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# THE PRODUCT OF TWO COUNTABLY COMPACT TOPOLOGICAL GROUPS

BY

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ABSTRACT. We use MA (= Martin's Axiom) to construct two countably compact topological groups whose product is not countably compact. To this end we first use MA to construct an infinite countably compact topological group which has no nontrivial convergent sequences.

**1. Introduction.** Novák [N] and Teresaka [T] have given an example of two countably compact spaces whose product is not countably compact, and in fact is not even pseudocompact. One may ask if these two spaces can be topological groups, but this is not the case since by a theorem of Comfort and Ross [**CR**, Theorem 1.4] the product of any number of pseudocompact topological groups is pseudocompact (see [**dV**] for an "elementary" proof). This leaves open the question, due to Comfort [C] (K. A. Ross has kindly informed me about the existence of this reference) and repeated by Saks [S, Question 7.3], of whether the product of countably compact topological groups is countably compact. We answer this question in the negative, but unfortunately need an additional axiom.

EXAMPLE 1. MA $\vDash$ *There exist two countably compact topological groups whose product is not countably compact.*

Inspired by the example of Novák and Teresaka mentioned above we construct Example 1 from our next example.

EXAMPLE 2. MA $\vDash$ *There is an infinite countably compact topological group which has no nontrivial convergent sequences.*

This example also shows how much countably compact topological groups differ from compact topological groups (at least with additional axioms): Every infinite compact topological group has a nontrivial convergent sequence. [Argument: By a theorem of Kuz'minov [**Kz**] every compact topological group is dyadic [I am indebted to K. A. Ross for preventing me from including a dubious reference here] and every infinite dyadic space has a nontrivial convergent sequence, cf. [**E1**, Theorem 15]]. In this connection we remind the reader that if $G$ is a topological group then $\beta G$ can be made a topological group in which $G$ is a subgroup if and only if $G$ is pseudocompact [**CR**, Theorem 4.1] (see [**dV**] for an "elementary" proof, and [**vD1**] for related results), hence in particular if $G$ is countably compact.

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That Example 2 exists under CH is a result of Hajnal and Juhász **[HJ2]**, see §9. Our construction is of independent interest, not because MA is strictly weaker than CH, but because it is a very natural way to construct Example 2. It also is interesting (but unpleasant) that our construction needs MA in an essential way, see §7. The question of whether or not the existence of our examples can be proved in ZFC remains open.

I am indebted to Victor Saks for carefully reading this paper.

**2. Conventions.** An ordinal is the set of smaller ordinals, and a cardinal is an initial ordinal. $\omega$ is $\omega_0$, $\mathfrak{c}$ is $2^\omega$. $\kappa$ denotes a cardinal, other Greek letters, except for $\omega$, denote ordinals.

As usual, for a set $S$ we define $[S]^{<\omega}$ and $[S]^\omega$ by

$$[S]^{<\omega} = \{I \subseteq S: |I| < \omega\}, \quad [S]^\omega = \{I \subseteq S: |I| = \omega\}.$$

We think of 2 as a cardinal but also as a discrete topological group ($C_2$ or $Z_2$ in algebraist's notation). This makes ${}^\alpha 2$, the set of functions $\alpha \to 2$, a (compact) topological group if $+$ is defined coordinatewise and ${}^\alpha 2$ carries the product topology. We do really think of ${}^\alpha 2$ as a set of functions, so a typical neighborhood of $x \in {}^\alpha 2$ has the form $\{y \in {}^\alpha 2: x \restriction L \subseteq y\}$, with $L \in [\alpha]^{<\omega}$, where $x \restriction L$ denotes the restriction of $x$ to $L$.

All our spaces are $T_1$ (so our topological groups are completely regular), hence a point $x$ is a cluster point of a set $K$ (= every neighborhood of $x$ contains infinitely many points of $K$) iff $x \in (K - \{x\})^-$.

All our groups are additively written, and we use 0 for the identity, the fact that the identity of ${}^\alpha 2$ really is the function $\alpha \times \{0\}$, where 0 is the identity of 2, will not bother us.

We put explanatory remarks between square brackets.

**3. Boolean groups.** A group $G$ will be called a *Boolean group* if every $x \in G$ is of order 2 (i.e. $x + x = 0$). For the convenience of the reader we recall some elementary facts about Boolean groups which will be used in the construction of our example, not always with explicit reference.

Let $G$ be a Boolean group. Then the solution of $a + x = b$ is $x = a + b$, and $G$ is abelian. The latter fact implies that if $A$ and $B$ are subgroups of $G$ then $A + B$ is the subgroup of $G$ generated by $A \cup B$.

The following simple lemma, and its trivial corollary, will be very important in the construction of Example 2.

