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*-VALUATIONS AND ORDERED *-FIELDS

BY

SAMUEL S. HOLLAND, JR.

Dedicated to the memory of Reinhold Baer

ABSTRACT. We generalize elementary valuation theory to *-fields (division rings with involution), apply the generalized theory to the task of ordering *-fields, and give some applications to Hermitian forms.

1. Introduction. By a *-field I mean a (not necessarily commutative) field $\mathcal{K}$ together with a one-to-one mapping $\alpha \to \alpha^*$ (called the *involution*) that satisfies these conditions: $(\alpha + \beta)^* = \alpha^* + \beta^*$, $(\alpha\beta)^* = \beta^*\alpha^*$, and $\alpha^{**} = \alpha$. R. Baer [3, Chapter IV, Appendix I] calls a *-field *ordered* when it contains a subset Π (the *domain of positivity*)

(1) consisting solely of symmetric elements ($\alpha^* = \alpha$),
(2) containing 1 but not 0,
(3) closed under sum,
(4) closed under $\lambda \to \rho^*\lambda\rho$ for $\rho \neq 0$, and
(5) containing either λ or $-\lambda$ for each nonzero symmetric λ.

The subset Π defines the positive elements ($\lambda > 0$), and we totally order the set of symmetric elements by setting $\lambda > \mu$ when $\lambda - \mu \in \Pi$. The real numbers **R**, the complex numbers **C**, and the quaternions **H** all have orderings–take the domain of positivity as the positive reals in all three cases.

For a commutative field with the identity involution, Baer's axioms specialize to A. Prestel's axioms for a "q-ordered" field [13]. These axioms differ from the usual (Artin-Schreier) postulates in requiring only closure of the domain of positivity under $\lambda \to \rho^2\lambda$ for nonzero ρ, in place of requiring that positive elements have a positive product. Thus an ordered field in the usual sense satisfies Prestel's, and hence Baer's, axioms. (But not conversely, as Prestel has constructed q-orderings that do not qualify as orderings in the usual sense [13].)

This paper continues the study of ordered *-fields begun in [7]. Ordered *-fields seem to merit such further consideration on at least four counts. First, they have intrinsic interest deriving from their above-mentioned close relationship to the classical commutative ordered fields of Artin and Schreier, and the commutative q-ordered fields of A. Prestel. Second, they show real promise of use in the theory of noncommutative Hermitian forms; see [5, Chapter I, Appendix I] and §§5 and 6

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of this paper. Third, the theory of ordered *-fields may help advance our knowledge of *-rings, especially the Baer *-rings. And fourth, they may help delineate the structure of orthomodular lattices and the lattice-based axiom systems of quantum physics, because *-fields appear in the representation theory of both systems.

Within the theory of ordered *-fields, the classical number fields **R**, **C**, and **H** have a special significance, so a word or two seems in order here to clarify how they fit into the theory.

We call *-fields $(\mathcal{K}_1, *)$ and $(\mathcal{K}_2, \#)$ **-isomorphic* when there exists an algebraic isomorphism ϕ of $\mathcal{K}_1$ onto $\mathcal{K}_2$ such that (using functional notation)

$$\phi \circ * (\alpha) = \# \circ \phi(\alpha)$$

for all α in $\mathcal{K}_1$. If $\mathcal{K}_1$ has a domain of positivity Π, then one checks routinely that $\phi(\Pi)$ serves as domain of positivity ordering $\mathcal{K}_2$. If we deal with a fixed representative field $\mathcal{K}$ and an automorphism ϕ of $\mathcal{K}$, then we speak of the involutions $*$ and $\#$ as *conjugate*: $\# = \phi \circ * \circ \phi^{-1}$. We generally regard conjugate involutions as abstractly identical.

The real numbers **R** admit only the identity involution. The classical quaternion field, with **R** as center, has nontrivial automorphisms, but the ordinary quaternionic involution stands conjugate only to itself. When I write the symbol "**H**", I mean the quaternion field with this involution. The quaternion field admits other involutions, but none of them ordered.

The complex number field, as a pure field, admits an up-to-isomorphism characterization as the only algebraically closed commutative field of transcendence degree c and characteristic 0. Select a particular field W with these attributes. W admits many involutions. For each involution $\#$ of W, not the identity, the fixed field F of $\#$ has a unique ordering, and we have $W = F(i)$, $i^2 = -1$, $i^\# = -i$. Hence every involution of W is "complex conjugation" with respect to its (real closed) fixed field F, and W is an ordered *-field for any choice of involution. Two involutions stand conjugate in $\mathrm{aut}(W)$ exactly when their fixed fields are isomorphic (and therefore order isomorphic, since uniquely ordered). These fixed fields may have various order properties: Dedekind complete, archimedean but not Dedekind complete, or nonarchimedean. The involutions with Dedekind complete fixed field, i.e. with $F = \mathbf{R}$, form an infinite conjugacy class in $\mathrm{aut}(W)$. When I refer to "**C**", the complex numbers as an ordered *-field, I mean W together with one of the involutions in this conjugacy class. Now each of these involutions determines a metric $|\alpha|^2 = \alpha^*\alpha$, and any of the other involutions in this same conjugacy class is a discontinuous linear map on this metric space. But it is probably unwise and unnecessary in most contexts to select one involution this way, then refer to another conjugate one as "discontinuous". In most circumstances, one probably need not distinguish conjugate involutions. The key distinguishing feature of an involution $\#$ on W is the order character of its fixed field, which determines $(W, \#)$ as an ordered *-field.

Corollary 3 of [7] characterizes **R**, **C**, and **H** as the only Dedekind complete ordered *-fields.

This paper is organized under the following headings:
§2. *-valuations;
§3. Lifting an ordering: examples of ordered *-fields;
§4. The order topology and order valuation of an ordered *-field;
§5. Hermitian forms;
§6. Generalization of Wilbur's theorem.

2. *-valuations. We deal in this section with a general *-field $\mathcal{K}$ (no ordering). A map w of $\mathcal{K}^{\cdot}$ (the multiplicative group of nonzero elements of $\mathcal{K}$) to an ordered group G that satisfies

(1) $w(\alpha\beta) = w(\alpha) + w(\beta)$,
(2) $w(\alpha + \beta) \geqslant \min(w(\alpha), w(\beta))$, $\alpha + \beta \neq 0$,
(3) w maps onto G,
(4) $w(\alpha^*) = w(\alpha)$,

we call a **-valuation* of $\mathcal{K}$. One need not assume G abelian, as this follows from axioms 1, 3, and 4.

Call the set

$$\Phi = \{\alpha \in \mathcal{K}^{\cdot}: w(\alpha) \geqslant 0\} \cup \{0\}$$

the **-valuation ring* of w; it determines w uniquely up to equivalence in the usual way. Φ has these two properties: (1) it contains every $\alpha^*\alpha^{-1}$, $\alpha \in \mathcal{K}^{\cdot}$, and (2) it contains at least one of α, α^{-1} for each $\alpha \in \mathcal{K}^{\cdot}$. A subring of $\mathcal{K}$ that satisfies these two conditions I call a **-valuation ring*, a terminology justified by the following result.

2.1. *Given a subring Φ of a *-field $\mathcal{K}$ that contains every $\alpha^*\alpha^{-1}$, $\alpha \in \mathcal{K}^{\cdot}$, and contains at least one of α, α^{-1} for each α in $\mathcal{K}^{\cdot}$, then there exist an ordered abelian group G and a *-valuation w: $\mathcal{K}^{\cdot} \to G$ such that Φ equals the *-valuation ring of w.*

PROOF. Let $\Phi^{\cdot}$ denote the multiplicative group of invertible elements in Φ. We have $\alpha^*\alpha^{-1} \in \Phi^{\cdot}$ for every α in $\mathcal{K}^{\cdot}$, because $(\alpha^*\alpha^{-1})^{-1} = \beta^*\beta^{-1}$ with $\beta = \alpha^*$. Hence $\alpha\beta\alpha^{-1}\beta^{-1} \in \Phi^{\cdot}$ for every α, β in $\mathcal{K}^{\cdot}$ by virtue of the identity

$$\alpha\beta\alpha^{-1}\beta^{-1} = \big((\alpha^*)^*(\alpha^*)^{-1}\big)\big((\beta^*\alpha)^*(\beta^*\alpha)^{-1}\big)\big(\beta^*\beta^{-1}\big).$$

Hence $\Phi^{\cdot}$ contains the commutator subgroup $[\mathcal{K}^{\cdot}, \mathcal{K}^{\cdot}]$ of $\mathcal{K}^{\cdot}$, and so lies normal in $\mathcal{K}^{\cdot}$ with abelian quotient $G = \mathcal{K}^{\cdot}/\Phi^{\cdot}$. Define w: $\mathcal{K}^{\cdot} \to G$ as the natural map, and order G by setting $a \geqslant 0 \Leftrightarrow a = w(\alpha)$, $\alpha \in \Phi$. That completes the proof.

The trivial *-valuation has $G = \{0\}$. A *-valuation is trivial exactly when its *-valuation ring equals all of $\mathcal{K}$. The nontrivial case we also refer to as *proper*.

The first condition defining a *-valuation ring, namely that it contain every $\alpha^*\alpha^{-1}$, $\alpha \in \mathcal{K}^{\cdot}$, already by itself has strong implications. I shall call a subring Φ of $\mathcal{K}$ that satisfies this condition *symmetric*.

2.2. *Any symmetric subring Φ of a *-field $\mathcal{K}$ has the following properties*:

(1) *it is *-closed* ($\alpha \in \Phi \Rightarrow \alpha^* \in \Phi$);

(2) $[\mathcal{K}^{\cdot}, \mathcal{K}^{\cdot}] \subseteq \Phi^{\cdot}$; *in particular $\beta^{-1}\Phi\beta = \Phi$ for all β in $\mathcal{K}^{\cdot}$*;

(3) *each ideal Δ in Φ is *-closed, two-sided, and has the property that $\alpha\beta \in \Delta \Rightarrow \beta\alpha \in \Delta$*;

(4) *if* Δ *is a prime ideal in* Φ, $\Delta \neq \Phi$, *then the sets*

$$\{\alpha\beta^{-1}: \alpha, \beta \in \Phi, \beta \notin \Delta\}, \qquad \{\beta^{-1}\alpha: \alpha, \beta \in \Phi, \beta \notin \Delta\}$$

are equal. This set, call it Φ_Δ, *is itself a *-subring of* $\mathcal{K}$ *containing* Φ, *and hence also symmetric.*

PROOF. The *-closure of Φ follows from the equation $\alpha^* = (\alpha^*\alpha^{-1})\alpha$. During the proof of 2.1 we proved every element of the commutator subgroup is invertible in Φ, which is item (2). As for (3), the *-closure of a left ideal follows from $\alpha^* = \alpha(\alpha^*\alpha^{-1})\alpha$, that of a right ideal from $\alpha^* = \alpha(\alpha^{-1}\alpha^*)$, and two-sidedness from *-closure. The equations $\alpha^*\beta^* = (\alpha^*\alpha^{-1})(\alpha\beta)(\beta^{-1}\beta^*)$ and $\beta\alpha = (\alpha^*\beta^*)^*$ prove the last assertion in (3). As for (4), we use the identity $\alpha\beta^{-1} = (\beta(\alpha^*\alpha^{-1}))^{-1}(\beta\alpha^*\beta^{-1})$ to prove the equality of the two sets. If α and β belong to Φ but β does not belong to the ideal Δ, then $\beta(\alpha^*\alpha^{-1})$ cannot belong to Δ owing to the invertibility of $\alpha^*\alpha^{-1}$ in Φ. Hence $\alpha\beta^{-1}$ has the form $\mu^{-1}\sigma$, $\mu, \sigma \in \Phi$, $\mu \notin \Delta$. That proves the equality of the two displayed sets in (4). To prove Φ_Δ closed under product, argue as follows:

$$\begin{aligned}(\alpha_1\beta_1^{-1})(\alpha_2\beta_2^{-1}) &= \alpha_1(\beta_1^{-1}\alpha_2)\beta_2^{-1} = \alpha_1(\alpha_3\beta_3^{-1})\beta_2^{-1}\\ &= (\alpha_1\alpha_3)(\beta_2\beta_3)^{-1},\end{aligned}$$

noting that $\beta_2\beta_3 \notin \Delta$ if both $\beta_1 \notin \Delta$ and $\beta_2 \notin \Delta$ (here we use the primeness). Finally, establish the closure of Φ_Δ under sum from the identity

$$\alpha_1\beta_1^{-1} + \alpha_2\beta_2^{-1} = (\alpha_1\beta_2 + \alpha_2(\beta_2^{-1}\beta_1\beta_2))(\beta_1\beta_2)^{-1}.$$

That completes the proof of 2.2.

If, in (4) of 2.2, we use 0 for the prime ideal Δ, then Φ_0 equals the quotient field of Φ. If Φ is a *-valuation ring, then $\Phi_0 = \mathcal{K}$.

A *-valuation ring satisfies the symmetry condition (1), so has all the properties in 2.2. It satisfies also the traditional condition (2), that it contain at least one of α, α^{-1} for each α in $\mathcal{K}$. From this latter condition we can deduce the usual properties of *-valuation rings, listed in 2.3 which follows (using Krull's terminology **[9]**). The omitted proof runs almost the same as in the commutative case. (A *right segment* in G means a subset consisting exclusively of nonnegative elements and containing with each a every b satisfying $b > a$.)

2.3. *Let* Φ *denote a *-valuation ring in the *-field* $\mathcal{K}$, *and let* w, G *denote the *-valuation and ordered group respectively belonging to* Φ.

