

ON A 4-MANIFOLD HOMOLOGY EQUIVALENT TO A BOUQUET OF SURFACES¹

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ABSTRACT. This paper gives some algebraic invariants for a piecewise linear imbedding of a surface into some 4-manifold inducing a Z or Q -homology isomorphism. Several examples are obtained by using these invariants.

Let F be a closed (possibly disconnected) oriented surface and let W be a compact connected oriented piecewise linear 4-manifold with an isomorphism $\varphi_q: H_q(F; R) \approx H_q(W; R)$ for all $q > 0$, where $R = Z$ or Q , and such that the intersection number of any two elements of $H_2(W; Z)/(\text{torsions})$ is 0. The purpose of this paper is to give some algebraic invariants which are necessary to find a piecewise linear imbedding $F \rightarrow W$ inducing this isomorphism φ_q for all $q > 0$. Such invariants come from some developed arguments of quadratic forms of 3-manifolds defined by the author in [9]. By using these invariants, we shall have some examples.

EXAMPLE 2.5. *For each $g \geq 1$ there are compact connected orientable 4-manifolds W such that W is homotopy equivalent to a closed connected orientable surface F_g of genus g , but there is no (possibly nonlocally flat) piecewise linear imbedding from F_g to W inducing any homology isomorphism.*

For $g = 1$ this gives an elementary proof of a spineless 4-manifold announced by Y. Matsumoto [15]. In the higher even dimensional case, S. E. Cappell and J. L. Shaneson [1] and Y. Matsumoto [16] have constructed spineless manifolds W^{n+2} for each even $n \geq 4$ whose statements are weaker than the above.

EXAMPLE 2.6. *For each $g \geq 0$ there are infinitely many relatively nonhomology cobordant compact connected orientable 4-manifolds W such that there is a homotopy equivalent piecewise linear imbedding $F_g \rightarrow W$, but there is no locally flat piecewise linear imbedding from F_g to W inducing any homology isomorphism.*

Let L be a link of s components with linking numbers 0 (i.e. any two components of L have the linking number 0) and let W_L be a 4-manifold obtained from a 4-cell D^4 by attaching s 2-handles along the link L in $S^3 = \partial D^4$ with null-homologous framings. Clearly, W_L is homotopy equivalent to a bouquet $S^2 \vee S^2 \vee \cdots \vee S^2$ of s 2-spheres and any two elements of $H_2(W_L; Z)$ have the

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intersection number 0 and every element of $H_2(W_L; \mathbb{Z})$ is represented by a piecewise linearly imbedded 2-sphere.

EXAMPLE 2.7. *For each $s \geq 2$ there is a boundary link L of s components such that a basis of $H_2(W_L; \mathbb{Z})$ coming from the components of L cannot be represented by mutually disjoint piecewise linearly imbedded 2-spheres. In particular, for each $s \geq 2$ there is a boundary link of s components which is not cobordant to a completely splittable link.*

In the higher odd-dimensional case, such a boundary link also exists. (See [6].)

EXAMPLE 2.10. *For each $s \geq 2$ there is a link L of s components (with linking numbers 0) such that any s elements of $H_2(W_L; \mathbb{Z})$ forming a basis of $H_2(W_L; \mathbb{Q})$ cannot be represented by mutually disjoint piecewise linearly imbedded 2-spheres.*

In fact, we shall show this for the links Q_0 and E_0 of 2-components in Figure 3 considered by H. Lambert [13]. So, neither Q_0 nor E_0 bounds two 2-cells mutually-disjointly and piecewise linearly imbedded in D^4 . This answers a question of H. Lambert [13]. Note that this assertion for Q_0 and E_0 is not derived from the arguments of the Robertello-Arf invariants of links. Cf. [13], [17].

EXAMPLE 2.12. *For each $s \geq 1$ there are compact 4-manifolds W homotopy equivalent to a bouquet of s 2-spheres such that a basis of $H_2(W; \mathbb{Z})$ is represented by s mutually disjoint piecewise linearly imbedded 2-spheres, but any s elements of $H_2(W; \mathbb{Z})$ forming a basis of $H_2(W; \mathbb{Q})$ cannot be represented by mutually disjoint locally flat 2-spheres.*

The case $s = 1$ is the 4-dimensional result of M. Kato [5, Theorem D].

§1 concerns a general argument of quadratic forms of closed oriented odd-dimensional manifolds, where we shall give Fundamental Theorems I and II. In §2, we shall give some algebraic invariants for a piecewise linear imbedding $F \rightarrow W$ inducing an isomorphism ϕ_q for all $q > 0$ and the above examples.

Spaces and maps are considered in the piecewise linear category throughout this paper and it seems that this category is essential for our invariants, because, for example, the Giffen's shift spinning construction produces a homotopy equivalent topological (wild) imbedding $F_g \rightarrow W$ for the manifold W in Example 2.5. (See C. H. Giffen [4], W. T. Eaton, C. P. Pixley and G. A. Venema [2] and Y. Matsumoto [18].)

1. Properties of quadratic forms of odd-dimensional manifolds. Let X be a finite complex with $\text{rank}_{\mathbb{Z}} H_1(X; \mathbb{Z}) \geq 1$. Every element of $H^1(X; \mathbb{Z})$ corresponds bijectively to a homomorphism from the free product $F(X)$ of the fundamental groups of the components of X (or $H_1(X; \mathbb{Z})$) to a fixed infinite cyclic group $\langle t \rangle$ with a specified generator t by the identification $H^1(X; \mathbb{Z}) = \text{Hom}[F(X), \langle t \rangle]$ (or $= \text{Hom}[H_1(X; \mathbb{Z}), \langle t \rangle]$). A nonzero element γ of $H^1(X; \mathbb{Z})$ is *indivisible* if the equation $\gamma = n\gamma'$ with $n \in \mathbb{Z}$ and $\gamma' \in H^1(X; \mathbb{Z})$ implies that $|n| = 1$. Indivisible elements of $H^1(X; \mathbb{Z})$ correspond bijectively to epimorphisms from $F(X)$ (or $H_1(X; \mathbb{Z})$) to $\langle t \rangle$. Let $\gamma \in H^1(X; \mathbb{Z})$ be an element and \tilde{X} be the infinite cyclic cover of X associated with γ , that is, \tilde{X} is the fibered product of a map $f: X \rightarrow S^1$ inducing γ and the exponential cover $\exp: \mathbb{R}^1 \rightarrow S^1$ (cf. [7, p. 437]). The covering translation group of \tilde{X} is identified with $\langle t \rangle$ by the evaluation map γ . For a

(possibly empty) subcomplex X' of X , let \tilde{X}' be the lift of X' by the covering projection $\tilde{X} \rightarrow X$. The rational homology group $H_*(\tilde{X}, \tilde{X}'; Q) = H_*(\tilde{X}, \tilde{X}'; Q)_\gamma$ is a finitely generated module over the rational group ring $Q\langle t \rangle$ of $\langle t \rangle$. $T_*(\tilde{X}, \tilde{X}') = T_*(\tilde{X}, \tilde{X}')_\gamma$ is the $Q\langle t \rangle$ -torsion part of $H_*(\tilde{X}, \tilde{X}'; Q)$ and $T^*(\tilde{X}, \tilde{X}') = T^*(\tilde{X}, \tilde{X}')_\gamma$ is the dual vector space over Q . $T^*(\tilde{X}, \tilde{X}')$ admits a $Q\langle t \rangle$ -module structure induced naturally from $T_*(\tilde{X}, \tilde{X}')$ so that $(fu)(x) = u(fx)$ ($f \in Q\langle t \rangle$, $u \in T^*(\tilde{X}, \tilde{X}')$ and $x \in T_*(\tilde{X}, \tilde{X}')$), since $Q\langle t \rangle$ is a commutative ring. The q th Alexander polynomial $A_q^\gamma(t)$ of the pair (X, X') with $\gamma \in H^1(X; Z)$ is the order ideal of the $Q\langle t \rangle$ -torsion module $T_q(\tilde{X}, \tilde{X}')$ (or equivalently $T^q(\tilde{X}, \tilde{X}')$) (cf. [9]). The q th $Q\langle t \rangle$ -Betti number $\beta_q^\gamma(X, X')$ of the pair (X, X') with $\gamma \in H^1(X; Z)$ is the $Q\langle t \rangle$ -rank of $H_q(\tilde{X}, \tilde{X}'; Q)$. If two nonzero elements $f_1(t)$ and $f_2(t)$ in $Q\langle t \rangle$ are equal up to units of $Q\langle t \rangle$, then the notation $f_1(t) \doteq f_2(t)$ is used.

We have the following reduction formula.

LEMMA 1.1. *Let $\gamma = n\bar{\gamma} \in H^1(X; Z)$ with nonzero $\bar{\gamma} \in H^1(X; Z)$ and nonzero $n \in Z$. Let*

$$T_q(\tilde{X}, \tilde{X}')_{\bar{\gamma}} \approx Q\langle t \rangle / (f_1(t)) \oplus \cdots \oplus Q\langle t \rangle / (f_r(t))$$

be a cyclic decomposition. We have that $\beta_q^\gamma(X, X') = \beta_q^{\bar{\gamma}}(X, X')$ and

$$T_q(\tilde{X}, \tilde{X}')_\gamma \approx Q\langle t \rangle / (f_1(t^n)) \oplus \cdots \oplus Q\langle t \rangle / (f_r(t^n))$$

and, in particular, $A_q^\gamma(t) \doteq A_q^{\bar{\gamma}}(t^n)$.

