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COMPOSITION FACTORS OF THE PRINCIPAL SERIES REPRESENTATIONS OF THE GROUP $Sp(n, 1)$

BY

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ABSTRACT. Using Vogan's algorithm the composition factors of any principal series representation of the group $Sp(n, 1)$ are determined.

0. Introduction. D. Vogan has discovered in **[14]** a certain algorithm for expressing the character of any (generalized) principal series representation of a semisimple Lie group G as the linear combination of irreducible characters. It is conjectured that this algorithm gives the final result for any G and any such representation, or, in other words, that it enables one to find all composition factors (together with the multiplicity) of any generalized principal series representation.

In this paper we use this algorithm to determine the composition factors of any principal series of the group $Sp(n, 1)$. In the same way (but much simpler) one can also proceed for the groups $Spin(n, 1)$ and $SU(n, 1)$, but in these cases the results were already obtained by other methods (see e.g. **[5]**, **[8]**, **[9]**, **[13]**). In the subsequent paper we will treat the remaining real rank one group–a real form of F_4. These results may be used to prove the existence of some complementary series as in **[17]**.

1. Notation and preliminaries. Throughout the paper G will denote the group $Sp(n, 1)$, $n \geqslant 2$, defined as the group of matrices in $Sp(n + 1, \mathbf{C})$ which leave invariant the hermitian form ${}^tZK_{n,1}\bar{Z}$ where $Z = (z_1, \ldots, z_{2n+2}) \in \mathbf{C}^{2n+1}$ and

$$K_{n,1} = \begin{bmatrix} -I_n & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -I_n & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad I_n \text{ the unit matrix of order } n.$$

Let $\mathfrak{g}$ be the Lie algebra of G. It will be identified with the Lie algebra $Sp(n, 1)$ of matrices:

$$Sp(n, 1) = \left\{ \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ {}^t\bar{Z}_{12} & Z_{22} & {}^tZ_{14} & Z_{24} \\ -\bar{Z}_{13} & \bar{Z}_{14} & \bar{Z}_{11} & -\bar{Z}_{12} \\ {}^t\bar{Z}_{14} & -\bar{Z}_{24} & -{}^tZ_{12} & \bar{Z}_{22} \end{bmatrix} \middle| \begin{array}{l} Z_{ij} \text{ complex matrix; } Z_{11} \text{ and } Z_{13} \text{ of} \\ \text{order } n,\ Z_{12} \text{ and } Z_{14}\ n \times 1 \text{ matrices,} \\ Z_{11} \text{ and } Z_{22} \text{ are skew hermitian, } Z_{13} \\ \text{and } Z_{24} \text{ are symmetric.} \end{array} \right\}.$$

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Let θ: $\mathfrak{g} \to \mathfrak{g}$ be the Cartan involution defined by $\theta(X) = K_{n,1}XK_{n,1}$ and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition. Let K be the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$ and let $\mathfrak{t}$ be the set of all diagonal matrices in $\mathfrak{g}$. $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ corresponding to a compact Cartan subgroup of G.

Denote by Δ the root system of the pair $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$ (the subscript $\mathbf{C}$ denotes the complexification). Then $\Delta = \{\pm\varepsilon_i \pm \varepsilon_j,\ 1 < i < j \leqslant n+1;\ \pm 2\varepsilon_i,\ 1 < i < n+1\}$, where

$$\varepsilon_i(\operatorname{diag}(a_1, \ldots, a_{n+1}, -a_1, \ldots, -a_{n+1})) = a_i.$$

Let Δ_k and Δ_n be the sets of compact and noncompact roots, respectively. Then $\Delta_k = \{\pm\varepsilon_i \pm \varepsilon_j,\ 1 \leqslant i \leqslant j \leqslant n;\ \pm 2\varepsilon_i,\ 1 \leqslant i \leqslant n+1\}$. Let W and W_k be the Weyl groups of Δ and Δ_k, respectively.

Let bar denote the conjugation of $\mathfrak{g}_{\mathbf{C}}$ with respect to $\mathfrak{g}$ and B the killing form of $\mathfrak{g}_{\mathbf{C}}$. Denote by (,) the dual of the Killing form restricted to it.

We shall need to make computations with root vectors and we fix a normalization of them. Namely [**4**, pp. 155–156] for each $\alpha \in \Delta$ we can select a root vector X_α in such a way that $B(X_\alpha, X_{-\alpha}) = 2/(\alpha, \alpha)$ and $\theta\overline{X_\alpha} = -X_\alpha$. Then it follows that H_α defined by $H_\alpha = [X_\alpha, X_{-\alpha}]$ satisfies $\alpha(H_\alpha) = 2$ and that

$$X_\alpha - X_{-\alpha},\ i(X_\alpha + X_{-\alpha}) \text{ are in } \mathfrak{g} \text{ if } \alpha \text{ is compact and}$$
$$X_\alpha + X_{-\alpha},\ i(X_\alpha - X_{-\alpha}) \text{ are in } \mathfrak{g} \text{ if } \alpha \text{ is noncompact.}$$

Let $\beta = \varepsilon_1 - \varepsilon_{n+1}$ and $\mathfrak{a} = \mathbf{R}(X_\beta + X_{-\beta})$.

$\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{p}$. Let M be the centralizer of $\mathfrak{a}$ in K and $\mathfrak{m}$ its Lie algebra. Let $\mathfrak{h}^-$ be the subalgebra of $\mathfrak{t}$ spanned over $\mathbf{R}$ by iH_α, $\alpha \in \Delta$ and $(\alpha, \beta) = 0$. Let $\mathfrak{h} = \mathfrak{h}^- \oplus \mathfrak{a}$. Then $\mathfrak{h}^-$ and $\mathfrak{h}$ are Cartan subalgebras of $\mathfrak{m}$ and $\mathfrak{g}$, respectively. Let (,) denote also the dual of the Killing form restricted to $i\mathfrak{h}^- \oplus \mathfrak{a}$. Denote by Φ (respectively Φ_m) the root system of $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ (respectively $(\mathfrak{m}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}}^-)$) and by $\tilde{W}$ the Weyl group of Φ. If $\gamma \in \Delta_n$ set $u_\gamma = \exp(\pi/4)(X_\gamma - X_{-\gamma})$. Then $\operatorname{Ad} u_\gamma$ is the Cayley transform corresponding to γ. In particular $\operatorname{Ad} u_\beta$ carries $\mathfrak{t}_{\mathbf{C}}$ onto $\mathfrak{h}_{\mathbf{C}}$ and $\Phi = \Delta \circ \operatorname{Ad}(u_\lambda)^{-1}$.

The roots of Φ_m may be identified with the roots $\gamma \in \Phi$ of the form $\gamma = \alpha \circ \operatorname{Ad} u_\beta^{-1}$, where $\alpha \in \Delta$ and $(\alpha, \beta) = 0$. Set

$$e_1 = \varepsilon_1 \circ \operatorname{Ad} u_\beta^{-1}, \quad e_2 = -\varepsilon_{n+1} \circ \operatorname{Ad} u_\beta^{-1},$$
$$e_i = \varepsilon_{i-1} \circ \operatorname{Ad} u_\beta^{-1}, \qquad 3 \leqslant i \leqslant n+1.$$

Then $\Phi = \{\pm e_i \pm e_j;\ 1 \leqslant i < j \leqslant n+1\} \cup \{\pm 2e_i;\ 1 \leqslant i \leqslant n+1\}$. The compact roots in Φ are

$$\Phi_m = \{\pm(e_1 - e_2)\} \cup \{\pm e_i \pm e_j;\ 3 \leqslant i < j \leqslant n+1\} \cup \{\pm 2e_i;\ 3 \leqslant i \leqslant n+1\}.$$

The real roots in Φ are $\tilde{\beta} = \beta \circ \operatorname{Ad} u_\beta^{-1} = e_1 + e_2$ and $-\tilde{\beta}$. The remaining roots in Φ are complex.

We now define systems of positive roots Δ^+, $\Delta_{\mathfrak{k}}^+$, Φ^+ and $\Phi_{\mathfrak{m}}^+$ in Δ, $\Delta_{\mathfrak{k}}$, Φ and $\Phi_{\mathfrak{m}}$, respectively, which will be fixed throughout the paper. They are defined so that the corresponding sets of simple roots are the following:

$$\Delta^+: \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_n - \varepsilon_{n+1}, 2\varepsilon_{n+1}\},$$
$$\Delta_{\mathfrak{k}}^+: \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n, 2\varepsilon_{n+1}\},$$
$$\Phi^+: \{e_1 - e_2, e_2 - e_3, \ldots, e_n - e_{n+1}, 2e_{n+1}\},$$
$$\Phi_{\mathfrak{m}}^+: \{e_1 - e_2, e_3 - e_4, e_4 - e_5, \ldots, e_n - e_{n+1}, 2e_{n+1}\}.$$

Let Λ be the system of restricted roots (i.e. the root system of the pair $(\mathfrak{g}, \mathfrak{a})$ and $\Lambda^+ = \{\alpha|_{\mathfrak{a}};\ \alpha \in \Phi^+,\ \alpha|_{\mathfrak{a}} \neq 0\}$. Let $\mathfrak{n}$ be the sum of the positive restricted root space. Let A (resp. N) be the analytic subgroup of G with Lie algebra $\mathfrak{a}$ (resp. $\mathfrak{n}$). Denote by ρ half the sum of the positive restricted roots, counted with multiplicities. We also set $\delta_{\mathfrak{m}} = \frac{1}{2}\sum_{\alpha \in \Phi_{\mathfrak{m}}^+} \alpha$. $W(\mathfrak{a})$ will denote the Weyl group of the root system ("the little Weyl group").

Let U be the universal enveloping algebra of $\mathfrak{g}_{\mathbf{C}}$ and $\mathfrak{z}$ its center. Let $\mathfrak{b}$ be a Cartan subalgebra of $\mathfrak{g}_{\mathbf{C}}$, Q the system of roots of the pair $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{b})$, $W_{\mathfrak{b}}$ the Weyl group of Q, Q^+ a system of positive roots in Q, $\delta_{Q^+} = \frac{1}{2}\sum_{\gamma \in Q^+} \gamma$. For $\lambda \in \mathfrak{b}^*$ let χ_λ denote the infinitesimal character of the Verma module for $\mathfrak{g}_{\mathbf{C}}$ (with respect to Q^+) with the highest weight $\lambda - \delta_{Q^+}$. As is well known, we have that $\chi_\lambda = \chi_\mu$ if and only if $\lambda = \sigma\mu$ for some $\sigma \in W_{\mathfrak{b}}$; furthermore, any homomorphism $\mathfrak{z} \to \mathbf{C}$ is of the form χ_λ for some $\lambda \in \mathfrak{b}^*$.

Let now $\hat{M}$ be the set of all equivalence classes of irreducible finite dimensional representations of M. It is well known **[15]** that $\hat{M}$ is in bijective correspondence with

$$D_M = \left\{ b_0(e_1 - e_2) + \sum_{i=3}^{n+1} b_i e_i;\ b_3 \geqslant \cdots \geqslant b_{n+1} \geqslant 0, \right.$$
$$\left. 2b_0 \in \mathbf{Z},\ b_i \in \mathbf{Z},\ 3 \leqslant i \leqslant n+1 \right\}.$$

This correspondence is obtained by attaching to each element ξ of $\hat{M}$ its highest weight Λ_ξ with respect to $\Phi_{\mathfrak{m}}^+$.

For each $\xi \in \hat{M}$ we fix an element (ξ, H^ξ) in the class ξ; it will be supposed that H^ξ carries an inner product such that ξ is unitary.

For $\xi \in \hat{M}$ and $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ we define the principal series representation $(\pi_{\xi,\nu}, H^{\xi,\nu})$ as follows:

(1) $H^{\xi,\nu}$ is the space of all (classes of) measurable functions f: $G \to H^\xi$ such that

$$f(gman) = e^{-(\nu+\rho)(\lg a)}\xi(m^{-1})f(g) \tag{i}$$

for every $g \in G$, $m \in M$, $a \in A$, $n \in N$ (here lg: $A \to \mathfrak{a}$ denotes the inverse of exp: $\mathfrak{a} \to A$);

$$\int_K \|f(k)\|^2\, dk < \infty. \tag{ii}$$

$H^{\xi,\nu}$ is a Hilbert space if we introduce the inner product

$$(f, g) = \int_K (f(k), g(k))\, dk.$$

(2) $(\pi_{\xi,\nu}(g)f)(x) = f(g^{-1}x)$, $g, x \in G, f \in H^{\xi,\nu}$. $\pi_{\xi,\nu}$ is an admissible representation of G of finite length called principal series representation. According to **[6]**, for example, the infinitesimal character of $\pi_{\xi,\nu}$ is $\chi_{\Lambda(\xi,\nu)}$ where $\Lambda(\xi, \nu)\mathfrak{h}_{\mathbf{C}}^*$ is defined by

$$\Lambda(\xi, \nu)|_{\mathfrak{h}_{\mathbf{C}}^-} = \Lambda_\xi + \delta_{\mathfrak{m}}, \quad \Lambda(\xi, \nu)|_{\mathfrak{a}} = \nu.$$

Set $D = \{\Lambda(\xi, \nu); \xi \in \hat{M}, \nu \in \mathfrak{a}_{\mathbf{C}}^*\}$. Then

$$D = \left\{ \sum_{i=1}^{n+1} a_i e_i;\ a_1 - a_2 \in \mathbf{Z}, a_1 - a_2 > 0, a_i \in \mathbf{Z}, \right.$$

$$\left. 3 \leqslant i \leqslant n + 1, a_j > a_{j+1}, 3 \leqslant j \leqslant n \right\}.$$

Obviously, the set D parametrizes the principal series representations. If $\gamma \in D$, $\gamma = \Lambda(\xi, \nu)$, we shall write $\pi(\gamma)$ instead of $\pi_{\xi,\nu}$. Let D^+ denote the set of all $\gamma \in D$ such that if $\gamma = \sum_{i=1}^{n+1} a_i e_i$ then either $\mathrm{Re}(a_1 + a_2) > 0$ or $\mathrm{Re}(a_1 + a_2) = 0$ and $\mathrm{Im}(a_1 + a_2) \geqslant 0$. D^+ is the intersection of D with a fundamental domain of $W(\mathfrak{a})$; therefore, D^+ parametrizes the principal series representations up to the action of $W(\mathfrak{a})$. Note that the character of a principal series representation is invariant under the action of $W(\mathfrak{a})$, i.e. this action only changes the order of the composition factors.

According to Theorem II.3.1 in **[11]** if $\gamma \in D$ is such that $\mathrm{Re}(a_1 + a_2) > 0$, then $\pi(\gamma)$ contains the unique irreducible quotient. It is usually called the Langlands quotient and will be denoted by $\bar{\pi}(\gamma)$.