(a) *An ideal in* Φ *with finite basis is already principal. If an ideal* Δ *is prime, then* $\alpha^n \in \Delta \Rightarrow \alpha \in \Delta$ *for any* $n = 2, 3, \ldots$. *Conversely, if for some fixed* $n = 2, 3, \ldots$ *the ideal* Δ *satisfies* $\alpha^n \in \Delta \Rightarrow \alpha \in \Delta$, *then* Δ *is prime.*

(b) *The set of noninvertible elements in* Φ *forms a proper ideal* $\mathcal{P}$ *that contains every other ideal. We have* $\mathcal{P} = \{\alpha \in \Phi: w(\alpha) > 0\} \cup \{0\}$.

(c) *The map* $\Delta \to w(\Delta\setminus 0)$ *is an order isomorphism of the multiplicative system of all proper ideals of* Φ *onto the additive system of all right segments of* G. *The ideals of* Φ *are totally ordered by set inclusion. If* $\Delta_1 \supset \Delta_2$, *then* Δ_1 *is a factor of* Δ_2 $(\Delta_2 = \Delta\Delta_1) \Leftrightarrow$ *either* Δ *is principal, or both* Δ_1 *and* Δ_2 *have no finite basis.*

(d) *The ideal* Δ *is principal* $\Leftrightarrow$ *the right segment* $w(\Delta\setminus 0)$ *has a least element.*

(e) *Let* Δ *be a proper ideal in* Φ, $C = w(\Delta \setminus 0)$ *its corresponding right segment. Then* Δ *is prime* $\Leftrightarrow \{g \in G\colon |g| < c$ *for all* $c \in C\}$ *forms an isolated subgroup of* G. (An *isolated* subgroup contains along with each positive g each d satisfying $0 < d < g$.)

With each *-valuation ring Φ with maximal ideal $\mathcal{P}$ we associate the *residue class *-field* $\mathcal{K}_0 = \Phi/\mathcal{P}$ with involution defined by $(\alpha + \mathcal{P})^* = \alpha^* + \mathcal{P}$. We shall use the symbol θ to stand for the natural *-homomorphism of Φ onto $\mathcal{K}_0$: $\theta(\alpha) = \alpha + \mathcal{P}$. Then, by definition of the involution on $\mathcal{K}_0$, $\theta(\alpha^*) = \theta(\alpha)^*$ for all α in Φ.

We define a **-place* as a mapping ϕ of a *-field $\mathcal{K}$ into a system $\Lambda \cup \{\infty\}$ consisting of a *-field Λ with involution #, and a separate symbol ∞ such that ϕ satisfies

(1) if $\alpha \neq 0$, then $\phi(\alpha) = \infty \Rightarrow \phi(\alpha^{-1}) = 0$,

(2) $\phi(\alpha^*\alpha^{-1}) \neq \infty$ for all nonzero α,

(3) if $\phi(\alpha) \neq \infty$ and $\phi(\beta) \neq \infty$, then $\phi(\alpha \pm \beta) = \phi(\alpha) \pm \phi(\beta)$, $\phi(\alpha\beta) = \phi(\alpha)\phi(\beta)$, and $\phi(\alpha^*) = \phi(\alpha)^{\#}$.

One then verifies routinely

2.4. *The set* $\Phi = \{\alpha \in \mathcal{K}\colon \phi(\alpha) \neq \infty\}$ *constitutes a *-valuation ring in* $\mathcal{K}$ *with maximal ideal* $\mathcal{P} = \{\alpha \in \mathcal{K}\colon \phi(\alpha) = 0\}$. *We have* Λ **-isomorphic to the residue class *-field* $\Phi/\mathcal{P}$.

We thus have the usual tie-up between *-valuations, *-valuation rings, and *-places.

We end this section with a generalization to *-valuations of an important theorem from commutative valuation theory [**9**, Satz 6].

2.5 THEOREM. *Any symmetric subring of a *-field* $\mathcal{K}$ *not itself a field extends to a proper *-valuation ring.*

With the help of A. R. Richardson's result on simultaneous linear equations over a division algebra [**16**, Theorem 15], and (4) of 2.2, one can adapt Krull's proof in [**9**] to prove Theorem 2.5. I forego the details.

3. Lifting an ordering: examples of ordered *-fields. Continue the terminology of the preceding section: $\mathcal{K}$ a *-field with *-valuation ring Φ and residue class *-field $\mathcal{K}_0 = \Phi/\mathcal{P}$, $\mathcal{P}$ the maximal ideal of Φ.

We now address this question: Given an ordering of the residue class *-field $\mathcal{K}_0$, can we lift this ordering to $\mathcal{K}$? The answer is "yes" (under a mild additional condition), by a direct generalization of Prestel's method [**14**].

Inasmuch as both the *-valuation ring Φ and its maximal ideal $\mathcal{P}$ are invariant under inner automorphisms of $\mathcal{K}$, any inner automorphism of $\mathcal{K}$, say $A_\rho(x) = \rho x \rho^{-1}$, induces an automorphism of $\mathcal{K}_0$ (not generally inner) which we denote by the same symbol:

$$A_\rho(\theta(x)) = \theta(\rho x \rho^{-1}), \qquad x \in \Phi.$$

Composing an inner automorphism A_ρ with our involution *, we get an antiautomorphism # of $\mathcal{K}$, $\# = A_\rho \circ *$ or $x^{\#} = \rho x^* \rho^{-1}$. This antiautomorphism qualifies as an involution exactly when $\rho^*\rho^{-1}$ belongs to the center of $\mathcal{K}$. Since $\rho^* + \rho = (\rho^*\rho^{-1} + 1)\rho$, we may replace ρ by the symmetric element $\rho^* + \rho$ when

$\rho^*\rho^{-1} \in \text{center}(\mathcal{K})$, unless $\rho^*\rho^{-1} + 1 = 0$. But this corresponds to skew ρ. *Hence any involution* $\#$ *on* $\mathcal{K}$ *of the form* $x^\# = \rho x^* \rho^{-1}$ *may be obtained already with* ρ *symmetric or skew.*

Given ρ symmetric or skew, the involution $\# = A_\rho \circ *$ induces an involution on $\mathcal{K}_0$. We call the element ρ *smooth* if this induced involution on $\mathcal{K}_0$ is conjugate to $*$. That is, ρ is smooth when there exists an automorphism Γ_ρ of $\mathcal{K}_0$ so that

$$\# = A_\rho \circ * = \Gamma_\rho \circ * \circ \Gamma_\rho^{-1} \quad \text{on } \mathcal{K}_0.$$

I call a $*$-valuation $w\colon \mathcal{K} \to G$ *smooth* when $\text{char}(\mathcal{K}) \neq 2$ and w fulfills the following conditions:

(1) $w(2) = 0$;

(2) each equivalence class $w^{-1}(g)$ contains a smooth symmetric element if it contains symmetric elements at all, otherwise it contains a smooth skew element.

With regard to condition (2) note this: *If condition* (1) *holds, then* $w^{-1}(g)$ *always contains either a symmetric element or a skew element.* Because, given α with $w(\alpha) = g$, then $\sigma = \alpha + \alpha^*$ is symmetric, $\delta = \alpha - \alpha^*$ skew, and both $w(\sigma) \geqslant g$, $w(\delta) \geqslant g$. But we have also $g = w(\alpha) = w(\frac{1}{2}(\sigma + \delta)) \geqslant \min(w(\sigma), w(\delta))$. Hence we must have either $w(\sigma) = g$ or $w(\delta) = g$, so either σ or δ belongs to $w^{-1}(g)$. Hence, in (2), smoothness is the issue; the symmetric and skew elements come free. And, as we shall see, smoothness represents a mild condition on w.

Given a $*$-valuation $w\colon \mathcal{K} \to G$, we define, with Prestel, a *presection* as a function $s\colon G \to \mathcal{K}$ that selects for each g in G an element $s(g)$ in $w^{-1}(g)$ such that

(1) $s(0) = 1$,

(2) $s(g)$ is symmetric if $w^{-1}(g)$ contains symmetric elements, otherwise it is skew,

(3) $s(2g) = \beta\beta^*$ for some β,

(4) $s(g + 2k) = \gamma s(g)\gamma^*$ for some γ.

We call the presection *smooth* if each $s(g)$ is smooth *and* the selected family of automorphisms of $\mathcal{K}_0$, $\{\Gamma_{s(g)}\colon g \in G\}$, has these properties.

(1′) $\Gamma_1 = I$,

(3′) $\Gamma_{s(2g)} = A_\beta$, given $s(2g) = \beta\beta^*$,

(4′) $\Gamma_{s(g+2k)} = A_\gamma \circ \Gamma_{s(g)}$, given $s(g + 2k) = \gamma s(g)\gamma^*$.

3.1. Lemma. *Any smooth valuation has a smooth presection.*

Proof. Start by selecting $s(0) = 1$, and $\Gamma_1 =$ identity of $\mathcal{K}_0$. For each $2g \neq 0$, the class $w^{-1}(2g)$ always contains a $\beta\beta^*$; select such an element as the value $s(2g)$, and select as an automorphism of $\mathcal{K}_0$ $\Gamma_{\beta\beta^*} = A_\beta$. Next select a representative a in each coset of $2G$, selecting 0 for $2G$ itself. Call A the set of representatives so selected. For each nonzero a in A, select $s(a)$ smooth symmetric if $w^{-1}(a)$ contains symmetric elements; otherwise select $s(a)$ smooth skew (possible because of the assumption of a smooth valuation). With each smooth $s(a)$ comes its automorphism $\Gamma_{s(a)}$ of $\mathcal{K}_0$. Then for each g in G, we have $g = a + 2h$ for a unique a in A and h in G. Define $s(g) = \beta s(a)\beta^*$ where $s(2h) = \beta\beta^*$. And for automorphism we may take $\Gamma_{s(g)} = A_\beta \circ \Gamma_{s(a)}$ by the following argument. First,

$$(A_\beta \circ \Gamma_{s(a)}) \circ * \circ (A_\beta \circ \Gamma_{s(a)})^{-1} = A_\beta \circ \Gamma_{s(a)} \circ * \circ \Gamma_{s(a)}^{-1} \circ A_\beta^{-1}.$$

Now $\Gamma_{s(a)} \circ * \circ \Gamma_{s(a)}^{-1} = A_{s(a)} \circ *$, and $* \circ A_{\beta^{-1}} = A_{\beta^*} \circ *$, so

$$(A_\beta \circ \Gamma_{s(a)}) \circ * \circ (A_\beta \circ \Gamma_{s(a)})^{-1} = A_\beta \circ A_{s(a)} \circ A_{\beta^*} \circ * = A_{\beta s(a)\beta^*} \circ *,$$

which shows $\Gamma_{\beta s(a)\beta^*} = A_\beta \circ \Gamma_{s(a)}$ as desired. That completes the definition of s and Γ on all of G. It remains to establish properties (4) and (4′). Given $m \in G$ of the form $m = g + 2k$, write $g = a + 2h$, unique a in A and h in G, so

$$m = a + 2(h + k).$$

Then, by definition, $s(g) = \beta s(a)\beta^*$ where $s(2h) = \beta\beta^*$, and $s(m) = \mu s(a)\mu^* = (\mu\beta^{-1})(\beta s(a)\beta^*)(\mu\beta^{-1})^* = \gamma s(g)\gamma^*$. Likewise, by definition, $\Gamma_{s(g)} = A_\beta \circ \Gamma_{s(a)}$ and $\Gamma_{s(m)} = A_\mu \circ \Gamma_{s(a)}$. Then $\Gamma_{s(m)} = A_\mu \circ A_\beta^{-1} \circ \Gamma_{s(g)} = A_\gamma \circ \Gamma_{s(g)}$. That proves Lemma 3.1.

3.2. THEOREM. *Given a *-field $\mathcal{K}$ with smooth valuation w, there exists a map N of $\mathcal{K}$ onto its residue class *-field $\mathcal{K}_0$ such that*

(1) $N(\alpha) = 0 \Leftrightarrow \alpha = 0$,

(2) *for each α and each symmetric λ in $\mathcal{K}$, there exists β in $\mathcal{K}_0$ so that* $N(\alpha\lambda\alpha^*) = \beta N(\lambda)\beta^*$,

(3)

$$N(\alpha^*) = \begin{cases} N(\alpha)^* & \text{when } w^{-1}(w(\alpha)) \text{ contains a symmetric element,} \\ -N(\alpha)^* & \text{otherwise,} \end{cases}$$

(4) *if $N(\alpha) + N(\beta) \neq 0$, then $N(\alpha + \beta) = N(\alpha), N(\beta)$, or $N(\alpha) + N(\beta)$ according as $w(\alpha) < w(\beta)$, $w(\alpha) > w(\beta)$, or $w(\alpha) = w(\beta)$.*

PROOF. According to Lemma 3.1, our smooth valuation w has a smooth presection $s: G \to \mathcal{K}$ with family $\{\Gamma_{s(g)}: g \in G\}$ of automorphisms of $\mathcal{K}_0$ having the properties listed above. Given $\alpha \in \mathcal{K}^\cdot$ with $w(\alpha) = g$, define $N(\alpha) = \Gamma_{s(g)}^{-1} \circ \theta(\alpha s(g)^{-1})$, θ the natural *-homomorphism of w's *-valuation ring Φ onto $\mathcal{K}_0$. Set also $N(0) = 0$.