PROOF. Clearly it suffices to prove this lemma for the case that $n \geq 2$ and X is connected and $\bar{\gamma}$ is indivisible. [Note that the case that $n = -1$ is obvious and $\gamma = (-n)(-\bar{\gamma})$ for $n \leq -2$.] Let (\tilde{X}, \tilde{X}') be the cover of (X, X') associated with $\gamma \in H^1(X; Z)$. \tilde{X} has just n components, since X is connected and $\gamma = n\bar{\gamma}$, $n \geq 2$, and $\bar{\gamma}$ is indivisible. The actions t, t^2, \dots, t^{n-1} are cyclic translations on the components of \tilde{X} and the subgroup $\text{Im } \gamma = \langle t^n \rangle$ of $\langle t \rangle$ acts on each component of \tilde{X} . Let \tilde{X}_0 be any component of \tilde{X} . \tilde{X}_0 is the infinite cyclic cover of X associated with the epimorphism $\gamma_0: \pi_1(X) \rightarrow \text{Im } \gamma = \langle t^n \rangle$ defined by γ . Let $\tilde{X}'_0 = \tilde{X}' \cap \tilde{X}_0$. By assumption we have

$$H_q(\tilde{X}_0, \tilde{X}'_0; Q)_{\gamma_0} \approx Q\langle t^n \rangle^\beta \oplus Q\langle t^n \rangle / (f_1(t^n)) \oplus \cdots \oplus Q\langle t^n \rangle / (f_r(t^n))$$

as $Q\langle t^n \rangle$ -modules, where $\beta = \beta_q^{\bar{\gamma}}(X, X')$. Hence we obtain that

$$\begin{aligned} H_q(\tilde{X}, \tilde{X}'; Q)_\gamma &\approx \sum_{i=0}^{n-1} t^i [Q\langle t^n \rangle^\beta \oplus Q\langle t^n \rangle / (f_1(t^n)) \oplus \cdots \oplus Q\langle t^n \rangle / (f_r(t^n))] \\ &\approx Q\langle t \rangle^\beta \oplus Q\langle t \rangle / (f_1(t^n)) \oplus \cdots \oplus Q\langle t \rangle / (f_r(t^n)) \end{aligned}$$

as $Q\langle t \rangle$ -modules. This completes the proof.

Let M be a compact oriented n -manifold with $\text{rank}_Z H_1(M; Z) \geq 1$. Let $\gamma \in H^1(M; Z)$ be a nonzero element and \tilde{M} be the infinite cyclic cover of M associated with γ . By Duality Theorem (II) and Remark 1.7 of [9], there is a nonsingular pairing

$$T^q(\tilde{M}, \partial\tilde{M}) \times T^{n-q-1}(\tilde{M}) \xrightarrow{\cup} T^{n-1}(\tilde{M}, \partial\tilde{M})$$

induced from the usual cup product pairing

$$H^q(\tilde{M}, \partial\tilde{M}; Q) \times H^{n-q-1}(\tilde{M}; Q) \xrightarrow{\cup} H^{n-1}(\tilde{M}, \partial\tilde{M}; Q).$$

Let M_1, \dots, M_c be the components of M such that the restrictions $\gamma|_{M_i}$ are nonzero, $i = 1, \dots, c$. Let $d = \sum_{i=1}^c |\langle t \rangle| / \text{Im}(\gamma|_{M_i})| < +\infty$. Note that \tilde{M} has just d noncompact components and each noncompact component of \tilde{M} is an infinite cyclic cover of some M_i associated with an indivisible element γ_i in $H^1(M_i; Z)$ specified uniquely by the equality $\gamma|_{M_i} = |\langle t \rangle| / \text{Im}(\gamma|_{M_i})| \gamma_i$. Using this infinite cyclic cover, we can have the quasi-fundamental class² for each noncompact component of \tilde{M} . It follows that $T_{n-1}(\tilde{M}, \partial\tilde{M})$ is the direct sum of d copies of Q with a basis consisting of the quasi-fundamental classes of the noncompact components of \tilde{M} . This basis (as a set) is t -invariant and can be specified uniquely by the orientation of M and the element $\gamma \in H^1(M; Z)$. Using the dual basis u_1, \dots, u_d of this basis, we define a map $\lambda: T^{n-1}(\tilde{M}, \partial\tilde{M}) \rightarrow Q$ by the equality

$$\lambda\left(\sum_{i=1}^d q_i u_i\right) = \sum_{i=1}^d q_i.$$

In other words, λ denotes the Kronecker product of an element $\sum_{i=1}^d q_i u_i \in T^{n-1}(\tilde{M}, \partial\tilde{M})$ with the sum of the quasi-fundamental classes of the noncompact components of \tilde{M} . Note that the following triangle

$$\begin{array}{ccc} T^{n-1}(\tilde{M}, \partial\tilde{M}) & \xrightarrow{\lambda} & \\ \uparrow \delta & & \searrow Q \\ T^{n-2}(\partial\tilde{M}) & \xrightarrow{\lambda} & \end{array}$$

is commutative, since the boundary operator $\partial: T_{n-1}(\tilde{M}, \partial\tilde{M}) \rightarrow T_{n-2}(\partial\tilde{M})$ sends the sum of the quasi-fundamental classes of the noncompact components of \tilde{M} to the sum of the quasi-fundamental classes of the noncompact components of $\partial\tilde{M}$, where δ is the dual homomorphism of the boundary operator ∂ . The pairing $T^q(\tilde{M}, \partial\tilde{M}) \times T^{n-q-1}(\tilde{M}) \rightarrow Q$ induced by the map λ and denoted also by \cup , is

²Let M_* be a compact connected oriented n -manifold with an indivisible element $\gamma_* \in H^1(M_*; Z)$. Let \tilde{M}_* be the infinite cyclic (connected) cover of M_* associated with γ_* . The quasi-fundamental class of \tilde{M}_* is an element of $T_{n-1}(\tilde{M}_*, \partial\tilde{M}_*) (\approx Q)$ which is the image of $1 \in H^0(M_*; Q)$ under a canonical composite monomorphism

$$H^0(\tilde{M}_*; Q) \xrightarrow{\delta} H_c^1(\tilde{M}_*; Q) \xrightarrow{\cap [\tilde{M}_*]} H_{n-1}(\tilde{M}_*, \partial\tilde{M}_*; Q)$$

(cf. [7, Definition 2.2] or [9, p. 181]), where note that for a lift $\tilde{f}: \tilde{M}_* \rightarrow R^1$ of a map $f: M_* \rightarrow S^1$ representing γ_* the following square

$$\begin{array}{ccc} H^0(\tilde{M}_*; Q) & \xrightarrow{\delta} & H_c^1(\tilde{M}_*; Q) \\ \approx \uparrow \tilde{f}^* & & \uparrow \tilde{f}^* \\ H^0(R^1; Q) & \xrightarrow{\delta} & H_c^1(R^1; Q) \end{array}$$

is commutative. The quasi-fundamental class of \tilde{M}_* is determined uniquely by the orientation of M_* and the element $\gamma_* \in H^1(M_*; Z)$.

still nonsingular, because the cup product splits into the components. By [9, Lemma 1.8], and the above remarks, the cup product pairing

$$H^q(\tilde{M}, \partial\tilde{M}; Q) \times H^{n-q-1}(\tilde{M}, \partial\tilde{M}; Q) \xrightarrow{\cup} H^{n-1}(\tilde{M}, \partial\tilde{M}; Q)$$

induces a pairing (evaluated by the map λ)

$$T^q(\tilde{M}, \partial\tilde{M}) \times T^{n-q-1}(\tilde{M}, \partial\tilde{M}) \xrightarrow{\cup} Q$$

satisfying the identity $x \cup y = j^*(x) \cup y = x \cup j^*(y)$, where $j^*: T^*(\tilde{M}, \partial\tilde{M}) \rightarrow T^*(\tilde{M})$ is a homomorphism induced by the inclusion $j: \tilde{M} \subset (\tilde{M}, \partial\tilde{M})$.

Let $n = 2m + 1$ ($m \geq 1$). A t -isometric symmetric bilinear Q -form $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_\gamma: T^m(\tilde{M}, \partial\tilde{M}) \times T^m(\tilde{M}, \partial\tilde{M}) \rightarrow Q$ is defined by the identity

$$\langle x, y \rangle = \begin{cases} x \cup (t - t^{-1})y & \text{if } m \text{ is odd,} \\ x \cup y & \text{if } m \text{ is even,} \end{cases}$$

for all $x, y \in T^m(\tilde{M}, \partial\tilde{M})$.

DEFINITION 1.2. The pair $(\langle \cdot, \cdot \rangle, t)$ is called the *quadratic form of M^{2m+1} with nonzero element $\gamma \in H^1(M; Z)$* (cf. [9, Remark 3.2]).

As an example, let $E(k)$ be the manifold obtained from S^3 by removing an open tubular neighborhood of an oriented knot k in the oriented 3-sphere S^3 and $\gamma \in H^1(E(k); Z)$ be the dual of a generator of $H_1(E(k); Z)$ linking the knot k in S^3 by the linking number $+1$. The quadratic form

$$(\langle \cdot, \cdot \rangle_k, t): T^1(\tilde{E}(k), \partial\tilde{E}(k)) \times T^1(\tilde{E}(k), \partial\tilde{E}(k)) \rightarrow Q$$

is necessarily nonsingular and called the *quadratic form of the knot k* (cf. J. W. Milnor [19]).

For simplicity, we shall henceforth consider a *closed* $(2m + 1)$ -manifold M^{2m+1} . Then in the case that m is even the quadratic form $\langle \cdot, \cdot \rangle$ is nonsingular by definition. In the case that m is odd the (*maximal*) *null subspace* $N = N(\tilde{M})$ of this quadratic form $\langle \cdot, \cdot \rangle$, whose Q -dimension is called the *nullity of M with γ* and denoted by $n^\gamma(M)$, has a $Q\langle t \rangle$ -module structure and is $Q\langle t \rangle$ -isomorphic to a direct sum of copies of $Q\langle t \rangle/(t - 1)$ and $Q\langle t \rangle/(t + 1)$, for N is precisely equal to the kernel of $t - t^{-1}: T^m(\tilde{M}) \rightarrow T^m(\tilde{M})$.³ The signature of the form $\langle \cdot, \cdot \rangle$ is called the *signature of M with γ* and denoted by $\sigma^\gamma(M)$. Note that $\sigma^\gamma(M)$ for odd m is equal to the signature of the nonsingular form $\langle \cdot, \cdot \rangle^*: \hat{T}^m(\tilde{M}) \times \hat{T}^m(\tilde{M}) \rightarrow Q$ with $\hat{T}^m(\tilde{M}) = T^m(\tilde{M})/N(\tilde{M})$ induced from the form $\langle \cdot, \cdot \rangle: T^m(\tilde{M}) \times T^m(\tilde{M}) \rightarrow Q$. We now describe the local signature $\sigma_\omega^\gamma(M)$ at any value ω in the interval $[-1, 1]$ (cf. J. W. Milnor [19]). Consider a nonsingular t -isometric symmetric Q -form $\langle \cdot, \cdot \rangle: T \times T \rightarrow Q$ with a finitely generated torsion $Q\langle t \rangle$ -module T . By tensoring with the real field R , we consider this form as a nonsingular t -isometric symmetric R -form $\langle \cdot, \cdot \rangle_R: T_R \times T_R \rightarrow R$. T_R splits into the $p(t)$ -primary components $C_{p(t)}$, where $p(t)$ ranges over all irreducible real polynomials of degree 1 or 2 with the leading coefficient 1 and $C_{p(t)} = 0$ except for a finite number of $p(t)$. Because of the