Let L denote the set of integral forms on $t_{\mathbf{C}}$, i.e. those that lift to characters of T (the Cartan subgroup of G with Lie algebra $\mathfrak{t}$). Then

$$L = \left\{ \lambda \in t_{\mathbf{C}}^*; \lambda = \sum_{i=1}^{n+1} a_i \varepsilon_i, a_i \in \mathbf{Z}, 1 \leqslant i \leqslant n + 1 \right\}.$$

We say that $\lambda \in \mathfrak{t}_{\mathbf{C}}^*$ (resp. $\gamma \in \mathfrak{h}_{\mathbf{C}}^*$) is nonsingular if $(\lambda, \alpha) \neq 0$ (resp. $(\gamma, \alpha) \neq 0$) for each $\alpha \in \Delta$ (resp. Φ). Otherwise, λ is called singular. Let L' be the set of nonsingular elements of L. To each $\lambda \in L'$ Harish-Chandra has attached certain invariant eigendistributions Θ_λ on G **[2]**. Two of these coincide precisely when their parameters are related by an element of $W_{\mathfrak{t}}$ **[3]**. Each Θ_λ is the character of a discrete series representation and in this way all such representations are exhausted (up to equivalence). The discrete series representation with character Θ_λ will be denoted by $\pi(\lambda)$. Its infinitesimal character is χ_λ.

2. Sufficient condition for irreducibility of a principal series representation. Recall our choice of the real positive root $\tilde{\beta} = e_1 + e_2$ and let $\gamma = \sum_{i=1}^{n+1} a_i e_i \in D$. Let ϕ be a homomorphism of SL(2, $\mathbf{R}$) into G such that (denoting by ϕ_* the tangential map)

$$\phi_*\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = H_{\tilde{\beta}}, \quad \phi_*\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = X_{\tilde{\beta}}, \quad \phi_*\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = X_{-\tilde{\beta}}.$$

Set $m = \phi_*(\left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right])$. Then m is in the center of M; hence it is represented by a scalar in any irreducible finite dimensional representation of M. Let $\gamma(m)$ be this scalar for the irreducible representation of M with the highest weight $\gamma|_{\mathfrak{h}^-} - \delta_{\mathfrak{m}}$. Fix a positive system P in Δ such that β is simple ($\beta = \varepsilon_1 - \varepsilon_{n+1}$). Let

$$\delta = \frac{1}{2}\sum_{\alpha \in P \cap \Delta_n} \alpha - \frac{1}{2}\sum_{\alpha \in P \cap \Delta_k} \alpha, \quad k = 2\frac{(\delta, \beta)}{(\beta, \beta)}, \quad l(\gamma) = 2\frac{(\gamma, \tilde{\beta})}{(\tilde{\beta}, \tilde{\beta})}.$$

We need the following technical lemma.

LEMMA 2.1. *Let $\gamma = \sum_{i=1}^{n+1} a_i e_i \in D$. Then the following two conditions are mutually equivalent*:

(i) $l(\gamma) \in \mathbf{Z}$ *and* $\gamma(m) = (-1)^{k+l(\gamma)}$.

(ii) $a_1, a_2 \in \mathbf{Z}$.

PROOF. By **[12]** (i) is independent of the choice of P. Choose P so that the corresponding simple roots are $\varepsilon_1 - \varepsilon_{n+1}, \varepsilon_{n+1} - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n$. Then $\delta = -(n-1)\varepsilon_1 - \sum_{j=2}^{n}(n-j+1)\varepsilon_j + (n-2)\varepsilon_{n+1}$ and $k = -2n+3$. Furthermore, $l(\gamma) = a_1 + a_2$.

Let us compute $\gamma(m)$. We have

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \exp\left(\pi\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)$$

and $X_{-\tilde{\beta}} - X_{\tilde{\beta}} = -iH_\beta$; hence $m = \exp(-i\pi H_\beta)$. Now,

$$H_\beta = \operatorname{diag}(1, 0, \ldots, 0, -1, -1, 0, \ldots, 0, 1)$$

and, for $\alpha = e_1 - e_2$, $H_\alpha = \operatorname{diag}(1, 0, \ldots, 0, 1, -1, 0, \ldots, 0, -1)$. Therefore, $m = \exp(-i\pi H_\alpha)$. In this way we obtain

$$\gamma(m) = \exp(-i\pi(\gamma - \delta_{\mathrm{m}})(H_\alpha)).$$

α is simple for Φ_{m}^+; hence $\delta_{\mathrm{m}}(H_\alpha) = 1$. Furthermore, $\gamma(H_\alpha) = a_1 - a_2$. Therefore, (i) takes the form

$$a_1 + a_2 \in \mathbf{Z} \quad \text{and} \quad \exp(-i\pi(a_1 - a_2)) = (-1)^{a_1 + a_2}.$$

As $a_1 - a_2 \in \mathbf{Z}$, this is obviously equivalent to (ii). Q.E.D.

THEOREM 2.2. *Let $\gamma = \sum_{j=1}^{n+1} a_j e_j \in D$ and suppose that the principal series representation $\pi(\gamma)$ is reducible. Then $a_j \in \mathbf{Z}$, $1 \leqslant j \leqslant n+1$.*

PROOF. As $\gamma \in D$, we have only to prove that the reducibility of $\pi(\gamma)$ implies that a_1 and a_2 are integers.

Suppose first that γ is nonsingular. By Proposition 6.1 in **[12]** the reducibility of $\pi(\gamma)$ implies that one of the following two possibilities holds true.

(1) There exists a complex root $\alpha \in \Phi$ such that $2(\alpha, \gamma)/(\alpha, \alpha) \in \mathbf{Z}$.

The complex roots in Φ are $\pm e_1 \pm e_i, \pm e_2 \pm e_i, 3 \leqslant i \leqslant n+1, \pm 2e_1, \pm 2e_2$. An easy computation shows that for any of these roots α, $2(\alpha, \gamma)/(\alpha, \alpha) \in \mathbf{Z}$ implies $a_1, a_2 \in \mathbf{Z}$.

(2) $l(\gamma)$ and $\gamma(m) = (-1)^{k+l(\gamma)}$ (notation as before). By Lemma 2.1 this is satisfied if and only if $a_1, a_2 \in \mathbf{Z}$.

In this way we have proved that if γ is nonsingular and if $\pi(\gamma)$ is reducible, then $a_j \in \mathbf{Z}$ for $1 \leqslant j \leqslant n+1$.

Suppose now that γ is singular. Let P be a positive system in Φ such that γ is dominant with respect to P. Let $\mu = \sum_{i=1}^{n+1} b_i e_i$ be the highest weight (with respect to P) of a finite dimensional irreducible representation of G, such that $\gamma + \mu$ is dominant with respect to P and nonsingular.

Let $\psi_\gamma^{\gamma+\mu}$ be the Zuckerman's functor as defined in **[16]**. Then $\psi_\gamma^{\gamma+\mu}(\pi(\gamma + \mu)) = \pi(\gamma)$ by Corollary 5.9 in **[12]**. $\psi_\gamma^{\gamma+\mu}$ is an exact functor by **[16]**. Therefore, the reducibility of $\pi(\gamma)$ implies the reducibility of $\pi(\gamma + \mu)$. By the first part of the proof, $a_j + b_j \in \mathbf{Z}$ for $1 \leqslant j \leqslant n + 1$. μ being a weight of a finite dimensional representation of G, we have $b_j \in \mathbf{Z}$ for $1 \leqslant j \leqslant n + 1$. Therefore, $a_j \in \mathbf{Z}$ for $1 \leqslant j \leqslant n + 1$. Q.E.D.

3. Composition factors for nonsingular infinitesimal characters. Let γ be in $\mathfrak{t}_{\mathbf{C}}^*$ (resp. $\mathfrak{h}_{\mathbf{C}_{n+1}}^*$). We shall say that γ is integral if $\gamma = \sum_{i=1}^{n+1} a_i \varepsilon_i$ (resp. $\gamma = \sum_{i=1}^{n+1} a_i e_i$) with $a_i \in \mathbf{Z}$, $1 \leqslant i \leqslant n + 1$. An infinitesimal character χ will be called integral if $\chi = \chi_\gamma$ for some integral γ.

If γ is a nonsingular integral element of $\mathfrak{t}_{\mathbf{C}}^*$ (resp. $\mathfrak{h}_{\mathbf{C}}^*$) we denote by Δ_γ^+ (resp. Φ_γ^+) the unique system of positive roots in Δ (resp. Φ) with respect to which γ is dominant.

By Theorem 2.2 if $\gamma \in D$ is not integral then $\pi(\gamma)$ is irreducible. Therefore, in the rest of the paper we can restrict ourselves to the representations with integral infinitesimal characters.

Throughout this section χ will denote a fixed nonsingular integral infinitesimal character. We need to parametrize all the discrete series and the principal series representations with infinitesimal character χ.

If C is a system of positive roots in Δ (resp. Φ) we shall denote by $\tilde{C}$ the corresponding closed Weyl chamber in $i\mathfrak{t}^*$ (resp. $(i\mathfrak{h}^- + \mathfrak{a})^*$).

For $0 \leqslant j \leqslant n$ let C_j be the system of positive roots in Δ defined by

$$\tilde{C}_j = \left\{ \sum_{i=1}^{n+1} a_i \varepsilon_i;\ a_1 \geqslant \cdots \geqslant a_j \geqslant a_{n+1} \geqslant a_{j+1} \geqslant \cdots a_n \geqslant 0 \right\}.$$

Let $\sigma_j \in W$ be defined by $\sigma_j C_n = C_j$. Then σ_n is the identity and for $0 \leqslant j \leqslant n - 1$:

$$\begin{aligned} \sigma_j \varepsilon_i &= \varepsilon_i, \qquad 1 \leqslant i \leqslant j, \\ \sigma_j \varepsilon_{j+1} &= \varepsilon_{n+1}, \\ \sigma_j \varepsilon_i &= \varepsilon_{i-1}, \qquad j + 2 \leqslant i \leqslant n + 1. \end{aligned}$$

For $j \in \{0, 1, \ldots, n\}$ let λ_j denote the unique element in $\tilde{C}_j$ such that $\chi = \chi_{\lambda_j}$. By **[3]** every discrete series representation with infinitesimal character χ is equivalent to $\pi(\lambda_j)$ for exactly one $j \in \{0, 1, \ldots, n\}$. We shall also use the notation $\pi_j(\chi) = \pi(\lambda_j)$.

Up to the action of $W(\mathfrak{a})$ all the principal series representations with infinitesimal character χ are $\pi(\gamma)$, $\gamma \in D^+(\chi)$, where $D^+(\chi) = \{\gamma \in D^+;\ \chi_\gamma = \chi\}$. We are going to parametrize the set $\{\Phi_\gamma^+;\ \gamma \in D^+(\chi)\}$.

For $0 \leqslant i < j \leqslant n$ let $P_{i,j}$ be the system of positive roots in Φ defined by

$$\tilde{P}_{i,j} = \left\{ \sum_{k=1}^{n+1} a_k e_k;\ a_3 \geqslant \cdots \geqslant a_{i+2} \geqslant a_1 \geqslant a_{i+3} \geqslant \cdots \right.$$
$$\left. \geqslant a_{j+1} \geqslant a_2 \geqslant a_{j+2} \geqslant \cdots \geqslant a_{n+1} \geqslant 0 \right\}.$$

For $0 \leqslant i \leqslant n-1, n+1 \leqslant j \leqslant 2n-i$, let $P_{i,j}$ be the system of positive roots in Φ defined by

$$\tilde{P}_{i,j} = \left\{ \sum_{k=1}^{n+1} a_k e_k;\ a_3 \geqslant \cdots \geqslant a_{i+2} \geqslant a_1 \geqslant a_{i+3} \geqslant \cdots \right.$$
$$\left. \geqslant a_{2n+2-j} \geqslant -a_2 \geqslant a_{2n+3-j} \geqslant \cdots \geqslant a_{n+1} \geqslant 0 \right\}.$$

Then $\{\Phi_\gamma^+;\ \gamma \in D^+(\chi)\} = \{P_{i,j};\ 0 \leqslant i \leqslant n-1, i+1 \leqslant j \leqslant 2n-i\}$.

Let $w_{i,j} \in \tilde{W}$ be defined by $w_{i,j}P_{0,1} = P_{i,j}$. Then we have for $0 \leqslant i < j \leqslant n$:

$$\begin{aligned} w_{i,j}e_k &= e_{k+2}, & 1 \leqslant k \leqslant i,\\ w_{i,j}e_{i+1} &= e_1, & \\ w_{i,j}e_k &= e_{k+1}, & i+2 \leqslant k \leqslant j,\\ w_{i,j}e_{j+1} &= e_2, & \\ w_{i,j}e_k &= e_k, & j+2 \leqslant k \leqslant n+1, \end{aligned}$$

and for $0 \leqslant i \leqslant n-1, n+1 \leqslant j \leqslant 2n-i$:

$$\begin{aligned} w_{i,j}e_k &= e_{k+2}, & 1 \leqslant k \leqslant i,\\ w_{i,j}e_{i+1} &= e_1, & \\ w_{i,j}e_k &= e_{k+1}, & i+2 \leqslant k \leqslant 2n+1-j,\\ w_{i,j}e_{2n+2-j} &= -e_2, & \\ w_{i,j}e_k &= e_k, & 2n+3-j \leqslant k \leqslant n+1. \end{aligned}$$

For $0 \leqslant i \leqslant n-1, i+1 \leqslant j \leqslant 2n-i$, let $\gamma_{i,j}$ be the unique element in $\tilde{P}_{i,j}$ such that $\chi = \chi_{i,j}$. We set $\pi_{i,j}(\chi) = \pi(\gamma_{i,j})$ and $\bar{\pi}_{i,j}(\chi) = \bar{\pi}(\gamma_{i,j})$.

By **[10]** every admissible irreducible representation of G with infinitesimal character χ is infinitesimally equivalent either to a discrete series representation or to $\bar{\pi}_{i,j}(\chi)$ for exactly one pair (i,j), $0 \leqslant i \leqslant n-1, i+1 \leqslant j \leqslant 2n-i$.