(1) is clear. As for (2), set $g = w(\alpha\lambda\alpha^*) = l + 2c$, $l = w(\lambda)$, $c = w(\alpha)$. Then $s(g) = s(l + 2c) = \gamma s(l)\gamma^*$ where $w(\gamma) = c = w(\alpha)$. Also $\Gamma_{s(g)} = A_\gamma \circ \Gamma_{s(l)}$. Then

$$\begin{aligned} N(\alpha\lambda\alpha^*) &= \Gamma_{s(g)}^{-1} \circ \theta(\alpha\lambda\alpha^* s(g)^{-1}) \\ &= \Gamma_{s(g)}^{-1} \circ \theta(\alpha\lambda\alpha^*\gamma^{*-1}s(l)^{-1}\gamma^{-1}) \\ &= \Gamma_{s(g)}^{-1} \circ A_\gamma \circ \theta(\gamma^{-1}\alpha\lambda\alpha^*\gamma^{*-1}s(l)^{-1}) \\ &= \Gamma_{s(l)}^{-1} \circ \theta(\gamma^{-1}\alpha\lambda\alpha^*\gamma^{*-1}s(l)^{-1}). \end{aligned}$$

Since $\gamma^{-1}\alpha \in \Phi^\cdot$, we have

$$\begin{aligned} &= (\Gamma_{s(l)}^{-1} \circ \theta(\gamma^{-1}\alpha))(\Gamma_{s(l)}^{-1} \circ \theta(\lambda\alpha^*\gamma^{*-1}s(l)^{-1})) \\ &= (\Gamma_{s(l)}^{-1} \circ \theta(\gamma^{-1}\alpha))(\Gamma_{s(l)}^{-1} \circ A_{s(l)} \circ \theta(s(l)^{-1}\lambda\alpha^*\gamma^{*-1})) \\ &= (\Gamma_{s(l)}^{-1} \circ \theta(\gamma^{-1}\alpha))(\Gamma_{s(l)}^{-1} \circ A_{s(l)} \circ \theta(s(l)^{-1}\lambda))(\Gamma_{s(l)}^{-1} \circ A_{s(l)} \circ \theta(\alpha^*\gamma^{*-1})) \\ &= (\Gamma_{s(l)}^{-1} \circ \theta(\gamma^{-1}\alpha))(\Gamma_{s(l)}^{-1} \circ \theta(\lambda s(l)^{-1}))(\Gamma_{s(l)}^{-1} \circ A_{s(l)} \circ * \circ \theta(\gamma^{-1}\alpha)). \end{aligned}$$

The middle term equals $N(\lambda)$. For the last term, apply the formula

$$A_{s(l)} \circ * = \Gamma_{s(l)} \circ * \circ \Gamma_{s(l)}^{-1},$$

to get $* \circ \Gamma_{s(l)}^{-1} \circ \theta(\gamma^{-1}\alpha)$. Thus $N(\alpha\lambda\alpha^*) = \beta N(\lambda)\beta^*$ where $\beta = \Gamma_{s(l)}^{-1} \circ \theta(\gamma^{-1}\alpha)$.

For (3) we argue as follows.

$$\begin{aligned} N(\alpha)^* &= * \circ \Gamma_{s(g)}^{-1} \circ \theta\big(\alpha s(g)^{-1}\big) \\ &= \Gamma_{s(g)}^{-1} \circ A_{s(g)} \circ * \circ \theta\big(\alpha s(g)^{-1}\big) \\ &= \pm\Gamma_{s(g)}^{-1} \circ A_{s(g)} \circ \theta\big(s(g)^{-1}\alpha^*\big) \\ &\qquad\qquad (+ \text{ for symmetric } s(g), - \text{ for skew}) \\ &= \pm\Gamma_{s(g)}^{-1} \circ \theta\big(\alpha^* s(g)^{-1}\big) = \pm N(\alpha^*). \end{aligned}$$

The proof of (4) follows very much the argument in [**13**, Lemma 3.8]; we leave the details to the reader.

3.3. COROLLARY. *Given a *-field $\mathcal{K}$ with smooth *-valuation w and residue class *-field $\mathcal{K}_0$, then every ordering of $\mathcal{K}_0$ lifts to $\mathcal{K}$.*

More precisely: Refer to the function N of Theorem 3.2. For nonzero symmetric α in $\mathcal{K}$, set $\alpha > 0 \Leftrightarrow N(\alpha) > 0$ in $\mathcal{K}_0$. This orders $\mathcal{K}$. The ordering is moreover compatible with w in the sense

$$0 < \alpha \leqslant \beta \Rightarrow w(\alpha) \geqslant w(\beta).$$

The proof is a direct application of Theorem 3.2.

We now apply Theorem 3.2 and its corollary to construct examples (and nonexamples) of ordered *-fields. The constructions use also this principle: *An ordered *-field is formally real* (refer to §5). Put another way, in an ordered *-field an expression $\alpha_1\rho\alpha_1^* + \cdots + \alpha_n\rho\alpha_n^*$ with ρ symmetric can vanish only trivially.

(a) *Generalized complex and quaternionic *-fields.* When $\alpha^* = \alpha$ for all α in the *-field $\mathcal{K}$, then $\mathcal{K}$ is necessarily commutative, and the definition of a domain of positivity Π reduces to this: Π contains 1 but not 0, is closed under sum, contains $\rho^2\lambda$ along with λ for any $\rho \neq 0$, and contains either λ or $-\lambda$ for each nonzero λ. This kind of ordering Prestel calls a "q-ordering", and in his papers [**13**], [**14**], he gives many examples and a detailed analysis.

Each example of a q-ordered commutative field $\mathfrak{X}_0$ gives rise to two different examples of ordered *-fields with nontrivial involution, namely the complex numbers $\mathfrak{X}_0(\sqrt{-\alpha})$ and the quaternions $\mathfrak{X}_0(-\alpha, -\beta)$ built on $\mathfrak{X}_0$. In fact the domain of positivity Π for $\mathfrak{X}_0$ also serves as domain of positivity for both $\mathfrak{X}_0(\sqrt{-\alpha})$ and $\mathfrak{X}_0(-\alpha, -\beta)$. (Here α and β represent positive elements of $\mathfrak{X}_0$, $\mathfrak{X}_0(\sqrt{-\alpha})$ carries the standard "complex conjugation" involution, and $\mathfrak{X}_0(-\alpha, -\beta)$ stands for the usual quaternion *-algebra over $\mathfrak{X}_0$ with basis $1, i, j, k$ where $i^2 = -\alpha, j^2 = -\beta$, $ij = k = -ji$, involution determined by $i^* = -i, j^* = -j$.)

Conversely, given a commutative field $\mathfrak{X}_0$, if either $\mathfrak{X}_0(\sqrt{-\alpha})$ or $\mathfrak{X}_0(-\alpha, -\beta)$ for α, β in $\mathfrak{X}_0$ admits an ordering, then this ordering induces a q-ordering on $\mathfrak{X}_0$, and in that induced ordering we have both $\alpha > 0$ and $\beta > 0$ as follows from the general condition $\zeta^*\zeta > 0$. Thus each example of an ordered complex or quaternionic *-field arises from a q-ordered commutative field in the manner described previously. Indeed, for such generalized complex and quaternionic *-fields, our theory of ordered *-fields reduces essentially to Prestel's theory of q-ordered fields.

(b) *Tensor product of quaternionic *-fields*. Surprisingly, the tensor product of quaternion *-fields with its usual involution never admits an ordering. If $\mathcal{K}_1$ and $\mathcal{K}_2$ are quaternion *-fields with respective bases $\{1, i, j, k\}$ and $\{1, u, v, w\}$ over a common center Λ, then the element $\rho = j \otimes u$ of $\mathcal{K} = \mathcal{K}_1 \otimes \mathcal{K}_2$ is symmetric with respect to the usual involution * on $\mathcal{K}$ induced by $\mathcal{K}_1$ and $\mathcal{K}_2$. But $\alpha^*\rho\alpha + \rho = 0$ for $\alpha = i \otimes 1$ as a simple calculation shows, so $\mathcal{K}$ cannot admit an ordering.

(c) *Nonabelian and abelian bicyclic *-fields*. Next we discuss a class of odd-dimensional examples (and nonexamples) that come under the "bicyclic" *-fields of Albert [1] whose method of construction goes like this. Start with a tower of commutative fields $\mathfrak{Z}_0 \subset \mathfrak{Z} \subset \mathfrak{M}$ of characteristic zero, with $\mathfrak{Z}/\mathfrak{Z}_0$ of degree 2, $\mathfrak{M}/\mathfrak{Z}$ cyclic of odd degree $n \geqslant 3$ with, say, S as generator of $\mathrm{Gal}(\mathfrak{M}/\mathfrak{Z})$, and $\mathfrak{M}/\mathfrak{Z}_0$ normal (Galois) with Galois group generated by S and an automorphism $*$ of period 2. Albert refers to this situation as *bicyclic*. Distinguish two cases according to the relation connecting $*$ and S: If $*S* = S^{-1}$ call this case *nonabelian bicyclic*; if $*S = S*$, *abelian bicyclic*. Now, select adroitly an element $\gamma \in \mathfrak{Z}$ that makes the cyclic algebra $\mathcal{K} = (\mathfrak{M}/\mathfrak{Z}, S, \gamma)$ a field, and that satisfies either $\gamma = \gamma^*$ in the nonabelian case or $\gamma\gamma^* = 1$ in the abelian case. The cyclic field $\mathcal{K}$ has center $\mathfrak{Z}$ and has $\mathfrak{M}$ as a maximal abelian subfield; we describe it in the usual way

$$\mathcal{K} = \left\{ \mu_0 + \mu_1 y + \cdots + \mu_{n-1} y^{n-1} : \mu_i \in \mathfrak{M}, y^n = \gamma, y\mu = S(\mu)y \right\}.$$

The formula $y^* = y$, respectively $y^* = y^{-1}$, then determines uniquely an involution on $\mathcal{K}$ in the nonabelian, respectively the abelian, case. These are the nonabelian and abelian bicyclic *-fields of Albert–all finite dimensional by definition.

We construct first an infinite family of nonabelian bicyclic ordered *-fields.

3.4. *Let p be a prime* $\equiv 3 \pmod 4$, *let* $\omega = \exp(2\pi i/p)$, *and let* $\Lambda = \mathbf{Q}(\omega + \bar{\omega})$ *be the maximal real subfield of the cyclotomic field* $\mathbf{Q}(\omega)$. *Let x be an indeterminate, set* $\mathfrak{Z}_0 = \Lambda(x^p)$, $\mathfrak{Z} = \mathbf{Q}(\omega)(x^p)$, *and* $\mathfrak{M} = \mathbf{Q}(\omega)(x)$. *Take* $S(x) = x\omega$ *as a generator of* $\mathrm{Gal}(\mathfrak{M}/\mathfrak{Z})$, *and define * on* $\mathfrak{M}$ *by setting* $\zeta^* = \bar{\zeta}$ *for* $\zeta \in \mathbf{Q}(\omega)$ *and* $x^* = x$. *Then the nonabelian bicyclic *-field* $\mathcal{K} = (\mathfrak{M}/\mathfrak{Z}, S, 2)$ *admits an ordering*.

PROOF. When $p \equiv 3 \pmod 4$, then $\mathbf{Q}(\omega) = \Lambda(\sqrt{-p}\,)$ has degree 2 over $\Lambda = \mathbf{Q}(\omega + \omega^*)$ [**10**, Article 183]; hence $\mathfrak{Z} = \mathbf{Q}(\omega)(x^p)$ has degree 2 over $\mathfrak{Z}_0 = \Lambda(x^p)$. Since $*S* = S^{-1}$ (as is easily checked), the tower of fields $\mathfrak{Z}_0 \subset \mathfrak{Z} \subset \mathfrak{M}$ stands in the nonabelian bicyclic case with $\deg(\mathfrak{M}/\mathfrak{Z}) = p$. Write the general nonzero element μ of $\mathfrak{M}$ as a formal Laurent series in x, coefficients in $\mathbf{Q}(\omega)$:

$$\mu = \zeta x^k(1 + \lambda x + \cdots), \qquad 0 \neq \zeta \in \mathbf{Q}(\omega).$$

Then deduce that if $2 \in \mathfrak{Z}$ were a norm from $\mathfrak{M}$, then $\mathbf{Q}(\omega)$ would contain the real pth root of 2, which is clearly not the case. Thus $\mathcal{K} = (\mathfrak{M}/\mathfrak{Z}, S, 2)$ is a field. Since $2^* = 2$, we can extend the involution (as before) from $\mathfrak{M}$ to $\mathcal{K}$, making $\mathcal{K}$ into a nonabelian bicyclic *-field.

Next represent the general nonzero element α of $\mathcal{K}$ as a formal Laurent series

$$\alpha = x^n(\zeta_0 + \zeta_1 y + \zeta_2 y^2 + \cdots + \zeta_{p-1}y^{p-1}) + x^{n+1}(\ldots) + \cdots$$

where $\zeta_i \in \mathbf{Q}(\omega)$ and not all ζ_i equal zero. One then checks easily that the function w: $\mathcal{K} \to \mathbf{Z}$ defined by $w(\alpha) = n$ qualifies as a *-valuation (§2). The residue class *-field $\mathcal{K}_0$ we can identify with $\mathbf{Q}(\omega)(2^{1/p})$, which under complex conjugation is an ordered *-field. To lift this ordering to $\mathcal{K}$, we need to verify smoothness of our *-valuation.