3.1 LEMMA. *Let $G$ and $H$ be Boolean groups, let $A$ and $B$ be subgroups of $G$ and let $p: A \to H$ and $q: B \to H$ be homomorphisms. If $p \restriction A \cap B = q \restriction A \cap B$, then there is a homomorphism $r: A + B \to H$ which extends both $p$ and $q$.*

□ For $a \in A$ and $b \in B$ put $r(a + b) = p(a) + q(b)$. Check that $r$ is well-defined. □

3.2 COROLLARY. *Let $G$ and $H$ be Boolean groups, let $A$ be a subgroup of $G$ and let $q: A \to H$ be a homomorphism. For every $y \in G - A$ and $b \in H$ there is a homomorphism $p: \{0, y\} + A \to H$ which extends $q$ such that $p(y) = b$.*

**4. Construction of Example 2.** Our Example, $E$, will be a subgroup of ${}^{\mathfrak{c}}2$. [This way we get the structure of a topological group for free.] We construct $E$ by finding

better and better approximations to the elements of larger and larger subgroups. (*)

[This we make precise below.] For bookkeeping purposes we have to index our approximations.

*What we will do.* We will construct subgroups $E_\alpha$ of ${}^{\alpha}2$ for $\omega \leqslant \alpha \leqslant \mathfrak{c}$, our example $E$ will be $E_{\mathfrak{c}}$. Each $E_\alpha$ will be indexed as $E_\alpha = \{x_{\alpha,\xi}\colon \xi < \sigma_\alpha\}$, where the ordinals $\sigma_\alpha$ are defined by

$$\sigma_\omega = \omega; \qquad \sigma_{\alpha+1} = \sigma_\alpha + |\sigma_\alpha| \qquad (\omega \leqslant \alpha < \mathfrak{c});$$

$$\sigma_\lambda = \sup\nolimits_{\omega \leqslant \alpha < \lambda} \sigma_\alpha \qquad (\omega < \lambda \leqslant \mathfrak{c}, \text{ and } \lambda \text{ a limit}).$$

[Note that $\sigma_\alpha < \mathfrak{c}$ for $\alpha < \mathfrak{c}$ (since $\sigma_\alpha < \kappa$ for $\alpha < \kappa$ if $\kappa \leqslant \mathfrak{c}$ is regular), and that $\sigma_{\mathfrak{c}} = \mathfrak{c}$.] We will require the following conditions, which make (*) precise:

(1) if $\xi < \eta < \sigma_\alpha$ then $x_{\alpha,\xi} \neq x_{\alpha,\eta}$ $(\omega \leqslant \alpha \leqslant \sigma_\alpha)$;

(2) if $\xi < \sigma_\alpha$ then $x_{\alpha,\xi} \subset x_{\beta,\xi}$ $(\omega \leqslant \alpha < \beta \leqslant \mathfrak{c})$; and

(3) if $\xi, \eta, \zeta < \sigma_\alpha$ satisfy $x_{\alpha,\xi} + x_{\alpha,\eta} = x_{\alpha,\zeta}$, then $x_{\beta,\xi} + x_{\beta,\eta} = x_{\beta,\zeta}$ $(\omega \leqslant \alpha < \beta \leqslant \mathfrak{c})$.

Since $\sigma_{\mathfrak{c}} = \mathfrak{c}$ we can enumerate $[\mathfrak{c}]^\omega$ as $\langle I_\xi\colon \omega \leqslant \xi < \mathfrak{c}\rangle$ in such a way that $I_\xi \subseteq \sigma_\xi$ $(\omega \leqslant \xi < \mathfrak{c})$. [This is why we specify the $\sigma_\alpha$'s in advance.] We will find $\lambda_\xi < \sigma_{\xi+1}$ such that

(4) if $\omega \leqslant \xi < \alpha$ then $x_{\alpha,\lambda_\xi}$ is a cluster point of $\{x_{\alpha,\eta}\colon \eta \in I_\xi\}$ $(\omega < \alpha \leqslant \mathfrak{c})$.

We also will make sure that

(5) $|\{\eta \in I_\xi\colon x_{\xi+1,\eta}(\xi) = i\}| = \omega$ for both $i = 0$ and $i = 1$ $(\omega \leqslant \xi < \mathfrak{c})$.

Then $E$ is countably compact by the special case $\alpha = \mathfrak{c}$ of (4), but has no nontrivial convergent sequences by (5) and the special case $\beta = \mathfrak{c}$ of (2).

*How we do it.* Let $E_\omega$ be any countably infinite subgroup of ${}^{\omega}2$, and index $E_\omega$, making sure that (1) holds for $\alpha = \omega$.

Next assume $E_\alpha$ to be known for $\omega \leqslant \alpha < \delta$, where $\omega < \delta \leqslant \mathfrak{c}$.

*Case* 1: $\delta$ is a limit ordinal.

Since for each $\xi < \sigma_\delta$ there is an $\alpha$ with $\omega \leqslant \alpha < \delta$ such that $\xi < \sigma_\alpha$, condition (2) forces us to define

$$x_{\delta,\xi} = \bigcup \{x_{\alpha,\xi}\colon \omega \leqslant \alpha < \delta \text{ and } \sigma_\alpha > \xi\} \qquad (\xi < \sigma_\delta).$$

The only nontrivial thing to check is that (4) holds for $\alpha = \delta$. Pick $\xi$ with $\omega \leqslant \xi < \delta$, and let $L \in [\delta]^{<\omega}$ be arbitrary. We have to find $\eta \in I_\xi$ such that $x_{\delta,\xi} \neq x_{\delta,\lambda_\xi}$ and $x_{\delta,\lambda_\xi} \restriction L \subset x_{\delta,\eta}$. There is a $\gamma$ with $\omega \leqslant \gamma < \delta$ such that $L \subseteq \gamma$ and $\xi < \sigma_\gamma$. Then by the case $\alpha = \gamma$ of (4) there is $\eta \in I_\xi$ with $x_{\gamma,\eta} \neq x_{\delta,\lambda_\xi}$ and $x_{\gamma,\lambda_\xi} \restriction L \subseteq x_{\gamma,\eta}$. This $\eta$ is as required, by the special case $\alpha = \gamma$ and $\beta = \delta$ of (2).