If π (resp. X) is an admissible representation of G of finite length (resp. a Harish-Chandra module of finite length **[16]**) we shall denote by $\Theta(\pi)$ (resp. $\Theta(X)$) its global character. For each (i,j), $\pi_{i,j}(\chi)$ is an admissible representation of G of finite length and has infinitesimal character χ. Therefore, every composition factor of $\pi_{i,j}(\chi)$ is infinitesimally equivalent to some $\pi_k(\chi)$, $0 \leqslant k \leqslant n$, or to some $\bar{\pi}_{k,t}(\chi)$, $0 \leqslant k \leqslant n-1$, $k+1 \leqslant t \leqslant 2n-k$. Hence there exist unique nonnegative integers $m(i,j;k)$, $m(i,j;k,t)$ such that

$$\Theta(\pi_{i,j}(\chi)) = \sum_{k=0}^{n} m(i,j;k)\Theta(\pi_k(\chi)) + \sum_{k=0}^{n-1} \sum_{t=k+1}^{2n} m(i,j;k,t)\Theta(\bar{\pi}_{k,t}(\chi)).$$

Our goal in this section is to determine all these numbers $m(i, j;\, k)$, $m(i, j;\, k, t)$. To do this we shall use Vogan's results **[14]**. For the convenience of the reader we shall state these results in our situation.

Let Π denote the set of simple roots in C_n, i.e. $\Pi = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_n - \varepsilon_{n+1}, 2\varepsilon_{n+1}\}$. For $\alpha \in \Pi$ let ϕ_α and ψ_α be the Zuckerman's functors as defined in **[15]** (after Lemma 3.1) with respect to λ_n. If π (resp. X) is an admissible representation (resp. a Harish-Chandra module) of finite length with infinitesimal character χ, let $\tau(\pi)$ (resp. $\tau(\chi)$) be the Borho-Jantzen-Duflo τ-invariant of π (resp. X).

It is defined in **[14]** as the set of all $\alpha \in \Pi$ such that $\psi_\alpha(\pi) = 0$ (resp. $\psi_\alpha(X) = 0$), or, equivalently, $\phi_\alpha\psi_\alpha(\pi) = 0$ (resp. $\phi_\alpha\psi_\alpha(X) = 0$).

Let X be an irreducible Harish-Chandra module with infinitesimal character χ. For each $\alpha \in \Pi \setminus \tau(X)$, $\phi_\alpha\psi_\alpha(X)$ contains X as the unique irreducible quotient and as its unique irreducible subrepresentation **[14]**. The remaining subquotient of $\phi_\alpha\psi_\alpha(X)$ will be denoted by $U_\alpha(X)$ as in **[14]**. Therefore, $\Theta(U_\alpha(X)) = \Theta(\phi_\alpha\psi_\alpha(X)) - 2\Theta(X)$. If π is an irreducible admissible representation of G and X is the corresponding Harish-Chandra module of K-finite vectors, we shall sometimes write $U_\alpha(\pi)$ instead of $U_\alpha(X)$.

Let Θ be a virtual character (i.e. an integral linear combination of irreducible characters) with infinitesimal character χ. If $\mu \in \mathfrak{t}_{\mathbf{C}}^*$ is a weight of a finite dimensional representation of G, let $S_\mu(\Theta)$ denote the coherent continuation of Θ as defined in **[12]** or **[14]**.

PROPOSITION 3.1 **[14**, PROPOSITION 3.2]. *Let X be an irreducible Harish-Chandra module with infinitesimal character χ.*

(i) *If $\alpha \in \tau(X)$ and $k = 2(\alpha, \lambda_n)/(\alpha, \alpha)$, then $S_{-k\alpha}(\Theta(X)) = -\Theta(X)$.*

(ii) *If $\alpha \in \Pi - \tau(X)$ and $k = 2(\alpha, \lambda_n)/(\alpha, \alpha)$, then $S_{-k\alpha}(\Theta(X)) = \Theta(X) + \Theta(U_\alpha(X))$.*

Using Corollary 4.13 in **[14]** we easily find

PROPOSITION 3.2.

(i)

$$\tau(\pi_i(X)) = \begin{cases} \Pi\setminus\{\varepsilon_1 - \varepsilon_2\}, & i = 0, \\ \Pi\setminus\{\varepsilon_i - \varepsilon_{i+1}, \varepsilon_{i+1} - \varepsilon_{i+2}\}, & 1 \leqslant i \leqslant n-1, \\ \Pi\setminus\{\varepsilon_n - \varepsilon_{n+1}\}, & i = n. \end{cases}$$

(ii)

$$\tau(\bar{\pi}_{i,j}(\chi)) = \begin{cases} \Pi, & i = 0, j = 1, \\ \Pi\setminus\{\varepsilon_j - \varepsilon_{j+1}\}, & i = 0, 2 \leqslant j \leqslant n, \\ \Pi\setminus\{2\varepsilon_{n+1}\}, & i = 0, j = n+1, \\ \Pi\setminus\{\varepsilon_{2n+2-j} - \varepsilon_{2n+3-j}\}, & i = 0, n+2 \leqslant j \leqslant 2n, \\ \Pi\setminus\{\varepsilon_i - \varepsilon_{i+1}\}, & i \geqslant 1, j = i+1, \\ \Pi\setminus\{\varepsilon_i - \varepsilon_{i+1}, \varepsilon_j - \varepsilon_{j+1}\}, & i \geqslant 1, i+2 \leqslant j \leqslant n, \\ \Pi\setminus\{\varepsilon_i - \varepsilon_{i+1}, 2\varepsilon_{n+1}\}, & i \geqslant 1, j = n+1, \\ \Pi\setminus\{\varepsilon_i - \varepsilon_{i+1}, \varepsilon_{2n+2-j} - \varepsilon_{2n+3-j}\}, & i \geqslant 1, n+2 \leqslant j \leqslant 2n-i. \end{cases}$$

Theorem 4.12 in [**14**] in our situation reads

THEOREM 3.3. *Let* $0 \leqslant i \leqslant n-1$, $i+1 \leqslant j \leqslant 2n-i$. *Let* $\alpha \to \bar{\alpha}$ *be the dual of the Cayley transform of* $\mathfrak{t}_{\mathbf{C}}$ *onto* $\mathfrak{t}_{\mathbf{C}}$ *such that* $\bar{C}_n = P_{i,j}$. *Let* $\alpha \in \Pi \setminus \tau(\bar{\pi}_{i,j}(\chi))$ *and let* $k = 2(\alpha, \lambda_n)/(\alpha, \alpha)$.

(i) *If* $\bar{\alpha}$ *is complex then* $\Theta(U_\alpha(\bar{\pi}_{i,j}(\chi))) = \Theta(\bar{\pi}(\gamma_{i,j} - k\bar{\alpha})) + \Theta_0$.

(ii) *If* $\bar{\alpha}$ *is real then* $\Theta(U_\alpha(\bar{\pi}_{i,j}(\chi))) = \Theta_0$.

In each case Θ_0 *is the character of a representation, every irreducible constituent of* Θ_0 *is the character of a subquotient of* $\pi_{i,j}(\chi)$ *which contains* α *in its* τ*-invariant.*

Following [**14**] we shall call $\Theta(U_\alpha(\bar{\pi}_{i,j}(\chi))) - \Theta_0$ the special constituent of $\Theta(U_\alpha(\bar{\pi}_{i,j}(\chi)))$.

Theorem 4.12 in [**14**] implies also an analogous asssertion for the discrete series representations. To describe it we have to attach to each $\alpha \in \Pi \setminus \tau(\pi_i(\chi))$ the parameter of a principal series representation.

Let $0 \leqslant i \leqslant n$ and let $\sigma_i \in W$ be defined as before ($\sigma_i C_n = C_i$). Let $\alpha \in \Pi \setminus \tau(\pi_i(\chi))$ and set $\bar{\alpha} = \sigma_i \alpha$, $\mathfrak{a}_{\bar{\alpha}} = \mathbf{R}(X_{\bar{\alpha}} + X_{-\bar{\alpha}})$ and $\mathfrak{h}_{\bar{\alpha}}^- = \{H \in \mathfrak{t};\ \bar{\alpha}(H) = 0\}$. Then $\mathfrak{h}_{\bar{\alpha}} = \mathfrak{h}_{\bar{\alpha}}^- + \mathfrak{a}_{\bar{\alpha}}$ is a Cartan subalgebra of $\mathfrak{g}$ and the corresponding roots are $\Psi = \Delta \circ \mathrm{Ad}(u_{\bar{\alpha}})^{-1}$. Let $\Psi^+ = C_i \circ \mathrm{Ad}(u_{\bar{\alpha}})^{-1}$. Denote by $\mathfrak{m}_{\bar{\alpha}}$ the centralizer of $\mathfrak{a}_{\bar{\alpha}}$ in $\mathfrak{k}$. Then the roots $\Psi_{\mathfrak{m}_{\bar{\alpha}}}$ of the pair $((\mathfrak{m}_{\bar{\alpha}})_{\mathbf{C}}, (\mathfrak{h}_{\bar{\alpha}}^-)_{\mathbf{C}})$ may be identified with the roots of $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$ which are orthogonal to $\bar{\alpha}$. Set $\Psi_{\mathfrak{m}_{\bar{\alpha}}}^+ = \Psi^+ \cap \Psi_{\mathfrak{m}_{\bar{\alpha}}}$. Let $\tilde{\alpha} \in \Psi^+$ be the real root (Ψ^+ contains exactly one real root). Define $\tilde{\lambda}_i^\alpha \in (\mathfrak{h}_{\bar{\alpha}})_{\mathbf{C}}^*$ by

$$\tilde{\lambda}^\alpha_{i|\mathfrak{h}_{\bar{\alpha}}^-} = \lambda_{i|\mathfrak{h}_{\bar{\alpha}}^-}, \qquad (\tilde{\lambda}_i^\alpha, \tilde{\alpha}) = (\lambda_i, \bar{\alpha}).$$

Finally, let ϕ be the dual of the restriction to $\mathfrak{h}_{\mathbf{C}}$ of an inner automorphism of $\mathfrak{g}_{\mathbf{C}}$ which carries $\mathfrak{h}_{\mathbf{C}}$ to $(\mathfrak{h}_{\bar{\alpha}})_{\mathbf{C}}$ and such that $\phi(\Psi^+) = P_{0,1}$. Set $\lambda_i^\alpha = \phi(\tilde{\lambda}_i^\alpha)$.

THEOREM 3.4. *Let* $0 \leqslant i \leqslant n$, $\alpha \in \Pi \setminus \tau(\pi_i(\chi))$. *With the notation introduced above we have*

$$\Theta(U_\alpha(\pi_i(\chi))) = \Theta(\bar{\pi}(\lambda_i^\alpha)).$$

As we shall see, the proof of the main theorem in this section will be immediately reduced to the case when χ is the trivial infinitesimal character (i.e. the infinitesimal character of the trivial one-dimensional representation). In this case

$$\lambda_i = \frac{1}{2} \sum_{\alpha \in C_i} \alpha = \sum_{j=1}^{i} (n+2-j)\varepsilon_j + \sum_{j=i+1}^{n} (n+1-j)\varepsilon_j + (n+1-i)\varepsilon_{n+1}, \quad 0 \leqslant i \leqslant n, \tag{1}$$

$$\gamma_{i,j} = \frac{1}{2} \sum_{\alpha \in P_{i,j}} \alpha = (n+1-i)e_1 + (n+1-j)e_2 + \sum_{k=3}^{i+2} (n+4-k)e_k + \sum_{k=i+3}^{j+1} (n+3-k)e_k + \sum_{k=j+2}^{n+1} (n+2-k)e_k, \qquad 0 \leqslant i < j \leqslant n, \tag{2}$$

$$\gamma_{i,j} = (n+1-i)e_1 - je_2 + \sum_{k=3}^{i+2}(n+4-k)e_k + \sum_{k=i+3}^{n+2-j}(n+3-k)e_k$$
$$+ \sum_{k=n+3-j}^{n+1}(n+2-k)e_k, \qquad 0 \leqslant i \leqslant n-1, n+1 \leqslant j \leqslant 2n-i. \quad (3)$$

Direct computation gives

PROPOSITION 3.5. *Let* $0 \leqslant i \leqslant n$ *and let* χ *be the trivial infinitesimal character. Let* $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ *if* $1 \leqslant i \leqslant n$ *and* $\beta_i = \varepsilon_{i+1} - \varepsilon_{i+2}$, $0 \leqslant i \leqslant n-1$. *Then*

$$\lambda_i^{\alpha_i} = \gamma_{i-1,2n+1-i}, \qquad \lambda_i^{\beta_i} = \gamma_{i,2n-i},$$

($\gamma_{j,k}$ *are given by* (2) *and* (3)).

THEOREM 3.6. *Let* χ *be an integral nonsingular infinitesimal character.*

(i) *For* $0 \leqslant i \leqslant n-1$

$$\Theta(\pi_{i,2n-i}(\chi)) = \Theta(\bar{\pi}_{i,2n-i}(\chi)) + \Theta(\pi_i(\chi)) + \Theta(\pi_{i+1}(\chi)).$$

(ii) *For* $0 \leqslant i \leqslant n-2$

$$\Theta(\pi_{i,2n-i-1}(\chi)) = \Theta(\bar{\pi}_{i,2n-i-1}(\chi)) + \Theta(\bar{\pi}_{i,2n-i}(\chi))$$
$$+ \Theta(\bar{\pi}_{i+1,2n-i-1}(\chi)) + \Theta(\pi_{i+1}(\chi)).$$

(iii) $\Theta(\pi_{n-1,n}(\chi)) = \Theta(\bar{\pi}_{n-1,n}(\chi)) + \Theta(\bar{\pi}_{n-1,n+1}(\chi))$.

(iv) $\Theta(\pi_{n-2,n}(\chi)) = \sum_{a,b=0}^{1} \Theta(\bar{\pi}_{n-2+a,n+b}(\chi)) + \Theta(\pi_n(\chi))$.

(v) $\Theta(\pi_{n-2,n-1}(\chi)) = \Theta(\bar{\pi}_{n-2,n-1}(\chi)) + \Theta(\bar{\pi}_{n-2,n}(\chi)) + \Theta(\pi_n(\chi))$.

Let $0 \leqslant i \leqslant n-1$, $i+1 \leqslant j \leqslant 2n-i-2$ *and set* $k = i+j$.

(vi) *If* k *is odd and* $0 \leqslant i \leqslant [(k-1)/2]-1$ *or if* k *is even and* $0 < i < (k/2)-2$,

$$\Theta(\pi_{i,j}(\chi)) = \sum_{a,b=0}^{1} \Theta(\bar{\pi}_{i+a,j+b}(\chi)).$$

(vii) *If* k *is odd and* $i = (k-1)/2$, *then*

$$\Theta(\pi_{i,j}(\chi)) = \Theta(\bar{\pi}_{i,j}(\chi)) + \Theta(\bar{\pi}_{i,j+1}(\chi)) + \Theta(\bar{\pi}_{i+2,j+2}(\chi)).$$

(viii) *If* k *is even and* $i = (k/2)-1$, *then*

$$\Theta(\pi_{i,j}(\chi)) = \sum_{a,b=0}^{1} \Theta(\bar{\pi}_{i+a,j+b}(\chi)) + \Theta(\bar{\pi}_{i+2,j+1}(\chi)).$$

PROOF. Using the results on the Zuckerman's functor (Theorem 1.2 in **[16]**, Theorem 6.18 and Corollary 5.12 in **[12]**) it is easy to see that it suffices to prove the theorem in the case that χ is the trivial infinitesimal character. Then λ_i, $\gamma_{i,j}$ are given by (1), (2), (3). For convenience we shall write π_i, $\pi_{i,j}$, $\bar{\pi}_{i,j}$ instead of $\pi_i(\chi)$, $\pi_{i,j}(\chi)$, $\bar{\pi}_{i,j}(\chi)$, respectively.