³Thus, in the case that m is odd, if $A_m^\gamma(\pm 1) \neq 0$ for the m th Alexander polynomial $A_m^\gamma(t)$ of M with γ , then $N = 0$, i.e., the quadratic form $\langle \cdot, \cdot \rangle$ is nonsingular.

equality $\langle f(t)x, y \rangle_R = \langle x, f(t^{-1})y \rangle_R$, the signature of the form \langle , \rangle_R (and hence of \langle , \rangle) is the sum of the signatures of the forms restricted to $C_{p(t)}$ with $p(t) = t - 1, t + 1$ or $t^2 - 2\omega t + 1, -1 < \omega < 1$ (cf. J. W. Milnor [19, p. 129]). [Note that if $p_1(t) \neq rt^a p_2(t^{-1}), r \in R$, then $C_{p_i(t)}, i = 1, 2$, are orthogonal, i.e., $C_{p_1(t)} \perp C_{p_2(t)}$, and if $p_1(t) = rt^a p_2(t^{-1}), r \in R$, but $p_1(t) \neq p_2(t)$, then $C_{p_i(t)} \perp C_{p_j(t)}, i = 1, 2$. (See J. W. Milnor [20, Lemma 3.1].) Further, if $p(t) = rt^a p(t^{-1}), r \in R$, then $p(t)$ must be $t - 1, t + 1$ or $t^2 - 2\omega t + 1, -1 < \omega < 1$.]

DEFINITION 1.3 (J. W. MILNOR [19]). For each value $\omega, -1 \leq \omega \leq 1$, σ_ω^γ is the signature of the restricted form of \langle , \rangle_R to the component $C_{p(t)}$, where $p(t) = t - 1, t + 1$ or $t^2 - 2\omega t + 1$ according as $\omega = 1, -1$ or $-1 < \omega < 1$, and called the local signature of the nonsingular form \langle , \rangle at $\omega \in [-1, 1]$. The local signature of the nonsingular form $\langle , \rangle: \hat{T}^m(\tilde{M}) \times \hat{T}^m(\tilde{M}) \rightarrow Q$ for odd m or $\langle , \rangle: T^m(\tilde{M}) \times T^m(\tilde{M}) \rightarrow Q$ for even m at $\omega \in [-1, 1]$ is called the local signature of M with $\gamma \in H^1(M; Z)$ at $\omega \in [-1, 1]$ and denoted by $\sigma_\omega^\gamma(M)$.

LEMMA 1.4 (J. W. MILNOR). The signature of a t -isometric symmetric nonsingular R -form of the following type

$$\langle , \rangle_R: R\langle t \rangle / (t^2 - 2\omega t + 1) \times R\langle t \rangle / (t^2 - 2\omega t + 1) \rightarrow R, \quad -1 < \omega < 1,$$

is ± 2 (cf. [19, Assertion 11]).

DEFINITION 1.5 (J. LEVINE [14]). The nonsingular t -isometric symmetric Q -form $(\langle , \rangle, t): T \times T \rightarrow Q$ with a finitely generated torsion $Q\langle t \rangle$ -module T is null-cobordant, if there is a $Q\langle t \rangle$ -submodule T_0 of T that is a self-orthogonal complement, i.e., $T_0^\perp = T_0$ with respect to this form \langle , \rangle . Two nonsingular t -isometric symmetric forms $(\langle , \rangle^i, t), i = 1, 2$, are cobordant, if the orthogonal sum $(\langle , \rangle^1 \perp -\langle , \rangle^2, t)$ is null-cobordant.

Clearly the signatures of cobordant forms are equal. We also have the following

LEMMA 1.6. The local signatures of cobordant forms at any $\omega \in [-1, 1]$ are equal.

PROOF. It suffices to show that the local signature at any $\omega \in [-1, 1]$ of a null-cobordant form is 0. Consider a null-cobordant form $(\langle , \rangle_R, t): T_R \times T_R \rightarrow R$ over R with $T_R^{0\perp} = T_R^0$. Let $p(t) = t - 1, t + 1$ or $t^2 - 2\omega t + 1, -1 < \omega < 1$. Note that the restricted form of \langle , \rangle_R to $C_{p(t)}$ is nonsingular, since for any different $q(t), C_{p(t)} \perp C_{q(t)}$. Let $y \in C_{p(t)}$. Suppose $\langle x, y \rangle = 0$ for all $x \in T_R^0 \cap C_{p(t)}$. Using that $C_{p(t)} \perp C_{q(t)}$ for a different $q(t)$, we obtain that $\langle x, y \rangle = 0$ for all $x \in T_R^0$ and hence $y \in T_R^0 \cap C_{p(t)}$. This implies that $(T_R^0 \cap C_{p(t)})^\perp = T_R^0 \cap C_{p(t)}$. Now the conclusion easily follows. This completes the proof.

Notations 1.7. Let \tilde{X} be an infinite cyclic cover of a finite complex X . The $Q\langle t \rangle$ -torsion module $T_*(\tilde{X})$ splits into the $p(t)$ -primary components over $Q\langle t \rangle$, where $p(t)$ is a primitive,⁴ nonconstant, irreducible polynomial. The $p(t)$ -primary component of $T_*(\tilde{X})$ with $p(t) = t - 1$ or $t + 1$ is denoted by ${}_1U_*(\tilde{X})$ or ${}_{-1}U_*(\tilde{X})$, respectively, and the direct summand consisting of all the $p(t)$ -primary components with $|p(1)| \neq 1$ is denoted by $U_*(\tilde{X})$. The corresponding parts of

⁴A nonzero element of $Q\langle t \rangle$ is primitive, if it is written as either the constant 1 or an integral polynomial $c_n t^n + c_{n-1} t^{n-1} + \cdots + c_0$ ($n > 0$) with $c_n (> 0), c_{n-1}, \dots, c_0 (\neq 0)$ coprime (cf. [9]).

$T^*(\tilde{X})$ to ${}_iU_*(\tilde{X})$ and $U_*(\tilde{X})$ are denoted by ${}_iU^*(\tilde{X})$ and $U^*(\tilde{X})$, respectively. Similar notations are used for a pair (\tilde{X}, \tilde{X}') .

The following Fundamental Theorem I is a generalization of Theorem 4.3 in [9] which is a main tool to show a failure of the 4-dimensional Whitney lemma in the piecewise linear category (cf. [9, Theorem C]).

FUNDAMENTAL THEOREM I. *Let X_i , $i = 1, 2$, be finite complexes with nonzero elements $\gamma_i \in H^1(X_i; Z)$ imbedded in a finite complex Y so that γ_i are the restrictions on the same element $\gamma \in H^1(Y; Z)$ to X_i . If $H_q(Y, X_i; Q) = H_{q+1}(Y, X_i; Q) = 0$, $i = 1, 2$, then $\beta_q^{\gamma_1}(X_1) = \beta_q^{\gamma_2}(X_2)$ and ${}_1U_q(\tilde{X}_1)$ is $Q\langle t \rangle$ -isomorphic to ${}_1U_q(\tilde{X}_2)$ and so the multiplicities of $t = 1$ in the roots of the q th Alexander polynomials $A_q^{\gamma_i}(t)$ are equal. Moreover, if $H_q(Y, X_i; Z) = H_{q+1}(Y, X_i; Z) = 0$, then $U_q(\tilde{X}_1)$ is $Q\langle t \rangle$ -isomorphic to $U_q(\tilde{X}_2)$ and, in particular, $A_q^{\gamma_1}(t) \doteq A_q^{\gamma_2}(t)$ (up to $f(t)$ with $f(t) \in Z\langle t \rangle$ and $|f(1)| = 1$).*

REMARK 1.8. A version using a finite field Z_p instead of Q is also possible.

The proof of Fundamental Theorem I is parallel to the proof of Theorem 4.3 in [9]. We sketch here only an outline of the proof. It proceeds as follows:

$$H_j(Y, X_i; Q) = 0$$

implies that $H_j(\tilde{Y}, \tilde{X}_i; Q) = T_j(\tilde{Y}, \tilde{X}_i)$ and ${}_1U_j(\tilde{Y}, \tilde{X}_i) = 0$. [Use the Wang exact sequence.] From the exact sequence of the pair (\tilde{Y}, \tilde{X}_i) , it follows that $\beta_q^{\gamma_i}(X_i) = \beta_q^{\gamma}(Y)$ and the sequence

$$T_{q+1}(\tilde{Y}, \tilde{X}_i) \xrightarrow{\partial} T_q(\tilde{X}_i) \xrightarrow{i^*} T_q(\tilde{Y}) \xrightarrow{j^*} T_q(\tilde{Y}, \tilde{X}_i)$$

is exact and so ${}_1U_q(\tilde{X}_i)$ is $Q\langle t \rangle$ -isomorphic to ${}_1U_q(\tilde{Y})$ ($i = 1, 2$). If

$$H_j(Y, X_i; Z) = 0,$$

then we have $U_j(\tilde{Y}, \tilde{X}_i) = 0$. Hence the above exact sequence implies that $U_q(\tilde{X}_i)$ is $Q\langle t \rangle$ -isomorphic to $U_q(\tilde{Y})$. In particular, $A_q^{\gamma_1}(t)f_1(t) \doteq A_q^{\gamma_2}(t)f_2(t)$ for some $f_i(t) \in Z\langle t \rangle$ with $|f_i(1)| = 1$. This completes the outlined proof of Fundamental Theorem I.