*Case* 2: $\delta$ is a successor ordinal, $\delta = \gamma + 1$ say.

Since ${}^{\gamma}2$ is compact the special case $\alpha = \gamma$ of (1) implies that $\{x_{\gamma,\eta}\colon \eta \in I_\gamma\}$ has a cluster point $c$. Pick $c' \in {}^{\gamma}2 - E_\gamma$, with $c' = c$ if $c \notin E_\gamma$, and let $H = \{0, c'\} + E_\gamma$. Since $E_\gamma \cap (c' + E_\gamma) = \varnothing$, we can index $H - E_\gamma$ in a one-to-one fashion as $\{x_{\gamma,\xi}\colon \sigma_\alpha \leqslant \xi < \sigma_{\gamma+1}\}$. Let $\lambda_\gamma$ be determined by $x_{\gamma,\lambda_\gamma} = c$. Then

(6) if $\omega \leqslant \xi \leqslant \gamma$ then $x_{\gamma,\lambda_\xi}$ is a cluster point of $\{x_{\gamma,\eta}: \eta \in I_\xi\}$.

We plan to find a function $h: H \to 2$ such that we can define $E_{\gamma+1}$ by

$$E_{\gamma+1} = \{x\hat{\ }h(x): x \in H\}.$$

[Recall that if $x \in {}^{\gamma}2$ and $i \in 2$ then $x\hat{\ }i$ is the element $x \cup \{\langle\gamma, i\rangle\}$ of ${}^{\gamma+1}2$.] It will be clear how to index $E_{\gamma+1}$. It also is clear that $h$ must satisfy the following conditions.

(A) $h$ is a homomorphism;

(B) if $\omega \leqslant \xi \leqslant \gamma$ then $x_{\gamma,\lambda_\xi}$ is a cluster point of

$$\{x_{\gamma,\eta}: \eta \in I_\xi \text{ and } h(x_{\gamma,\eta}) = h(x_{\gamma,\lambda_\xi})\};$$

(C) $|\{\eta \in I_\gamma: h(x_{\gamma,\eta}) = i\}| = \omega$ for both $i = 0$ and $i = 1$.

It is convenient to replace (B) by an easier condition. For $\omega \leqslant \xi \leqslant \gamma$ define

$$\mathcal{K}_\xi = \{\{x_{\gamma,\eta}: \eta \in I_\xi, x_{\gamma,\eta} \neq x_{\gamma,\lambda_\xi}, \text{ and } x_{\gamma,\lambda_\xi} \restriction L \subseteq x_{\gamma,\eta}\}: L \in [\gamma]^{<\omega}\}.$$

Next define

$$\mathcal{K} = \bigcup \{\{x_{\gamma,\lambda_\xi} + K: K \in \mathcal{K}_\xi\}: \omega \leqslant \xi \leqslant \gamma\}.$$

Then (B) is equivalent to

(B′) $\forall K \in \mathcal{K}\ \exists x \in K\ (h(x) = 0)$.

We construct $h$ in the next section, using the fact that each member of $\mathcal{K}$ is infinite. [This follows from (6).] This completes the construction.

**5. Tearing apart, but not too much.** The following lemma is the heart of the construction of Example 2. We show in §7 that MA is essential.

5.1 Lemma. MA $\models$ *Let $H$ be a Boolean group, let $I \in [H]^\omega$, and let $\mathcal{K} \subseteq [H]^\omega$. If $|H| < \mathfrak{c}$ and $|\mathcal{K}| < \mathfrak{c}$ then there is a homomorphism $h: H \to 2$ such that*

(a) *[We tear apart] $|I \cap h^{\leftarrow}\{i\}| = \omega$ for both $i = 0$ and $i = 1$.*

(b) *[But not too much] $K \cap h^{\leftarrow}\{0\} \neq \varnothing$ for $K \in \mathcal{K}$.*

Let $\mathcal{J}$ be an infinite pairwise disjoint collection of infinite subsets of $I$, and let $\mathcal{L} = \mathcal{J} \cup \mathcal{K}$. Then it suffices to find a homomorphism $h: H \to 2$ such that

(c) $L \cap h^{\leftarrow}\{i\} \neq \varnothing$ for each $L \in \mathcal{L}$, for both $i = 0$ and $i = 1$.

