligned}$$

5.5. PROPOSITION. *For $g \geq 2$, the multiplicity of V at 0 is $\mu_0(V) = 2g - 2$.*

PROOF. For $g = 2$, $y^2 - Q$ has degree $2 = 2g - 2$. Consider $g > 2$, and look at the tangent cone CV given by the equations (5.2). We use the technique of Whitney [8, Chapter 7, §7] for computing $\mu_0(V)$. Let $\Pi_1 = \{x_0 = x_{g-1} = 0\}$ and

$$\Pi_2 = \{x_1 = \cdots = x_{g-2} = y_0 = \cdots = y_{g-3} = 0\}.$$

Since $CV \cap \Pi_1 = \{0\}$, if ψ is the projection onto Π_1 with kernel Π_2 , ψ is (locally) proper on CV , and the multiplicity is the generic number of points in $\psi^{-1}(q) \cap V$ for $q \in \Pi_1$. Let $q = (a, b)$ with $ab \neq 0$. If

$$\psi(x_0, \dots, x_{g-1}, y_0, \dots, y_{g-3}) = q,$$

we must have

$$x_0 = a, \quad x_{g-1} = b, \quad x_1^{g-1} = a^{g-2}b \quad \text{and} \quad x_j = x_i^j / a^{j-1}, \quad j \neq 0, 1, g-1. \quad (5.6)$$

This system has $g - 1$ different solutions. We show that each one gives rise to precisely two points in V so long as $x_1 a^{-1}$ is not a root of $Q_0(1, z, \dots, z^{g-1}) = 0$ (still a generic condition). Then at a point of V above an x satisfying

$$y_0^2 = a^4 Q_0(1, x_1 a^{-1}, \dots, (x_1 a^{-1})^{g-1}) \neq 0,$$

y_0 admits two distinct values. For each choice let

$$y_j = (x_j / x_0) y_0, \quad 1 \leq j \leq g-3.$$

Then $(x_0, \dots, x_{g-1}, y_0, \dots, y_{g-1}) \in V$, and these are the only possible choices. Since there are $2(g-1)$ such choices, $\mu_0 = 2(g-1)$.

5.7. PROPOSITION. (i) *If $g = 2$, the minimal embedding dimension of V at 0 is 3.*

(ii) *For $g > 2$ the minimal embedding dimension of V is $2g - 2$.*

In other words V as constructed in §4 is minimally embedded.

PROOF. (i) For $g = 2$, since V is singular, the minimal embedding dimension must be at least 3, and since it is attained, it is 3.

(ii) Now take $g > 2$ and let

$$f(x, y) = \sum_{i=0}^{g-1} \lambda_i x_i + \sum_{j=0}^{g-3} \mu_j y_j + \sum a_{rs} x_r x_s + \text{other higher order terms}$$

be a function vanishing on V . Then $f \circ F$ vanishes on T . If we write $f \circ F$ as a series of homogeneous polynomials on T , each term must vanish independently on V . But the linear term is $\sum_{i=0}^{g-1} \lambda_i \omega_i$, so, since the ω_i are independent, $\lambda_0 = \cdots = \lambda_{g-1} = 0$. The second-order term is

$$\sum_{j=0}^{g-3} \mu_j \tau_j + \sum a_{rs} \omega_r \omega_s.$$

Thus $\sum \mu_j \tau_j \in S^2 H^0(K)$, but (following the proof of Theorem 3.15(II)(i)) the τ 's were chosen to be independent mod $S^2 H^0(K)$, so $\mu_0 = \cdots = \mu_{g-3} = 0$ also. Thus if f vanishes on V we have $df(0) \equiv 0$, so the minimal embedding dimension is not less than $2g - 2$.

6. The diagonal embedding. For R any compact Riemann surface, let $\Delta: R \rightarrow R \times R$ be the *diagonal embedding*. $\Delta(p) = (p, p)$. It is well known that the normal bundle to the diagonal is the tangent bundle TR to R . Thus if R has genus $g \geq 2$, $\Delta(R)$ is negatively embedded and $R \times R$ can be blown down along the diagonal. Let $\tau: R \times R \rightarrow W$ be the Grauert blowdown. In this section we shall be studying W , comparing it with the blowdown V of the zero-section of TR . In [4], we showed that these two embeddings are always first-order equivalent and are second-order equivalent if and only if $g = 1$. In the genus 1 case, they are in fact analytically equivalent.

