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L^p BEHAVIOR OF CERTAIN SECOND ORDER PARTIAL DIFFERENTIAL OPERATORS

BY

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ABSTRACT. We give examples of bounded inverses of polynomials in $\mathbf{R}^n$, $n > 1$, which are not Fourier multipliers of $L^p(\mathbf{R}^n)$ for any $p \neq 2$. Our main tool is the Kakeya set construction of C. Fefferman. Using these results, we relate the invertibility on L^p of a linear second order constant coefficient differential operator D on $\mathbf{R}^n$ to the geometric structure of quadratic surfaces associated to its symbol d. This work was motivated by multiplier conjectures of N. Rivière and R. Strichartz.

0. Introduction. Let D be a linear constant coefficient differential operator on $\mathbf{R}^n$, with symbol d. We assume throughout this paper that D has order at most two, and we shall examine the L^p spectrum and invertibility of D. (See §I for the precise definitions we use.) Our main tool in this analysis will be the Kakeya set construction of Charles Fefferman, through which we shall relate invertibility of D to the geometric structure of quadratic surfaces associated to d.

If D is elliptic and d has no real zeroes, D is L^p invertible for all p, $1 < p < \infty$. As we examine less trivial cases, great complexity seems to arise. The operator

$$\partial^2/\partial x_1^2 - \partial^2/\partial x_2^2 + i \tag{1}$$

is made tractable through the multiplier theorems of Marcinkiewicz **[10]** or Hörmander **[6]**; it is $L^p(\mathbf{R}^2)$ invertible if and only if $1 < p < \infty$. But a simple generalization

$$\partial^2/\partial x_1^2 - \sum_{j=2}^{n} \partial^2/\partial x_j^2 + i \tag{2}$$

cannot be $L^p(\mathbf{R}^n)$ invertible if $1 \leqslant p < 2(n-1)/n$. Again, Littman, McCarthy and Rivière **[9]** showed that

$$i\partial/\partial x_1 + \sum_{j=2}^{n} \partial^2/\partial x_j^2 + i \tag{3}$$

cannot be $L^p(\mathbf{R}^n)$ invertible if $1 \leqslant p \leqslant 2n/(n+1)$.

The complicated indices in the above results are illusory; it is the purpose of this paper to show that operators such as (2) and (3) above are $L^p(\mathbf{R}^n)$ invertible if and only if $p = 2$. The distinction between operators like (1) and those like (2) or (3)

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will appear as a consequence of the distinct geometric properties of the surfaces $\xi_1^2 - \xi_2^2 = 0$ versus $\xi_2 - \xi_1^2 = 0$ or $\xi_1^2 - \xi_2^2 - \xi_3^2 = 0$.

In §I we collect preliminary results. We shall state and prove our main results in §§II and III; in §IV we discuss applications and motivations.

The work presented here was motivated by multiplier conjectures of N. Rivière [12] and R. Strichartz. Weaker versions of some of these results appeared previously in [16] and [17]. The main results of this paper were announced in [8].

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I. Preliminaries. We begin with some facts on Fourier multipliers. A more extensive discussion may be found in Stein and Weiss [15].

DEFINITION. Let m be a bounded measurable function. The operator T defined by $\widehat{Tf}(\xi) = m(\xi)\hat{f}(\xi)$ is said to be an L^p multiplier if T has a bounded extension to L^p.

LEMMA 1. *m is a multiplier of L^2 if and only if m is in L^∞.*

LEMMA 2. *Let m be a multiplier of L^p. Then m is a multiplier of $L^{p'}$ and of all L^r for r between p and p'. In particular, m is in L^∞.*

LEMMA 3. *m is a multiplier of L^1 if and only if there exists a finite Borel measure μ with $\hat{\mu} = m$.*

LEMMA 4 ([4] AND [18]). *Let m be continuous and a multiplier of $L^p(\mathbf{R}^n)$ with operator norm C. Then*

(a) *the restriction of m to any k-dimensional hyperplane is a multiplier of $L^p(\mathbf{R}^n)$, with multiplier norm not exceeding C.*

(b) *if A is any nonsingular linear transformation of $\mathbf{R}^n$, $m(A\xi)$ is a multiplier of $L^p(\mathbf{R}^n)$ with norm C.*

We now recall some definitions and facts on L^p-behavior of constant coefficient partial differential operators D. A more extensive discussion may be found in Schechter [14].

DEFINITION. Let X be a Banach space, and let D_0 be a (possibly unbounded) linear operator from X to itself. A scalar λ is said to be in the resolvent set $\rho(D_0)$ of D_0 if $R(D_0 - \lambda)$ is dense in X and there is a constant C such that

$$\|X\| \leqslant C\|(D_0 - \lambda)X\|, \qquad x \in D(D_0).$$

The spectrum $\sigma(D_0)$ of D_0 consists of those scalars not in $\rho(D_0)$. If $D(D_0)$ is dense in X, and $0 \in \rho(D_0)$, we say that D_0 is invertible.