We consider the obvious partial order $P$ defined by

$$P = \{p: \text{there is a finite subgroup } A \text{ of } H \text{ such that } p \text{ is a homomorphism } A \to 2\};$$

$p \leqslant q$ if $p \supseteq q$. It is clear that if $F \subseteq P$ is a filter [i.e. $\forall p, q \in F\ \exists r \in F\ (r \leqslant p, q)$] then $S = \bigcup_{p \in F} \operatorname{dom}(p)$ is a subgroup of $H$ and $\bigcup F$ is a homomorphism $S \to 2$. Our $h$ is going to be $\bigcup F$ for a suitable filter $F$.

We force that $\operatorname{dom}(h) = H$ by letting $F$ intersect every member of

$$\mathcal{D} = \{D_x: x \in H\},$$

where

$$D_x = \{p \in P: x \in \operatorname{dom}(p)\}.$$

We force that (1) holds by letting $F$ intersect every member of

$$\mathfrak{E} = \{E_{L,i}: L \in \mathfrak{L}, i < 2\},$$

where

$$E_{L,i} = \{p \in P: \text{there is } y \in L \cap \operatorname{dom}(p) \text{ with } p(y) = i\}.$$

We leave it to the reader to check that this works.

Since we assume MA, and since $|\mathfrak{D} \cup \mathfrak{E}| \leqslant |H| + 2 \cdot |\mathfrak{K}| < \mathfrak{c}$, there is a filter $F$ which intersects every member of $\mathfrak{D} \cup \mathfrak{E}$ provided $P$ satisfies the countable antichain condition and each member of $\mathfrak{D} \cup \mathfrak{E}$ is dense in $P$. We complete the proof by giving the (straightforward) verification of these facts.

FACT 1. $P$ satisfies the countable antichain condition.

Let $Q \subseteq P$ be uncountable. We have to find distinct $p, q \in Q$, and $r \in P$, such that $r \leqslant p, q$. For each finite $A \subseteq H$ there are only finitely many functions $A \to 2$, hence the collection $\{\operatorname{dom}(p): p \in Q\}$ is uncountable. It follows from the $\Delta$-system lemma that there are an infinite (even uncountable) $Q' \subseteq Q$ and a (necessarily) finite subgroup $\Delta$ of $H$ such that $\operatorname{dom}(p) \cap \operatorname{dom}(q) = \Delta$ for any two distinct $p, q \in Q'$. As above, there are only finitely many functions $\Delta \to 2$, hence there are distinct $p, q \in Q'$ such that $p \restriction \Delta = q \restriction \Delta$. By Lemma 3.1 there is a homomorphism $r$: $\operatorname{dom}(p) + \operatorname{dom}(q) \to 2$ which extends both $p$ and $q$. These $p$, $q$ and $r$ are as required. $\square$

FACT 2. Each member of $\mathfrak{D}$ is dense.

Let $q \in P$ and $y \in H$ be arbitrary. We have to find $p \in P$ such that $p \leqslant q$ and $y \in \operatorname{dom}(p)$. If $y \in \operatorname{dom}(q)$ let $p = q$, otherwise use Corollary 3.2 to find a homomorphism $p$: $\{0, y\} + \operatorname{dom}(q) \to 2$ which extends $q$. $\square$

FACT 3. Each member of $\mathfrak{E}$ is dense.

Let $q \in P$, $L \in \mathfrak{L}$ and $i < 2$ be arbitrary. We have to find $p \in P$ with $p \leqslant q$ such that $p(y) = i$ for some $y \in L \cap \operatorname{dom}(p)$. This is another application of Corollary 3.2, based on the fact that there is a $y \in L - \operatorname{dom}(q)$. $\square$

5.2 REMARK. Actually, information about $|H|$ is irrelevant, but we won't bother.

**6. Construction of Example 1 from Example 2.** We omit the proof of the following known lemma since it is a trivial modification of the proof of the Čech-Pospíšil theorem that a compact Hausdorff space has cardinality at least $2^\kappa$ if no point has a neighborhood base of cardinality less than $\kappa$ **[CP]** (See **[E**, Problem 3.12.11(a)**]** for a recent reference.)

6.1 LEMMA. *Let $X$ be a countably compact regular space. If $I$ is an infinite subset of $X$ such that no infinite subset of $I$ has precisely one cluster point, then $I$ has at least* $\mathfrak{c}$ *cluster points.* $\square$

6.2 *The construction.* Let $E$ be any infinite countably compact topological Boolean group without nontrivial convergent sequences. We will construct countably compact subgroups $E_0$ and $E_1$ of $E$ such that $|E_0 \cap E_1| = \omega$. Then $E_0 \times E_1$ has a countably infinite closed subgroup, namely the "diagonal",

$$\Delta = \{\langle x, y\rangle \in E_0 \times E_1: x = y\} = \{\langle x, x\rangle: x \in E_0 \cap E_1\},$$

for $E$ is Hausdorff. But no countably infinite topological group is (countably) compact. This follows from the Čech-Pospíšil theorem quoted above, as is well known. (In our particular case we also can argue as follows: $\Delta$ has no nontrivial convergent sequences because $E$ has none, but every countably infinite (countably) compact regular space has a nontrivial convergent sequence.)

We now proceed to the construction of $E_0$ and $E_1$.