FUNDAMENTAL THEOREM II. *Let M be the boundary of a compact oriented $(2m + 2)$ -manifold W with a nonzero element $\bar{\gamma} \in H^1(W; Z)$ extending a nonzero element $\gamma \in H^1(M; Z)$. Suppose that the sequence*

$$T^m(\tilde{W}) \xrightarrow{i^*} T^m(\tilde{M}) \xrightarrow{\delta} T^{m+1}(\tilde{W}, \tilde{M})$$

is exact at $T^m(\tilde{M})$. Then for even m the quadratic form $(\langle, \rangle, t): T^m(\tilde{M}) \times T^m(\tilde{M}) \rightarrow Q$ is null-cobordant. In particular, $A_m^{\gamma}(t) \doteq f(t)f(t^{-1})$ for some $f(t) \in Q\langle t \rangle$ and $\sigma_\omega^{\gamma}(M) = 0$ at all values $\omega \in [-1, 1]$ and $\sigma^{\gamma}(M) = 0$. For odd m $A_m^{\gamma}(t) \doteq f(t)f(t^{-1})$ for some $f(t) \in Q\langle t \rangle$ and $\sigma_\omega^{\gamma}(M) = 0$ at all values $\omega \in (-1, 1)$. Further, for odd m if the kernel of $t - t^{-1}: \text{Im } \delta \rightarrow \text{Im } \delta$ is equal to the image of $N(\tilde{M})$ by the map δ , then $N(\tilde{M})$ is $Q\langle t \rangle$ -isomorphic to the direct double $\delta(N(\tilde{M})) \oplus \delta(N(\tilde{M}))$ and, in particular, $n^{\gamma}(M) \equiv 0 \pmod{2}$ and the induced nonsingular form $(\langle, \rangle^*, t): \hat{T}^m(\tilde{M}) \times \hat{T}^m(\tilde{M}) \rightarrow Q$ is null-cobordant. In particular, $\sigma_\omega^{\gamma}(M) = 0$ at all values $\omega \in [-1, 1]$ and $\sigma^{\gamma}(M) = 0$.

1.9. PROOF OF FUNDAMENTAL THEOREM II. Given $y \in T^m(\tilde{M})$, suppose $i^*(x) \cup y = 0$ for all $x \in T^m(\tilde{M})$. By [9, Lemma 1.8], $x \cup \delta(y) = 0$ for all x . Hence $\delta(y) = 0$, for the cup product pairing $T^m(\tilde{W}) \times T^{m+1}(\tilde{W}, \tilde{M}) \xrightarrow{\cup} Q$ is nonsingular ([9, Duality Theorem (II)]). Using the exact sequence

$$T^m(\tilde{W}) \xrightarrow{i^*} T^m(\tilde{M}) \xrightarrow{\delta} T^{m+1}(\tilde{W}, \tilde{M}),$$

we see that y lies in $\text{Im } i^*$. It follows that $\text{Im } i^*$ is self-orthogonal, i.e., $(\text{Im } i^*)^\perp = \text{Im } i^*$ concerning the nonsingular cup product pairing $T^m(\tilde{M}) \times T^m(\tilde{M}) \xrightarrow{\cup} Q$. Since the sequence $0 \rightarrow \text{Im } i^* \rightarrow T^m(\tilde{M}) \xrightarrow{\delta} \text{Im } \delta \rightarrow 0$ is exact, from this fact and the equality $(tx) \cup y = x \cup (t^{-1}y)$ we obtain that $A_m^\gamma(t) \doteq f(t)f(t^{-1})$, where $f(t)$ is the order ideal of $\text{Im } i^*$. For even m , we also obtain that the form $(\langle \cdot, \cdot \rangle, t): T^m(\tilde{M}) \times T^m(\tilde{M}) \rightarrow Q$ is null-cobordant and in particular $\sigma_\omega^\gamma(M) = 0$ at all $\omega \in [-1, 1]$ and $\sigma^\gamma(M) = 0$. Let m be odd. Let T be the direct summand of $T^m(\tilde{M})$ consisting of the $p(t)$ -primary components such that $p(\pm 1) \neq 0$. The quadratic form $(\langle \cdot, \cdot \rangle, t): T^m(\tilde{M}) \times T^m(\tilde{M}) \rightarrow Q$ then restricts to the nonsingular form $(\langle \cdot, \cdot \rangle, t): T \times T \rightarrow Q$, since the order ideal $P(t)$ of T satisfies $P(\pm 1) \neq 0$. [Note that this asserts an isomorphism $t - t^{-1}: T \approx T$.]

We have $(T \cap \text{Im } i^*)^\perp = T \cap \text{Im } i^*$ concerning this nonsingular pairing. By Lemma 1.6 this implies that $\sigma_\omega^\gamma(M) = 0$ at $\omega \in (-1, 1)$.

Assume that the kernel of $t - t^{-1}: \text{Im } \delta \rightarrow \text{Im } \delta$ is equal to $\delta(N(\tilde{M}))$. Then we have that the kernel sequence of the following morphism

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{Im } i^* & \rightarrow & T^m(\tilde{M}) & \xrightarrow{\delta} & \text{Im } \delta \rightarrow 0 \\ & & \downarrow t - t^{-1} & & \downarrow t - t^{-1} & & \downarrow t - t^{-1} \\ 0 & \rightarrow & \text{Im } i^* & \rightarrow & T^m(\tilde{M}) & \xrightarrow{\delta} & \text{Im } \delta \rightarrow 0 \end{array}$$

is exact. Dividing by this kernel exact sequence, we obtain the quotient exact sequence $0 \rightarrow \text{Im } i^* \rightarrow \hat{T}^m(\tilde{M}) \xrightarrow{\delta} \text{Im } \delta \rightarrow 0$. Now it is easy to see that $(\text{Im } i^*)^\perp = \text{Im } i^*$ concerning the nonsingular form $\langle \cdot, \cdot \rangle: \hat{T}^m(\tilde{M}) \times \hat{T}^m(\tilde{M}) \rightarrow Q$. Therefore the form $(\langle \cdot, \cdot \rangle, t)$ is null-cobordant and $\sigma_\omega^\gamma(M) = 0$ at all $\omega \in [-1, 1]$ and $\sigma^\gamma(M) = 0$. Since the induced pairing $\text{Im } i^* \times T^m(\tilde{M})/\text{Im } i^* \xrightarrow{\cup} Q$ is nonsingular and $T^m(\tilde{M})/\text{Im } i^* \approx \text{Im } \delta$, the kernel of $t - t^{-1}: \text{Im } i^* \rightarrow \text{Im } i^*$ is $Q\langle t \rangle$ -isomorphic to the kernel of $t - t^{-1}: \text{Im } \delta \rightarrow \text{Im } \delta$ that is $\delta(N(\tilde{M}))$ by assumption. Hence $N(\tilde{M})$ is $Q\langle t \rangle$ -isomorphic to the duplication $\delta(N(\tilde{M})) \oplus \delta(N(\tilde{M}))$, since $N(\tilde{M})$ is $Q\langle t \rangle$ -isomorphic to a direct sum of some copies of $Q\langle t \rangle/(t - 1)$ and $Q\langle t \rangle/(t + 1)$. In particular, $n^\gamma(M) \equiv 0 \pmod{2}$. This completes the proof.

DEFINITION 1.10. Two closed connected oriented n -manifolds M_i , $i = 1, 2$, with nonzero elements $\gamma_i \in H^1(M_i; \mathbb{Z})$ are *homology cobordant* (or *rationally homology cobordant*, respectively), if there is a compact connected oriented cobordism W^{n+1} with $\partial W = M_1 \cup -M_2$ such that $H_*(W, M_i; \mathbb{Z}) = 0$ (or $H_*(W, M_i; \mathbb{Q}) = 0$, respectively) and γ_i are the restrictions of the same element $\gamma \in H^1(W; \mathbb{Z})$ to M_i .

We have two standard consequences of Fundamental Theorems I and II.

COROLLARY 1.11. *Suppose (M_1^{2m+1}, γ_1) and (M_2^{2m+1}, γ_2) are homology cobordant. Then*

- (a) $\beta_*^{\gamma_1}(M_1) = \beta_*^{\gamma_2}(M_2)$,
- (b) $U_*(\tilde{M}_1) \approx U_*(\tilde{M}_2)$, $N(\tilde{M}_1) \approx N(\tilde{M}_2)$ and $n^{\gamma_1}(M_1) = n^{\gamma_2}(M_2)$,
- (c) $A_q^{\gamma_1}(t) \doteq A_q^{\gamma_2}(t)$ (up to $f(t)$ with $f(t) \in Z\langle t \rangle$, $|f(1)| = 1$ if $q \neq m$ or up to $f(t)f(t^{-1})$ with $f(t) \in Z\langle t \rangle$, $|f(1)| = 1$ if $q = m$),
- (d) for even m the quadratic forms $(\langle, \rangle_i, t): T^m(\tilde{M}_i) \times T^m(\tilde{M}_i) \rightarrow Q$ are cobordant and for odd m the induced nonsingular forms $(\langle, \rangle_i, t): \hat{T}^m(\tilde{M}_i) \times \hat{T}^m(\tilde{M}_i) \rightarrow Q$ are cobordant, $i = 1, 2$,
- (e) $\sigma_\omega^{\gamma_1}(M_1) = \sigma_\omega^{\gamma_2}(M_2)$ at all $\omega \in [-1, 1]$ and $\sigma^{\gamma_1}(M_1) = \sigma^{\gamma_2}(M_2)$.

PROOF. Note that $T_{m+1}(\tilde{W}, \partial\tilde{W}) \xrightarrow{\partial} T_m(\partial\tilde{W}) \xrightarrow{i_*} T_m(\tilde{W})$ is exact, since the canonical homomorphism $j_*: H_{m+1}(\tilde{W}; Q) \rightarrow H_{m+1}(\tilde{W}, \partial\tilde{W}; Q)$ is the composite $H_{m+1}(\tilde{W}; Q) \rightarrow H_{m+1}(\tilde{W}, \tilde{M}_i; Q) \rightarrow H_{m+1}(\tilde{W}, \partial\tilde{W}; Q)$ and $T_{m+1}(\tilde{W}, \tilde{M}_i) = H_{m+1}(\tilde{W}, \tilde{M}_i; Q)$. Hence the dual sequence $T^m(\tilde{W}) \xrightarrow{i^*} T^m(\partial\tilde{W}) \xrightarrow{\delta} T^{m+1}(\tilde{W}, \partial\tilde{W})$ is exact. Then (a), (b), (c), (d) ($m = \text{even}$) and (e) ($m = \text{even}$) follow from Fundamental Theorems I and II. [Note that the order ideal of $T^m(\partial\tilde{W})$ is the product $A_m^{\gamma_1}(t)A_m^{\gamma_2}(t)$ and $A_m^{\gamma_i}(t) \doteq A_m^{\gamma_i}(t^{-1})$. So, $A_m^{\gamma_1}(t)A_m^{\gamma_2}(t^{-1}) \doteq g(t)g(t^{-1})$ for some $g(t) \in Q\langle t \rangle$ and hence $A_m^{\gamma_1}(t)g_1(t)g_1(t^{-1}) \doteq A_m^{\gamma_2}(t)g_2(t)g_2(t^{-1})$ for some $g_i(t) \in Z\langle t \rangle$ with $|g_i(1)| = 1$, since $A_m^{\gamma_1}(t) \doteq A_m^{\gamma_2}(t)$ up to $f(t) \in Z\langle t \rangle$ with $|f(1)| = 1$. Further, note that $\langle, \rangle = \langle, \rangle_1 \perp -\langle, \rangle_2$ for the form $\langle, \rangle: T^m(\partial\tilde{W}) \times T^m(\partial\tilde{W}) \rightarrow Q$.] Next, note that the above exact sequence induces a short exact sequence