As is well known, (see [14]) constant coefficient differential operators D are closable on $L^p(\mathbf{R}^n)$, $1 \leqslant p \leqslant \infty$, and thus, have a minimal closed extension on

$L^p(\mathbf{R}^n)$, which we denote by $D_{0,p}$. When no confusion is likely to arise, we denote $D_{0,p}$ simply by D_0.

LEMMA 5 (SEE THEOREM 4.1 OF [**14**]). *$\lambda \in \rho(D_{0,p})$ if and only if $(d(\xi) - \lambda)^{-1}$ is a multiplier of L^p for $1 < p < \infty$.*

COROLLARY. (a) *D_0 is invertible on L^2 if and only if $|d(\xi)| \geqslant C_0$ for all $\xi \in \mathbf{R}^n$.*

(b) *If D_0 is invertible on L^p, then $|d(\xi)| \geqslant C_0$ for all $\xi \in \mathbf{R}^n$.*

(c) *If $n = 1$, then either D_0 is invertible on $L^p(\mathbf{R}^n)$ for all p, $1 < p < \infty$, or for no p.*

(d) *If D is an elliptic operator of order m, $m > 1$, and $d(\xi) \neq 0$ for all $\xi \in \mathbf{R}^n$, then D_0 is invertible in $L^p(\mathbf{R}^n)$ for $1 < p < \infty$.*

The remaining results are the essential constituents of our theorem. Lemma 6 is due to Marcinkiewicz and Zygmund [**11**]; Lemmas 7 and 8 are variants of the construction in Fefferman [**2**].

LEMMA 6. *Let T be a linear operator with $\|Tf\|_p \leqslant C\|f\|_p$. Then*

$$\left\|\left(\sum |Tf_j|^2\right)^{1/2}\right\| \leqslant C\left\|\left(\sum |f_j|^2\right)^{1/2}\right\|_p.$$

LEMMA 7. *Let $N_k = 2^k \log k$. For each large k, there exists a set of $K \subset \mathbf{R}^2$ and a collection of 2^k disjoint rectangles $\{R_j\}$, such that for every $\gamma > 0$:*

(a) *The shorter and longer sides of each R_j are bounded above by N_k, N_k^2, and below by $N_k/4$, $N_k^2/4$.*

(b) $|K| \leqslant 20(\log\log N_k)^{-1}\Sigma|R_j|$.

(c) *Let $\bar{v}_j = (\cos\theta_j, \sin\theta_j)$ be the direction of the longer side of R_j. Then $|\theta_j - \pi/4| < \pi/8$.*

(d) *Let $\bar{R}_j$ denote $R_j + (1+\gamma)\bar{v}_j$. Then $|\bar{R}_j|/4 \leqslant |\bar{R}_j \cap K|$.*

LEMMA 8. *Let $N_k = 2^k \log k$, and let α, β, γ be given reals. For each $k > k_0(\alpha, \beta)$ there exists a set of $K \subset \mathbf{R}^2$ and a collection of 2^k disjoint rectangles R_j with*

(a) *the longer and shorter sides of each R_j are bounded above by βN_k^2, αN_k, and below by $C\beta N_k^2$, $C\alpha N_k$.*

(b) $|K| \leqslant 20(C\log\log N_k)^{-1}\Sigma|R_j|$.

(c) *Let $\bar{v}_j$ denote the direction of the longer side of R_j. Let $\bar{R}_j = R_j + (1+\gamma)\bar{v}_j$. Then $|\bar{R}_j \cap K| \geqslant \frac{1}{4}|\bar{R}_j|$. Here $C = C(\alpha, \beta) = \frac{2}{3}\beta(\alpha^2 + \beta^2)^{-1/2}$.*

II. Parabolic invariance.

THEOREM A. *Let φ be in $L^s \cap L^\infty(\mathbf{R}^1)$ for some s, $0 < s < \infty$, and assume φ is not identically zero. Then the multiplier $m(\xi_1, \xi_2) = \varphi(\xi_2 - \xi_1^2)$ is $L^p(\mathbf{R}^2)$ bounded if and only if $p = 2$.*

REMARKS. (1) The theorem is easily extended to higher dimensions using Lemma 4(a).

(2) The requirement $\varphi \in L^s$ is natural if we expect m to correspond to the inverse of a differential operator, and some condition is needed to prevent m from being constant. But arguments similar to those of [**2**] easily show that $\varphi(r) = |r|^{it}$

does not yield a multiplier of L^p if $p \neq 2$, and Jodeit has remarked that a similar situation occurs if $\varphi(+\infty) \neq \varphi(-\infty)$. It seems likely that the requirement $\varphi \in L^s$ is not essential.