Clearly $w(2) = 0$. As to the second condition, we need to check that each $w^{-1}(n)$ contains a smooth element ρ. To say that ρ is smooth means that the involution $\# = A_\rho \circ *$, where $A_\rho(\cdot) = \rho(\cdot)\rho^{-1}$, induces on $\mathcal{K}_0$ an involution conjugate to its *. Now note that if $\rho = \beta\beta^*\sigma$ where $\sigma = \sigma^* \in \text{center}(\mathcal{K})$, then $\# = A_\rho \circ *$ is conjugate to * on $\mathcal{K}$ itself because then $\# = A_\beta \circ * \circ A_\beta^{-1}$. Hence such a ρ is clearly smooth. In our just-constructed nonabelian bicyclic example, $x = (x^{(p+1)/2})^2 x^{-p}$ with x^{-p} central. Hence x is smooth, and clearly also x^n for any $n \in \mathbf{Z}$. Thus our valuation is smooth, and by Corollary 3.3 we can lift the ordering of $\mathcal{K}_0$ to $\mathcal{K}$. That completes the proof of 3.4.

The abelian bicyclic case presents a sharp contrast to the nonabelian.

3.5. *No abelian bicyclic *-field admits an ordering.*

Proof. Maintain the notation used in the discussion preceding 3.4, and symbolize the general abelian bicyclic *-field $\mathcal{K}$ as $(\mathfrak{M}/\mathfrak{Z}, S, \gamma)$. We may assume the maximal abelian subfield $\mathfrak{M}$ generated over the center $\mathfrak{Z}$ by a symmetric element ρ; $\mathfrak{M} = \mathfrak{Z}(\rho)$, $\rho^* = \rho$. Then $\zeta = \text{Trace}_{\mathfrak{M}/\mathfrak{Z}}(\rho)$ lies in the center $\mathfrak{Z}$ and satisfies $\zeta^* = \zeta$. Let $\sigma = \zeta - n\rho$. We have $0 \neq \sigma = \sigma^*$, yet

$$\begin{aligned}\sigma + y\sigma y^* + \cdots + y^{n-1}\sigma(y^{n-1})^* &= \sigma + y\sigma y^{-1} + \cdots + y^{n-1}\sigma y^{-(n-1)}\\ &= \sigma + S(\sigma) + \cdots + S^{n-1}(\sigma)\\ &= \text{Trace}(\sigma) = 0.\end{aligned}$$

Thus our general abelian bicyclic *-field never admits an ordering.

(d) **-fields of fractions of ordered Ore *-domains.* The definition of ordered *-field makes no reference to inverses–it applies to any *-ring. The construction of our next class of examples uses this fact and the following result.

3.6. *An ordering of an Ore *-domain $\mathfrak{P}$ extends uniquely to its field of fractions $\mathcal{K}$.*

Proof. An involution on $\mathfrak{P}$ extends uniquely to $\mathcal{K}$, and the equivalence class containing $\alpha\beta^{-1}$ is symmetric with respect to the extended involution if and only if $\beta^*\alpha$ is symmetric in $\mathfrak{P}$. The definition

$$\alpha\beta^{-1} > 0 \text{ in } \mathcal{K} \Leftrightarrow \beta^*\alpha > 0 \text{ in } \mathfrak{P}$$

then effectively extends a given ordering on $\mathfrak{P}$ uniquely to $\mathcal{K}$.

The Weyl algebra is an Ore domain [**4**, p. 137] which we shall use to construct an example, illustrating the application of 3.6. We shall in fact construct two examples in parallel, a "real case" and a "complex case".

3.7. *Let $\mathfrak{Z}_0$ denote a q-ordered commutative field, set $\mathfrak{Z} = \mathfrak{Z}_0$ in the real case, $\mathfrak{Z} = \mathfrak{Z}_0(i)$, $i^2 = -1$, in the complex case. Let $W(\mathfrak{Z})$ denote the Weyl *-algebra over*

$\mathfrak{Z}$ generated by indeterminates x and u subject to the relations $xu - ux = 1$, $x^* = x$, $u^* = -u$ in the real case, $xu - ux = i$, $x^* = x$, $u^* = u$, $i^* = -i$ in the complex. Then $W(\mathfrak{Z})$ admits an ordering, and thus also its Ore field of fractions Λ.

PROOF. Write the general nonzero $\alpha \in W$ in the normal form

$$\alpha = \rho_0 + u\rho_1 + \cdots + u^l\rho_l, \qquad \rho_i \in \mathfrak{Z}[x],\ 1 \leqslant i \leqslant l,\ \rho_l \neq 0, \tag{1}$$

where we add term-by-term, and multiply according to the rule

$$\rho u^l = u^l\rho + \binom{l}{1}u^{l-1}\Delta\rho + \binom{l}{2}u^{l-2}\Delta^2\rho + \cdots + \Delta^l\rho, \qquad \rho \in \mathfrak{Z}[x],$$

where $\Delta = \partial/\partial x$ in the real case, $\Delta = i\partial/\partial x$ in the complex.

For an α as given by (1), define $N(\alpha) = \zeta_q$, where ζ_q is the first nonzero coefficient of the polynomial ρ_l,

$$\rho_l = \zeta_q x^q + \zeta_{q+1}x^{q+1} + \cdots, \qquad \zeta_i \in \mathfrak{Z},\ i \geqslant q,\ \zeta_q \neq 0.$$

Set also $N(0) = 0$. The map N so defined maps W onto $\mathfrak{Z}$, and has the following properties:

(i) $N(\alpha) = 0 \Leftrightarrow \alpha = 0$;

(ii) $N(\alpha\beta) = N(\alpha)N(\beta)$;

(iii)

$$N(\alpha^*) = \begin{cases} (-1)^l N(\alpha)^* & \text{in the real case,} \\ N(\alpha)^* & \text{in the complex case;} \end{cases}$$

(iv) if $N(\alpha) + N(\beta) \neq 0$, then $N(\alpha + \beta) =$ one of $N(\alpha)$, $N(\beta)$, $N(\alpha) + N(\beta)$.

Then the definition $\alpha > 0 \Leftrightarrow N(\alpha) > 0$ for symmetric α orders W. Thus, in this case we have constructed directly the function N of Theorem 3.2.

(e) *Power series *-fields*. The power series construction of P. M. Cohn detailed in [7] provides another example of an ordered *-field, and at the same time provides a useful representation of the field of fractions Λ of our Weyl *-algebra. This construction goes as follows. Start with $\mathfrak{Z}_0$ and $\mathfrak{Z}$ as in the previous construction. Construct the field $\mathfrak{Z}((x))$ of $\mathfrak{Z}$-coefficient formal Laurent series in the single indeterminate x. Write the typical element in $\mathfrak{Z}((x))$ as

$$\phi = \sum_{n=0}^{\infty} \zeta_n x^{n+q}, \tag{2}$$

q an integer, positive, negative, or zero, the ζ_n in $\mathfrak{Z}$, and $\zeta_0 \neq 0$ understood unless $\phi = 0$. Endow $\mathfrak{Z}((x))$ with the natural involution determined by $x^* = x$. Next form the set $\mathcal{K}$ of formal Laurent series in an indeterminate y, coefficients in $\mathfrak{Z}((x))$ written on the left. Write the typical element α in $\mathcal{K}$ as

$$\alpha = \sum_{n=0}^{\infty} \phi_n y^{n+p}, \tag{3}$$

p an integer, positive, negative, or zero, $\phi_n \in \mathfrak{Z}((x))$, and $\phi_0 \neq 0$ understood unless $\alpha = 0$. Add term-by-term, and multiply according to the relation $xy^{-1} - y^{-1}x = 1$ in the real case, $xy^{-1} - y^{-1}x = i$ in the complex. To determine the general formula for multiplication, note that products of the form $y^n\phi$ determine, through distributivity, the general expression for the product of elements of the form (3).

Then, as in [7], show that

$$y^n\phi = (I - R\Delta)^{-n}(\phi y^n)$$

where R symbolizes the operation of right multiplication by y, $R(\alpha) = \alpha y$, and $\Delta = \partial/\partial x$ in the real case, $\Delta = i\partial/\partial x$ in the complex. Thus

$$y^n\phi = \phi y^n + \varepsilon n\frac{\partial\phi}{\partial x}y^{n+1} + \varepsilon^2\,\frac{n(n+1)}{2}\,\frac{\partial^2\phi}{\partial x^2}y^{n+2} + \cdots$$

where $\varepsilon = 1$ in the real case, $\varepsilon = i$ in the complex.

Make $\mathcal{K}$ into a *-field by setting $y^* = -y$ in the real case, $y^* = y$, $i^* = -i$ in the complex. Thus, with α as in (3),

$$\alpha^* = (-1)^p\left[\phi_0 y^p + \left(p(\partial\phi_0/\partial x) - \phi_1\right)y^{p+1} + \cdots\right] \quad \text{in the real case,}$$

and

$$\alpha^* = \phi_0^* y^p + \left(ip(\partial\phi_0^*/\partial x) + \phi_1^*\right)y^{p+1} + \cdots \quad \text{in the complex.}$$

Let $G = \mathbf{Z} \times \mathbf{Z}$ as an abelian group under componentwise addition, ordered lexicographically by

$$(m, n) \gtrless 0 \quad \text{according as } m \gtrless 0$$

and

$$(0, n) \gtrless 0 \quad \text{according as } n \gtrless 0.$$

Given nonzero α represented as (3) with ϕ_0 represented as (2), define $w(\alpha) = (p, q) \in G$. This is a *-valuation on $\mathcal{K}$ whose residue class *-field $\mathcal{K}_0$ we can identify with the center $\mathfrak{Z}$. The natural *-homomorphism θ of Φ onto $\mathcal{K}_0$ has the form

$$\theta((\zeta + \cdots) + \phi_1 y + \cdots) = \zeta.$$

One checks easily this property of θ: Given any nonzero elements $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $\mathcal{K}$, if the product $\alpha_1\alpha_2 \ldots \alpha_n$ lies in the *-valuation ring Φ, then the product in any order lies in Φ, and the element $\theta(\alpha_1\alpha_2 \ldots \alpha_n)$ of $\mathcal{K}_0$ does not depend on the order. Hence $\theta(\rho x\rho^{-1}) = \theta(x)$ for every nonzero ρ and every $x \in \Phi$. Consequently all symmetric and skew elements of $\mathcal{K}$ are smooth, and we may therefore by application of 3.3 lift the natural ordering of $\mathcal{K}_0 = \mathfrak{Z}$ to $\mathcal{K}$. (In this case the function N of Theorem 3.2 is easily given explicitly: $N(\alpha) = \theta(y^{-p}x^{-q}\alpha)$ when $w(\alpha) = (p, q)$.)

Return to the Weyl algebra $W(\mathfrak{Z})$ and a typical element α in normal form $\alpha = \rho_0 + u\rho_1 + \cdots + u^l\rho_l$. By substituting $u = y^{-1}$ in this expression, we get an element of the just constructed *-field $\mathcal{K}$. This identification embeds $W(\mathfrak{Z})$ *-isomorphically in $\mathcal{K}$. The subfield of $\mathcal{K}$ generated by $W(\mathfrak{Z})$ equals Λ, the Ore field of fractions of $W(\mathfrak{Z})$.

4. The order topology and order valuation of an ordered *-field. I summarize first some elementary properties of ordered *-fields [13], [7]. If $\lambda > 0$ and $(m/n) > 0$, then $(m/n)\lambda > 0$; if $\lambda > 0$ then $\lambda^{-1} > 0$; if $0 < \lambda < 1$ then $0 < \lambda^2 < \lambda < 1$; if $\lambda > 1$ then $\lambda^{-1} < 1$; if $0 < \lambda < 1$, then $\lambda^{-1} > 1$; and if $\lambda > 1$ then $\lambda^2 > \lambda > 1$. The theorem of [7] asserts that an archimedean ordered *-field is *- and order isomorphic to a subfield of the real numbers **R**, the complex numbers **C**, or the real quaternions **H**.

Given now an ordered *-field $\mathcal{K}$ with domain of positivity Π, define a norm $\|\cdot\|$: $\mathcal{K} \to \Pi$ by $\|\alpha\| = \alpha^*\alpha$. Clearly $\|\alpha\| > 0$ when $\alpha \neq 0$, and $\|0\| = 0$. One checks easily the following properties of the norm.

$$\|\alpha + \beta\| \leqslant 2(\|\alpha\| + \|\beta\|), \qquad \|\alpha\beta\| = \beta^*\|\alpha\|\beta,$$

$$\|\alpha^{-1}\| = \|\alpha^*\|^{-1} \quad \text{and} \quad \|\lambda\| = \lambda^2 \qquad \text{when } \lambda = \lambda^*.$$

Given $\varepsilon > 0$ in $\mathcal{K}$ set $N(\varepsilon) = \{\alpha \in \mathcal{K}: \|\alpha\| < \varepsilon\}$. This system of neighborhoods of 0 defines a Hausdorff topology on $\mathcal{K}$, the *order topology*, which makes continuous the following operations: addition and subtraction, $\alpha \to \alpha\alpha_0$ at 0 (α_0 fixed), $\alpha \to \alpha^2$ at 0, and $\alpha \to \alpha^{-1}$ at 1. To make $\mathcal{K}$ a topological *-field I need to assume, in addition, the continuity of the involution, that is, I need to assume that given $\varepsilon > 0$ in $\mathcal{K}$ there exists $\delta > 0$ so that $\|\alpha\| < \delta \Rightarrow \|\alpha^*\| < \varepsilon$.