(i) follows by an application of Schmid's identities (**[12]**; see also **[1]**). The method of the proof of the rest is based on the following algorithm. Suppose we know all the computation factors of $\pi_{i,j}$ for every i, j such that $i+j=k$.

Let $i'+j'=k-1$. Then we find a pair i, j such that $i+j=k$ and a simple root $\bar{\alpha}$ in $P_{i,j}$ such that $\gamma_{i',j'} = \gamma_{i,j} - \bar{\alpha}$. Let $\phi_{i,j}$ be the dual of the restriction of an inner automorphism of $\mathfrak{g}_C$ transforming $\mathfrak{h}_C$ into $\mathfrak{t}_C$ and such that $\phi_{i,j}(C_n) = P_{i,j}$. Set $\alpha = \phi_{i,j}^{-1}(\bar{\alpha})$. By Theorem 4.3 in [14] we have $\Theta(\pi_{i',j'}) = S_{-\alpha}\Theta(\pi_{i,j})$. Using Proposition 3.1 and 3.5 and Theorems 3.3 and 3.4 together with Theorem 4.14 in [14] (which will be called the multiplicity theorem) we compute $S_{-\alpha}\Theta(\pi)$ for each composition factor π of $\pi_{i,j}$ and we get the desired result.

We shall need $\phi_{i,j}$'s explicitly. If $0 \leqslant i < j \leqslant n$, we have

$$\phi_{i,j}(\varepsilon_k) = \begin{cases} e_{k+2}, & 1 \leqslant k \leqslant i, \\ e_1, & k = i+1, \\ e_{k+1}, & i+2 \leqslant k \leqslant j, \\ e_2, & k = j+1, \\ e_k, & j+2 \leqslant k \leqslant n+1. \end{cases}$$

If $0 \leqslant i \leqslant n-1, n+1 \leqslant j \leqslant 2n-i$, we have

$$\phi_{i,j}(\varepsilon_k) = \begin{cases} e_{k+2}, & 1 \leqslant k \leqslant i, \\ e_1, & k = i+1, \\ e_{k+1}, & i+2 \leqslant k \leqslant 2n+1-j, \\ -e_2, & k = 2n+2-j, \\ e_k, & 2n+3-j \leqslant k \leqslant n+1. \end{cases}$$

(ii) We have $\gamma_{i,2n-i-1} = \gamma_{i+1,2n-i-1} - \bar{\alpha}$, where $\bar{\alpha} = e_{i+3} - e_1$ is simple for $P_{i+1,2n-i-1}$. By (i) it follows

$$\Theta(\pi_{i,2n-i-1}) = S_{-\alpha}\Theta(\bar{\pi}_{i+1,2n-i-1}) + S_{-\alpha}\Theta(\pi_{i+1}) + S_{-\alpha}\Theta(\pi_{i+2}), \tag{4}$$

with $\alpha = (\phi_{i+1,2n-i-1})^{-1}(\bar{\alpha}) = \varepsilon_{i+1} - \varepsilon_{i+2}$.

By Proposition 3.2 $\alpha \in \Pi \setminus \tau(\bar{\pi}_{i+1,2n-i-1})$; thus by Theorem 3.3(i) the special constituent of $\Theta(U_\alpha(\bar{\pi}_{i+1,2n-i-1}))$ is $\Theta(\bar{\pi}_{i,2n-i-1})$. All the other constituents must occur in $\pi_{i+1,2n-i-1}$ and must have α in their τ-invariants. By (i) and by Proposition 3.2 the only candidate is $\Theta(\pi_{i+2})$. Let $\beta = \varepsilon_{i+2} - \varepsilon_{i+3}$. Then $\beta \in \tau(\bar{\pi}_{i+1,2n-i-1})$, $\beta \notin \tau(\pi_{i+2})$; so by that multiplicity theorem the multiplicity of $\Theta(\bar{\pi}_{i+1,2n-i-1})$ in $\Theta(U_\beta(\pi_{i+2}))$ is equal to the multiplicity of $\Theta(\pi_{i+2})$ in $\Theta(U_\alpha(\bar{\pi}_{i+1,2n-i-1}))$. But one checks easily that $\Theta(\bar{\pi}_{i+1,2n-i-1})$ is the special constituent of $\Theta(U_\beta(\pi_{i+2}))$, so this multiplicity is one. Hence

$$\Theta(U_\alpha(\bar{\pi}_{i+1,2n-i-1})) = \Theta(\bar{\pi}_{i,2n-i-1}) + \Theta(\pi_{i+2})$$

and, therefore,

$$S_{-\alpha}\Theta(\bar{\pi}_{i+1,2n-i-1}) = \Theta(\bar{\pi}_{i+1,2n-i-1}) + \Theta(\bar{\pi}_{i,2n-i-1}) + \Theta(\pi_{i+2}). \tag{5}$$

Since $\alpha \notin \tau(\pi_{i+1})$ and $\sigma_{i+1}\alpha = \varepsilon_{i+1} - \varepsilon_{n+1}$, using Theorem 3.4 and Proposition 3.5 we get

$$S_{-\alpha}\Theta(\pi_{i+1}) = \Theta(\bar{\pi}_{i,2n-i}) + \Theta(\pi_{i+1}). \tag{6}$$

Finally, $\alpha \in \tau(\pi_{i+2})$ so

$$S_{-\alpha}\Theta(\pi_{i+2}) = -\Theta(\pi_{i+2}). \tag{7}$$

Now, (4), (5), (6), and (7) give (ii).

(iii) We have $\gamma_{n-1,n} = \gamma_{n-1,n+1} - \bar{\alpha}$, where $\bar{\alpha} = -2e_2$ is simple for $P_{n-1,n+1}$.

Furthermore $\alpha = \phi_{n-1,n+1}^{-1}(\bar{\alpha}) = 2\varepsilon_{n+1}$ and we get from (i)

$$\Theta(\pi_{n-1,n}) = S_{-\alpha}\Theta(\bar{\pi}_{n-1,n+1}) + S_{-\alpha}\Theta(\pi_{n-1}) + S_{-\alpha}\Theta(\pi_n). \tag{8}$$

Since $\alpha \in \tau(\pi_{n-1})$ and $\alpha \in \tau(\pi_n)$ we have

$$S_{-\alpha}\Theta(\pi_{n-1}) = -\Theta(\pi_{n-1}), \tag{9}$$

$$S_{-\alpha}\Theta(\pi_n) = -\Theta(\pi_n). \tag{10}$$

$\alpha \notin \tau(\bar{\pi}_{n-1,n+1})$ and the special constituent of $\Theta(U_\alpha(\bar{\pi}_{n-1,n+1}))$ is $\Theta(\bar{\pi}_{n-1,n})$. All the other constituents of $\Theta(U_\alpha(\bar{\pi}_{n-1,n+1}))$ must occur in $\pi_{n-1,n+1}$ and must have α in their τ-invariants. The only candidates are $\Theta(\pi_{n-1})$ and $\Theta(\pi_n)$.

Let $\beta = \varepsilon_n - \varepsilon_{n+1}$. Then $\beta \in \tau(\bar{\pi}_{n-1,n+1})$ and $\beta \notin \tau(\pi_n)$, $\beta \notin \tau(\pi_{n-1})$. Since $\sigma_n\beta = \varepsilon_n - \varepsilon_{n+1}$ and $\sigma_{n-1}\beta = \varepsilon_{n+1} - \varepsilon_n$ then by Proposition 3.5 and Theorem 3.4 it follows easily that

$$\Theta(U_\beta(\pi_{n-1})) = \Theta(\bar{\pi}_{n-1,n+1}),$$

$$\Theta(U_\beta(\pi_n)) = \Theta(\bar{\pi}_{n-1,n+1}).$$

By a multiplicity argument as before we conclude that both $\Theta(\pi_n)$ and $\Theta(\pi_{n-1})$ occur once in $\Theta(U_\alpha(\bar{\pi}_{n-1,n+1}))$. So

$$\begin{aligned} S_{-\alpha}\Theta(\bar{\pi}_{n-1,n+1}) &= \Theta(\bar{\pi}_{n-1,n+1}) + \Theta(U_\alpha(\bar{\pi}_{n-1,n+1})) \\ &= \Theta(\bar{\pi}_{n-1,n+1}) + \Theta(\bar{\pi}_{n-1,n}) + \Theta(\pi_n) + \Theta(\pi_{n+1}). \end{aligned} \tag{11}$$

The equations (8), (9), (10) and (11) imply (iii).

(iv) We have $\gamma_{n-2,n} = \gamma_{n-1,n} - \bar{\alpha}$ where $\bar{\alpha} = e_{n+1} - e_1$ and $\alpha = (\phi_{n-1,n})^{-1}(\bar{\alpha}) = \varepsilon_{n-1} - \varepsilon_n$. Therefore, by (iii)

$$\Theta(\pi_{n-2,n}) = S_{-\alpha}\Theta(\pi_{n-1,n}) = S_{-\alpha}\Theta(\bar{\pi}_{n-1,n}) + S_{-\alpha}\Theta(\bar{\pi}_{n-1,n+1}). \tag{12}$$

$\alpha \notin \tau(\gamma_{n-1,n})$; therefore the special constituent of $\Theta(U_\alpha(\bar{\pi}_{n-1,n}))$ is $\Theta(\bar{\pi}_{n-2,n})$. The other constituents must occur in $\pi_{n-1,n}$ and have α in their τ-invariants, but there are no such constituents, and hence

$$S_{-\alpha}\Theta(\bar{\pi}_{n-1,n}) = \Theta(\bar{\pi}_{n-1,n}) + \Theta(\bar{\pi}_{n-2,n}). \tag{13}$$

$\phi_{n-1,n+1}\alpha = e_{n+1} - e_1$ and $\gamma_{n-1,n+1} - (e_{n+1} - e_1) = \gamma_{n-2,n+1}$. Therefore $\Theta(U_\alpha(\bar{\pi}_{n-1,n+1})) = \Theta(\bar{\pi}_{n-2,n+1}) + \Theta_0$, where Θ_0 is as in Theorem 3.3. The only candidate for Θ_0 is $\Theta(\pi_n)$.

Using $\beta = \varepsilon_n - \varepsilon_{n+1}$ and the usual multiplicity argument it follows that $\Theta_0 = \Theta(\pi_n)$ and hence

$$S_{-\alpha}\Theta(\bar{\pi}_{n-1,n+1}) = \Theta(\bar{\pi}_{n-2,n}) + \Theta(\bar{\pi}_{n-2,n+1}) + \Theta(\pi_n). \tag{14}$$

The equations (12), (13) and (14) imply (iv).

(v) We have $\gamma_{n-,n-1} = \gamma_{n-2,n} - \bar{\alpha}$, where $\bar{\alpha} = e_{n+1} - e_2$ and $\alpha = (\phi_{n-2,n})^{-1}(\bar{\alpha}) = \varepsilon_n - \varepsilon_{n+1}$. Therefore, by (iv)

$$\begin{aligned} \Theta(\pi_{n-2,n-1}) = S_{-\alpha}\Theta(\pi_{n-2,n}) &= S_{-\alpha}\Theta(\bar{\pi}_{n-2,n}) + S_{-\alpha}\Theta(\bar{\pi}_{n-1,n}) \\ &+ S_{-\alpha}\Theta(\bar{\pi}_{n-2,n+1}) + S_{-\alpha}\Theta(\bar{\pi}_{n-1,n+1}) + S_{-\alpha}\Theta(\pi_n). \end{aligned} \tag{15}$$

Since α is in the τ-invariants for $\pi_{n-2,n+1}, \pi_{n-1,n+1}, \pi_{n-1,n}$, we have

$$S_{-\alpha}\Theta(\bar{\pi}_{n-2,n+1}) = -\Theta(\bar{\pi}_{n-2,n+1}), \tag{16}$$

$$S_{-\alpha}\Theta(\bar{\pi}_{n-1,n+1}) = -\Theta(\bar{\pi}_{n-1,n+1}), \tag{17}$$

$$S_{-\alpha}\Theta(\bar{\pi}_{n-1,n}) = -\Theta(\bar{\pi}_{n-1,n}). \tag{18}$$

$\alpha \notin \tau(\bar{\pi}_{n-2,n})$; therefore $\Theta(U_\alpha(\bar{\pi}_{n-2,n})) = \Theta(\bar{\pi}_{n-2,n-1}) + \Theta_0$. Candidates for Θ_0 are $\Theta(\bar{\pi}_{n-2,n+1})$, $\Theta(\bar{\pi}_{n-1,n+1})$ and $\Theta(\bar{\pi}_{n-1,n})$. Let $\beta = \varepsilon_{n-1} - \varepsilon_n$; then $\beta \in \tau(\bar{\pi}_{n-2,n})$, and $\beta \notin \tau(\bar{\pi}_{n-1,n})$, $\beta \notin \tau(\bar{\pi}_{n-1,n+1})$. By the usual multiplicity argument we may conclude that Θ_0 contains $\Theta(\bar{\pi}_{n-1,n})$ once and does not contain $\Theta(\bar{\pi}_{n-1,n+1})$. For the fact that the multiplicity is one we need to use the first three character identities. Using $\beta = 2\varepsilon_{n+1}$ for $\bar{\pi}_{n-2,n+1}$ we also get that Θ_0 contains $\Theta(\bar{\pi}_{n-2,n+1})$ once. Hence

$$\Theta(U_\alpha(\bar{\pi}_{n-2,n})) = \Theta(\bar{\pi}_{n-2,n-1}) + \Theta(\bar{\pi}_{n-1,n}) + \Theta(\bar{\pi}_{n-2,n+1}),$$

and, therefore,

$$S_{-\alpha}\Theta(\bar{\pi}_{n-2,n}) = \Theta(\bar{\pi}_{n-2,n}) + \Theta(\bar{\pi}_{n-2,n-1}) + \Theta(\bar{\pi}_{n-1,n}) + \Theta(\bar{\pi}_{n-2,n+1}). \tag{19}$$

By a similar but much easier argument one gets

$$S_{-\alpha}\Theta(\pi_n) = \Theta(\pi_n) + \Theta(\bar{\pi}_{n-1,n+1}). \tag{20}$$

Now, putting together (15)–(20) we get (v).