Define ordinals $\sigma_\alpha$ for $\alpha \leqslant \mathfrak{c}$ by

$$\sigma_0 = \omega; \qquad \sigma_{\alpha+1} = \sigma_\alpha + |\sigma_\alpha| \qquad (\alpha < \mathfrak{c}); \qquad \sigma_\lambda = \sup_{\alpha<\lambda} \sigma_\alpha \qquad (\lambda \leqslant \mathfrak{c} \text{ a limit}).$$

As in §4 we have $\sigma_\alpha < \mathfrak{c}$ for $\alpha < \mathfrak{c}$, and $\sigma_\mathfrak{c} = \mathfrak{c}$. Enumerate $[\mathfrak{c}]^\omega$ as $\langle I_\alpha: \alpha < \mathfrak{c}\rangle$ in such a way that $I_\alpha \subseteq \sigma_\alpha$ $(\alpha < \mathfrak{c})$.

We will construct two (strictly) increasing transfinite sequences $\langle E_{i,\alpha}: \alpha \leqslant \mathfrak{c}\rangle$ of subgroups of $E$; our $E_0$ and $E_1$ will be $E_{0,\mathfrak{c}}$ and $E_{1,\mathfrak{c}}$. Each $E_{i,\alpha}$ will be indexed as $E_{i,\alpha} = \{y_{i,\xi}: \xi < \sigma_\alpha\}$, and we make sure that the following holds:

(1) if $\xi < \eta < \sigma_\alpha$ then $y_{i,\xi} \neq y_{i,\eta}$ $(i < 2,\ \alpha \leqslant \mathfrak{c})$,

(2) $E_{0,\alpha} \cap E_{1,\alpha} = E_{0,0}$ $(\alpha \leqslant \mathfrak{c})$ [This ensures $|E_0 \cap E_1| = \omega$.],

(3) $\{y_{i,\xi}: \xi \in I_\alpha\}$ has a cluster point in $E_{i,\alpha+1}$ $(\alpha < \mathfrak{c})$. [This ensures that $E_0$ and $E_1$ are countably compact.]

The construction and indexing of $E_{0,\lambda}$ and $E_{1,\lambda}$ for limit ordinals $\lambda \leqslant \mathfrak{c}$, including $\lambda = 0$, is too easy to talk about.

Now let $\alpha < \mathfrak{c}$, and assume $E_{i,\alpha}$ to be known. For $i < 2$ choose a cluster point $c_i$ of $\{y_{i,\xi}: \xi \in I_\alpha\}$ such that $c_0 \notin E_{0,\alpha} + E_{1,\alpha}$ and $c_1 \notin \{0, c_0\} + E_{0,\alpha} + E_{1,\alpha}$; this is possible since $\{y_{i,\xi}: \xi \in I_\alpha\}$ is infinite by (1), hence has at least $\mathfrak{c}$ cluster points by Lemma 6.1, and since $|\sigma_\alpha| < \mathfrak{c}$. Define

$$E_{i,\alpha+1} = \{0, c_i\} + E_{i,\alpha} \qquad (i < 2).$$

Since $|E_{i,\alpha}| = |\sigma_\alpha|$ and since $c_i \notin E_{i,\alpha}$, we have

$$|E_{i,\alpha+1} - E_{i,\alpha}| = |c_i + E_{i,\alpha}| = |\sigma_\alpha| \qquad (i < 2)$$

hence $E_{i,\alpha+1} - E_{i,\alpha}$ can be indexed in a one-to-one fashion as $\{y_{i,\alpha}: \sigma_\alpha \leqslant \xi < \sigma_{\alpha+1}\}$. This takes care of condition (1).

Next we observe that $(c_0 + E_{0,\alpha}) \cap E_{1,\alpha} = \varnothing$ since $c_0 \notin E_{0,\alpha} \cap E_{1,\alpha}$, and $E_{0,\alpha+1} \cap (c_1 + E_{1,\alpha}) = \varnothing$ since $c_1 \notin E_{0,\alpha+1} + E_{1,\alpha}$, hence

$$E_{0,\alpha+1} \cap E_{1,\alpha+1} = E_{0,\alpha+1} \cap E_{1,\alpha} = E_{0,\alpha} \cap E_{1,\alpha} = E_{0,0}.$$

This takes care of (2). Finally, it is clear that (3) holds.

This completes the construction.

6.3 Remark. $E_0 \times E_1$ is pseudocompact since as mentioned in the introduction, Comfort and Ross have shown that the product of any number of pseudocompact topological groups is pseudocompact. But $E_0 \times E_1$ has a countable closed subgroup, hence a closed nonpseudocompact subgroup. Comfort and Saks, **[CS**, Theorem 2.4], have shown in ZFC that a pseudocompact topological group can have a countable closed subgroup.

6.4 Remark. Victor Saks has asked if one can find groups $E_i$ $(i \leqslant n)$ with $\Pi_{i\leqslant n}E_i$ not countably compact but $\Pi_{i\leqslant n, i\neq j}E_i$ countably compact for each $j \leqslant n$, if $n \geqslant 2$, and if one can find groups $E_i$ $(i < \omega)$ such that $\Pi_i E_i$ is not countably compact but

$\Pi_{i<j}E_i$ is countably compact for each $j < \omega$ (personal communication). I believe that I can modify the construction so as to get groups $E_i$ ($i < \omega$) such that $\Pi_i E_i$ is not countably compact but $\Pi_{i \neq j} E_i$ is countably compact for each $j < \omega$, but the construction of these $E_i$'s would be at least twice as long, and I do not think this is worth the effort.