$$0 \rightarrow U^m(\tilde{W}) \xrightarrow{i^{**}} U^m(\partial\tilde{W}) \xrightarrow{\delta'} U^{m+1}(\tilde{W}, \partial\tilde{W}) \rightarrow 0$$

by using $U^m(\partial\tilde{W}) \approx U^m(\tilde{M}_1) \oplus U^m(\tilde{M}_2)$ (by (b)) and $U^m(\tilde{W}) \approx U^m(\tilde{M}_1)$ and (by [9, Duality Theorem (II)]) $U^{m+1}(\tilde{W}, \partial\tilde{W}) \approx U^m(\tilde{W}) \approx U^m(\tilde{M}_1)$. Noting these isomorphisms again, this short exact sequence also induces a short exact sequence

$$0 \rightarrow N^m(\tilde{W}) \xrightarrow{i^{***}} N(\partial\tilde{W}) \xrightarrow{\delta''} N^{m+1}(\tilde{W}, \partial\tilde{W}) \rightarrow 0,$$

where $N^m(\tilde{W})$ and $N^{m+1}(\tilde{W}, \partial\tilde{W})$ are the kernels of $t - t^{-1}: U^m(\tilde{W}) \rightarrow U^m(\tilde{W})$ and $t - t^{-1}: U^{m+1}(\tilde{W}, \partial\tilde{W}) \rightarrow U^{m+1}(\tilde{W}, \partial\tilde{W})$, respectively. This implies that the kernel of $t - t^{-1}: \text{Im } \delta \rightarrow \text{Im } \delta$ is precisely equal to $N^{m+1}(\tilde{W}, \partial\tilde{W}) = \delta(N(\partial\tilde{W}))$. Therefore by Fundamental Theorem II the induced nonsingular form $(\langle, \rangle_i, t): \hat{T}^m(\partial\tilde{W}) \times \hat{T}^m(\partial\tilde{W}) \rightarrow Q$ for odd m is null-cobordant. But $\langle, \rangle_i = \langle, \rangle_1 \perp -\langle, \rangle_2$. This shows (d) ($m = \text{odd}$). From Lemma 1.6, (e) ($m = \text{odd}$) also follows. This completes the proof.

REMARK 1.12. The property (c) can be also generalized to the many-variable Alexander polynomials (cf. [10]).

EXAMPLE 1.13. Let $D_1 \cup D_2 \cup \cdots \cup D_n$ be the disjoint union of 2-cells properly and piecewise linearly and locally-flatly imbedded in a 4-cell D^4 . Let M be the boundary of the 4-manifold obtained from D^4 by removing a regular neighborhood of $D_1 \cup \cdots \cup D_n$ in D^4 meeting ∂D^4 regularly. Clearly, M with an arbitrary nonzero $\gamma \in H^1(M; Z)$ is homology cobordant to the connected sum $\#^n S^1 \times S^2$

with some $\gamma' \in H^1(\#^n S^1 \times S^2; Z)$. By Corollary 1.11, we have that $\beta_1^\gamma(M) = n - 1$, $n^\gamma(M) = 0$, $A_1^\gamma(t) \doteq f(t)f(t^{-1})$, $f(t) \in Z\langle t \rangle$, $|f(1)| = 1$, and $\sigma^\gamma(M) = \sigma_\omega^\gamma(M) = 0$, all $\omega \in [-1, 1]$, and the quadratic form $(\langle, \rangle, t): T^1(\tilde{M}) \times T^1(\tilde{M}) \rightarrow Q$ is nonsingular and null-cobordant for all nonzero $\gamma \in H^1(M; Z)$ (cf. K. Murasugi [21], A. Kawauchi [9], [10]).

COROLLARY 1.14. *Suppose (M_1^{2m+1}, γ_1) and (M_2^{2m+1}, γ_2) are rationally homology cobordant. Then*

- (a) $\beta_{*}^{\gamma_1}(M_1) = \beta_{*}^{\gamma_2}(M_2)$,
- (b) ${}_1U_{*}(\tilde{M}_1) \approx {}_1U_{*}(\tilde{M}_2)$,
- (c) $A_q^{\gamma_1}(t) \doteq A_q^{\gamma_2}(t)$ (up to $f(t) \in Q\langle t \rangle$ with $f(1) \neq 0$ if $q \neq m$ or up to $f(t)f(t^{-1})$ with $f(t) \in Q\langle t \rangle$ and $f(1) \neq 0$ if $q = m$),
- (d) if m is even, then the quadratic forms $(\langle, \rangle, t): T^m(\tilde{M}_i) \times T^m(\tilde{M}_i) \rightarrow Q$, $i = 1, 2$, are cobordant and, in particular, $\sigma_\omega^{\gamma_1}(M_1) = \sigma_\omega^{\gamma_2}(M_2)$ at all $\omega \in [-1, 1]$ and $\sigma^{\gamma_1}(M_1) = \sigma^{\gamma_2}(M_2)$,
- (e) if m is odd, then $\sigma_\omega^{\gamma_1}(M_1) = \sigma_\omega^{\gamma_2}(M_2)$ at all $\omega \in (-1, 1]$.
Moreover, for odd m assume that ${}_1U^m(\tilde{M}_i) \subset N(\tilde{M}_i)$, $i = 1, 2$. Then
- (f) $N(\tilde{M}_1) \oplus N(\tilde{M}_2)$ is $Q\langle t \rangle$ -isomorphic to a direct double $N' \oplus N'$ and $n^{\gamma_1}(M_1) \equiv n^{\gamma_2}(M_2) \pmod{2}$,
- (g) the induced forms $(\langle, \rangle, t): \hat{T}^m(\tilde{M}_i) \times \hat{T}^m(\tilde{M}_i) \rightarrow Q$, $i = 1, 2$, are cobordant,
- (h) $\sigma^{\gamma_1}(M_1) = \sigma^{\gamma_2}(M_2)$.

PROOF. By an analogous method of Corollary 1.11, we obtain (a), (b), (c) and (d). The sequence $0 \rightarrow \text{Im } i^* \rightarrow T^m(\partial \tilde{W}) \xrightarrow{\delta} \text{Im } \delta \rightarrow 0$ and the induced sequence $0 \rightarrow {}_1U^m(\tilde{W}) \xrightarrow{i^*} {}_1U^m(\partial \tilde{W}) \xrightarrow{\delta'} {}_1U^{m+1}(\tilde{W}, \partial \tilde{W}) \rightarrow 0$ are exact, and the kernel sequence of the following morphism is also exact.

$$\begin{array}{ccccccc} 0 & \rightarrow & {}_1U^m(\tilde{W}) & \xrightarrow{i^*} & {}_1U^m(\partial \tilde{W}) & \xrightarrow{\delta'} & {}_1U^{m+1}(\tilde{W}, \partial \tilde{W}) \rightarrow 0 \\ & & \downarrow t - t^{-1} & & \downarrow t - t^{-1} & & \downarrow t - t^{-1} \\ 0 & \rightarrow & {}_1U^m(\tilde{W}) & \xrightarrow{i^*} & {}_1U^m(\partial \tilde{W}) & \xrightarrow{\delta'} & {}_1U^{m+1}(\tilde{W}, \partial \tilde{W}) \rightarrow 0 \end{array}$$

Thus, $\sigma_\omega^{\gamma_1}(M_1) - \sigma_\omega^{\gamma_2}(M_2) = 0$ at all $\omega \in (-1, 1]$ for odd m , showing (e). Let m be odd. If ${}_1U^m(\tilde{M}_i) \subset N(\tilde{M}_i)$ ($i = 1, 2$), then combining this with the above remarks we obtain that the kernel of $t - t^{-1}: \text{Im } \delta \rightarrow \text{Im } \delta$ is equal to $\delta(N(\partial \tilde{W}))$. Therefore, by Fundamental Theorem II we have (f), (g) and (h). This completes the proof.

The above two corollaries will be positively used in the next section and an intended paper [12].

2. A piecewise linear imbedding of a surface into some 4-manifold. Let F be a closed oriented surface with s (≥ 1) components F_1, \dots, F_s of genera g_1, \dots, g_s . Consider a compact connected oriented piecewise linear 4-manifold W with an isomorphism $\varphi_q: H_q(F; Z) \approx H_q(W; Z)$ for all $q > 0$ and such that the intersection number of any two elements of $H_2(W; Z)$ is 0. It follows that ∂W is connected and the sequence

$$0 \rightarrow H_2(W, \partial W; Z) \xrightarrow{\partial} H_1(\partial W; Z) \xrightarrow{i_*} H_1(W; Z) \rightarrow 0$$

is exact. Let $\varphi_2[F_1]^*, \dots, \varphi_2[F_s]^*$ be the basis of $H^2(W; Z)$ dual to the basis $\varphi_2[F_1], \dots, \varphi_2[F_s]$ of $H_2(W; Z)$, where $[F_i]$ is the fundamental class of F_i . Let a_1, \dots, a_s be the elements of $H_1(\partial W; Z)$ obtained from $\varphi_2[F_1]^*, \dots, \varphi_2[F_s]^*$ by the composite monomorphism

$$H^2(W; Z) \xrightarrow{\cap [W]} H_2(W, \partial W; Z) \xrightarrow{\partial} H_1(\partial W; Z).$$

Next, let $\varphi_1 H_1(F_1; Z)^*, \dots, \varphi_1 H_1(F_s; Z)^*$ be the direct summands of $H^1(W; Z)$ dual to $\varphi_1 H_1(F_1; Z), \dots, \varphi_1 H_1(F_s; Z)$ by using the decomposition $H_1(W; Z) = \varphi_1 H_1(F_1; Z) \oplus \dots \oplus \varphi_1 H_1(F_s; Z)$. Let G_1, \dots, G_s be the subgroups of $H^1(\partial W; Z)$ obtained from $\varphi_1 H_1(F_1; Z)^*, \dots, \varphi_1 H_1(F_s; Z)^*$ by the canonical monomorphism $i^*: H^1(W; Z) \rightarrow H^1(\partial W; Z)$.