6.1. PROPOSITION. *If R is an elliptic curve ($g = 1$), then the embeddings z_T and Δ are analytically equivalent.*

PROOF. Since $g = 1$, TR is holomorphically trivial: $TR = R \times \mathbb{C}$ and $Z = R \times \{0\}$. Let $\nu: \mathbb{C} \rightarrow R$ be the universal covering of R . Define a map $\Theta: R \times \mathbb{C} \rightarrow R \times R$ by

$$\Theta(\nu(x), y) = (\nu(x), \nu(x + y)).$$

Θ is obviously locally one-to-one, and has differential of maximal rank,

$$d\Theta = \begin{pmatrix} d\nu & 0 \\ d\nu & d\nu \end{pmatrix}.$$

Θ maps $R \times \{0\}$ one-to-one onto $\Delta(R)$, so Θ must be biholomorphic in a neighborhood of $R \times \{0\}$, thus providing the necessary equivalence.

Now we restrict attention to the case of genus $g \geq 2$. Let R be a compact Riemann surface of genus $g \geq 2$. Let $j: R \rightarrow X$ be an embedding of R in a two-dimensional manifold X with normal bundle the tangent bundle TR . We shall henceforth call this *the standard situation*. Let $\pi: X \rightarrow Y$ be the Grauert blowdown of R to a point, denoted 0. $\mathcal{I}$ is the ideal sheaf of R in X , and $\mathfrak{m}$ the maximal ideal at 0.

6.2. LEMMA. $H^1(R, \mathcal{G}^k) = 0$ for $k \geq 2$.

PROOF. If U, U' are two strictly pseudoconvex neighborhoods of R , then

$$H^v(U, \mathcal{F}) = H^v(U', \mathcal{F}), \quad \text{all } v \geq 1, \quad (6.3)$$

for any coherent sheaf $\mathcal{F}$. Since R has a neighborhood basis of strictly pseudoconvex domains, it suffices to show $H^1(U, \mathcal{G}^k) = 0$ for U such a domain and $k \geq 2$. We have the exact sequence

$$\cdots \rightarrow H^1(U, \mathcal{G}^{j+1}) \xrightarrow{\alpha_j} H^1(U, \mathcal{G}^j) \rightarrow H^1(R, K^j) \rightarrow \cdots$$

If $j \geq 2$, $H^1(R, \mathcal{G}^j) = 0$, so the map α_j is surjective for all $j \geq 2$. Composing such maps, we find that all maps

$$H^1(R, \mathcal{G}^{j+k}) \rightarrow H^1(R, \mathcal{G}^k), \quad j \geq 0, k \geq 2, \quad (6.4)$$

are surjective. But, by [2, §4, Satz 1], for any k , there is a j sufficiently large so that (6.3) is the zero map. So (6.3) is both surjective and identically zero; thus $H^1(R, \mathcal{G}^k) = 0$, $k \geq 2$.

6.5. PROPOSITION. In the standard situation, $\mathcal{G}^k = \hat{\mathfrak{m}}^k$.

PROOF. For $p \in R \subset X$, we have to show that $\mathcal{O}_p \pi^{-1} \mathfrak{m}_0^k = \mathcal{G}_p^k$. Clearly $\mathcal{O}_p \pi^{-1} \mathfrak{m}_0^k \subset \mathcal{G}_p^k$. By Theorem 1.6 there is an $\omega \in H^0(R, K)$ with $\omega(p) \neq 0$. By Lemma 6.2 the map β in the exact sequence

$$\cdots \rightarrow H^0(U, \mathcal{G}) \xrightarrow{\beta} H^0(R, K) \rightarrow H^1(U, \mathcal{G}^2) \rightarrow \cdots$$

is surjective, so there is an $f \in H^0(U, \mathcal{G}) = \mathfrak{m}_0$ with $\beta f = \omega$. Then f vanishes to order exactly one at p , so for all k , f^k generates $\mathcal{G}_p^k$ over $\mathcal{O}_p$. But $f^k \in \pi^{-1}(\mathfrak{m}_0^k)$.

6.6. PROPOSITION. In the standard situation, suppose R is hyperelliptic of genus $g > 2$. Then the natural injection

$$\mathfrak{m}_0^2 \rightarrow H^0(R, \mathcal{G}^2) \quad (6.7)$$

is not surjective.