**7. Why we need MA.** We let KA (for Kunen's Axiom) be the statement

KA: Some free ultrafilter on $\omega$ is generated by $\omega_1$ sets.

Clearly CH $\to$ KA. The following is due to Kunen, cf. [**K**, Remark on p. 303].

7.1 THEOREM. *If ZFC is consistent, then so is ZFC + KA +* $2^\omega = 2^{\omega_1} = \omega_2$.

Now if in Lemma 5.1 we replace conditions (a) and (b) by (c), then it is very easy to see that we need MA, even if we only consider a countable $H$, and drop the condition that the function $h: H \to 2$ be a homomorphism. For then we get

if $\mathcal{K} \subseteq [\omega]^\omega$ has $|\mathcal{K}| < \mathfrak{c}$, then there is $T \subseteq \omega$ such that $K \cap T \neq \emptyset$ and $K - T \neq \emptyset$ for all $K \in \mathcal{K}$, $(*)$

but clearly KA + $\neg$CH $\to \neg(*)$. On the other hand, if we keep conditions (a) and (b) in Lemma 5.1, and only drop the condition that the function $h: H \to 2$ be a homomorphism, then we get a statement which is true in ZFC. Indeed, we have the following easy

7.2 PROPOSITION. *Let $H$ be any set, let $I \in [H]^\omega$ and let $\mathcal{K} \subseteq [H]^\omega$. If $|\mathcal{K}| < \mathfrak{c}$, then there is $T \subseteq H$ such that*

(a) $|I \cap T| = |I - T| = \omega$, *and*

(b) $|K \cap T| = \omega$ *for all* $K \in \mathcal{K}$.

□ There is $\mathcal{A} \subseteq [I]^\omega$ with $|\mathcal{A}| = \mathfrak{c}$ such that $|A \cap B| < \omega$ for distinct $A, B \in \mathcal{A}$. There is $A \in \mathcal{A}$ such that $T = H - A$ works. □

We now proceed to the proof that (the statement in) Lemma 5.1 is false under KA + $2^\omega = 2^{\omega_1} = \omega_2$; in view of Theorem 7.1 this shows that MA is essential in Lemma 5.1.

Recall that a space is called *initially $\kappa$-compact* if every open cover of cardinality at most $\kappa$ has a finite subcover.

7.3 THEOREM. KA $\vDash$ *The product of any number of initially $\omega_1$-compact spaces is countably compact.*

Recall that if $\mathcal{D}$ is a free ultrafilter on $\omega$ then a space $X$ is called *$\mathcal{D}$-compact* if every sequence $\langle x_n \rangle_n$ in $X$ has a $\mathcal{D}$-limit, i.e. there is a point $y \in X$ such that $\{n \in \omega: x_n \in U\} \in \mathcal{D}$, for every neighborhood $U$ of $y$, [**B**, Definition 3.2]. Also recall that the product of any number of $\mathcal{D}$-compact spaces again is $\mathcal{D}$-compact [**B**, Theorem 4.2] (see [**GS**, Theorem 1.4] for a non-nonstandard proof), hence is countably compact. Hence it suffices to prove

Let $\mathcal{D}$ be a free ultrafilter on $\omega$ which is generated by a family $\mathcal{G} \subseteq \mathcal{D}$ with $|\mathcal{G}| < \kappa$. Then every initially $\kappa$-compact space is $\mathcal{D}$-compact. (†)

So let $X$ be an initially $\kappa$-compact space, and let $\langle x_n \rangle_n$ be a sequence in $X$. Then $\mathcal{F} = \{\{x_n: n \in G\}^-: G \in \mathcal{G}\}$ is a collection of closed sets with the finite intersection property and $|\mathcal{F}| \leqslant \kappa$. Hence $\bigcap \mathcal{F} \neq \varnothing$. It is easy to see, using the fact that $\mathcal{G}$ generates $\mathcal{D}$, that every point of $\bigcap \mathcal{F}$ is a $\mathcal{D}$-limit of $\langle x_n \rangle_n$. $\square$

Let S be the statement in Lemma 5.1.

7.4 EXAMPLE. S $+ 2^\omega = 2^{\omega_1} \models$ *There are two initially $\omega_1$-compact topological groups whose product is not countably compact.*

$\square$ Recall that if $X$ is a space and $A \subseteq X$ then $x$ is called a *complete accumulation* point of $A$ if $|A \cap U| = |A|$ for every neighborhood $U$ of $x$. Also recall that $X$ is initially $\omega_1$-compact iff every $A \subseteq X$ with $\omega \leqslant |A| \leqslant \omega_1$ has a complete accumulation point, cf. [**AU**, p. 17].

*Step* 1. We construct an initially $\omega_1$-compact subgroup $E$ of ${}^{\mathfrak{c}}2$ which has no convergent sequences.