4.1. THEOREM. *An ordered *-field $\mathcal{K}$ with continuous involution is a topological *-field under its order topology.*

PROOF. Verification of the continuity of addition and subtraction presents no difficulty. As for multiplication, check first that given $\alpha_0 \in \mathcal{K}$, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\alpha\| < \delta \Rightarrow \|\alpha\alpha_0\| < \varepsilon$. Here just set $\delta = \alpha_0^{-1*}\varepsilon\alpha_0^{-1}$. Next use the continuity of $*$ to verify the continuity of $\alpha \to \alpha_0\alpha$. Then handle the general case with the identity

$$\alpha\beta - \alpha_0\beta_0 = (\alpha - \alpha_0)(\beta - \beta_0) + \alpha_0(\beta - \beta_0) + (\alpha - \alpha_0)\beta_0$$

with its consequent inequality

$$\|\alpha\beta - \alpha_0\beta_0\| \leqslant 2\|(\alpha - \alpha_0)(\beta - \beta_0)\| + 4\|\alpha_0(\beta - \beta_0)\| + 4\|(\alpha - \alpha_0)\beta_0\|.$$

Given $\varepsilon > 0$, then (by the arguments just given) there exist δ_1 and δ_2 so that $\|\beta - \beta_0\| < \delta_1 \Rightarrow \|\alpha_0(\beta - \beta_0)\| < \varepsilon/12$ and $\|\alpha - \alpha_0\| < \delta_2 \Rightarrow \|(\alpha - \alpha_0)\beta_0\| < \varepsilon/12$. Set $\delta = \min(\delta_1, \delta_2, 1, \varepsilon/6)$. Then $\|\alpha - \alpha_0\| < \delta$ and $\|\beta - \beta_0\| < \delta$ together imply

$$\begin{aligned}\|(\alpha - \alpha_0)(\beta - \beta_0)\| &= (\beta - \beta_0)^*\|\alpha - \alpha_0\|(\beta - \beta) < (\beta - \beta)^* \cdot 1 \cdot (\beta - \beta_0)\\ &= \|\beta - \beta_0\| < \delta < \varepsilon/6,\end{aligned}$$

so that $\|\alpha\beta - \alpha_0\beta_0\| < 2(\varepsilon/6) + 4(\varepsilon/12) + 4(\varepsilon/12) = \varepsilon$.

As for the inverse, first check its continuity at $\alpha_0 = 1$. Note that if $\|1 - \alpha\| \leqslant 1/n$ for some integer $n \geqslant 3$, then $\|\alpha\| \geqslant (n-2)/2n$. Now given $\varepsilon > 0$, select $\delta < \min(1/4, \varepsilon/4)$. Then $\|1 - \alpha\| < \delta \Rightarrow \|\alpha\| \geqslant 1/4$ or $1 \leqslant 4\|\alpha\|$ whence

$$\|\alpha^{-1}\| = \alpha^{-1*}\alpha^{-1} \leqslant \alpha^{-1*}(4\|\alpha\|)\alpha^{-1} = 4.$$

So $\|1 - \alpha\| < \delta$ implies

$$\begin{aligned}\|\alpha^{-1} - 1\| &= \|\alpha^{-1}(1 - \alpha)\| = (1 - \alpha)^*\alpha^{-1*}\alpha^{-1}(1 - \alpha)\\ &= (1 - \alpha)^*\|\alpha^{-1}\|(1 - \alpha) \leqslant 4\|1 - \alpha\| < \varepsilon.\end{aligned}$$

Next check continuity of the inverse at the nonzero α_0 in the following steps. First choose $\delta_1 = \frac{1}{2}\|\alpha_0\|$. Then $\|\alpha - \alpha_0\| < \delta_1 \Rightarrow \|\alpha - \alpha_0\| \leqslant \frac{1}{2}\|\alpha_0\|$ so $\alpha \neq 0$ and

α^{-1} exists. Now, given $\varepsilon > 0$, by continuity of left multiplication there exists $\delta_2 > 0$ such that $\|\alpha^{-1}\alpha_0 - 1\| < \delta_2 \Rightarrow \|(\alpha^{-1}\alpha_0 - 1)\alpha_0^{-1}\| < \varepsilon$. Note that $(\alpha^{-1}\alpha_0 - 1)\alpha_0^{-1} = \alpha^{-1} - \alpha_0^{-1}$. By continuity of the inverse at 1, there exist $\delta_3 > 0$ such that $\|\alpha_0^{-1}\alpha - 1\| < \delta_3 \Rightarrow \|(\alpha_0^{-1}\alpha)^{-1} - 1\| < \delta_2$; and by continuity of right multiplication there exists $\delta_4 > 0$ such that $\|\alpha - \alpha_0\| < \delta_4 \Rightarrow \|\alpha_0^{-1}(\alpha - \alpha_0)\| < \delta_3$. Set $\delta = \min(\delta_1, \delta_4)$. If $\|\alpha - \alpha_0\| < \delta$, then $\|\alpha - \alpha_0\| < \delta_1$ so α^{-1} exists, and $\|\alpha - \alpha_0\| < \delta_4$ so $\|\alpha_0^{-1}(\alpha - \alpha_0)\| = \|\alpha_0^{-1}\alpha - 1\| < \delta_3$. From this latter inequality deduce $\|(\alpha_0^{-1}\alpha)^{-1} - 1\| = \|\alpha^{-1}\alpha_0 - 1\| < \delta_2$, whence $\|(\alpha^{-1}\alpha_0 - 1)\alpha_0^{-1}\| = \|\alpha^{-1} - \alpha_0^{-1}\| < \varepsilon$. That completes the proof.

4.2. COROLLARY [13, SATZ 1.4]. *A commutative q-ordered field is a topological field under the interval topology induced by the usual absolute value.*

PROOF. As already pointed out, when $*$ = identity, Baer's ordering reduces to the q-ordering of A. Prestel which differs from the common concept of commutative ordered field in that the condition $\lambda > 0 \Rightarrow \mu^2\lambda > 0$ ($\mu \neq 0$) replaces the familiar condition that positive elements have a positive product. To derive the corollary from the theorem, we need only to show that the norm $\|\lambda\| = \lambda^2$ and the usual absolute value $|\lambda|$ ($|\lambda| = \pm\lambda$ according as $\lambda \geqslant 0$ or < 0) define the same topology. Given $\varepsilon > 0$, set $\delta = \min(\varepsilon, 1)$. Then one checks easily that $|\alpha| < \delta \Rightarrow \|\alpha\| < \varepsilon$. For the converse, take $\varepsilon < 1$ and choose $\delta = \varepsilon^4$. Then $\|\alpha\| < \delta \Rightarrow |\alpha| < \varepsilon^2 < \varepsilon$. That completes the proof of the corollary.

We take up now the order valuation of an ordered $*$-field. This valuation measures "orders of magnitude".

As above, we let $\mathcal{K}$ denote an ordered $*$-field, $\mathcal{K}$ its multiplicative group of nonzero elements. Call $\alpha \in \mathcal{K}$ *medial* provided that $0 < r < \alpha^*\alpha < s$ for some positive rationals r and s; call α *infinitesimal* if $\alpha^*\alpha < 1/n$ for all $n = 1, 2, \ldots$; and call it *infinite* if $\alpha^*\alpha > n$ for all $n = 1, 2, \ldots$. Use $\mathfrak{M}$ (medium) to stand for the set of medial elements, Σ (small) for the set of infinitesimal elements, and Λ (large) for the set of infinite elements. Clearly $\mathcal{K}$ equals the disjoint union of Σ, $\mathfrak{M}$, and Λ. Using the elementary properties listed in the first paragraph of this section, one may verify the following properties of these sets:

Σ: closed under $*$ and multiplication; $\Sigma \cup \{0\}$ also closed under $\pm$;

$\mathfrak{M}$: a $*$-closed multiplicative subgroup of $\mathcal{K}$ closed under sums of positive elements;

Λ: closed under $*$, multiplication, and sums of positive elements.

Also we have

$$\Sigma^{-1} = \Lambda, \quad \Lambda^{-1} = \Sigma, \quad \Sigma\mathfrak{M} = \mathfrak{M}\Sigma = \Sigma, \quad \Lambda\mathfrak{M} = \mathfrak{M}\Lambda = \Lambda,$$
$$\Sigma + \mathfrak{M} = \mathfrak{M}, \quad \mathfrak{M} + \Lambda = \Lambda, \quad \Sigma + \Lambda = \Lambda.$$

I shall prove a few typical cases, starting with $\alpha \in \Sigma \Rightarrow \alpha^* \in \Sigma$. If $\alpha \in \Sigma$, then $\alpha^*\alpha < 1/n$, $n = 1, 2, \ldots$. Let $\lambda = \alpha^*\alpha$. Then $n\lambda < 1$ so $(n\lambda)^2 < n\lambda < 1$, whence $\lambda^2 < \lambda/n$ or $\alpha^*\alpha\alpha^*\alpha < \alpha^*\alpha/n$. Then

$$(\alpha^{-1})^*(\alpha^*\alpha\alpha^*\alpha)\alpha^{-1} < \frac{1}{n}(\alpha^{-1})^*(\alpha^*\alpha)(\alpha^{-1}),$$

or $\alpha\alpha^* < 1/n$, $n = 1, 2, \ldots,$ which shows $\alpha^* \in \Sigma$. One proves in a similar way the *-closure of $\mathfrak{M}$ and Λ.

Closure of $\mathfrak{M}$ under multiplication: Suppose $\alpha, \beta \in \mathfrak{M}$ so that $0 < r_1 < \alpha^*\alpha < s_1$ and $0 < r_2 < \beta^*\beta < s_2$ for rational r_1, s_1, r_2, s_2. Then from the first inequality we get $0 < r_1\beta^*\beta < \beta^*\alpha^*\alpha\beta < s_1\beta^*\beta$, which yields in turn $0 < r_1r_2 < (\alpha\beta)^*(\alpha\beta) < s_1s_2$. Thus $\alpha\beta \in \mathfrak{M}$.

One may check similarly the remaining statements, and the bulk of the following.

4.3. Theorem. *$\mathcal{K}$ denotes an ordered *-field, Σ and $\mathfrak{M}$ its sets of infinitesimal and medial elements respectively. Then $\Phi = \Sigma \cup \mathfrak{M} \cup \{0\}$ is a *-subring of $\mathcal{K}$ containing for each α in $\mathcal{K}$ at least one of α, α^{-1}. $\mathcal{P} = \Sigma \cup \{0\}$ is a *-closed, two-sided ideal in Φ that contains every proper ideal. In the quotient field $\mathcal{K}_0 = \Phi/\mathcal{P}$, endowed with the natural involution $(\alpha + \mathcal{P})^* = \alpha^* + \mathcal{P}$, the definition $\alpha + \mathcal{P} > 0 \Leftrightarrow \alpha > 0$ $(\alpha = \alpha^* \notin \mathcal{P})$ defines effectively a domain of positivity making $\mathcal{K}_0$ into an archimedean ordered *-field, *- and order isomorphic therefore to a subfield of* **R**, **C** *or* **H**.

The proof follows the lines given previously. To check, for example, that we have an ordering on $\mathcal{K}_0$, we must verify that every symmetric element of $\mathcal{K}_0$ equals a $\sigma + \mathcal{P}$ where $\sigma = \sigma^*$. But if $\alpha^* + \mathcal{P} = \alpha + \mathcal{P}$, then $\alpha - \alpha^* \in \mathcal{P}$, and $\sigma = \frac{1}{2}(\alpha + \alpha^*)$ satisfies $\sigma = \sigma^*$ and $\alpha - \sigma = \frac{1}{2}(\alpha - \alpha^*) \in \mathcal{P}$, so $\sigma + \mathcal{P} = \alpha + \mathcal{P}$. As for the archimedean character of the ordering of $\mathcal{K}_0$, if $0 < \sigma + \mathcal{P} < 1/n + \mathcal{P}$, $n = 1, 2, \ldots,$ then $0 < \sigma < 1/n$ in $\mathcal{K}$ for $n = 1, 2, \ldots,$ whence $\sigma \in \mathcal{P}$ contradicting $\sigma + \mathcal{P} > 0$. I leave the remaining details to the reader.

The *-subring Φ consists of those α which satisfy $\alpha^*\alpha < n$ for some positive integer n (depending on α in general), and the maximal *-ideal $\mathcal{P}$ consists of those α satisfying $\alpha^*\alpha < 1/n$ for all $n = 1, 2, \ldots$. In keeping with the usual terminology, I call Φ the *valuation ring of finite elements*, $\mathcal{P}$ the *ideal of infinitesimal elements*, and $\mathcal{K}_0 = \Phi/\mathcal{P}$ the *residue class *-field* associated to the given ordering. As noted, we may regard $\mathcal{K}_0$ as a *-subfield of **R**, **C**, or **H**. We use θ for the natural *-homomorphism of Φ on $\mathcal{K}_0$: $\theta(\alpha) = \alpha + \mathcal{P}$.

The appropriate notion of valuation to go with our general kind of (noncommutative) valuation ring has been found by Rado **[15]**; see also the paper of Mathiak **[11]**.

4.4. Theorem. *The formula $\alpha \sim \beta \Leftrightarrow \beta\alpha^{-1} \in \mathfrak{M}$ defines an equivalence relation on $\mathcal{K}$, and the formula $[\alpha] < [\beta] \Leftrightarrow \beta\alpha^{-1} \in \Sigma$ defines effectively a total ordering of the set $G = \{[\alpha]: \alpha \in \mathcal{K}\}$ of equivalence classes. The map w: $\mathcal{K} \to G$ defined by $w(\alpha) = [\alpha]$ satisfies these conditions*:

(1′) $w(\alpha) \leqslant w(\beta) \Rightarrow w(\alpha\gamma) \leqslant w(\beta\gamma)$ *for all* γ *in* $\mathcal{K}$;

(2) $w(\alpha + \beta) \geqslant \min(w(\alpha), w(\beta))$ $(\alpha + \beta \neq 0)$;

(3) *w maps onto G*;

(4′) $w(\alpha) > w(1) \Rightarrow w(\alpha^*) > w(1)$.