DEFINITION 2.1. Let H be an abelian group and let $H^* = \text{Hom}[H, \langle t \rangle]$ and G be a subgroup of H^* . An element γ of H^* is independent of G , if $\gamma(x) = 1$ for $x \in H$ such that $\gamma'(x) \neq 1$ for some $\gamma' \in G$.

Suppose a piecewise linear imbedding $\tilde{\varphi}: F \rightarrow W$ inducing this isomorphism φ_q , $q > 0$. Then $\tilde{\varphi}$ is homotopic to a piecewise linear embedding $\tilde{\varphi}': F \rightarrow W$ such that for each i , $\tilde{\varphi}'(F_i)$ has just one locally knotted point (cf. R. H. Fox and J. W. Milnor [3]). So, assume that for our imbedding $\tilde{\varphi}$, $\tilde{\varphi}(F_i)$ has always just one locally knotted point with knot type denoted by k_i .

THEOREM 2.2. Let $\delta_1, \dots, \delta_s$ be 0 or 1 such that $\delta_1 + \dots + \delta_s \geq 1$. If there exists a piecewise linear imbedding $\tilde{\varphi}: F \rightarrow W$ inducing the isomorphism φ_q for all $q \geq 0$, then $n^\gamma(\partial W) = 2g_1\delta_1 + \dots + 2g_s\delta_s$, $\beta_1^\gamma(\partial W) = s - 1 + 2g_1(1 - \delta_1) + \dots + 2g_s(1 - \delta_s)$ and the induced nonsingular form $(\langle, \rangle, t): \hat{T}^1(\partial \tilde{W}) \times \hat{T}^1(\partial \tilde{W}) \rightarrow Q$ is cobordant to the orthogonal sum $(\langle, \rangle_{k_1}\delta_1, t) \perp \dots \perp (\langle, \rangle_{k_s}\delta_s, t)$ and in particular $\sigma_\omega^\gamma(\partial W) = \sigma_\omega(k_1)\delta_1 + \dots + \sigma_\omega(k_s)\delta_s$ at all $\omega \in [-1, 1]$ and $\sigma^\gamma(\partial W) = \sigma(k_1)\delta_1 + \dots + \sigma(k_s)\delta_s$, and further

$$A_1^\gamma(t) \doteq (t - 1)^{2g_1\delta_1 + \dots + 2g_s\delta_s} \Delta_{k_1}(t^{\delta_1}) \dots \Delta_{k_s}(t^{\delta_s})$$

(up to $f(t)f(t^{-1})$ with $f(t) \in Z\langle t \rangle$ and $|f(1)| = 1$) for all indivisible elements $\gamma \in H^1(\partial W; Z)$ sending a_1, \dots, a_s to $t^{\delta_1}, \dots, t^{\delta_s}$ and independent of the subgroup $(1 - \delta_1)G_1 + \dots + (1 - \delta_s)G_s$.

COROLLARY 2.3. The cobordism class of the quadratic form $(\langle, \rangle_{k_i}, t)$ of the knot k_i representing the locally knotted point of $\tilde{\varphi}(F_i)$ and its invariants: the knot signature $\sigma(k_i)$ and the local knot signature $\sigma_\omega(k_i)$ at all $\omega \in [-1, 1]$ and the knot polynomial $\Delta_{k_i}(t)$ (up to $f(t)f(t^{-1})$ with $f(t) \in Z\langle t \rangle$ and $|f(1)| = 1$) do not depend on a particular choice of piecewise linear imbeddings inducing the isomorphism φ_q for all $q > 0$.

This corollary easily follows from Theorem 2.2 by taking $\delta_1 + \dots + \delta_s = 1$.

2.4. PROOF OF THEOREM 2.2. Let N be a disk sum in $\text{Int } W$ of mutually disjoint regular neighborhoods N_i of $\tilde{\varphi}(F_i)$ in $\text{Int } W$. N_i is obtained from $F_i^0 \times D^2$ and a 4-cell D^4 by pasting $(\partial F_i^0) \times D^2$ to a knotted solid torus T_i in ∂D^4 with knot type k_i , where F_i^0 is a surface obtained from $\tilde{\varphi}(F_i)$ by removing an open 2-cell. Since the self-intersection number of $\tilde{\varphi}(F_i)$ in W is 0, the framing of the knotted solid torus T_i is the null-homologous framing. Let $a_i^0 = [p_i \times \partial D] \in H_1(\partial N_i; Z)$, $i = 1, 2, \dots, s$,

where $p_i \in F_i^0$ and let $p_i \times \partial D$ have the orientation so that the linking number with $\tilde{\varphi}(F_i)$ in W is $+1$. Let $W' = W - \text{Int } N$. Clearly, $(W'; \partial W, \partial N)$ is a homology cobordism. Using this cobordism, the element $a_i \in H_1(\partial W; Z)$ corresponds to the element $a_i^0 \in H_1(\partial N_i; Z)$. Without loss of generality, we can assume that $\delta_1 = \cdots = \delta_r = 1$ and $\delta_{r+1} = \cdots = \delta_s = 0$. Then $\gamma^0 \in H^1(\partial N; Z)$ corresponding to γ sends a_1^0, \dots, a_r^0 to t and a_{r+1}^0, \dots, a_s^0 to 1 and is independent of the subgroup $G_{r+1}^0 + \cdots + G_s^0$, where G_i^0 is the image of the canonical monomorphism $i^*: H^1(N; Z) \rightarrow H^1(\partial N; Z)$ of the direct summand $H^1(N_i; Z)$ of

$$H^1(N; Z) = H^1(N_1; Z) \oplus \cdots \oplus H^1(N_s; Z).$$

It is easily shown that $\gamma' \in H^1(\partial N; Z)$ is independent of $G_{r+1}^0 + \cdots + G_s^0$ and $\gamma'(a_{r+1}^0) = \cdots = \gamma'(a_s^0) = 1$ if and only if $\gamma'|H_1(\partial N_{r+1}; Z) \oplus \cdots \oplus H_1(\partial N_s; Z) = 1$. [Note that $\gamma'(a_{r+1}^0) = \cdots = \gamma'(a_s^0) = 1$ implies $\gamma'|H_1(\partial N_{r+1}; Z) \oplus \cdots \oplus H_1(\partial N_s; Z) \in G_{r+1}^0 + \cdots + G_s^0$.] Hence by direct computations,⁵

$$T_1(\widetilde{\partial N})_{\gamma^0} \approx \oplus [Q\langle t \rangle / (t-1)]^{2g_1 + \cdots + 2g_r} \oplus T_1(\tilde{E}(k_1)) \oplus \cdots \oplus T_1(\tilde{E}(k_r))$$

and $\beta_1^{\gamma^0}(\partial N) = s - 1 + 2g_{r+1} + \cdots + 2g_s$. Further, the induced nonsingular form $(\langle \cdot, \cdot \rangle, t): \hat{T}^1(\widetilde{\partial N}) \times \hat{T}^1(\widetilde{\partial N}) \rightarrow Q$ is equal to the orthogonal sum $(\langle \cdot, \cdot \rangle_{k_1}, t) \perp \cdots \perp (\langle \cdot, \cdot \rangle_{k_r}, t)$ (cf. A. Kawauchi [8, p. 578]). The desired results now follow from Corollary 1.11. This completes the proof.

EXAMPLE 2.5. Consider the Matsumoto's 4-manifold W_1 (cf. Y. Matsumoto [15]) constructed as follows: Let $S^1 \times D^2$ be a standard solid torus in S^3 . Take the imbedding $h: S^1 \rightarrow S^1 \times D^2$ illustrated in Figure 1. Extend h to a framed imbedding $\bar{h}: S^1 \times D^2 \rightarrow \text{Int } S^1 \times D^2$ so that the framing is trivial in S^3 via inclusion $S^1 \times D^2 \subset S^3$. Let W_1 be the mapping torus of \bar{h} : $W_1 = S^1 \times D^2 \times [0, 1]/\bar{h}$. For $g \geq 2$, further remove from W_1 a 4-cell D^4 which is a regular neighborhood of $p \times D^2 \times (1/2)$, $p \in S^1$, in W_1 meeting the boundary ∂W_1 regularly and then replace D^4 by $F_{g-1}^0 \times D^2$ so that $(\partial F_{g-1}^0) \times D^2$ is identified with the unknotted solid torus $(W_1 - \text{Int } D^4) \cap D^4$ in ∂D^4 with trivial framing, where F_{g-1}^0 is an

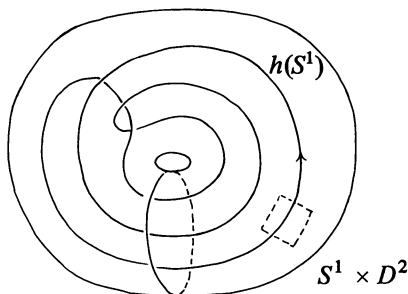


FIGURE 1

⁵Note the fact that

$$H_1(\widetilde{F^0 \times S^1}; Z) \approx \oplus [Z\langle t \rangle / (t-1)]^{2g}$$

for all indivisible elements $\gamma \in H^1(F^0 \times S^1; Z)$ sending $[p \times S^1] \in H_1(F^0 \times S^1; Z)$ to t , where F^0 is an oriented surface of genus g with an open 2-cell removed.

oriented surface of genus $g - 1$ with an open 2-cell removed. Clearly the resulting 4-manifold W_g ($g \geq 1$) is orientable and (simple) homotopy equivalent to a surface of genus g . We show that *there exists no piecewise linear imbedding from F_g to W_g inducing a homology isomorphism.*