PROOF. Suppose (6.7) were surjective. Let $\sigma \in H^0(R, K^2)$. By Lemma 6.2 ($k = 3$), there is $f \in H^0(R, \mathcal{G}^2)$ whose initial term (in the Taylor expansion along R) is σ . If (6.7) were surjective, we could write $f = \sum a_{ij} g_i g_j$ with $g_i \in \mathfrak{m}_0 = H^0(R, \mathcal{G})$. Letting $\sigma(g_i) \in H^0(R, K)$ be the initial term of the Taylor expansion of g_i , we have

$$\sigma = \sigma(f) = \sum a_{ij} \sigma(g_i) \sigma(g_j),$$

so that σ is in the image of $S^2 H^0(R, K)$. But the map $S^2 H^0(R, K) \rightarrow H^0(R, K^2)$ is not surjective (Theorem 3.15(II)(i)).

Similarly, by using Theorem 3.15(I)(ii) we obtain

6.8. PROPOSITION. In the standard situation, if $g = 2$, the natural injection $\mathfrak{m}_0^3 \rightarrow H^0(R, \mathcal{G}^3)$ is not surjective.

REMARK. In the general situation of the Grauert blowdown $X \rightarrow \tilde{X}$ we shall always have (using Theorem 1.3) $H^0(R, \mathcal{G}^k) \subset H^0(R, \mathcal{G})^{F(k)}$ where $F(k) \rightarrow \infty$ as $k \rightarrow \infty$. We shall make this more precise for the standard situation being considered here.

6.9. PROPOSITION. *In the standard situation, if R is not hyperelliptic, the natural injection $m_0^k \rightarrow H^0(R, \mathcal{G}^k)$ is an isomorphism for all k .*

PROOF. We have the following commutative diagram:

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
 & & m_0^k & \xrightarrow{\alpha} & H^0(R, \mathcal{G}^{k+1}) \\
 \alpha_1 \nearrow & & \searrow \alpha_2 & & \downarrow \varepsilon \\
 S^k m_0 & \xrightarrow{\alpha} & & & H^0(R, \mathcal{G}^k) \\
 \downarrow \delta & & & & \downarrow \gamma \\
 S^k H^0(R, K) & \xrightarrow{\beta} & & & H^0(R, K^k) \\
 \downarrow & & & & \downarrow \\
 0 & & & & 0
 \end{array} \quad (6.10)$$

The maps δ , γ , α_1 are surjective; α_2 and ε are injective. By Noether's theorem, β is surjective.

Let $f \in H^0(R, \mathcal{G}^k)$. Then $\gamma(f) = \beta\delta\Theta$ for some $\Theta \in S^k m_0$. By commutativity, $\gamma(f - \alpha\Theta) = 0$, so $f - \alpha\Theta = g$ for some $g \in H^0(R, \mathcal{G}^{k+1})$. Thus

$$H^0(R, \mathcal{G}^k) \subset m_0^k + H^0(R, \mathcal{G}^{k+1}).$$

Proceeding inductively for $j > k$ (since $m_0^j \subset m_0^k$), we have

$$H^0(R, \mathcal{G}^k) \subset m_0^k + H^0(R, \mathcal{G}^j) \subset m_0^k + m_0^{F(j)}, \quad \text{all } j > k,$$

by the preceding remark. Since $F(j) \rightarrow \infty$, eventually $F(j) > k$, thus $H^0(R, \mathcal{G}^k) \subset m_0^k$, proving the proposition.

In the hyperelliptic case the best we can do is

6.11. PROPOSITION. *In the standard situation with R hyperelliptic, we have*

(i) $g = 2$:

$$H^0(R, \mathcal{G}^{k+1}) \subset H^0(R, \mathcal{G}^3)m_0^{k-2} \subset m_0^{k-1}, \quad k > 3,$$

(ii) $g > 2$:

$$H^0(R, \mathcal{G}^{k+1}) \subset H^0(R, \mathcal{G}^2)m_0^{k-1} \subset m_0^k, \quad k > 2.$$

PROOF. We shall prove (i); the proof of (ii) is similar. Diagram (6.10) has to be replaced by

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
 & & H^0(K, \mathcal{G}^3)m_0^k & \xrightarrow{\alpha} & H^0(R, \mathcal{G}^{k+4}) \\
 \alpha_1 \nearrow & & \searrow \alpha_2 & & \downarrow \varepsilon \\
 S^k m_0 \times H^0(R, \mathcal{G}^3) & \xrightarrow{\alpha} & & & H^0(R, \mathcal{G}^{k+3}) \\
 \searrow \delta & & & & \downarrow \\
 S^k H^0(R, K) \otimes H^0(R, K^3) & \xrightarrow{\beta} & & & H^0(R, K^{k+3}) \\
 \downarrow & & & & \downarrow \\
 0 & & & & 0
 \end{array} \quad (6.12)$$

and the proof is now the same as that of Proposition 6.9 (β is surjective now because of Theorem 3.15(I)(iv)).