Proof. The reflexivity, symmetry, and transitivity of the relation $\sim$ follows from the fact $\mathfrak{M}$ is a multiplicative subgroup of $\mathcal{K}$. To show the effectiveness of the definition of the relation $<$ in G, we need to prove that $\beta\alpha^{-1} \in \Sigma$, $\alpha_1 \sim \alpha$, and

$\beta_1 \sim \beta$ together imply that $\beta_1\alpha_1^{-1} \in \Sigma$. This follows from the identity $\beta_1\alpha_1^{-1} = (\beta_1\beta^{-1})(\beta\alpha^{-1})(\alpha\alpha_1^{-1})$ which represents $\beta_1\alpha_1^{-1}$ as the product of two medial elements and an infinitesimal one, thus infinitesimal. If $[\alpha] < [\beta]$ and $[\beta] < [\gamma]$, then $[\alpha] < [\gamma]$ because $\gamma\alpha^{-1} = (\gamma\beta^{-1})(\beta\alpha^{-1}) \in \Sigma$. Thus we have an ordering on the set G, in which any two unequal elements are comparable, because if $\beta\alpha^{-1} \notin \mathfrak{M}$, then either $\beta\alpha^{-1}$ or $(\beta\alpha^{-1})^{-1} = \alpha\beta^{-1}$ is infinitesimal. As to the conditions on the map w, condition (1′) is obtained simply because $(\beta\gamma)(\alpha\gamma)^{-1} = \beta\alpha^{-1}$. In (2) we may assume $w(\alpha) \leqslant w(\beta)$, so that $\min(w(\alpha), w(\beta)) = w(\alpha)$. Then $(\alpha + \beta)\alpha^{-1} = 1 + \beta\alpha^{-1}$ is finite, hence $w(\alpha) \leqslant w(\alpha + \beta)$. Condition (3) follows by definition, and condition (4′) from the closure of Σ under $*$. That completes the proof.

The map $w: \mathcal{K} \to G$ constructed in Theorem 4.4 I call the *order valuation* of our ordered $*$-field $\mathcal{K}$. For the ordered $*$-fields constructed in §3 (which constitute all examples known at present), their order valuations have the additional property that $w(\alpha^*) = w(\alpha)$ and therefore qualify as $*$-valuations in the sense of §2 (see 4.6). However, I do not know that $w(\alpha^*) = w(\alpha)$ holds in general for the order valuation (it appears unlikely), and pending availability of a larger class of examples the question remains open.

For an archimedean ordered $\mathcal{K}$, which we may consider as a $*$-subfield of **R**, **C**, or **H**, our norm $\|\alpha\| = \alpha^*\alpha$ equals the square of the usual norm, and the order topology coincides with the usual topology. The order valuation, on the other hand, becomes trivial in the archimedean case, with the value set reducing to the single element $[1]$.

In the nonarchimedean case, the order valuation can be used in the usual way to define the order topology.

4.5. *Consider a nonarchimedean ordered $*$-field $\mathcal{K}$ with order valuation $w: \mathcal{K} \to G$* (*see Theorem* 4.4).

(a) *The topology defined on $\mathcal{K}$ by the system of neighborhoods $N_g = \{\alpha: w(\alpha) > g\} \cup \{0\}$, $g \in G$, equals the order topology.*

(b) *If we have $w(\alpha^*) = w(\alpha)$ for all α in $\mathcal{K}$; then the involution is continuous in the order topology.*

Proof. (a) Given the order topology neighborhood $N_\varepsilon = \{\alpha: \alpha^*\alpha < \varepsilon\}$, $\varepsilon > 0$, set $\delta = \min(\varepsilon, 1)$, $g = w(\delta)$.

If $\alpha \in N_g$, so that $w(\alpha) > g = w(\delta)$, then $\alpha\delta$ is infinitesimal, i.e. $(\alpha\delta^{-1})^*(\alpha\delta^{-1}) < 1/n$, so $\alpha^*\alpha < \delta^2 \leqslant \delta \leqslant \varepsilon$. Thus $\alpha \in N_\varepsilon$, so $N_g \subseteq N_\varepsilon$.

Conversely, suppose given $g = w(\beta) \in G$. The desired inclusion, $N_\varepsilon \subseteq N_g$, will follow from $\varepsilon < \beta^*\beta/n$, $n = 1, 2, \ldots$. If β is medial or infinite, we may choose ε as any symmetric infinitesimal; if β is infinitesimal, choose $\varepsilon = (\beta^*\beta)^2$. That proves (a), and (b) then follows easily.

For the next result, refer to §2 and the numbered conditions after Theorem 4.4.

4.6. *If the order valuation w of the ordered $*$-field $\mathcal{K}$ satisfies* (4) $w(\alpha^*) = w(\alpha)$, *then the value set G becomes a commutative ordered group under the addition* $[\alpha] + [\beta] = [\alpha\beta]$, *and the valuation w satisfies* (1) $w(\alpha\beta) = w(\alpha) + w(\beta)$, *thus qualifies as a $*$-valuation in the sense of* §2.

Proof. The hypothesis, in its equivalent form $\alpha^*\alpha^{-1} \in \mathfrak{M}$, forces the symmetry of our valuation ring of finite elements Φ, whence the theory of §2 applies. Statement 4.6 then follows directly from 2.1, as the ordered set G here coincides with the G there.

5. Hermitian forms. Given a (finite- or infinite-dimensional) left vector space E over a (general) *-field $\mathcal{K}$, we define a Hermitian form on E as a map $(\cdot, \cdot)$ from $E \times E$ to $\mathcal{K}$ satisfying the usual axioms

$$(\lambda a + \mu b, c) = \lambda(a, c) + \mu(b, c), \qquad (a, b)^* = (b, a).$$

In this paper, "form" means a Hermitian form on a left vector space over a *-field. We shall use, by and large, the terminology of [5].

We write $\langle\lambda\rangle$ for the one-dimensional form on $\mathcal{K}e$ with $(e, e) = \lambda$, and write $\langle\lambda_1\rangle + \cdots + \langle\lambda_n\rangle$ for the n-dimensional form with orthogonal basis $e_1, e_2, \ldots, e_n$ such that $(e_i, e_i) = \lambda_i$. When $\lambda_1 = \lambda_2 = \cdots = \lambda_n$, we write $n\langle\lambda\rangle$ for $\langle\lambda\rangle + \cdots + \langle\lambda\rangle$ (n times).

Call the *-field $\mathcal{K}$ *formally real* if for each nonzero symmetric λ in $\mathcal{K}$, and each positive integer n, the form $n\langle\lambda\rangle$ is anisotropic. This supercedes the definition given in [7] which required anisotropy only when $\lambda = 1$. One of the major open questions in this young subject: *Does every formally real *-field admit an ordering*? An affirmative answer here would follow from an affirmative answer to this question: *In a formally real *-field, if $n\langle 1\rangle$ represents μ, does $m\langle\mu\rangle$ represent 1 for some positive integer m*?

When the underlying *-field carries an ordering, then the usual definitions of *positive definite* form and *positive semidefinite* form apply. Much of the basic theory dealing with forms over commutative ordered fields carries over to this case. In the noncommutative case, the Schwarz inequality can be phrased this way.

5.1. *For a positive semidefinite form we have*

$$(y, x)(x, x)^{-1}(x, y) \leqslant (y, y)$$

for any anisotropic x and any y.

We get as a consequence

5.2. *A nonsingular positive semidefinite form is positive definite.*

And Sylvester's "law of inertia" goes over.

5.3. *Given a finite-dimensional form $\{(\cdot, \cdot), E\}$ over an ordered *-field, we may write E as the direct sum of three orthogonal subspaces $E = L + P + N$ such that the form restricted to L vanishes identically, the form restricted to P is positive definite, and restricted to N is negative definite. The dimensions of P and N are uniquely determined, and $L = E^{\perp}$.*

We deal now with a positive definite form on a finite or infinite-dimensional left vector space E over an ordered *-field $\mathcal{K}$, and shall show how we can consider this form as an extension, in a sense, of a positive definite form over one of the three classical number fields.

Continue the notation of §4: Φ stands for the valuation ring of finite elements in $\mathcal{K}$, $\mathcal{P}$ for the ideal of infinitesimal elements, and $\mathcal{K}_0 = \Phi/\mathcal{P}$ for the residue class *-field of $\mathcal{K}$. Denote by V the set of vectors x in E for which $(x, x) \in \Phi$. Call the elements in V *finite vectors*. We can then prove without difficulty: *The set V of* *finite vectors qualifies as a left Φ-module, and the form restricted to V takes its values in Φ.*

Denote by R the set of all x in E for which $(x, x) \in \mathcal{P}$, and call these *infinitesimal vectors*. R forms a Φ-submodule of V, and $(V, R) = (R, V) \subseteq \mathcal{P}$. Let ϕ stand for the natural homomorphism of V onto the quotient space H, $H = V/R$, $\phi(a) = a + R$. Continue to use θ for the natural homomorphism of Φ onto $\mathcal{K}_0$, $\theta(\alpha) = \alpha + \mathcal{P}$. One checks routinely that H forms a $\mathcal{K}_0$-vector space under the canonical scalar multiplication: given $\eta \in \mathcal{K}_0$ and $x \in H$, select $\alpha \in \Phi$ with $\theta(\alpha) = \eta$ and select $a \in V$ with $\phi(a) = x$. Then define $\eta x = \phi(\alpha a)$. Given x, y in H, select a, b in V with $\phi(a) = x$, $\phi(b) = y$, and define $f(x, y) = \theta((a, b))$. Straightforward computations verify that this defines a positive definite form on H. Note that the map ϕ: $V \to H$ takes orthonormal sets to orthonormal sets, so that if the form represents 1 on every one-dimensional subspace of E, we shall have $\dim(E) = \dim(H)$. We have proved

5.4. THEOREM. *Suppose given a left vector space E over an ordered $*$-field $\mathcal{K}$, together with a positive definite form on E. Then the quotient space H of finite vectors modulo the infinitesimal vectors, with its natural inherited structure, becomes a classical positive definite inner product space over $\mathcal{K}_0$, the residue class $*$-field of $\mathcal{K}$, $\mathcal{K}_0$ a $*$-subfield of* $\mathbf{R}$, $\mathbf{C}$ *or* $\mathbf{H}$. *If the form represents* 1 *on each one-dimensional subspace of E, then* $\dim(E) = \dim(H)$.

Suppose now that $\mathcal{K}$ is a general $*$-field, not necessarily ordered, but carrying a nontrivial $*$-valuation w: $\mathcal{K} \to G$. One has available in this case the following useful construction (compare [**6**]).

5.5. *Given a $*$-field $\mathcal{K}$ with nontrivial $*$-valuation w: $\mathcal{K} \to G$, and given a left $\mathcal{K}$-vector space E (of any dimension) with an anisotropic Hermitian form $(\cdot, \cdot)$ that satisfies "Schwarz's inequality",*

$$2w((x, y)) \geqslant w((x, x)) + w((y, y))$$

for any nonorthogonal vectors x, y in E, then the system of neighborhoods

$$N_g = \{x \in E: w((x, x)) > g\} \cup \{0\}, \qquad g \in G,$$

makes E into a topological vector space.

The omitted verification follows routine lines.

As to the hypothesized "Schwarz inequality", we can single out two quite different and separately useful sets of circumstances in which it holds.

5.6. *If $\mathcal{K}$ carries an ordering compatible with the valuation in the sense that $0 < \alpha \leqslant \beta \Rightarrow w(\alpha) \geqslant w(\beta)$, and the form is positive definite, then Schwarz's inequality holds.*

This follows directly from 5.1. All the examples constructed in §3 have compatible orderings.

5.7. *If the form is anisotropic and satisfies the following condition: for any pair of nonzero orthogonal vectors x, y we have $w((x, x)) \neq w((y, y))$, then Schwarz's inequality holds.*

One checks easily that the stated condition is equivalent to: $x \neq 0$, $y \neq 0$, $(x, y) = 0$ together imply $w((x, x)) \not\equiv w((y, y)) \bmod 2G$. We therefore describe this property of the form by saying that *orthogonal vectors belong to different square classes*.

PROOF OF 5.7. Under this assumption we wish to prove $2w((x, y)) > w((x, x)) + w((y, y))$ when $(x, y) \neq 0$. Select $0 \neq e \in \mathcal{K}x + \mathcal{K}y$ so that $(x, e) = 0$. Then $y = \lambda x + \mu e$, $(x, y) = (x, x)\lambda^*$, and

$$2w((x, y)) = 2w((x, x)) + 2w(\lambda^*) = w((x, x)) + w(\lambda(x, x)\lambda^*).$$

Now $(y, y) = \lambda(x, x)\lambda^* + \mu(e, e)\mu^*$, so our desired inequality reduces to $w(\lambda(x, x)\lambda^*) \geqslant w(\lambda(x, x)\lambda^* + \mu(e, e)\mu^*)$. But $\lambda x \perp \mu e$, so $w((\lambda x, \lambda x)) \neq w((\mu e, \mu e))$, hence

$$\begin{aligned} w(\lambda(x, x)\lambda^* + \mu(e, e)\mu^*) &= \min\{w(\lambda(x, x)\lambda^*), w(\mu(e, e)\mu^*) \\ &\leqslant w(\lambda(x, x)\lambda^*), \end{aligned}$$

as desired.