PROOF. From construction the self-intersection number of a generator of $H_2(W_g; \mathbb{Z})$ is 0. So, the sequence

$$0 \rightarrow H_2(W_g, \partial W_g; \mathbb{Z}) \xrightarrow{\partial} H_1(\partial W_g; \mathbb{Z}) \xrightarrow{i_*} H_1(W_g; \mathbb{Z}) \rightarrow 0$$

is exact. Choose a basis $a, x_1, y_1, \dots, x_g, y_g$ of $H_1(\partial W_g; \mathbb{Z})$ so that a is the image of a generator of $H_2(W_g, \partial W_g; \mathbb{Z})$ by the boundary operator ∂ and x_1 is the homology class represented by a longitude curve in $S^1 \times D^2$ which is null-homologous in $S^3 - \text{Int } S^1 \times D^2$ and y_1 is a generator resulting from the mapping torus of \bar{h} and the other generators $x_2, y_2, \dots, x_g, y_g$ are obtained from a homology basis for $F_{g-1}^0 \times q$, $q \in \partial D^2$. Suppose there exists a piecewise linear imbedding $\tilde{\varphi}: F \rightarrow W$ inducing a homology isomorphism. We consider two kinds of elements γ' and γ'' in $H^1(\partial W_g; \mathbb{Z})$ such that γ' sends a, x_1, y_1 to t and $x_2, y_2, \dots, x_g, y_g$ to 1, and γ'' sends a to t and $x_1, y_1, x_2, y_2, \dots, x_g, y_g$ to 1. By direct calculations,⁶

$$H_1(\widetilde{\partial W_g}; Q)_{\gamma'} \approx \oplus [Q\langle t \rangle / (t - 1)]^{2g} \oplus Q\langle t \rangle / (2t^2 - 3t + 2)$$

and

$$H_1(\widetilde{\partial W_g}; Q)_{\gamma''} \approx \oplus [Q\langle t \rangle / (t - 1)]^{2g}.$$

Hence $A_1^{\gamma'}(t) \doteq (t - 1)^{2g}(2t^2 - 3t + 2)$ and $A_1^{\gamma''}(t) \doteq (t - 1)^{2g}$. Then by Theorem 2.2, $2t^2 - 3t + 2$ must be of type $f(t)f(t^{-1})(f(t) \in \mathbb{Z}\langle t \rangle, |f(1)| = 1)$, which is obviously impossible, since $2t^2 - 3t + 2$ is irreducible. This completes the proof.⁷

Here is another proof. By Lemma 1.4 we have $\sigma^{\gamma'}(\partial W_g) = \sigma_{3/4}^{\gamma'}(\partial W_g) = \pm 2$ (since $2t^2 - 3t + 2 \doteq t^2 - 2(3/4)t + 1$) and $\sigma^{\gamma''}(\partial W_g) = 0$. This contradicts Theorem 2.2.

Using various imbeddings $S^1 \rightarrow \text{Int } S^1 \times D^2$ obtained from the imbedding h by tying various knots in the dotted square in Figure 1, one can obtain infinitely many 4-manifolds with similar properties.

EXAMPLE 2.6. Let F_g^0 be a surface obtained from a closed connected oriented surface F_g of genus $g \geq 0$ by removing an open 2-cell. Taking for each positive integer m a knot k_m with Alexander polynomial $\Delta^{(m)}(t) = mt^2 - (2m - 1)t + m$ in the boundary ∂D^4 of a 4-cell D^4 , we construct a 4-manifold $W_g^{(m)}$ from $F_g^0 \times D^2$ and D^4 by attaching $(\partial F_g^0) \times D^2$ to D^4 along a knotted solid torus in ∂D^4 with knot type k_m and with null-homologous framing. Clearly, there is a homotopy equivalent piecewise linear imbedding $\tilde{\varphi}: F_g \rightarrow W_g^{(m)}$ whose locally knotted point has

⁶For example, find a finite presentation of $\pi_1(\partial W_1)$ by using the van Kampen theorem and then apply [9, Corollary 2.5] for this group. For $g > 1$ from construction we have

$$H_1(\partial \tilde{W}_g; Q) = \oplus [Q\langle t \rangle / (t - 1)]^{2g-2} \oplus H_1(\partial \tilde{W}_1; Q)$$

for all $\gamma \in H^1(\partial W_g; \mathbb{Z})$ sending a to t .

⁷The proof of the case that $n = 1$ and $g = 1$ gives an elementary proof of Y. Matsumoto's result [15].

the type k_m . However, *there exists no locally flat, piecewise linear imbedding from F_g to $W_g^{(m)}$ inducing a homology isomorphism.*

PROOF. Note that the self-intersection number of a generator of $H_2(W_g^{(m)}; Z)$ is 0. Let $a = [p \times \partial D^2] \in H_1(\partial W_g^{(m)}; Z)$, $p \in F_g^0$. By Theorem 2.2 we have $\sigma_\omega^\gamma(\partial W_g^{(m)}) = \sigma_\omega(k_m)$ at all $\omega \in [-1, 1]$ and for all $\gamma \in H^1(\partial W_g^{(m)}; Z)$ sending a to t . Hence, by Lemma 1.4, $\sigma_\omega^\gamma(\partial W_g^{(m)}) = \pm 2$ if $\omega = (2m - 1)/2m$ and $\sigma_\omega^\gamma(\partial W_g^{(m)}) = 0$ if $\omega \neq (2m - 1)/2m$ (cf. J. W. Milnor [19]). By Theorem 2.2 there exists no locally flat piecewise linear imbedding from F_g to $W_g^{(m)}$ inducing a homology isomorphism. This completes the proof.

Further, we obtain that *no two manifolds in the 4-manifolds $W_g^{(m)}$, $m = 1, 2, 3, \dots$ are relatively homology cobordant.*⁸ To see this, note that by Theorem 2.2 $\sigma_\omega^\gamma(\partial W_g^{(m)})$ at each $\omega \in [-1, 1]$ is invariant under all $\gamma \in H^1(\partial W_g^{(m)}; Z)$ sending a to $t^{\pm 1}$ and hence that $\sigma_\omega^\gamma(\partial W_g^{(m)})$ at each $\omega \in [-1, 1]$ is invariant under relative homology cobordisms. It follows from the above calculation of the local signature that no two manifolds are relatively homology cobordant.

Of course, any $W_g^{(m)}$ is not relatively homology cobordant to $F_g \times D^2$.

EXAMPLE 2.7. Consider a link L_0 of 2-components illustrated in Figure 2 in a 3-cell $D^3 \subset S^3$. For each $s \geq 2$, let L be a link of s components in S^3 obtained from L_0 by adding a trivial (i.e. unknotted and unlinked) link of $(s - 2)$ components in $S^3 - D^3$. (If $s = 2$, then $L = L_0$.) Clearly, this link L is a boundary link (that is, the components of L bound mutually disjoint orientable surfaces in S^3). Let e_1, e_2, \dots, e_s be a basis of $H_2(W_L; Z)$ coming from the components of L , where e_1, e_2 come from L_0 . We show that e_1, e_2, \dots, e_s are not represented by mutually disjoint piecewise linearly imbedded 2-spheres.

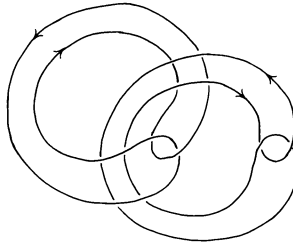


FIGURE 2

PROOF. Let a_1, \dots, a_s be a basis of $H_1(\partial W_L; Z)$ dually related to the basis e_1, \dots, e_s of $H_2(W_L; Z)$. Let γ, γ_1 and γ_2 be the indivisible elements of $H^1(\partial W_L; Z)$ such that $\gamma = \gamma_1 + \gamma_2$ and $\gamma_i(a_i) = t$ and $\gamma_i(a_j) = 1$, $i \neq j$, $i = 1, 2$. Suppose the basis e_1, \dots, e_s are represented by s mutually disjoint piecewise linearly imbedded 2-spheres in W_L with locally knotted points of types k_1, \dots, k_s . By direct computations,⁹ $A_1^\gamma(t) \doteq (t^2 - t + 1)(t^2 - 3t + 1)$, $A_1^{\gamma_i}(t) \doteq 1$, $i = 1, 2$.

⁸Two bounded oriented manifolds W_1 and W_2 are relatively homology cobordant, if there is a relative oriented cobordism \bar{W} between W_1 and W_2 such that $H_*(\bar{W}, W_i; Z) = H_*(\partial \bar{W}, \partial W_i; Z) = 0$, $i = 1, 2$.

⁹Compute a finite presentation of $\pi_1(\partial W_L)$ by using the van Kampen theorem and then apply [9, Corollary 2.5].

By Lemma 1.4 and Theorem 2.2, $\sigma_{1/2}(k_1) + \sigma_{1/2}(k_2) = \sigma_{1/2}^\gamma(\partial W_L) = \pm 2$ and $\sigma_{1/2}(k_i) = \sigma_{1/2}^\gamma(\partial W_L) = 0, i = 1, 2$, a contradiction.

In particular, *this boundary link L is not cobordant to a completely splittable link* (i.e. a link whose components C_1, \dots, C_s are contained in mutually disjoint 3-cells B_1, B_2, \dots, B_s in S^3 so that $C_i \subset B_i, i = 1, \dots, s$).

To derive a further property of a piecewise linear imbedding of 2-spheres into a 4-manifold, let F be a surface consisting of s 2-spheres S_1, \dots, S_s and let W be a compact connected oriented 4-manifold with $H_q(W; Q) \approx H_q(F; Q)$ for all $q > 0$ and such that any two elements of $H_2(W; Z)/(\text{torsions})$ have the intersection number 0. Then ∂W is connected and we have an isomorphism

$$\partial: H_2(W, \partial W; Q) \approx H_1(\partial W; Q),$$

so $H_1(\partial W; Z)$ has the rank s .