Let q be a point in an analytic space Y with structure sheaf $\mathcal{O}$. Let $\mathfrak{m}$ be the maximal ideal at q . The associated graded ring at q is

$$\mathrm{gr}_{\mathfrak{m}}(\mathcal{O}_q) = \bigoplus_{k=0}^{\infty} \mathfrak{m}^k / \mathfrak{m}^{k+1}.$$

This is a homogeneous ring which is in fact the ring of polynomials on the tangent cone $C_q Y$ of Y at q . The dimensions of the finite-dimensional vector spaces $\mathfrak{m}^k / \mathfrak{m}^{k+1}$ determine the Hilbert polynomial of $\mathcal{O}_q$ and hence the multiplicity of Y at q . The dimension of $\mathfrak{m} / \mathfrak{m}^2$ is the minimal embedding dimension of Y at q .

6.13. THEOREM. *In the standard situation suppose R is nonhyperelliptic of genus $g > 2$. Then*

$$\mathrm{gr}_{\mathfrak{m}}(\mathcal{O}_0) \cong \bigoplus_{k=0}^{\infty} H^0(R, K^k) \quad (6.14)$$

as graded rings.

PROOF. By Proposition 6.9,

$$\mathfrak{m}^k / \mathfrak{m}^{k+1} \cong H^0(R, \mathcal{G}^k) / H^0(R, \mathcal{G}^{k+1}),$$

and by Lemma 6.2

$$H^0(R, K^k) \cong H^0(R, \mathcal{G}^k) / H^0(R, \mathcal{G}^{k+1}).$$

Thus (6.14) is true as graded vector spaces. The map $\mathfrak{m}^k / \mathfrak{m}^{k+1} \rightarrow H^0(R, K^k)$ is induced by the map $\mathfrak{m}^k \rightarrow H^0(R, K^k)$ taking an $f \in \mathfrak{m}_0^k$ to the initial term $\sigma(f)$ in the Taylor expansion of $f \circ \pi$ along R . Since $\sigma(fg) = \sigma(f)\sigma(g)$, (6.14) is also a ring isomorphism.

6.15. THEOREM. *Let R be nonhyperelliptic of genus $g > 2$. Let $R \rightarrow X$ be an embedding of R in a smooth surface so that the normal bundle is the tangent bundle of R . Let $\pi: X \rightarrow \tilde{X}$ be the Grauert blowdown of R to Q and $\tau: TR \rightarrow V$ the Grauert blowdown of the normal bundle. Then*

- (a) $\pi: X \rightarrow \tilde{X}$ is a monoidal transformation,
- (b) V is the tangent cone $C_0(\tilde{X})$.

PROOF. (a) Follows from [4, Theorem 5.7] and (b) is a corollary of Theorem 6.13.

In particular, in the nonhyperelliptic case, the Grauert blowdown of the diagonal in $R \times R$ has the same minimal embedding dimension, Hilbert polynomial and multiplicity on the blowdown of the zero section of the tangent bundle.

6.16. THEOREM. *In the standard situation, suppose R is hyperelliptic of genus $g > 2$. Then, for all k , $\mathfrak{m}^k / \mathfrak{m}^{k+1}$ is independent of the embedding (i.e., is the same as for the embedding in the tangent bundle).*

PROOF. For each $k \geq 0$, let $I_k = \prod_{j \geq k} H^0(R, K^j)$. Then I_0 is the ring of formal power series in the fiber coordinate of TR , and the Taylor expansion provides a

map $H^0(Z, \mathcal{O}) \rightarrow I_0$ which takes $H^0(R, \mathcal{G}^k) \rightarrow I_k$ (where $\mathcal{G}$ is the ideal sheaf of the zero section of TR). Since

$$H^0(R, \mathcal{G}^{k+2}) \subset \mathfrak{m}_0^{k+1} \subset \mathfrak{m}_0^k \subset H^0(R, \mathcal{G}^k),$$

the map $H^0(R, \mathcal{G}^k) \rightarrow I_k$ induces an injection

$$\mathfrak{m}_0^k / \mathfrak{m}_0^{k+1} \rightarrow I_k / I_{k+2} = H^0(R, K^k) + H^0(R, K^{k+1}). \quad (6.17)$$