We indicate the modifications in the construction of Example 2. Since $2^\omega = 2^{\omega_1}$ (hence $cf(\mathfrak{c}) > \omega_1$), we can enumerate $[\mathfrak{c}]^\omega \cup [\mathfrak{c}]^{\omega_1}$ as $\langle I_\xi: \omega \leqslant \xi < \mathfrak{c}\rangle$ in such a way that $I_\xi \subseteq \sigma_\xi$ $(\omega \leqslant \xi < \mathfrak{c})$. Everywhere we replace "cluster point" by "complete accumulation point". [This requires an obvious modification in the verification in Case 1.] Finally, at the end of the construction, for every uncountable $K \in \mathcal{K}$ we choose an uncountable pairwise disjoint collection $c(K) \subseteq [K]^\omega$, and replace $\mathcal{K}$ by

$$\{K \in \mathcal{K}: |K| = \omega\} \cup \bigcup \{c(K): K \in \mathcal{K}, \text{and } |K| = \omega_1\}.$$

It is easy to see that (B) (with "condensation point" instead of "cluster point") follows from (B′) (with the new $\mathcal{K}$).

Then every $A \subseteq E$ with $\omega \leqslant |A| \leqslant \omega_1$ has a complete accumulation point, hence $E$ is initially $\omega_1$-compact.

*Step* 2. We construct two initially $\omega_1$-compact subgroups of $E$ whose product is not countably compact.

This requires some obvious modifications in §6. $\square$

This completes the proof that MA is essential in Lemma 5.1. I do not know a simple direct proof. Since $2^\omega = 2^{\omega_1}$ under MA $+ \neg$CH we have the following:

7.5 COROLLARY. MA $+ \neg$CH $\models$ *There are two initially $\omega_1$-compact topological groups whose product is not countably compact.*

7.6 REMARK. The fact that Theorem 7.3 is true under CH for regular spaces is an immediate consequence of the following result of Saks and Stephenson [**SS**, Theorem 2.1]: Any regular initially $2^\kappa$-compact space is $\kappa$-bounded ($\equiv$ every set of cardinality at most $\kappa$ has compact closure). The idea of the proof is the same, and also occurs in the proof of the following result of Bernstein [**B**, Theorem 3.5]: A regular space is $\omega$-bounded if (and, trivially, only if) it is $\mathcal{D}$-compact for every free ultrafilter $\mathcal{D}$ on $\omega$. (Bernstein assumes complete regularity, but as pointed out by Juhász [**J**] (semi-) regularity suffices.)

SUBREMARK. Saks has asked if this result is true for Hausdorff spaces [**S**, 7.4]; in an "added in proof" he uses Kunen's $wP$-points to give a counterexample [**S**, p. 92]. Here we give a much more elementary counterexample: Let $K$ be a nowhere dense noncompact $\omega$-bounded subspace of ${}^{\mathfrak{c}}2$, e.g. a homeomorph of the countable cardinals. Let $S$ be the space with underlying set ${}^{\mathfrak{c}}2$ which has {open sets of ${}^{\mathfrak{c}}2$} $\cup$ $\{{}^{\mathfrak{c}}2 - K\}$ as a subbase. Then $S$ is separable, since ${}^{\mathfrak{c}}2$ is, but not compact, hence is not $\omega$-bounded. But one easily checks that every countable set in $S$ has the same closure in $S$ as it has in ${}^{\mathfrak{c}}2$, hence $S$ is $\mathfrak{D}$-compact for every free ultrafilter $\mathfrak{D}$ on $\omega$ since ${}^{\mathfrak{c}}2$ is.

7.7 REMARK. Corollary 7.5 shows that it is consistent with ZFC that initial $\omega_1$-compactness is not finitely productive. See **[vD2]** for further results of this sort.

7.8 REMARK. It is not true that one can show that MA is essential in Example 2 by proving that under KA + $\neg$CH every countably compact topological group is initially $\omega_1$-compact (and then use Theorem 7.3): the subgroup $\{x \in {}^{\omega_1}2\colon |\{\alpha \in \omega_1\colon x(\alpha) = 1\}| \leqslant \omega\}$ of ${}^{\omega_1}2$ is countably compact but not initially $\omega_1$-compact.

**8. Variations of Example 2.** It is obvious that $E$, Example 2, can be made nondense in ${}^{\mathfrak{c}}2$; just make sure that $x(0) = 0$ for $x \in E$. The following two variations are more interesting.

8.1 EXAMPLE. MA$\vDash$ ${}^{\mathfrak{c}}2$ *has a separable dense countably compact subgroup which has no convergent sequences.*

□ The shortest construction is as follows. Let $G$ be a separable closed subgroup of Example 2, as constructed in §4. Then $\overline{G}$ is a compact Boolean group, and $\overline{G}$ has weight $\mathfrak{c}$ since $G$ has. Therefore there is a topological isomorphism $\overline{G} \to {}^{\mathfrak{c}}2$ **[HR**, 9.15].

I believe it is more interesting to modify the construction of $E$ as follows: we assume that in addition to the other conditions the following holds:

(a) $D_\alpha = \{x_{\alpha,k}\colon k < \omega\}$ is dense in ${}^{\alpha}2$.