(vi) Suppose now $i + j = k$, $1 \leqslant k \leqslant 2n - 2$. We want to prove (vi) by backwards induction on k. To start the induction we have to prove that if $k = 2n - 2$ then $0 \leqslant i \leqslant n - 3$,

$$\begin{aligned}\Theta(\pi_{i,2n-2-i}) = \Theta(\bar{\pi}_{i,2n-i-2}) + \Theta(\bar{\pi}_{i,2n-i-1}) \\ + \Theta(\bar{\pi}_{i+1,2n-i-2}) + \Theta(\bar{\pi}_{i+1,2n-i-1}).\end{aligned} \tag{21}$$

We have $\gamma_{i,2n-i-2} = \gamma_{i,2n-i-1} - \bar{\alpha}$ with $\bar{\alpha} = -e_2 - e_{i+4}$ and $\alpha = (\phi_{i,2n-i-1})^{-1}(\bar{\alpha})$ $= \varepsilon_{i+3} - \varepsilon_{i+4}$.

By (ii) we have

$$\begin{aligned}\Theta(\pi_{i,2n-i-2}) = S_{-\alpha}\Theta(\pi_{i,2n-i-1}) = S_{-\alpha}\Theta(\bar{\pi}_{i,2n-i-1}) + S_{-\alpha}\Theta(\bar{\pi}_{i,2n-i}) \\ + S_{-\alpha}\Theta(\bar{\pi}_{i+1,2n-i-1}) + S_{-\alpha}\Theta(\pi_{i+1}).\end{aligned} \tag{22}$$

$\alpha \notin \tau(\pi_{i,2n-i-1})$; thus $\Theta(U_\alpha(\bar{\pi}_{i,2n-i-1})) = \Theta(\bar{\pi}_{i,2n-i-2}) + \Theta_0$. Candidates for Θ_0 are $\Theta(\bar{\pi}_{i,2n-i})$ and $\Theta(\pi_{i+1})$. Using $\beta = \varepsilon_{i+2} - \varepsilon_{i+3}$ we conclude easily that $\Theta_0 =$ $\Theta(\bar{\pi}_{i,2n-i})$. Therefore

$$S_{-\alpha}\Theta(\bar{\pi}_{i,2n-i-1}) = \Theta(\bar{\pi}_{i,2n-i-1}) + \Theta(\bar{\pi}_{i,2n-i-2}) + \Theta(\bar{\pi}_{i,2n-i}). \tag{23}$$

By a similar argument we get

$$S_{-\alpha}\Theta(\bar{\pi}_{i+1,2n-i-1}) = \Theta(\bar{\pi}_{i+1,2n-i-1}) + \Theta(\bar{\pi}_{i+1,2n-i-2}) + \Theta(\pi_{i+1}).$$

Since α is in the τ-invariants for $\bar{\pi}_{i,2n-i}$ and π_{i+1} we get the required results.

Suppose now that we know that (vi) is true up to a fixed $1 \leqslant k \leqslant 2n - 2$ and let us prove it for $k - 1$. We have to distinguish two possibilities, $k - 1$ even or $k - 1$ odd.

We first prove it in the case $k - 1$ even. So we want to prove that $0 \leqslant i \leqslant$ $[(k-1)/2] - 2$, $\Theta(\pi_{i,j}) = \Theta(\pi_{i,j}) + \Theta(\pi_{i,j+1}) + \Theta(\pi_{i+1,j}) + \Theta(\pi_{i+1,j+1})$ where $j =$ $k - 1 - i$, knowing that it is true with t instead of $k - 1$, $t \geqslant k$. (Note that $5 \leqslant k \leqslant 2n - 3$.)

We have $\gamma_{i,k-i-1} = \gamma_{i+1,k-i-1} - \bar{\alpha}$ with $\bar{\alpha} = e_{i+3} - e_1$ and

$$\alpha = (\phi_{i+1,k-i-1})^{-1}(\bar{\alpha}) = \varepsilon_{i+1} = \varepsilon_{i+2}.$$

Hence, by the induction hypothesis

$$\Theta(\pi_{i,k-i-1}) = S_{-\alpha}\Theta(\pi_{i+1,k-i-1}) = S_{-\alpha}\Theta(\bar{\pi}_{i+1,k-i-1}) + S_{-\alpha}\Theta(\bar{\pi}_{i+1,k-i}) + S_{-\alpha}\Theta(\bar{\pi}_{i+2,k-i-1}) + S_{-\alpha}\Theta(\bar{\pi}_{i+2,k-i}).$$

α is in the τ-invariants for $\bar{\pi}_{i+2,k-i}$, and $\bar{\pi}_{i+2,k-i-1}$, while it is not in the τ-invariants for $\bar{\pi}_{i+1,k-i-1}$, $\bar{\pi}_{i+1,k-i}$. Thus $\Theta(U_\alpha(\bar{\pi}_{i+1,k-i-1})) = \Theta(\bar{\pi}_{i,k-i-1}) + \Theta_0$, where Θ_0 has the usual meaning. By means of the root $\beta = \varepsilon_{i+2} - \varepsilon_{i+3}$ we may conclude that $\Theta_0 = \Theta(\bar{\pi}_{i+2,k-i-1})$.

$\phi_{i+1,k-i}(\alpha) = e_{i+3} - e_1$ and $\gamma_{i+1,k-i} - (e_{i+3} - e_1) = \gamma_{i,k-i}$; thus $\Theta(U_\alpha(\bar{\pi}_{i+1,k-i})) = \Theta(\bar{\pi}_{i,k-i}) + \Theta_0$. Since $k+1$ is even and $1 \leqslant i+1 \leqslant (k+1)/2 - 2$ by the induction hypothesis we know that the constituents of $\bar{\pi}_{i+1,k-i}$ are $\bar{\pi}_{i+1,k-i}$, $\bar{\pi}_{i+1,k-i+1}$, $\bar{\pi}_{i+2,k-i}$ and $\bar{\pi}_{i+2,k-i+1}$. α is in the τ-invariants of only the last two; hence the candidates for Θ_0 are $\Theta(\bar{\pi}_{i+2,k-i})$ and $\Theta(\bar{\pi}_{i+2,k-i+1})$. Using $\beta = \varepsilon_{i+2} - \varepsilon_{i+3}$ and the multiplicity theorem we get that $\Theta_0 = \Theta(\bar{\pi}_{i+2,k-i})$.

With all this information we get the result.

The proof in the case $k-1$ odd is a little more complicated; we indicate the line of proof. We have to distinguish two cases: (a) $0 \leqslant i \leqslant (k/2) - 3$ and (b) $i = (k/2) - 2$.

Let us consider (a) first.

$$\gamma_{i,k-i-1} = \gamma_{i+1,k-i-1} - \bar{\alpha}, \qquad \bar{\alpha} = e_{i+3} - e_1,$$
$$\alpha = (\phi_{i+1,k-i-1})^{-1}(\bar{\alpha}) = \varepsilon_{i+1} - \varepsilon_{i+2}.$$

Then by the induction hypothesis we have

$$\Theta(\pi_{i,k-i-1}) = S_{-\alpha}\Theta(\pi_{i+1,k-i-1}) = S_{-\alpha}\Theta(\bar{\pi}_{i+1,k-i-1}) + S_{-\alpha}\Theta(\bar{\pi}_{i+1,k-i}) + S_{-\alpha}\Theta(\bar{\pi}_{i+2,k-i-1}) + S_{-\alpha}\Theta(\bar{\pi}_{i+2,k-i}).$$

Since $\alpha \in \tau(\bar{\pi}_{i+2,k-i})$, $\alpha \in \tau(\bar{\pi}_{i+2,k-i-1})$, $\alpha \notin \tau(\bar{\pi}_{i+1,k-i-1})$, $\alpha \notin \tau(\bar{\pi}_{i+1,k-i})$ exactly as in the case $k-1$ even we may conclude

$$S_{-\alpha}\Theta(\bar{\pi}_{i+2,k-i}) = -\Theta(\bar{\pi}_{i+2,k-i}),$$
$$S_{-\alpha}\Theta(\bar{\pi}_{i+2,k-i-1}) = -\Theta(\bar{\pi}_{i+2,k-i-1}),$$
$$S_{-\alpha}\Theta(\bar{\pi}_{i+1,k-i-1}) = \Theta(\bar{\pi}_{i+1,k-i-1}) + \Theta(\bar{\pi}_{i,k-i-1}) + \Theta(\bar{\pi}_{i+2,k-i-1}).$$

Also $\Theta(U_\alpha(\bar{\pi}_{i+1,k-i})) = \Theta(\bar{\pi}_{i,k-i}) + \Theta_0$ where Θ_0 has the usual meaning. To find candidates for Θ_0 we have to know the constituents of $\pi_{i+1,k-i}$. If $k < 2n-2$ again we do not have any problem and we may conclude by the induction hypothesis that

$$S_{-\alpha}\Theta(\bar{\pi}_{i+1,k-i}) = \Theta(\bar{\pi}_{i+1,k-i}) + \Theta(\bar{\pi}_{i,k-i}) + \Theta(\bar{\pi}_{i+2,k-i}). \tag{24}$$

If $k = 2n-2$, then by (ii) we have that

$$\Theta(\pi_{i+1,k-i}) = \Theta(\pi_{i+1,2n-i-2}) = \Theta(\bar{\pi}_{i+1,2n-i-2}) + \Theta(\bar{\pi}_{i+2,2n-i-2}) + \Theta(\bar{\pi}_{i+1,2n-i-1}) + \Theta(\pi_{i+2}).$$

$\alpha \in \tau(\bar{\pi}_{i+2,2n-i-2})$, $\alpha \in \tau(\pi_{i+2})$, $\alpha \notin \tau(\bar{\pi}_{i+1,2n-i-2})$, $\alpha \notin \tau(\bar{\pi}_{i+1,2n-i-1})$. Therefore the candidates for Θ_0 are $\Theta(\bar{\pi}_{i+2,2n-i-2})$ and $\Theta(\pi_{i+2})$. Using the root $\beta = \varepsilon_{i+2} - \varepsilon_{i+3}$ we may conclude $\Theta_0 = \Theta(\bar{\pi}_{i+2,2n-i-2})$ and $\Theta(U_\alpha(\bar{\pi}_{i+1,2n-i-2})) = \Theta(\bar{\pi}_{i,2n-i-2}) + \Theta(\bar{\pi}_{i+2,2n-i-2})$. Therefore (24) is true also for $k = 2n - 2$ and so the result holds true for (a).

Let us now consider (b), i.e. $i = k/2 - 2$. Recall that $2 \leqslant k \leqslant 2n - 2$. We need to distinguish two subcases

(b_1) $2 \leqslant k \leqslant 2n - 4$.

(b_2) $k = 2n - 2$.

For (b_1) using $\gamma_{i,k-i-1} = \gamma_{i,k-i} - (e_{k-i+1} - e_2)$ the proof goes on without any problem and we omit it; we have only to be careful in the choice of the root to apply the multiplicity theorem; it will be different depending on whether $k - i + 1 = n + 1$ or $k - i + 1 \leqslant n$. (If $k - i + 1 = n + 1$ choose $\beta = 2\varepsilon_{n+1}$; if $k - i + 1 \leqslant n$ choose $\beta = \varepsilon_{k-i+1} - \varepsilon_{k-i+2}$.)

Let us consider (b_2). In this case $i = n - 3$ and $k - i - 1 = n$, so we want to prove

$$\Theta(\pi_{n-3,n}) = \Theta(\bar{\pi}_{n-3,n}) + \Theta(\bar{\pi}_{n-3,n+1}) + \Theta(\bar{\pi}_{n-2,n}) + \Theta(\bar{\pi}_{n-2,n+1}). \qquad (25)$$

$\gamma_{n-3,n} = \gamma_{n-3,n+1} - \bar{\alpha}$, $\bar{\alpha} = -2e_2$ and $\alpha = (\phi_{n-3,n+1})^{-1}(\bar{\alpha}) = 2\varepsilon_{n+1}$. By the induction hypothesis, we have

$$\Theta(\pi_{n-3,n}) = S_{-\alpha}\Theta(\pi_{n-3,n+1}) = S_{-\alpha}\Theta(\bar{\pi}_{n-3,n+1}) + S_{-\alpha}\Theta(\bar{\pi}_{n-3,n+2}) + S_{-\alpha}\Theta(\bar{\pi}_{n-2,n+1}) + S_{-\alpha}\Theta(\bar{\pi}_{n-2,n+2}).$$

It follows immediately that

$$S_{-\alpha}\Theta(\bar{\pi}_{n-3,n+2}) = -\Theta(\bar{\pi}_{n-3,n+2}), \qquad S_{-\alpha}\Theta(\bar{\pi}_{n-2,n+2}) = -\Theta(\bar{\pi}_{n-2,n+2}).$$

Furthermore, we find easily

$$S_{-\alpha}\Theta(\bar{\pi}_{n-3,n+1}) = \Theta(\bar{\pi}_{n-3,n+1}) + \Theta(\bar{\pi}_{n-3,n}) + \Theta(\bar{\pi}_{n-3,n+2}).$$

For the last equality we use the root $\beta = \varepsilon_n - \varepsilon_{n+1}$ in the multiplicity argument. To compute $S_{-\alpha}\Theta(\bar{\pi}_{n-2,n+1})$ we use (ii) and the root $\beta = \varepsilon_n - \varepsilon_{n+1}$ in the multiplicity argument to get

$$S_{-\alpha}\Theta(\bar{\pi}_{n-2,n+1}) = \Theta(\bar{\pi}_{n-2,n+1}) + \Theta(\bar{\pi}_{n-2,n}) + \Theta(\bar{\pi}_{n-2,n+2}).$$

From these equations (25) follows.