Let $R \rightarrow X$ be another embedding in the standard situation, with Y the Grauert blowdown, $\mathfrak{m}_Y$ the maximal ideal at the singularity, and $\mathcal{G}_Y$ the ideal sheaf of R in X . Let σ_k be the natural map

$$\sigma_k: H^0(R, \mathcal{G}_Y^k) \rightarrow H^0(R, K^k).$$

Since σ_k is surjective, there is a map $\omega_k: H^0(R, K^k) \rightarrow H^0(R, \mathcal{G}_Y^k)$ such that $\sigma_k \circ \omega_k = \text{identity}$. Define

$$\tau_k: H^0(R, \mathcal{G}_Y^k) \rightarrow I_k / I_{k+2}$$

by

$$\tau_k(f) = (\sigma_k(f), \sigma_{k+1}(f - \omega_k \sigma_k f)).$$

We shall show that $\tau_k|_{\mathfrak{m}_Y^k}$ induces an isomorphism $\tilde{\tau}_k$ of $\mathfrak{m}_Y^k / \mathfrak{m}_Y^{k+1}$ with the image R^k of $\mathfrak{m}_0^k / \mathfrak{m}_0^{k+1}$ under (6.17).

(a) Let $f \in \mathfrak{m}_Y^k, f = \sum f_{i_1} \dots f_{i_k}$ with $\{f_{i_j}\} \subset \mathfrak{m}_Y$. Then

$$\sigma_k(f) = \sum \sigma_1(f_{i_1}) \dots \sigma_1(f_{i_k}) \in \mathfrak{m}_0^k,$$

and $\sigma_k(f - \omega_k \sigma_k f) = 0$. Thus, the initial form of $f - \omega_k \sigma_k f$ is $\sigma_\nu(f - \omega_k \sigma_k f)$, $\nu > k$. As a section of K^ν it is in $H^0(R, \mathcal{G}^\nu)$, where this group is associated to the canonical embedding $R \rightarrow TR$. Applying Proposition 6.11(ii) to this embedding, we obtain $\sigma_{k+1}(f - \omega_k \sigma_k f) \in \mathfrak{m}_0^k$ (if $\nu = k + 1$; if $\nu > k + 1$ this is 0, so certainly in $\mathfrak{m}_0^k$). Thus $\tau_k(f) \in \mathfrak{m}_0^k$ and the map $\tilde{\tau}_k$ is well defined and has image in R^k .

(b) τ_k is injective. If $\tau_k(f) = 0$, then $\sigma_k(f) = 0$ and $\sigma_{k+1}(f) = 0$, so $f \in H^0(R, \mathcal{G}_Y^{k+2}) \subset \mathfrak{m}_Y^{k+1}$.

(c) τ_k is surjective. Let $g = (g_0, g_1) \in \mathfrak{m}_0^k / \mathfrak{m}_0^{k+1}$. Then $g_0 = \sum g_{i_1} \dots g_{i_k}$ with $g_{i_j} \in \mathfrak{m}_0$. Let

$$f = \sum \omega_1 \sigma_1(g_{i_1}) \dots \omega_k \sigma_k(g_{i_k}).$$

Then $f \in \mathfrak{m}_Y^k$ and $\sigma_k(f) = g_0$. Let $\tau_k(f) = (g_0, g'_1)$. Choose $f' \in H^0(R, \mathcal{G}_Y^{k+1})$ such that $\sigma_{k+1}(f') = g_1 - g'_1$. Then $f + f' \in \mathfrak{m}_Y^k$ and $\tau_k(f + f') = (g_0, g_1)$.

6.18. COROLLARY. *Let R be hyperelliptic with genus $g > 2$. Let R be embedded in X with normal bundle TR . Then the Hilbert polynomial of the Grauert blowdown is independent of the embedding, as is the minimal embedding dimension. In particular the blowdown of $\Delta \subset R \times R$ and $Z \subset TR$ have the same multiplicity and minimal embedding dimension at the singular point.*

REMARK. Since nothing is said of the multiplicative structure of the ring $\text{gr}(\mathcal{O}_Y)$, we cannot conclude that all such blowdowns have the same tangent cone. Furthermore, the above argument breaks down for $g = 2$, and we do not know whether or not the same result holds.

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