This requires an obvious modification in Case 1. In Case 2 we make sure that $D_{\gamma+1}$ is dense in ${}^{\gamma+1}2$ as follows: We define

$$\mathfrak{M} = \{\{y \in D_\gamma - \{x\}\colon x \restriction L \subseteq y\}\colon x \in D_\gamma \text{ and } L \in [\gamma]^{<\omega}\}$$

and then require

(D) $\forall M \in \mathfrak{M}\ \forall i < 2[|\{y \in M\colon h(y) = i\}| = \omega]$.

Since $D_\gamma \times 2$ is dense in ${}^{\gamma+1}2$, it follows that $D_{\gamma+1}$ is dense in ${}^{\gamma+1}2$. In order to get (D) we replace $\mathfrak{L}$ by $\mathfrak{L} \cup M$ in (1) of §5. □

8.2 EXAMPLE. MA$\vDash$ ${}^{\mathfrak{c}}2$ *has a nonseparable dense countably compact subgroup which has no nontrivial convergent sequences.*

□ We modify the construction in §4 as follows. We list $[\mathfrak{c}]^\omega$ as $\langle I_\xi\colon \omega \leqslant \xi < \mathfrak{c}$ and $\xi$ even$\rangle$, again with $I_\xi \subseteq \sigma_\xi$ ($\omega \leqslant \xi < \mathfrak{c}$, $\xi$ even), and let $D_\alpha$ be a countable dense subset of ${}^{\alpha}2$. We make $E$ dense in ${}^{\mathfrak{c}}2$ by ensuring that $D_\alpha \subseteq E_\alpha$ ($\omega \leqslant \alpha < \mathfrak{c}$, $\alpha$ even, $\alpha$ not a limit).

If $\lambda$ is a limit we construct $E_\lambda$ as before. If $\gamma < \mathfrak{c}$ is even then we first construct $E_{\gamma+1}$ as before and then let $E_{\gamma+2}$ be the subgroup of ${}^{\gamma+2}2$ generated by $E_\gamma \times \{0\} \cup D_\gamma$. This makes $E$ nonseparable. □

As we saw in §7, we used MA in an essential way in the construction of the above two examples; this does not give any information about the question of whether or not MA is essential for their existence. We now consider the question of whether MA is still essential if we only look for a subspace, not for a subgroup.

8.3 EXAMPLE. *${}^{\mathfrak{c}}2$ has a nonseparable dense countably compact subspace, which has no nontrivial convergent sequences.*

□ We proceed as in Example 8.2, but omit every algebraic statement. Then we need the version of Lemma 5.1 in which $H$ is just a set and $h$ is just a function. By Proposition 7.2, this version is true in ZFC. □

8.4 REMARK. Note that one cannot modify Example 7.4 in this way, and get two initially $\omega_1$-compact spaces, not necessarily topological groups, whose product is not countably compact, assuming only that $2^\omega = 2^{\omega_1}$. This follows from Theorems 7.1 and 7.3. Note that the reason that the construction breaks down is that $|T| = \omega$, i.e. that one of $h^{\leftarrow}\{0\}$ and $h^{\leftarrow}\{1\}$ is countable in the version of Lemma 5.1 that we used above. I do not know if a similar modification of Example 8.1 can be constructed in ZFC, but I did not seriously try. Note that Example 8.1 implies:

8.5 EXAMPLE. MA$\vdash$ *${}^{\mathfrak{c}}2$ has a countable dense subgroup $G$ such that no point of ${}^{\mathfrak{c}}2$ is the limit of a nontrivial sequence in $G$.*

However, one can construct such a $G$ in ZFC. (This result has also been obtained, independently, by Victor Saks.) The construction uses different ideas and does not seem to give Example 8.1. Earlier, Priestley **[P]** has constructed in ZFC a nonalgebraic version of Example 8.5 (this reference was contributed by K. A. Ross):

8.6 EXAMPLE. *${}^{\mathfrak{c}}2$ has a countable dense subset $S$ such that no point of ${}^{\mathfrak{c}}2$ is the limit of a nontrivial sequence in $S$.*

Such an example has also been constructed, for a totally different purpose, by McKenzie and Monk, as one can see with some effort from [**McKM**, Theorem 3.1].

**9. The Hajnal-Juhász example.** Recall that a subset $S$ of ${}^{\mathfrak{c}}2$ is called *hereditarily finally dense* (or HFD for short) if for every countable $T \subseteq S$ there is an $\alpha < \mathfrak{c}$ such that $\{t \upharpoonright (\mathfrak{c} - \alpha)\colon t \in T\}$ is dense in ${}^{\mathfrak{c}-\alpha}2$ [**HJ1**, p. 153] and that an HFD subspace of ${}^{\mathfrak{c}}2$ is hereditarily separable [**HJ1**, Theorem 2] and hereditarily normal, [**HJ1**, Theorem 4].

Hajnal and Juhász show that under CH there is a countably compact HFD subgroup in ${}^{\mathfrak{c}}2$ [**HJ2**]. Since clearly no HFD subset of ${}^{\mathfrak{c}}2$ can have a nontrivial convergent sequence, this shows that under CH, Example 2 can be made hereditarily separable and hereditarily normal.

ADDENDUM. I now can construct from MA one single countably compact group whose square is not countably compact. I intend to publish this elsewhere.

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