(vii) We will use again the backwards induction on k. Suppose $k = 2n - 4$; then we want to prove that

$$\Theta(\pi_{n-3,n-1}) = \Theta(\bar{\pi}_{n-3,n-1}) + \Theta(\bar{\pi}_{n-2,n-1}) + \Theta(\bar{\pi}_{n-3,n}) + \Theta(\bar{\pi}_{n-2,n}) + \Theta(\bar{\pi}_{n-1,n}).$$

$\gamma_{n-3,n-1} = \gamma_{n-3,n} - \bar{\alpha}$ with $\bar{\alpha} = e_{n+1} - e_2$ and $\alpha = (\phi_{n-3,n})^{-1}$, $\bar{\alpha} = \varepsilon_n - \varepsilon_{n+1}$. Then, by (vi)

$$\Theta(\pi_{n-3,n-1}) = S_{-\alpha}\Theta(\pi_{n-3,n}) = S_{-\alpha}\Theta(\bar{\pi}_{n-3,n}) + S_{-\alpha}\Theta(\bar{\pi}_{n-3,n+1}) + S_{-\alpha}\Theta(\bar{\pi}_{n-2,n}) + S_{-\alpha}\Theta(\bar{\pi}_{n-2,n+1}).$$

α is in the τ-invariants of $\bar{\pi}_{n-3,n+1}$ and $\bar{\pi}_{n-2,n+1}$ while it is not in those of $\bar{\pi}_{n-3,n}$, $\bar{\pi}_{n-2,n}$. Thus $\Theta(U_\alpha(\bar{\pi}_{n-3,n})) = \Theta(\bar{\pi}_{n-3,n-1}) + \Theta_0$. Candidates for Θ_0 are $\Theta(\bar{\pi}_{n-3,n+1})$

and $\Theta(\bar{\pi}_{n-2,n+1})$. Using $\beta = 2\varepsilon_{n+1}$ we get that $\Theta_0 = \Theta(\bar{\pi}_{n-3,n+1})$. $\phi_{n-2,n}(\alpha) = e_{n+1} - e_2$ and $\gamma_{n-2,n} - (\phi_{n-2,n}\alpha) = \gamma_{n-2,n-1}$ so $\Theta(U_\alpha(\bar{\pi}_{n-2,n})) = \Theta(\bar{\pi}_{n-2,n-1}) + \Theta_0$. Combining the character identity for $\Theta(\bar{\pi}_{n-2,n})$ and Proposition 3.2 we conclude that candidates for Θ_0 are $\Theta(\bar{\pi}_{n-1,n})$, $\Theta(\bar{\pi}_{n-2,n+1})$ and $\Theta(\bar{\pi}_{n-1,n+1})$. Using $\beta = 2\varepsilon_{n+1}$ for $\bar{\pi}_{n-1,n+1}$ and $\bar{\pi}_{n-2,n+1}$ and $\beta = \varepsilon_{n-1} - \varepsilon_n$ for $\bar{\pi}_{n-1,n}$ and $\bar{\pi}_{n-2,n}$ we conclude that $\Theta_0 = \Theta(\bar{\pi}_{n-2,n+1}) + \Theta(\bar{\pi}_{n-1,n})$ and $\Theta(U_\alpha(\bar{\pi}_{n-2,n})) = \Theta(\bar{\pi}_{n-2,n-1}) + \Theta(\bar{\pi}_{n-2,n+1}) + \Theta(\bar{\pi}_{n-1,n})$.

The desired result is now straightforward.

Suppose now that (vii) is true for $t \geqslant k$ and prove it for $k-1$. We have to consider the two possibilities

(a) $k-1$ odd.

(b) $k-1$ even.

We first consider (a); $1 \leqslant k-1 \leqslant 2n-5$ and $i = k/2 - 1$ and we want to prove $\Theta(\pi_{i,k-i-1}) = \Theta(\bar{\pi}_{i,k-i-1}) + \Theta(\bar{\pi}_{i,k-i}) + \Theta(\bar{\pi}_{i+2,k-i+1})$.

$\gamma_{i,k-i-1} = \gamma_{i,k-i} - \bar{\alpha}$, where $\bar{\alpha} = e_{k-i+1} - e_2$ and $\alpha = (\phi_{i,k-i})^{-1}\bar{\alpha} = \varepsilon_{k-i} - \varepsilon_{k-i+1}$. Therefore, by the induction hypothesis

$$\begin{aligned}\Theta(\pi_{i,k-i-1}) = S_{-\alpha}\Theta(\pi_{i,k-1}) &= S_{-\alpha}\Theta(\bar{\pi}_{i,k-i}) + S_{-\alpha}\Theta(\bar{\pi}_{i+1,k-i})\\ &+ S_{-\alpha}\Theta(\bar{\pi}_{i,k-i+1}) + S_{-\alpha}\Theta(\bar{\pi}_{i+1,k-i+1}) + S_{-\alpha}\Theta(\bar{\pi}_{i+2,k-i+1}).\end{aligned}$$

Since α is in the τ-invariants of $\bar{\pi}_{i+1,k-i}$, $\bar{\pi}_{i,k-i+1}$ and $\bar{\pi}_{i+1,k-i+1}$ we have

$$\begin{aligned}S_{-\alpha}\Theta(\bar{\pi}_{i+1,k-i}) &= -\Theta(\bar{\pi}_{i+1,k-i}),\\ S_{-\alpha}\Theta(\bar{\pi}_{i,k-i+1}) &= -\Theta(\bar{\pi}_{i,k-i+1}),\\ S_{-\alpha}\Theta(\bar{\pi}_{i+1,k-i+1}) &= -\Theta(\bar{\pi}_{i+1,k-i+1}),\end{aligned}$$

$\alpha \notin (\bar{\pi}_{i,k-i})$; hence

$$\Theta(U_\alpha(\bar{\pi}_{i,k-i})) = \Theta(\bar{\pi}_{i,k-i-1}) + \Theta_0.$$

Candidates for Θ_0 are $\Theta(\bar{\pi}_{i+1,k-i})$, $\Theta(\bar{\pi}_{i,k-i+1})$ and $\Theta(\bar{\pi}_{i+1,k-i+1})$. Using $\beta = \varepsilon_{i+1} - \varepsilon_{i+2}$ for $\bar{\pi}_{i+1,k-i}$ and $\bar{\pi}_{i+1,k-i+1}$, and $\beta = \varepsilon_{k-i+1} - \varepsilon_{k-i+2}$ for $\bar{\pi}_{i,k-i+1}$ we get that $\Theta_0 = \Theta(\bar{\pi}_{i,k-i+1}) + \Theta(\bar{\pi}_{i+1,k-i})$; hence $S_{-\alpha}\Theta(\bar{\pi}_{i,k-i}) = \Theta(\bar{\pi}_{i,k-i}) + \Theta(\bar{\pi}_{i,k-i-1}) + \Theta(\bar{\pi}_{i,k-i+1}) + \Theta(\bar{\pi}_{i+1,k-i})$.

Let us now compute $\Theta(U_\alpha(\bar{\pi}_{i+2,k-i+1}))$. It is easily obtained that

$$\Theta(U_\alpha(\bar{\pi}_{i+2,k-i+1})) = \Theta(\bar{\pi}_{i+1,k-i+1}) + \Theta_0. \tag{26}$$

To find candidates for Θ_0 we must distinguish three possibilities

(1) $k = 2n-4$,

(2) $k = 2n-6$,

(3) $k \leqslant 2n-8$.

If $k = 2n-4$ then (26) becomes $\Theta(U_\alpha(\bar{\pi}_{n-1,n})) = \Theta(\bar{\pi}_{n-2,n}) + \Theta_0$ and $\alpha = \varepsilon_{n-1} - \varepsilon_n$.

By (iii) together with Proposition 3.2 we are forced to conclude that $\Theta_0 = 0$. If $k = 2n-6$ then (26) becomes $\Theta(U_\alpha(\bar{\pi}_{n-2,n-1})) = \Theta(\bar{\pi}_{n-3,n-1}) + \Theta_0$ and $\alpha = \varepsilon_{n-2} - \varepsilon_{n-1}$.

By (v) together with Proposition 3.2, the only candidate for Θ_0 is $\Theta(\pi_n)$. Using $\beta = \varepsilon_n - \varepsilon_{n+1}$ the usual multiplicity argument gives $\Theta_0 = 0$. If $k \leqslant 2n-8$ then by the induction hypothesis the only candidate for Θ_0 is $\Theta(\bar{\pi}_{i+4,k-i+3})$. Using $\beta = \varepsilon_{i+4} - \varepsilon_{i+5}$ we get again $\Theta_0 = 0$.

So in each case $\Theta(U_\alpha(\bar{\pi}_{i+2,k-i+1})) = \Theta(\bar{\pi}_{i+1,k-i+1})$. It is now a simple matter to get the required results.

The proof in the case $k - 1$ even goes on without any problem and we omit it. The theorem is thus completely proved.

4. Composition factors for singular infinitesimal characters. In this section we shall consider the reducible principal series representation $\pi(\gamma)$ such that $\gamma \in D^+$ is singular. By Theorem 2.2, the reducibility of $\pi(\gamma)$ implies that γ is integral. So we have to consider γ, contained in at least one wall, such that

$$\gamma = \sum_{j=1}^{n+1} a_j e_j, \tag{1}$$

$$a_j \in \mathbf{Z}, \qquad 1 \leqslant j \leqslant n+1, \tag{2}$$

$$a_1 > a_2, \quad a_1 \geqslant -a_2, \quad a_3 > \cdots > a_{n+1} > 0. \tag{3}$$

If γ satisfies (1), (2) and (3), then it is easy to see that γ can be contained in at most two walls.

In the proof of the character identities we shall use very often a result from **[12]**. For the convenience of the reader we shall state this in terms of our notation. Let $\lambda \in \mathfrak{t}_{\mathbf{C}}^*$ be integral and contained in the wall separating $\tilde{C}_i$ from $\tilde{C}_{i+1}$ $(0 \leqslant i \leqslant n)$. We denote by $\pi^+(\lambda)$ (resp. $\pi^-(\lambda)$) the limit of discrete series representation with infinitesimal character χ_λ and determined by $\tilde{C}_{i+1}$ (resp. $\tilde{C}_i$) (see [7] and **[16]**).

Let $\tilde{C}$ be a Weyl chamber in $i\mathfrak{t}^*$ (or $(i\mathfrak{h}^- + \mathfrak{a})^*$). Let γ be dominant with respect to C. Let γ be the highest weight (with respect to C) of an irreducible finite dimensional representation of G. Suppose that $\gamma + \mu$ is strictly dominant with respect to C. Let $\psi_\lambda^{\lambda+\mu}$ denote the Zuckerman's functor as introduced in §2. Recall that for the dual τ of the restriction to $\mathfrak{t}_{\mathbf{C}}$ (or $\mathfrak{h}_{\mathbf{C}}$) of any inner automorphism of $\mathfrak{g}_{\mathbf{C}}$, $\psi_{\tau\gamma}^{\tau\gamma+\tau\mu} = \psi_\gamma^{\gamma+\mu}$. Especially, for w in the Weyl group $\psi_{w\gamma}^{w\gamma+w\mu} = \psi_\gamma^{\gamma+\mu}$.

THEOREM 4.1 [**12**, THEOREM 6.18 AND COROLLARY 5.12]. (a) *Let $\tilde{Q}$ be a chamber in $\mathfrak{h}_{\mathbf{C}}^*$, $\gamma \in \mathfrak{h}_{\mathbf{C}}^*$ integral and dominant* (*with respect to Q*). *Let μ be highest weight* (*with respect to Q*) *of an irreducible finite dimensional representation of G such that $\gamma + \mu$ is strictly dominant with respect to Q. Then*

$$\psi_\gamma^{\gamma+\mu}(\pi(\gamma + \mu)) = \pi(\gamma).$$

If there exists a root α simple for Q such that $(\alpha, \gamma) = 0$ and either compact imaginary, or real, or complex with $\Theta\alpha$ negative, then

$$\psi_\gamma^{\gamma+\mu}(\bar{\pi}(\gamma + \mu)) = 0.$$

If such a root does not exist, then $\psi_\gamma^{\gamma+\mu}(\bar{\pi}(\gamma + \mu)) = \bar{\pi}(\gamma)$.

(b) *Let $\lambda \in \mathfrak{t}_{\mathbf{C}}^*$ integral and contained in the wall separating $\tilde{C}_i$ from $\tilde{C}_{i+1}$ $(0 \leqslant i \leqslant n-1)$. Let μ be the highest weight* (*with respect to C_i, C_{i+1}, respectively*) *of an irreducible finite dimensional representation of G, such that $\lambda + \mu$ is strictly dominant with respect to C_i, C_{i+1}, respectively. If there is a compact root α simple for C_i, C_{i+1},* *respectively, such that* $(\alpha, \lambda) = 0$, *then*

$$\psi_\lambda^{\lambda+\mu}(\pi(\lambda + \mu)) = 0.$$

If such a root does not exist, then $\psi_\lambda^{\lambda+\mu}(\pi(\lambda + \mu)) = \pi^-(\lambda)$, $\psi_\lambda^{\lambda+\mu}(\pi(\lambda + \mu)) = \pi^+(\lambda)$, *respectively.*

Let now $\gamma = \sum_{i=1}^{n+1} a_i e_i \in D^+$ be integral and contained in only one wall. We have the following possibilities:

(I) $a_2 = -a_1$, $a_1 \neq a_j$, $3 \leq j \leq n+1$, $a_1 > 0$.

(II) $a_2 = 0$, $a_1 > 0$, $a_1 \neq a_j$, $3 \leq j \leq n+1$.

(III) $a_1 = a_i$ for some $i \in \{3, \ldots, n+1\}$, $a_2 > 0$, $a_2 \neq a_j$, $3 \leq j \leq n+1$, $a_1 > a_2$.

(IV) $a_1 = a_i$ for some $i \in \{3, \ldots, n+1\}$, $a_2 < 0$, $-a_2 \neq a_j$, $3 \leq j \leq n+1$, $a_1 > -a_2$.

(V) $a_2 = a_i$ for some $i \in \{3, \ldots, n+1\}$, $a_1 \neq a_j$, $3 \leq j \leq n+1$, $a_1 > a_2$.

(VI) $a_2 = -a_i$ for some $i \in \{3, \ldots, n+1\}$, $a_1 \neq a_j$; $3 \leq j \leq n+1$, $a_1 > -a_2$.

Let such γ be given. If γ is of type (I), (II), (III), (IV), (V), (VI), let b denote a_1, a_1, a_2, $-a_2$, a_1, a_1, respectively. Let $j \in \{1, \ldots, n\}$ be defined as follows: if $b > a_3$ we set $j = 1$; if $a_{n+1} > b$ we set $j = n$; otherwise, j is the unique integer such that $a_{j+1} > b > a_{j+2}$.

In what follows, τ (or τ_1) will always denote the dual of the restrictions to $t_{\mathbf{C}}$ of an inner automorphism of $\mathfrak{g}_{\mathbf{C}}$ carrying $t_{\mathbf{C}}$ to $\mathfrak{h}_{\mathbf{C}}$ and such that $\tau(P) = C$, where P and C will be specified systems of positive roots in ϕ and Δ.

If α is a root, w_α will denote the reflection with respect to α.