THEOREM 2.8. *If there exists a piecewise linear imbedding $\tilde{\varphi}: F \rightarrow W$ inducing a monomorphism $\tilde{\varphi}_*: H_q(F; Z) \rightarrow H_q(W; Z)$ for all $q > 0$, then we have $\beta_1^\gamma(\partial W) = s - 1$ and $A_1^\gamma(1) \neq 0$ for all nonzero elements $\gamma \in H^1(\partial W; Z)$ (cf. [9, Theorem C]).*

PROOF. By Lemma 1.1 (the reduction formula), it suffices to prove this theorem for all indivisible elements $\gamma \in H^1(\partial W; Z)$. Let $\gamma \in H^1(\partial W; Z)$ be an indivisible element. We can find a basis b_1, \dots, b_s of $H_1(\partial W; Z)/(\text{torsions})$ such that $\gamma(b_i) = t, i = 1, \dots, s$. Let N be a disk sum in $\text{Int } W$ of mutually disjoint regular neighborhoods N_i of S_i in $\text{Int } W$. Since the self-intersection number of S_i in W is 0, $H_1(\partial N_i; Z) = Z$, that is, ∂N_i is a homology orientable handle (cf. [11]). Note that ∂N is homeomorphic to the connected sum $(\partial N_1) \# (\partial N_2) \# \dots \# (\partial N_s)$. Let $W' = W - \text{Int } N$. Clearly, $(W'; \partial W, \partial N)$ is a rational homology cobordism. Note that $I: H_1(\partial W; Z)/(\text{torsions}) \rightarrow H_1(W'; Z)/(\text{torsions})$ induced by inclusion is a monomorphism of a finite index. There are bases $\{x_1, \dots, x_s\}, \{y_1, \dots, y_s\}$ for $H_1(\partial W; Z)/(\text{torsions}), H_1(W'; Z)/(\text{torsions})$, respectively, and nonzero integers n_1, \dots, n_s such that $I(x_i) = n_i y_i, i = 1, 2, \dots, s$. Let n be the least common multiple of n_1, \dots, n_s and let $n = n_i \tilde{n}_i$. Further let (m_{ij}) be an invertible matrix with $(b_1, \dots, b_s) = (x_1, \dots, x_s)(m_{ij})$. Let d_1, \dots, d_s be integers such that $(1, \dots, 1) = (d_1, \dots, d_s)(m_{ij})$. Define a map $\tilde{\gamma}: H_1(W'; Z)/(\text{torsions}) \rightarrow \langle t \rangle$ by the identities $\tilde{\gamma}(y_i) = t^{\tilde{n}_i d_i}, i = 1, \dots, s$. Let $\gamma_n = \tilde{\gamma}I: H_1(\partial W; Z)/(\text{torsions}) \rightarrow \langle t \rangle$. We have

$$\gamma_n(b_j) = \gamma_n\left(\sum_i m_{ij} x_i\right) = \tilde{\gamma}\left(\sum_i m_{ij} n_i y_i\right) = t^{(\sum_i m_{ij} d_i) n} = t^n,$$

$j = 1, \dots, s$. By Lemma 1.1 it suffices to prove the assertion for this map γ_n . By Corollary 1.14, $\beta_1^{\gamma_n}(\partial W) = \beta_1^{\gamma'}(\partial N)$, and $A_1^{\gamma_n}(1) \neq 0$ is equivalent to $A_1^{\gamma'}(1) \neq 0$, where γ' is the restriction of $\tilde{\gamma}$ to ∂N . Since ∂N is the connected sum of the homology orientable handles $\partial N_i, i = 1, \dots, s$, it is not difficult to see that $\beta_1^{\gamma'}(\partial N) = s - 1$ and $A_1^{\gamma'}(1) \neq 0$ for all indivisible elements $\gamma' \in H^1(\partial N; Z)$.¹⁰ It follows from Lemma 1.1 that $\beta_1^{\gamma_n}(\partial W) = \beta_1^{\gamma'}(\partial N) = s - 1$ and $A_1^{\gamma_n}(1) \neq 0$. This completes the proof.

¹⁰Note that there is a homomorphism from $\pi_1(\partial N)$ onto the free group of rank s .

Let L be a link of s (≥ 2) components. $\beta_1(L)$ is defined to be $\beta_1^{\gamma^0}(E(L))$, where $E(L)$ is a 3-manifold obtained from S^3 by removing an open tubular neighborhood of L and $\gamma^0 \in H^1(E(L); \mathbb{Z})$ is an indivisible element sending each oriented meridian curve of L to t (cf. [9]). We have that $\beta_1(L) \leq s - 1$. (See [10, Corollary 2.3].) Suppose L has the linking numbers 0.

COROLLARY 2.9. *If $\beta_1(L) \leq s - 2$, then any s elements of $H_2(W_L; \mathbb{Z})$ forming a basis of $H_2(W_L; \mathbb{Q})$ cannot be represented by mutually disjoint, piecewise linearly imbedded 2-spheres.*

PROOF. We may consider that ∂W_L is obtained from $E(L)$ and s solid tori T_1, \dots, T_s by attaching ∂T_i to one component of $\partial E(L)$. From the construction of W_L we have an isomorphism $i_*: H_1(E(L); \mathbb{Z}) \approx H_1(\partial W_L; \mathbb{Z})$. Hence the indivisible element $\gamma^0 \in H^1(E(L); \mathbb{Z})$ canonically determines an indivisible element $\gamma \in H^1(\partial W_L; \mathbb{Z})$. Take an infinite cyclic cover $(\tilde{\partial W}_L, \tilde{E}(L))$ associated with γ . Since

$$H_2(\tilde{\partial W}_L, \tilde{E}(L); \mathbb{Q}) \rightarrow H_1(\tilde{E}(L); \mathbb{Q}) \xrightarrow{i_*} H_1(\tilde{\partial W}_L; \mathbb{Q}) \xrightarrow{j_*} H_1(\tilde{\partial W}_L, \tilde{E}(L); \mathbb{Q})$$

is exact and, by excision, $H_*(\tilde{\partial W}_L, \tilde{E}(L); \mathbb{Q}) = \bigoplus_{i=1}^s H_*(\tilde{T}_i, \partial \tilde{T}_i; \mathbb{Q})$ is a $\mathbb{Q}\langle t \rangle$ -torsion module, it follows that $\beta_1(L) = \beta_1^{\gamma}(\partial W_L)$. By Theorem 2.8 we obtain a desired result.

EXAMPLE 2.10. Consider the Whitehead link L_0 (of 2-components) (see, for example, [9, Figure 1]) and the links Q_0 and E_0 illustrated in Figure 3. By direct computations,¹¹ $\beta_1(L_0) = \beta_1(Q_0) = \beta_1(E_0) = 0$. Let L^2 be L_0 , Q_0 or E_0 . For each $s \geq 3$, let L^s be a link of s components obtained from the link L^2 in a 3-cell $D^3 \subset S^3$ by adding a trivial link of $(s - 2)$ components in $S^3 - D^3$. We obtain $\beta_1(L^s) = s - 2$, since $\beta_1(L^2) = 0$. By Corollary 2.9, for each $s \geq 2$ any s elements of $H_2(W_{L^s}; \mathbb{Z})$ forming a basis of $H_2(W_{L^s}; \mathbb{Q})$ cannot be represented by mutually disjoint piecewise linearly imbedded 2-spheres. For $L^2 = L_0$ we obtain a stronger assertion than a result in [9, Application 3]. For $L^2 = Q_0$ or E_0 this answers enough a question of H. Lambert [13] who asked whether either of the links Q_0 , E_0 bounds two 2-cells mutually-disjointly and piecewise linearly imbedded in D^4 . By applying, more directly, Theorem 2.8 we have that the link L^s cannot bound s 2-cells mutually-disjoint and piecewise linearly imbedded in any rational homology 4-cell.

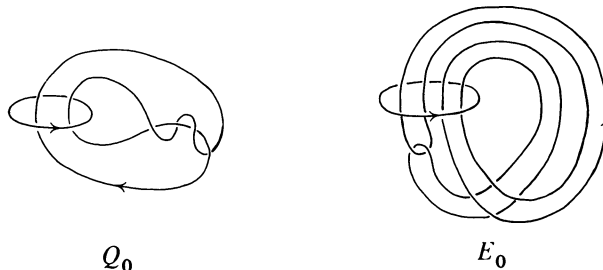


FIGURE 3

¹¹See [9, Corollary 2.4].

One may note that Corollary 2.9 cannot apply for boundary links, since $\beta_1(L) = s - 1$ for all boundary links L of s components (cf. N. Smythe [22] and [9, Corollary 2.4]).

Now suppose F and W satisfy the assumptions of Theorem 2.8. For an integer n and an element $\gamma \in H^1(\partial W; Z)$ we let $\gamma_n = n\gamma \in H^1(\partial W; Z)$. The proof of Theorem 2.8 combined with Corollary 1.14 also implies the following.

COROLLARY 2.11. *If there exists a locally flat, piecewise linear imbedding $\tilde{\varphi}: F \rightarrow W$ inducing a monomorphism $\tilde{\varphi}_*: H_q(F; Z) \rightarrow H_q(W; Z)$ for all $q > 0$, then for each indivisible element $\gamma \in H^1(\partial W; Z)$ there exists a positive integer n such that $\sigma_\omega^{\gamma_n}(\partial W) = 0$ at all $\omega \in (-1, 1]$ and $A_1^{\gamma_n}(t) \doteq f(t)f(t^{-1})$ for some $f(t) \in Q\langle t \rangle$ with $f(1) \neq 0$.*

EXAMPLE 2.12. Let $W_0^{(m)}$ be a 4-manifold in Example 2.6 with $g = 0$. $W_0^{(m)}$ is homotopy equivalent to S^2 and any element of $H_2(W_0^{(m)}; Z)$ can be represented by a piecewise linearly imbedded 2-sphere. However, any nonzero element of $H_2(W_0^{(m)}; Z)$ cannot be represented by a locally flat 2-sphere.

PROOF. $H_1(\partial \widetilde{W}_0^{(m)}; Z)_{\gamma_n} \approx Q\langle t \rangle / (\Delta^{(m)}(t^n))$, $\Delta^{(m)}(t^n) = mt^{2n} - (2m - 1)t^n + m$ by Lemma 1.1. $\Delta^{(m)}(t^n)$ has $2n$ distinct roots of complex numbers of norm 1. [Take the derivative $d\Delta^{(m)}(t^n)/dt$.] Let $t^2 - 2\omega t + 1$, $-1 < \omega < 1$, be any real irreducible factor of $\Delta^{(m)}(t^n)$. We have $\sigma_\omega^{\gamma_n}(\partial W_0^{(m)}) = \pm 2$. Therefore by Corollary 2.11, we obtain a desired result.

This fact was also obtained by M. Kato [5, Theorem D].¹²

Let ${}_s W^{(m)}$ be a disk sum of $W_0^{(m)}$ and $(s - 1)$ copies of $S^2 \times D^2$. By Corollary 2.11 the above proof also implies that any s elements of $H_2({}_s W^{(m)}; Z)$ forming a basis for $H_2({}_s W^{(m)}; Q)$ cannot be represented by mutually disjoint locally flat 2-spheres.

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¹²We can also obtain a result of Theorem D in the higher even dimensional case as an analogy of the above proof by using Corollary 1.14 and considering a $(2r - 1)$ -knot with the r th Alexander polynomial $mt^2 - (2m - 1)t + m$ or $mt^4 - (2m - 1)t^2 + m$ according as r is odd or even. Note that Kato's proof contains some erroneous point. [Let W be a compact connected orientable $(2r + 2)$ -manifold with $H_*(W; Q) \approx H_*(S^1; Q)$ and such that $H_*(\partial W; Z) \approx H_*(S^1 \times S^{2r}; Z)$. The r th Alexander polynomial of ∂W with a generator $\gamma \in H^1(\partial W; Z)$ is not necessarily of a slice type.]

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