The following general construction of a class of forms satisfying 5.7 comes from a remarkable paper by Hans Keller **[8]**.

5.8. THEOREM. *Suppose given a *-field $\mathcal{K}$ complete with respect to a nontrivial *-valuation w: $\mathcal{K} \to G$. Select from $\mathcal{K}$ a sequence X_n, $n = 1, 2, \ldots$, of nonzero symmetric elements, and set $p_n = w(X_n)$.*

(1) *The set E of sequences $\{\xi_1, \xi_2, \ldots\}$ of elements from $\mathcal{K}$ such that the series $\sum_{i=1}^{\infty} \xi_i X_i \xi_i^*$ converges in $\mathcal{K}$ forms a left vector space over $\mathcal{K}$ (under the usual componentwise operations), and the formula $(x, y) = \sum_{i=1}^{\infty} \xi_i X_i \eta_i^*$ where $x = \{\xi_i\}$, $y = \{\eta_i\}$ defines a Hermitian form on E.*

(2) *If the p_n satisfy this condition: $m \neq n \Rightarrow p_m \not\equiv p_n \bmod 2G$, then the form defined in* (1) *is anisotropic and in E nonzero orthogonal vectors always belong to different square classes* (5.7 *satisfied*). *Moreover the space E is complete with respect to the topology described in* 5.5, *and in E any orthogonal family of nonzero vectors is countable.*

(3) *Suppose that the sequence p_n also has this property: $p_n \to \infty$ and for any bounded-below sequence q_n for which $q_n \equiv p_n \bmod 2G$ also $q_n \to \infty$. Then, given any maximal orthogonal set $\{f_i\}$ of nonzero vectors, $x = \sum_{i=1}^{\infty}(x, f_i)(f_i, f_i)^{-1} f_i$ for any $x \in E$ (every maximal orthogonal set is a basis); every topologically closed subspace M of E is $\perp$-closed (satisfies $M = M^{\perp\perp}$); and for any closed subspace M, $E = M + M^{\perp}$.*

The condition on the p_n given in (3), which replaces an incorrect condition I had used earlier, comes from a letter from Professor H. Gross. Mrs. A. Faessler caught the original mistake, and I thank her and Professor Gross for their help.

Keller in [8] deals specifically with the commutative field Λ obtained by adjoining countably many indeterminates $X_1, X_2, \ldots$ to the rational field $\mathbf{Q}$, and takes for his field $\mathcal{K}$ the completion of Λ with respect to the usual valuation w with value group $G = \mathbf{Z} \times \mathbf{Z} \times \ldots$. Under this valuation, p_n $(= w(X_n))$ equals the element of G that has 1 in the nth place and 0 elsewhere. This selection satisfies the conditions in (2) and (3) of Theorem 5.8.

Keller's construction finally settled the long open problem as to the existence of a "nonclassical Hilbert space": an infinite dimensional form $\{(\cdot, \cdot), E\}$, different from real, complex, and quaternionic Hilbert space, yet satisfying $E = M + M^{\perp}$ for every $\perp$-closed subspace M. (The nonstandard Hilbert spaces do not necessarily have this property, see **[12]**.)

Proof of Theorem 5.8. (1) In the complete nonarchimedean field $\mathcal{K}$, convergence of a series $\Sigma\, \alpha_n$ is equivalent to $w(\alpha_n) \to \infty$. From the identity $2w(\xi_i X_i \eta_i^*) = w(\xi_i X_i \xi_i) + w(\eta_i X_i \eta_i)$ conclude then that the series used to define the form in (1) always converges.

(2) From the assumption that the p_n are all incongruent mod $2G$, deduce that $w(\xi_i X_i \xi_i^*) = w(\eta_j X_j \eta_j^*) \Leftrightarrow i = j$, so that, in particular, the terms in $(x, x) = \Sigma\, \xi_i X_i \xi_i^*$ have distinct values. Hence if nonzero entries occur on the right at all, then there is exactly one index $i = n$ where $w(\xi_i X_i \xi_i^*)$ has its minimum value. If $(x, x) = 0$, then we would have $-\xi_n X_n \xi_n^* = \Sigma_{i \neq n}\, \xi_i X_i \xi_i^*$ with $w(\xi_i X_i \xi_i^*) > w(\xi_n X_n \xi_n^*)$, $i \neq n$, a contradiction. Hence our form is anisotropic, and, for $x \neq 0$, $w((x, x)) = \min w(\xi_i X_i \xi_i^*) = \min 2w(\xi_i) + p_i$, the minimum occurring at exactly one index $i = n$. Set $N(x) = w((x, x)) = 2w(\xi_n) + p_n$, and $T(x) = n$. Call $N(x) \in G$ the *norm* of x, and the positive integer $T(x)$ the *type* of x. Check that $T(x) = T(y) \Leftrightarrow N(x) \equiv N(y) \bmod 2G$, and that $(x, y) = 0 \Rightarrow T(x) \neq T(y)$. Thus orthogonal vectors belong to different square classes (5.7 satisfied) which validates Schwarz's inequality and legitimatizes the use of the topology described in 5.5. The proof of the completeness of E under this topology follows the pattern of the usual proof of completeness of l_2. The map $x \to T(x)$ maps each orthogonal family of E one-to-one onto a subset of the positive integers. In particular then, every orthogonal family is countable. A series $\Sigma\, x_i$ converges in $E \Leftrightarrow N(x_i) \to \infty$. Using the easily proved continuity of the form, check that if $x = \Sigma\, \lambda_i f_i$ for an orthogonal family f_i, then necessarily $\lambda_i = (x, f_i)(f_i, f_i)^{-1}$. Finally note that the specific orthogonal sequence e_i which has 1 in the ith place and 0 elsewhere forms an orthogonal basis for our space $x = \Sigma(x, e_i)(e_i, e_i)^{-1} e_i = \Sigma\, \xi_i e_i$ for every $x = \{\xi_i\} \in E$.

(3) Assuming now the condition stated in (3), check that for any orthogonal sequence f_i and any x, the series $\Sigma(x, f_i)(f_i, f_i)^{-1} f_i$ converges in E. This implies that every maximal orthogonal set is an orthogonal basis. Given a topologically closed subspace M of E, choose a maximal orthogonal subset f_i of M. Then, given $x \in E$, set $m = \Sigma(x, f_i)(f_i, f_i)^{-1} f_i$, and write $x = m + (x - m)$ to show $E = M + M^{\perp}$. The equation $M = M^{\perp\perp}$ follows from this.

6. Generalization of Wilbur's theorem. Continue the terminology of the preceding section, except for brevity call a subspace M closed when $M = M^{\perp\perp}$. Following Kaplansky, we call a nonsingular form on a space E *orthomodular* when $M + M^{\perp} = E$ for every closed subspace M. As noted in the previous section, Hans Keller has constructed an example of a nonclassical infinite-dimensional orthomodular form **[8]**. On the other hand, one can characterize the classical forms by adding some reasonable conditions. The following result generalizes Wilbur's Theorem 5.8

from his paper [17]. For other results along this line see [6].

6.1. THEOREM. *Suppose the ordered *-field $\mathcal{K}$ has this property: For every symmetric infinitesimal ε, $1 + \varepsilon = \alpha\alpha^*$ for some $\alpha \in \mathcal{K}$. Let E stand for an infinite-dimensional left vector space over such an ordered *-field, and suppose given on E an orthomodular form $(\cdot, \cdot)$ that represents 1 on every one-dimensional subspace. Then $\mathcal{K}$ is either* **R**, **C**, *or* **H**, *and $\{E, (\cdot, \cdot)\}$ is the corresponding classical Hilbert space.*

We use Theorem 5.4 to prove this result, and shall use the notations in the discussion preceding that result. Note that an orthomodular form that represents 1 on every one-dimensional subspace is necessarily positive definite, so we can apply Theorem 5.4.

Given a Φ-submodule M of the Φ-module V of finite vectors, set $M' = \{x \in V: (x, M) = 0\}$, the orthocomplement of M in V. Clearly $M' = M^{\perp} \cap V$. Given any subspace M of E, formal considerations show that $(M \cap V)^{\perp} \supseteq M^{\perp}$. Conversely, given $x \in (M \cap V)^{\perp}$, then for any $m \in M$ either $m \in M \cap V$ or $(m, m)^{-1}m \in M \cap V$. In either case we conclude that $(x, m) = 0$ so $x \in M^{\perp}$. Thus $(M \cap V)^{\perp} = M^{\perp}$, and it follows that $M'' = M^{\perp\perp} \cap V$ for any Φ-submodule M. Call a Φ-submodule M of V *closed* when $M = M''$.

6.2. LEMMA. The map $g(M) = M \cap V$ *takes the lattice of all closed subspaces of E one-to-one onto the lattice of all closed Φ-submodules of V such that $M \subseteq N \Leftrightarrow g(M) \subseteq g(N)$. Also $M = M'' \Rightarrow V = M + M'$ for Φ-submodules M of V.*

All verifications follow routinely.

6.3. LEMMA. *For an orthomodular form, the natural map ϕ: $V \to H$ takes closed Φ-submodules of V onto closed subspaces of H, and for a closed Φ-submodule M of V we have $\phi(M)^{\perp} = \phi(M')$ and $\phi(M) + \phi(M)^{\perp} = H$.*

PROOF. For any Φ-submodule M of V, $\phi(M') \subseteq \phi(M)^{\perp}$, because if $x = \phi(a)$, $a \in M'$, then for any $m \in M$

$$f(x, \phi(m)) = f(\phi(a), \phi(m)) = \theta((a, m)) = \theta(0) = 0$$

so $x \in \phi(M)^{\perp}$. Suppose conversely that $x \in \phi(M)^{\perp}$ so that $f(x, \phi(m)) = 0$ for all $m \in M$.

Now $x = \phi(a)$ for some $a \in V$, so

$$0 = f(x, \phi(m)) = f(\phi(a), \phi(m)) = \theta((a, m))$$

for all $m \in M$. Thus $(a, m) \in \mathcal{P}$ for all $m \in M$. Assuming M closed, we have $M + M' = V$, so that $a = a_1 + a_2$, $a_1 \in M$, $a_2 \in M'$. Then $(a, m) = (a_1, m)$ for all $m \in M$. Putting $m = a_1$ we get $(a_1, a_1) \in \mathcal{P}$ which means $a_1 \in R$ so $\phi(a_1) = 0$. Thus $x = \phi(a_2)$, $a_2 \in M'$, so $x \in \phi(M')$. Hence $\phi(M)^{\perp} \subseteq \phi(M')$ which proves $\phi(M)^{\perp} = \phi(M')$ for every closed subspace M of V. Then

$$\phi(M)^{\perp\perp} = \phi(M')^{\perp} = \phi(M'') = \phi(M),$$

establishing the closure of $\phi(M)$ in H. Finally, given $x \in H$, $x = \phi(a)$ some $a \in V$, whence $a = a_1 + a_2$, $a_1 \in M$, $a_2 \in M'$. So

$$x = \phi(a) = \phi(a_1) + \phi(a_2), \phi(a_1) \in \phi(M), \phi(a_2) \in \phi(M') = \phi(M)^{\perp}.$$

Thus $H = \phi(M) + \phi(M)^{\perp}$, which completes the proof.

6.4. LEMMA. *Given a left vector space E over an ordered *-field $\mathcal{K}$, and given a positive definite orthomodular form on E that has an infinite orthonormal sequence, then the residue class *-field of $\mathcal{K}$ equals* **R**, **C**, *or* **H**.

Note that we can prove that the residue class *-field of our *-field equals **R**, **C**, or **H** under considerably weaker hypotheses than those used in the theorem itself. This suggests that the theorem probably holds under weaker hypotheses.

PROOF. With $e_1, e_2, \ldots$ the given orthonormal sequence, define $\rho_i = \varepsilon_i 2^{1-i}$, $\varepsilon_i = \pm 1$, then define

$$c_n = \sum_{i=1}^{2n-1} \rho_i e_i + (\sigma - \sigma_{2n-1})\rho_{2n}^{-1} e_{2n}, \qquad n = 1, 2, \ldots, \tag{1}$$

$$d_m = \sigma\rho_1^{-1} e_1 + (\sigma_{2m} - \sigma)\rho_{2m+1}^{-1} e_{2m+1} - \sum_{i=1}^{2m} \rho_i e_i, \qquad m = 1, 2, \ldots, \tag{2}$$

where $\sigma_n = \sum_{i=1}^{n} \rho_i^2$, $\sigma = \sum_{i=1}^{\infty} \rho_i^2 = 4/3$. Then verify by direct computation that $(c_n, d_m) = 0$, $n, m = 1, 2, \ldots,$ and that all (c_n, c_m), (d_n, d_m) lie in the rational field **Q**. Clearly $\mathbf{Q} \subseteq \Phi$ the valuation ring of finite elements of $\mathcal{K}$, so all vectors e_i, c_n, d_m lie in V.

Define a closed subspace M of V by $M = \{c_n \colon n = 1, 2, \ldots\}''$ ($'$ equals orthocomplementation in the space V). Owing to the orthomodularity of V, we have $M + M' = V$. In particular,

$$\tfrac{4}{3}\rho_1^{-1} e_1 = c + d, \qquad c \in M, d \in M'.$$

Note that all the vectors d_m lie in $M' = \{c_n\}'$.