THEOREM 4.2. *Let* $\gamma = \sum_{j=1}^{n+1} a_j e_j \in D^+$ *be integral and contained in exactly one wall. Let j be defined as above.*

(i) *If* γ *is type* I, *then*

$$\Theta(\pi(\gamma)) = \Theta(\pi^-(\tau\gamma)) + \Theta(\pi^+(\tau\gamma)), \qquad \tau(P_{j-1,2n-j-1}) = C_{j-1}.$$

(ii) *If* γ *is of type* II *and* $j = n$, $\pi(\gamma)$ *is irreducible.*

(iii) *If* γ *is of type* V, $i = n+1$, $j = n-1$, *then*

$$\Theta(\pi(\gamma)) = \Theta(\bar{\pi}(\gamma)) + \Theta(\pi^+(\tau\gamma)), \qquad \tau(P_{n-2,n}) = C_n.$$

(iv) *If* γ *is of type* V, $i \leq n$, $j = i-2$, *then*

$$\Theta(\pi(\gamma)) = \Theta(\bar{\pi}(\gamma)) + \Theta(\bar{\pi}(w\gamma))$$

where $w = w_{e_1-e_i} w_{e_2-e_{i+1}}$ (*i.e.* $wP_{i-3,i-1} = P_{i-1,i}$).

(v) *If* γ *is of type* VI *and* $j = i-2$, *then*

$$\Theta(\pi(\gamma)) = \Theta(\bar{\pi}(\gamma)) + \Theta(\pi^-(\tau\gamma)), \qquad \tau P_{i-3,2n+2-i} = C_{i-2}.$$

(vi) *If* γ *is of type* IV *and* $j = i-1$, *then*

$$\Theta(\pi(\gamma)) = \Theta(\bar{\pi}(\gamma)) + \Theta(\pi^+(\tau\gamma)), \qquad \tau P_{i-2,2n+2-i} = C_{i-2}.$$

(vii) *In all the other cases*

$$\Theta(\pi(\gamma)) = \Theta(\bar{\pi}(\gamma)) + \Theta(\bar{\pi}(w\gamma))$$

where

$$w = \begin{cases} w_{e_1 - e_{j+2}}, & \gamma \textit{ of type } \mathrm{II, V, VI}, \\ w_{e_2 - e_{j+2}}, & \gamma \textit{ of type } \mathrm{III}, j \leqslant n-1, \\ w_{2e_2}, & \gamma \textit{ of type } \mathrm{III}, j = n, \\ w_{e_{j+1}+e_2}, & \gamma \textit{ of type } \mathrm{IV}. \end{cases}$$

PROOF. The method of the proof is the following. For each such γ we specify a system of positive roots P in Φ with respect to which γ is dominant. Then we choose an irreducible finite dimensional representation of G with highest weight μ (with respect to P) so that $\gamma + \mu$ is strictly dominant with respect to P. Using Theorem 3.6 we know the composition factors of $\pi(\gamma + \mu)$. All of them are of the form $\bar{\pi}_{i,j}(\chi)$ or $\pi_k(\chi)$, where $\chi = \chi_{\gamma+\mu}$. For each composition factor π we use Theorem 4.1 to compute $\psi_\gamma^{\gamma+\mu}(\pi)$. Since $\psi_\gamma^{\gamma+\mu}(\pi(\gamma + \mu)) = \pi(\gamma)$, we shall get the required character identities due to the exactness of the functor $\psi_\gamma^{\gamma+\mu}$.

Let γ be of type I. Take $P = P_{j-1,2n-j+1}$, $\mu = e_1 + \sum_{k=3}^{j+1} e_k$ ($\mu = e_1$ if $j = 1$). By Theorem 3.6(i) we have (with $\chi = \chi_{\gamma+\mu}$)

$$\Theta(\pi(\gamma + \mu)) = \Theta(\pi_{j-1,2n-j+1}(\chi)) = \Theta(\bar{\pi}_{j-1,2n-j+1}(\chi)) + \Theta(\bar{\pi}_{j-1}(\chi)) + \Theta(\bar{\pi}_j(\chi)).$$

Therefore,

$$\Theta(\pi(\gamma)) = \Theta\big(\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{j-1,2n-j+1}(\chi))\big) + \Theta\big(\psi_\gamma^{\gamma+\mu}(\pi_{j-1}(\chi))\big) + \Theta\big(\psi_\gamma^{\gamma+\mu}(\pi_j(\chi))\big).$$

$\tilde{\beta} = e_1 + e_2$ is simple for $P_{j-1,2n-j-1}$, it is real and orthogonal to γ. Therefore, $\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{j-1,2n-j+1}(\chi)) = \psi_\gamma^{\gamma+\mu}(\bar{\pi}(\gamma + \mu)) = 0$.

Let $\tau P_{j-1,2n-j-1} = C_{j-1}$. Then $\tau e_1 = \varepsilon_{n+1}$, $\tau e_2 = \varepsilon_j$, $\tau e_k = \varepsilon_{k-2}$, $3 \leqslant k \leqslant j+1$, $\tau e_k = \varepsilon_{k-1}, j+2 \leqslant k \leqslant n+1$; hence

$$\tau\gamma = \sum_{k=1}^{j-1} a_{k+2}\varepsilon_k - a_2\varepsilon_j + \sum_{k=j+1}^{n} a_{k+1}\varepsilon_k + a_1\varepsilon_{n+1}.$$

$\varepsilon_{n+1} - \varepsilon_j$ is the only root simple for C_{j-1} and orthogonal to $\tau\gamma$. But $\varepsilon_{n+1} - \varepsilon_j$ is noncompact; therefore

$$\psi_\gamma^{\gamma+\mu}\big(\pi_{j-1}(\chi)\big) = \psi_{\tau\gamma}^{\tau(\gamma+\mu)}(\pi(\tau(\gamma + \mu))) = \pi^-(\tau\gamma).$$

Let $\tau_1 P_{j-1,2n-j+1} = C_j$. Then $\tau_1\gamma = \tau\gamma$. $\varepsilon_j - \varepsilon_{n+1}$ is the only root simple for C_j and orthogonal to $\tau_1\gamma$. But $\varepsilon_j - \varepsilon_{n+1}$ is noncompact; therefore

$$\psi_\gamma^{\gamma+\mu}\big(\pi_j(\chi)\big) = \psi_{\tau_1\gamma}^{\tau_1(\gamma+\mu)}(\pi(\tau_1(\gamma + \mu))) = \pi^+(\tau_1\gamma) = \pi^+(\tau\gamma).$$

In this way the theorem is proven if γ is of type I.

Let γ be of type II. Take $P = P_{j-1,n}$, $\mu = \sum_{k=1}^{n+1} e_j$, $\chi = \chi_{\gamma+\mu}$. We distinguish three cases:

(a) $j \leqslant n-2$. By Theorem 3.6(vi) we get

$$\Theta(\pi(\gamma)) = \sum_{a,b=0}^{1} \Theta\big(\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{j-1+a,n+b}(\chi))\big).$$

$2e_2$ is simple for $P_{j-1,n}$, complex, $\theta(2e_2) = -2e_1$ is negative for $P_{j-1,n}$ and

$(2e_2, \gamma) = 0$. Therefore,

$$\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{j-1,n}(\chi)) = \psi_\gamma^{\gamma+\mu}(\bar{\pi}(\gamma+\mu)) = 0.$$

Similarly, $\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{j,n}(\chi)) = \psi_{w\gamma}^{w(\gamma+\mu)}(\bar{\pi}(w(\gamma+\mu))) = 0$ where $w = w_{j,n}w_{j-1,n}^{-1} = w_{e_1-e_{j+2}}$ (use again the root $2e_2$).

$\bar{\pi}_{j-1,n+1}(\chi) = \bar{\pi}(w(\gamma+\mu))$, where $w = w_{j-1,n+1}w_{j-1,n}^{-1} = w_{2e_2}$. The only root simple for $P_{j-1,n+1}$ and orthogonal to $w\gamma = \gamma$ is $-2e_2$. Since $\theta(-2e_2) = 2e_1$ is positive for $P_{j-1,n+1}$, we get $\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{j-1,n+1}(\chi)) = \psi_{w\gamma}^{w(\gamma+\mu)}(\bar{\pi}(w(\gamma+\mu))) = \bar{\pi}(w\gamma) = \bar{\pi}(\gamma)$.

$\bar{\pi}_{j,n+1}(\chi) = \bar{\pi}(w_1(\gamma+\mu))$, where $w_1 = w_{e_1-e_{j+2}}w_{2e_2}$. Exactly as above we find

$$\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{j,n+1}(\chi)) = \psi_{w_1\gamma}^{w_1(\gamma+\mu)}(\bar{\pi}(w_1(\gamma+\mu))) = \bar{\pi}(w_1\gamma) = \bar{\pi}(w\gamma)$$

where $w = w_{e_1-e_{j+2}}$.

(b) $j = n-1$. By Theorem 3.6(iv) we get

$$\Theta(\pi(\gamma)) = \sum_{a,b=0}^{1} \Theta(\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{n-2+a,n+b}(\chi))) + \Theta(\psi_\gamma^{\gamma+\mu}(\pi_n(\chi))).$$

Exactly as before we obtain the following identities:

$$\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{n-2,n}(\chi)) = 0,$$
$$\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{n-1,n}(\chi)) = 0,$$
$$\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{n-2,n+1}(\chi)) = \bar{\pi}(\gamma),$$
$$\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{n-1,n+1}(\chi)) = \bar{\pi}(w_1\gamma) = \bar{\pi}(w\gamma),$$

where $w_1 = w_{e_1-e_{n+1}}w_{2e_2}$, $w = w_{e_1-e_{n+1}}$.

$\pi_n(\chi) = \pi(\tau(\gamma+\mu))$, where $\tau P_{n-2,n} = C_n$. Then

$$\tau\gamma = \sum_{i=1}^{n-2} a_{i+2}\varepsilon_i + a_1\varepsilon_{n-1} + a_{n+1}\varepsilon_n + a_2\varepsilon_{n+1}.$$

The compact root $2\varepsilon_{n+1}$ is simple for C_n and orthogonal to $\tau\gamma$; therefore

$$\psi_\gamma^{\gamma+\mu}(\pi_n(\chi)) = \psi_{\tau\gamma}^{\tau(\gamma+\mu)}(\pi(\tau(\gamma+\mu))) = 0.$$

(c) $j = n$. By Theorem 3.6(iii) we get

$$\Theta(\pi(\gamma)) = \Theta(\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{n-1,n}(\chi))) + \Theta(\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{n-1,n+1}(\chi))).$$

It is easy to see that $\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{n-1,n}(\chi)) = 0$ and $\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{n-1,n+1}(\chi)) = \bar{\pi}(\gamma)$. Therefore $\Theta(\pi(\gamma)) = \Theta(\bar{\pi}(\gamma))$, i.e. $\pi(\gamma)$ is irreducible.

Therefore, the theorem is true in case γ is of type II.

Let γ be of type III. Take $P = P_{i-3,j}$, $\mu = \sum_{k=3}^{i-1} e_k + e_1$, $\chi = \chi_{\gamma+\mu}$. We have to distinguish three cases:

(a) $j \geqslant i$. By Theorem 3.6(vi) we get

$$\Theta(\pi(\gamma)) = \sum_{a,b=0}^{1} \Theta(\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{i-3+a,j+b}(\chi))).$$

Since $e_1 - e_i$ is a complex root simple for $P_{i-3,j}$, orthogonal to γ and $\theta(e_1 - e_i) \notin P_{i-3,j}$, it follows that $\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{i-3,j}(\chi)) = 0$.

$\bar{\pi}_{i-3,j+1}(\chi) = \bar{\pi}(w(\gamma+\mu))$ where $w = w_{e_2-e_{j+2}}$ if $j \leqslant n-1$ and $w = w_{2e_2}$ if $j = n$. By means of the root $e_1 - e_i$ we easily get $\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{i-3,j+1}(\chi)) = 0$.

$\bar{\pi}_{i-2,j}(\chi) = \bar{\pi}(w(\gamma + \mu))$ where $w = w_{e_1-e_i}$ and $w\gamma = \gamma$. The only root simple for $P_{i-2,j}$ and orthogonal to $w\gamma$ is $e_i - e_1$. But it is complex and $\theta(e_i - e_1) \in P_{i-2,j}$. Therefore,

$$\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{i-2,j}(\chi)) = \bar{\pi}(w\gamma) = \bar{\pi}(\gamma).$$

$\bar{\pi}_{i-2,j+1}(\chi) = \bar{\pi}(w_1(\gamma + \mu))$, where $w_1 = w_{e_2-e_{j+2}}w_{e_1-e_i}$ if $j \leqslant n-1$ and $w_1 = w_{2e_2}w_{e_1-e_i}$ if $j = n$. Set $w = w_{e_2-e_{j+2}}$ if $j \leqslant n-1$ and $w = w_{2e_2}$ if $j = n$. We have $w_1\gamma = w\gamma$. The only root simple for $P_{i-2,j+1}$ and orthogonal to $w_1\gamma$ is $e_i - e_1$.

It is complex and $\theta(e_i - e_1) \in P_{i-2,j+1}$; hence

$$\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{i-2,j+1}(\chi)) = \bar{\pi}(w_1\gamma) = \bar{\pi}(w\gamma).$$

(b) $j = i-1$, $i \leqslant n$. By Theorem 3.6(vii) we get

$$\Theta(\pi(\gamma)) = \sum_{a,b=0}^{1} \Theta(\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{i-3+a,i-1+b}(\chi))) + \Theta(\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{i-1,i}(\chi))).$$

One easily obtains that

$\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{i-3,i-1}(\chi)) = 0$ (use the root $e_1 - e_i$),

$\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{i-3,i}(\chi)) = 0$ (use the root $e_1 - e_i$ simple for $P_{i-3,i} = w_{e_2-e_{i+1}}P_{i-3,i-1}$),

$\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{i-2,i-1}(\chi)) = \bar{\pi}(\gamma)$,

$\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{i-2,i}(\chi)) = \bar{\pi}(w_1\gamma) = \bar{\pi}(w\gamma)$, $w_1 = w_{e_2-e_{i+1}}w_{e_i-e_1}$, $w = w_{e_2-e_{i+1}}$.

$\bar{\pi}_{i-1,i}(\chi) = \bar{\pi}(w(\gamma + \mu))$ for $wP_{i-3,i-1} = P_{i-1,i}$. The compact root $e_i - e_{i+1}$ is simple for $P_{i-1,i}$ and orthogonal to $w\gamma$. Therefore $\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{i-1,i}(\chi)) = 0$.

(c) $j = n$, $i = n+1$. By Theorem 3.6(iv)

$$\Theta(\pi(\gamma)) = \sum_{a,b=0}^{1} \Theta(\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{n-2+a,n+b}(\chi))) + \Theta(\psi_\gamma^{\gamma+\mu}(\pi_n(\chi))).$$

In the same way as in the case (b) (only using $2e_2$ everywhere instead of $e_2 - e_{i+1}$) we get

$$\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{n-2,n}(\chi)) = 0,$$
$$\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{n-2,n+1}(\chi)) = 0,$$
$$\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{n-1,n}(\chi)) = \bar{\pi}(\gamma),$$
$$\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{n-1,n+1}(\chi)) = \bar{\pi}(w\gamma), \qquad w = w_{2e_2}.$$

We have $\pi_n(\chi) = \pi(\tau(\gamma + \mu))$ where $\tau P_{n-2,n} = C_n$. The compact root $\varepsilon_{n-1} - \varepsilon_n$ is simple for C_n and orthogonal to $\tau\gamma$. Therefore, $\psi_\gamma^{\gamma+\mu}(\pi_n(\chi)) = 0$.

Therefore, the theorem holds true also if γ is of type III.

Let γ be of type IV. Set $P = P_{i-3,2n+1-j}$, $\mu = e_1\Sigma_{k=3}^{i-1} e_k$, $\chi = \chi_{\gamma+\mu}$. We have to distinguish two cases.

(a) $j \geqslant i$. In this case we have by Theorem 3.6(vi)

$$\Theta(\pi(\gamma)) = \sum_{a,b=0}^{1} \Theta(\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{i-3+a,2n+1-j+b}(\chi))).$$

The complex root $e_1 - e_i$ is simple for $P_{i-3,2n+1-j}$, orthogonal to γ and $\theta(e_1 - e_i) \notin P_{i-3,2n+1-j}$. Therefore, $\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{i-3,2n+1-j}(\chi)) = 0$.

$\bar{\pi}_{i-3,2n+2-j}(\chi) = \bar{\pi}(w(\gamma + \mu))$, $w = w_{e_2+e_{j+1}}$. $e_1 - e_i$ is again simple for $P_{i-3,2n+2-j}$, orthogonal to $w\gamma$ and $\theta(e_1 - e_i) \notin P_{i-3,2n+2-j}$. Therefore, $\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{i-3,2n+2-j}(\chi)) = 0$.

$\bar{\pi}_{i-2,2n+1-j}(\chi) = \bar{\pi}(w(\gamma + \mu))$ where $w = w_{e_1-e_i}$. We have $w\gamma = \gamma$. $e_i - e_1$ is the only root simple for $P_{i-2,2n+1-j}$ and orthogonal to γ. But $e_i - e_1$ is complex and $\theta(e_i - e_1) \in P_{i-2,2n+1-j}$. Hence, $\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{i-2,2n+1-j}(\chi)) = \bar{\pi}(\gamma)$.

$\bar{\pi}_{i-2,n+2-j}(\chi) = \bar{\pi}(w_1(\gamma + \mu))$, where $w_1 = w_{e_1-e_i}w_{e_1+e_{j+1}}$. Set $w = w_{e_2+e_{j+1}}$. Since $w_1\gamma = w\gamma$ we get easily that

$$\psi_\gamma^{\gamma+\mu}\big(\bar{\pi}_{i-2,2n+2-j}(\chi)\big) = \bar{\pi}(w_1\gamma) = \bar{\pi}(w\gamma).$$

(b) $j = i - 1$. By Theorem 3.6(ii) we have

$$\begin{aligned}\Theta(\pi(\gamma)) &= \Theta\big(\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{i-3,2n+2-i}(\chi))\big) + \Theta\big(\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{i-3,2n+3-i}(\chi))\big)\\ &\quad + \Theta\big(\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{i-2,2n+2-i}(\chi))\big) + \Theta\big(\psi_\gamma^{\gamma+\mu}(\pi_{i-2}(\chi))\big).\end{aligned}$$

In a quite similar way as above we get:

$$\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{i-3,2n+2-i}(\chi)) = 0, \qquad \psi_\gamma^{\gamma+\mu}(\bar{\pi}_{i-3,2n+3-i}(\chi)) = 0$$

(use the real root $e_1 + e_2$ which is simple for $P_{i-3,2n+3-i} = w_{e_i+e_2}P_{i-3,2n+2-i}$)

$$\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{i-2,2n+2-i}(\chi)) = \bar{\pi}(\gamma).$$

$\pi_{i-2}(\chi) = \pi(\tau(\gamma + \mu))$, where $\tau P_{i-3,2n+2-i} = C_{i-2}$, we have

$$\tau\gamma = \sum_{k=1}^{i-3} a_{k+2}\varepsilon_k + a_1\varepsilon_{i-2} - a_2\varepsilon_{i-1} + \sum_{k=i}^{n} a_{k+1}\varepsilon_k + a_i\varepsilon_{n+1}.$$

$\varepsilon_{i-2} - \varepsilon_{n+1}$ is the only root simple for C_{i-2} and orthogonal to $\tau\gamma$. But it is noncompact; hence $\psi_\gamma^{\gamma+\mu}(\pi_{i-2}(\chi)) = \pi^+(\tau\gamma)$.

Therefore, the theorem is proven in case IV too.

Let γ be of type V. Set $P = P_{j-1,i-1}$, $\mu = e_1 + \sum_{k=3}^{i} e_k$, $\chi = \chi_{\gamma+\mu}$. We have to distinguish three cases.

(a) $j < i - 2$. By Theorem 3.6(vi) we have

$$\Theta(\pi(\gamma)) = \sum_{a,b=0}^{1} \Theta\big(\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{j-1+a,i-1+b}(\chi))\big).$$

The only root simple for P and orthogonal to γ is the complex root $e_i - e_2$. But $\theta(e_i - e_2)$ is in P; therefore $\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{j-1,i-1}(\chi)) = \bar{\pi}(\gamma)$.

$\bar{\pi}_{j,i-1}(\chi) = \bar{\pi}(w(\gamma + \mu))$ where $w = w_{e_1-e_{j+2}}$. The only root simple for $P_{j,i-1}$ and orthogonal to $w\gamma$ is $e_i - e_2$. But $\theta(e_i - e_2) \in P_{j,i-1}$; therefore $\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{j,i-1}(\chi)) = \bar{\pi}(w\gamma)$.

Finally, we easily find

$$\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{j-1,i}(\chi)) = 0, \qquad \psi_\gamma^{\gamma+\mu}(\bar{\pi}_{j,i}(\chi)) = 0.$$

(In both cases use the root $e_i - e_{i+1}$ if $i \leqslant n$ and $e_2 + e_{n+1}$ if $i = n + 1$.)

(b) $j = i - 2, i \leqslant n$. By Theorem 3.6(vii) we get

$$\Theta(\pi(\gamma)) = \sum_{a,b=0}^{1} \Theta\big(\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{i-3+a,i-1+b}(\chi))\big) + \Theta\big(\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{i-1,i}(\chi))\big).$$

We easily get as before

$$\begin{aligned} \psi_\gamma^{\gamma+\mu}(\bar{\pi}_{i-3,i-1}(\chi)) &= \bar{\pi}(\gamma), \\ \psi_\gamma^{\gamma+\mu}(\bar{\pi}_{i-3,i}(\chi)) &= 0, \\ \psi_\gamma^{\gamma+\mu}(\bar{\pi}_{i-2,i}(\chi)) &= 0 \qquad \text{(use the complex root } e_1 - e_{i+1}), \\ \psi_\gamma^{\gamma+\mu}(\bar{\pi}_{i-2,i-1}(\chi)) &= 0 \qquad \text{(use the compact root } e_1 - e_2). \end{aligned}$$

Finally, $\bar{\pi}_{i-1,i}(\chi) = \bar{\pi}(w(\gamma + \mu))$, $w = w_{e_1 - e_i} w_{e_2 - e_{i+1}}$. The only root simple for $P_{i-1,i}$ and orthogonal to $w\gamma$ is the complex root $e_{i+1} - e_1$. But $\theta(e_{i+1} - e_1) \in P_{i-1,i}$; therefore $\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{i-1,i}(\chi)) = \bar{\pi}(w\gamma)$.

(c) $j = n - 1, i = n + 1$. By Theorem 3.6(iv)

$$\Theta(\pi(\gamma)) = \sum_{a,b=0}^{1} \Theta(\psi^{\gamma+\mu}(\bar{\pi}_{n-2+a,n+b}(\chi))) + \Theta\big(\psi_\gamma^{\gamma+\mu}(\pi_n(\chi))\big).$$

By similar arguments we easily get:

$$\begin{aligned} \psi_\gamma^{\gamma+\mu}(\bar{\pi}_{n-2,n}(\chi)) &= \bar{\pi}(\gamma), \\ \psi_\gamma^{\gamma+\mu}(\bar{\pi}_{n-2,n+1}(\chi)) &= 0 \qquad \text{(use the root } e_{n+1} + e_2), \\ \psi_\gamma^{\gamma+\mu}(\bar{\pi}_{n-1,n}(\chi)) &= 0 \qquad \text{(use the root } e_1 - e_2), \\ \psi_\gamma^{\gamma+\mu}(\bar{\pi}_{n-1,n+1}(\chi)) &= 0 \qquad \text{(use the root } e_1 + e_2). \end{aligned}$$

$\pi_n(\chi) = \pi(\tau(\gamma + \mu))$ where $\tau P_{n-2,n} = C_n$. Then

$$\tau\gamma = \sum_{k=1}^{n-2} a_{k+2}\varepsilon_k + a_1\varepsilon_{n-1} + a_{n+1}\varepsilon_n + a_2\varepsilon_{n+1}.$$

The noncompact root $\varepsilon_n - \varepsilon_{n+1}$ is the only root simple for C_n and orthogonal to $\tau\gamma$. Therefore $\psi_\gamma^{\gamma+\mu}(\pi_n(\chi)) = \pi^+(\tau\gamma)$.

Therefore, in case V the theorem is also true.

Finally, let γ be of type VI. Set $P = P_{j-1,2n+2-i}$, $\mu = e_1 + \sum_{k=3}^{i} e_i$, $\chi = \chi_{\gamma+\mu}$. We have two cases.

(a) $j < i - 2$. By Theorem 3.6(vi) we have

$$\Theta(\pi(\gamma)) = \sum_{a,b=0}^{1} \Theta\big(\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{j-1+a,2n+2-i+b}(\chi))\big).$$

Since $e_i + e_2$ is a complex root simple for P, orthogonal to γ and such that $\theta(e_i + e_2) \notin P$, we conclude that $\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{j-1,2n+2-i}(\chi)) = 0$.

We have $\bar{\pi}_{j,2n+2-i}(\chi) = \bar{\pi}(w(\gamma + \mu))$, $w = w_{e_1 - e_{j+2}}$. Using the root $e_i + e_2$ we find $\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{j,2n+2-i}(\chi)) = 0$.

We have $\bar{\pi}_{j-1,2n+3-i}(\chi) = \bar{\pi}(w(\gamma + \mu))$ where $w = w_{e_2+e_i}$. Then $w\gamma = \gamma$. $-e_2 - e_i$ is the only root simple for $P_{j-1,2n+3-i}$ and orthogonal to γ. $-e_2 - e_i$ is complex and $\theta(-e_2 - e_i) \in P_{j-1,2n+3-i}$. Therefore $\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{j-1,2n+3-i}(\chi)) = \bar{\pi}(w\gamma) = \bar{\pi}(\gamma)$.

Finally $\bar{\pi}_{j,2n+3-i}(\chi) = \bar{\pi}(w_1(\gamma + \mu))$ where $w_1 = w_{e_1-e_{j+2}} w_{e_i+e_2}$. Set $w = w_{e_1-e_{j+2}}$. Then $w_1\gamma = w\gamma$. The only root simple for $P_{j,2n+3-i}$ and orthogonal to $w\gamma$ is the complex root $-e_2 - e_i$, but $\theta(-e_2 - e_i) \in P_{j,2n+3-i}$; therefore

$$\psi_\gamma^{\gamma+\mu}\big(\bar{\pi}_{j,2n+3-i}(\chi)\big) = \bar{\pi}(w_1\gamma) = \bar{\pi}(w\gamma).$$

(b) $j = i - 2$. By Theorem 3.6(ii) we get

$$\Theta(\pi(\gamma)) = \Theta\big(\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{i-3,2n+2-i}(\chi))\big) + \Theta\big(\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{i-3,2n+3-i}(\chi))\big)$$
$$+\Theta\big(\psi_\gamma^{\gamma+\mu}(\bar{\pi}_{i-2,2n+2-i}(\chi))\big) + \Theta\big(\psi_\gamma^{\gamma+\mu}(\pi_{i-2}(\chi))\big).$$

Exactly as before we get

$$\psi_\gamma^{\gamma+\mu}\big(\bar{\pi}_{i-3,2n+2-i}(\chi)\big) = 0,$$
$$\psi_\gamma^{\gamma+\mu}\big(\bar{\pi}_{i-2,2n+2-i}(\chi)\big) = 0 \qquad (\text{use } e_1 + e_2),$$
$$\psi_\gamma^{\gamma+\mu}\big(\bar{\pi}_{i-3,2n+3-i}(\chi)\big) = \bar{\pi}(\gamma).$$

Finally, $\pi_{i-2}(\chi) = \pi(\tau(\gamma + \mu))$, where $\tau P_{i-3,2n+2-i} = C_{i-2}$. Then

$$\tau\gamma = \sum_{k=1}^{i-3} a_{k+2}\varepsilon_k + a_1\varepsilon_{i-2} - a_2\varepsilon_{i-1} + \sum_{k=i}^{n} a_{k+1}\varepsilon_k + a_i\varepsilon_{n+1}.$$

$\varepsilon_{n+1} - \varepsilon_{i-1}$ is the only root simple for C_{i-2} and orthogonal to $\tau\gamma$. Therefore, $\psi_\gamma^{\gamma+\mu}(\pi_{i-2}(\chi)) = \pi^-(\tau\gamma)$.

By this the theorem is completely proven.

THEOREM 4.3. *If $\gamma \in D^+$ is integral and contained in two walls, then $\pi(\gamma)$ is irreducible.*

PROOF. The possibilities for γ are as follows.

(a) $a_2 = 0$, $a_1 = a_i$ for some $i \in \{3, \ldots, n+1\}$.

(b) $a_1 = a_i$, $a_2 = a_j$ for some i, j, $3 \leqslant i < j \leqslant n + 1$.

(c) $a_1 = a_i = -a_2$ for some $i \in \{3, \ldots, n+1\}$.

(d) $a_1 = a_i$, $a_2 = -a_j$ for some i, j, $3 \leqslant i < j \leqslant n + 1$.

The method of the proof is the same as that in the proof of Theorem 4.2. In each case we show that $\psi_\gamma^{\gamma+\mu}(\pi) = 0$ for every but one composition factor π of $\pi(\gamma + \mu)$. But this is an immediate consequence of the proof of Theorem 4.2; for each of the cases (a), (b), (c), (d) (or its subcases) one should only combine the two parts of the proof of Theorem 4.2 which correspond to the two walls containing γ.

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