In $\mathcal{K}_0 = \Phi/\mathcal{P}$, the residue class *-field of $\mathcal{K}$, the rationals **Q** form the unique subfield generated by 1; use the same symbol for a rational in $\mathcal{K}$ as for one in $\mathcal{K}_0$, and regard the natural homomorphism $\theta\colon \Phi \to \mathcal{K}_0$ as the identity on **Q**. The natural homomorphism $\phi\colon V \to H$ preserves orthogonality, preserves rational-valued inner products because $f(\phi(a), \phi(b)) = \theta((a, b))$ and $\theta =$ identity on **Q**, thus preserves all the relationships holding among the vectors e_i, c_n, d_m. Also ϕ maps the closed subspace M onto a closed subspace N of H; we have $N^{\perp} = \phi(M')$, and $N + N^{\perp} = H$ (Lemma 6.3). So we have the equation

$$\tfrac{4}{3}\rho_1^{-1}\phi(e_1) = \phi(c) + \phi(d), \qquad \phi(c) \in N, \phi(d) \in N^{\perp}.$$

Now embed the space $\{H, \mathcal{K}_0\}$ into the classical Hilbert space $\{H^{\sim}, \mathcal{K}_0^{\sim}\}$ in the usual way, $\mathcal{K}_0^{\sim} =$ **R**, **C**, or **H**. The extended form on $H^{\sim}$ induces an orthocomplementation on $H^{\sim}$ that we shall denote by $\#$. Given a subset S of H, we have the relation $S^{\perp} = S^{\#} \cap H$, where $S^{\perp}$ equals the H-orthocomplement of S, and $S^{\#}$ its $H^{\sim}$ orthocomplement.

The subspaces N, $N^{\perp}$ in H remain orthogonal subsets of $H^{\sim}$. Construct first the $\mathcal{K}_0^{\sim}$-subspace of $H^{\sim}$ generated by N, then construct its metric closure L, a closed subspace of $H^{\sim}$. At both stages of this construction, you preserve the existing orthogonality to $N^{\perp}$, so have $N^{\perp} \subseteq L^{\#}$, and of course $N \subseteq L$. We have $H^{\sim} = L + L^{\#}$; in particular

$$\sigma\rho^{-1}\phi(e_1) = \phi(c) + \phi(d), \qquad \phi(c) \in N \subseteq L, \phi(d) \in N^{\perp} \subseteq L^{\#}.$$

Let $f_i = \phi(e_i)$. The f_i form an orthonormal sequence in H and also in $H^{\sim}$. The vector $x = \sum_{i=1}^{\infty} \rho_i f_i$ belongs to $H^{\sim}$ because $\sum_1^{\infty} \rho_i^2 = \sigma < \infty$ (in fact $\sigma = 4/3$). Let $y = \sigma\rho_1^{-1}f_1 - x$. We have $f(x, y) = 0$ and $\sigma\rho_1^{-1}f_1 = x + y$.

I shall prove that $x = \phi(c) \in N$.

We have

$$\begin{aligned}\|x - \phi(c_n)\|^2 &= \left\|\sum_1^{\infty} \rho_i f_i - \phi(a_n) - \sigma\rho_{2n}^{-1}f_{2n}\right\|^2 \\ &= \left\|\sum_1^{\infty} \rho_i f_i - \sum_1^{2n-1} \rho_i f_i + \sigma_{2n-1}\rho_{2n}^{-1}f_{2n} - \sigma\rho_{2n}^{-1}f_{2n}\right\|^2 \\ &= \left\|\sum_{2n}^{\infty} \rho_i f_i + \sigma_{2n-1}\rho_{2n}^{-1}f_{2n} - \sigma\rho_{2n}^{-1}f_{2n}\right\|^2 \\ &= \left\|\sum_{2n+1}^{\infty} \rho_i f_i + \left(\rho_{2n} + \sigma_{2n-1}\rho_{2n}^{-1} - \sigma\rho_{2n}^{-1}\right)f_{2n}\right\|^2.\end{aligned}$$

Now $\rho_{2n} + \sigma_{2n-1}\rho_{2n}^{-1} - \sigma\rho_{2n}^{-1} = (\sigma_{2n} - \sigma)\rho_{2n}^{-1} = -\rho_{2n}^{-1}\sum_{2n+1}^{\infty} \rho_i^2$;

$$\|x - \phi(c_n)\|^2 = \sum_{2n+1}^{\infty} \rho_i^2 + \left(\rho_{2n}^{-1}\sum_{2n+1}^{\infty} \rho_i^2\right)^2 = \frac{1}{3}\frac{1}{4^{2n-1}} + \left(\frac{1}{3}\frac{1}{2^{2n-1}}\right)^2 \to 0.$$

Since $\phi(c_n) \in \phi(M) = N$, the vector x lies in the metric closure of N, hence $x \in L$.

A similar calculation shows that $y = \lim \phi(d_m)$. Since all d_m lie in M', all $\phi(d_m)$ lie in $\phi(M') = N^{\perp}$. The metrically closed subspace $L^{\#}$ contains $N^{\perp}$, hence contains y. Now we have

$$\sigma\rho_1^{-1}f_1 = \phi(c) + \phi(d), \qquad \phi(c) \in L, \phi(d) \in L^{\#},$$

and $\sigma\rho_1^{-1}f_1 = x + y$, $x \in L, y \in L^{\#}$. Thus, by uniqueness, $x = \phi(c)$. Since $\phi(c) \in N$, we have $x \in N \subseteq H$.

Hence, for any choice of $\varepsilon_i = \pm 1$, the vector

$$x = \sum_1^{\infty} (\varepsilon_i/2^{i-1})f_i$$

belongs to H. Since the inner product on H takes its values in $\mathcal{K}_0$, $\mathcal{K}_0$ contains every

$$f\left(\sum_1^{\infty}(1/2^{i-1})f_i, \sum_1^{\infty}(\varepsilon_i/2^{i-1})f_i\right) = \sum_1^{\infty} \varepsilon_i/4^{i-1} = \varepsilon_1 + \frac{\varepsilon_2}{4} + \frac{\varepsilon_3}{4^2} + \cdots$$

for every choice of $\varepsilon_i = \pm 1$. Any real number, in its base 4 representation, equals a rational combination of such numbers. Hence $\mathbf{R} \subseteq \mathcal{K}_0$, so we have $\mathcal{K}_0 = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$. That proves Lemma 6.4.

To show that $\mathcal{K}$ contains no infinitesimal elements, I adapt with very slight modifications the ingenious arguments of Wilbur [17, Lemmas 5.5 through 5.7].

6.5. LEMMA. *Let $(\cdot, \cdot)$ be a nonsingular orthomodular form over a (general) *-field, and let $\{e_i\}$ and $\{f_i\}$ be two mutually orthogonal orthonormal sequences. If $x \in \{e_i\}^{\perp\perp}$, then there exists $y \in \{f_i\}^{\perp\perp}$ with $(y, f_i) = (x, e_i)$, $i = 1, 2, \ldots,$ and $(y, y) = (x, x)$.*

PROOF. Argue as in [17, Lemma 5.5] using $\frac{3}{5}e_i + \frac{4}{5}f_i$, $\frac{4}{5}e_i - \frac{3}{5}f_i$ in place of the vectors $\alpha u_i + \alpha v_i$, $\alpha u_i - \alpha v_i$ there.

6.6. LEMMA. *Let $(\cdot, \cdot)$ be a positive definite orthomodular form over an ordered *-field, and let $\{e_i\}$, $\{f_i\}$, $\{g_i\}$ be mutually orthogonal orthonormal sequences. Given $x \in \{e_i\}^{\perp\perp}$ and given field elements β_i, γ_i that satisfy $\beta_i\beta_i^* + \gamma_i\gamma_i^* = 1$, $i = 1, 2, \ldots,$ then there exists $y \in \{e_i\}^{\perp\perp}$ with $(y, e_i) = (x, e_i)\beta_i$, $i = 1, 2, \ldots,$ and $(y, y) \leqslant (x, x)$.*

PROOF. Essentially Lemma 5.7 of [17].

Refer now to the proof of Lemma 6.4. Take $\varepsilon_i = +1$, $i = 1, 2, \ldots,$ so that $\rho_i = 2^{1-i}$, $i = 1, 2, \ldots,$ and take the vectors c_n, d_m as given there. Maintain the notation $M = \{c_n\}''$ (a closed subspace of the space V of finite vectors) and set

$$\tfrac{4}{3}e_1 = c + d, \qquad c \in M, d \in M'. \tag{3}$$

Define $\alpha_i = (c, e_i)$, the Fourier coefficients of c with respect to the orthonormal set e_i. From (3) we find that $\frac{4}{3}\alpha_1 = (c, c)$ so that α_1 is symmetric. From (1) we find $(e_1, c_n) = 1$, $n = 1, 2, \ldots,$ and from (3), $(4/3)(e_1, c_n) = (c, c_n)$ so $(c, c_n) = 4/3$, $n = 1, 2, \ldots$. Equation (1) allows us to compute (c, c_n) and we get

$$\frac{4}{3} = \sum_{i=1}^{2n-1} \rho_i\alpha_i + (\sigma - \sigma_{2n-1})\rho_{2n}^{-1}\alpha_{2n}, \qquad n = 1, 2, \ldots. \tag{4}$$

The condition $(c, d_n) = 0$, $n = 1, 2, \ldots,$ yields

$$0 = \sigma\rho_1^{-1}\alpha_1 - \sum_{i=1}^{2n} \rho_i\alpha_i + (\sigma_{2n} - \sigma)\rho_{2n+1}^{-1}\alpha_{2n+1}, \qquad n = 1, 2, \ldots. \tag{5}$$

Equations (4) and (5) show that the knowledge of α_1 determines all the α_n. Since α_1 is symmetric, we see that all the α_i are symmetric and mutually commute.

As we establish in the proof of Lemma 6.4,

$$\phi(c) = \sum_{i=1}^{\infty} \rho_i f_i \in H,$$

where ϕ is the natural map of the space V of finite vectors onto the quotient space H, and $f_i = \phi(e_i)$. Hence

$$\rho_i = f(\phi(c), f_i) = \theta((c, e_i)) = \theta(\alpha_i),$$

θ the natural map of the finite scalars Φ onto **R**, **C**, or **H**. In particular, $\theta(\alpha_1) = \rho_1 = \theta(\rho_1)$, so $\theta(\alpha_1 - \rho_1) = 0$. It follows that $\alpha_1 - \rho_1$ is either 0 or a symmetric infinitesimal ε. If $\alpha_1 = \rho_1$, then $\alpha_i = \rho_i = 2^{i-1}$ for all $i = 1, 2, \ldots$. In the second case, the recursions (4) and (5) show that $\alpha_i = \rho_i + \zeta_i\varepsilon$ where $\zeta_i \in \mathbf{Q}$, $i = 1, 2, \ldots$.

We shall assume that $\mathcal{K}$ has infinitesimals, and shall work to a contradiction. If $\alpha_1 = \rho_1$ select any symmetric infinitesimal ε; it will commute with all the rational

$\alpha_i = \rho_i$. Otherwise set $\varepsilon = \alpha_1 - \rho_1$; this symmetric infinitesimal will likewise commute with all the α_i. In either case the α_i are all nonzero medial elements (finite, noninfinitesimal).

Now apply Lemma 6.6. We may assume that we have, in advance, broken up our original orthonormal sequence into three mutually orthogonal orthonormal sequences so that Lemma 6.6 applies. Set $\beta_i = \varepsilon\alpha_i^{-1}$, $i = 1, 2, \ldots$. The β_i are all infinitesimal symmetric, and commute with all the α_i and ε. By the first hypothesis of Theorem 6.1, there exist γ_i in $\mathcal{K}$ so that $1 - \beta_i^2 = \gamma_i\gamma_i^*$, $i = 1, 2, \ldots$, so by Lemma 6.6 there exists $a \in \{e_i\}^{\perp\perp}$ with $(a, e_i) = (c, e_i)\beta_i = \varepsilon$, $i = 1, 2, \ldots$. Since our form represents 1 on every one-dimensional subspace, there exists α so that $(\alpha a, \alpha a) = 1$. Set $b = \alpha a$, and $\eta = (b, e_i) = \alpha\varepsilon$. Since $1 = (b, b) \geqslant \sum_{i=1}^n (b, e_i)(b, e_i)^* = n(\eta\eta^*)$, $n = 1, 2, \ldots$, η is infinitesimal. Select some positive integer m, and set $d = mb$. Then $d \in \{e_i\}^{\perp\perp}$, $(d, d) = m^2$, and $(d, e_i) = m\eta$, $i = 1, 2, \ldots$. Now set $\lambda_i = \alpha_i^{-1}m\eta$. The λ_i are infinitesimal, so according to Lemma 6.6 again (λ_i in place of β_i), there exists $e \in \{e_i\}^{\perp\perp}$, $(e, e_i) = \alpha_i\lambda_i = m\eta$, and $(e, e) \leqslant (c, c) \leqslant \frac{4}{3} + \varepsilon < 2$. Since both d and e belong to $\{e_i\}^{\perp\perp}$ and $(d, e_i) = (e, e_i)$, $i = 1, 2, \ldots$, we have $d = e$. Thus $m^2 = (d, d) < 2$ for every positive integer m, our desired contradiction.

Hence $\mathcal{K}_0 = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$, and $E = H$. Now by the theorem of Amemiya-Araki-Piron [**2**], E is Hilbert space.

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Department of Mathematics, University of Massachusetts, Amherst, Massachusetts 01003