The proof is conceptually straightforward. Let $m_t(\xi_1, \xi_2) = m(t\xi_1, t^2\xi_2)$, $m_\infty = \operatorname{sign}(\xi_2 - \xi_1^2)$, and let K_t, K_∞ be the corresponding convolution kernels. It is an immediate consequence of C. Fefferman's work on the disk that m_∞ is not a multiplier of L^p. This is shown using a variant of Lemma 7, due to Yves Meyer and by showing that $e^{i\bar{\omega}_j \cdot \bar{x}}K_\infty(\bar{x})$ is large on Kakeya set counterexample functions, for appropriate choices of $\bar{\omega}_j$. To prove Theorem A, we show that t may be chosen so that $e^{i\bar{\omega}_j \cdot \bar{x}}(K_t - K_\infty)$ is small on the above counterexamples.

Proof of Theorem A. We shall assume m is a multiplier of $L^p(\mathbf{R}^2)$ for $p \neq 2$, and derive a contradiction. By Lemma 2, we may assume $p > 2$. The conjugate $\bar{m}$ is a multiplier, as are $m \pm \bar{m}$; we may assume m is real: Powers of m are also multipliers, as these correspond to iterates of the operator: we may assume m is positive. Given that φ is in $L^s \cap L^\infty$, some power of φ is in L^1, so we may assume φ is in L^1.

In all, we may assume $\varphi \geqslant 0$ and $\int_{\mathbf{R}'} \varphi = 1$; this requires φ not identically zero.

From Lemma 4 we see that dilates $m(t\xi_1, t^2\xi_2)$ are uniformly bounded multipliers; if $\hat{K}_t = m_t$, we see from Lemma 6 that

$$\left\| \left(\sum |K_t * f_j|^2 \right)^{1/2} \right\|_p \leqslant A \left\| \left(\sum |f_j|^2 \right)^{1/2} \right\|_p.$$

Let $K_t^j(\bar{x}) = e^{i\bar{\omega}_j \cdot \bar{x}}K_t(\bar{x})$; then

$$\left\| \left(\sum |K_t^j * f_j|^2 \right)^{1/2} \right\|_p \leqslant A \left\| \left(\sum |f_j|^2 \right)^{1/2} \right\|_p \tag{4}$$

where A is independent of $\bar{\omega}_j$ and t.

Let K, R_j, $\bar{R}_j$ be as in the construction of Lemma 7. Let $f_j(\bar{x}) = \chi_{R_j}(\bar{x})$; we shall show that for each N_k, and $t = \gamma\delta N_{k_j}^2$

$$|K_t^j * f_j(x)| \geqslant C_{\gamma\delta} \quad \text{when } \bar{x} \text{ is in } \bar{R}_j.$$

As in Fefferman [2], it immediately follows that

$$A \geqslant \frac{C_{\gamma\delta}}{2} \left(\frac{\log \log N_k}{20} \right)^{1/2 - 1/p}. \tag{5}$$

As N_k tends to infinity, we find that (4) cannot hold uniformly in t for $p > 2$, which is the desired contradiction.

To prove (5) note that

$$\int_K \sum |K_t^j * f_j(x)|^2 \geqslant \sum \int_{\bar{R}_j \cap K} |K_t^j * f_j(x)|^2 \geqslant C_{\gamma\delta}^2 \sum |\bar{R}_j \cap K| \geqslant C_{\gamma\delta}^2 \sum |R_j|.$$

But

$$\begin{aligned} \int_K \sum |K_t^j * f_j(x)|^2 &\leqslant |K|^{1-2/p} \left\| \left(\sum |K_t^j * f_j|^2 \right)^{1/2} \right\|_p^2 \\ &\leqslant |K|^{1-2/p} A^2 \left\| \left(\sum |f_j|^2 \right)^{1/2} \right\|_p^2 \\ &= |K|^{1-2/p} A^2 \left(\sum |R_j| \right)^{2/p} \quad \text{as the } R_j \text{ are disjoint.} \end{aligned}$$

Then

$$A^2 \geq \frac{C_{\gamma\delta}^2}{4}\left(\frac{\Sigma|R_j|}{|K|}\right)^{1-2/p}$$

and estimate (b) of Lemma 7 completes the proof.

To estimate $K_t^j * f_j(\bar{x})$ on $\bar{R}_j$, choose $\bar{\omega}_j = (-\frac{1}{2}\cot\theta_j, \frac{1}{4}\cot^2\theta_j)$, where θ_j is as in Lemma 7. Note

$$K_t^j * f_j(\bar{x}) = t^{-1}\int_{R_j}(x_2 - y_2)^{-1/2}\exp\{i\phi(\bar{x}-\bar{y})\}\hat{\varphi}((x_2 - y_2)/t)\,d\bar{y}$$

where $\phi(\bar{x}-\bar{y}) = (x_2 - y_2)/4[(x_1 - y_1)/(x_2 - y_2) - \cot\theta_j]^2$.

The quantity in brackets is $\cot t - \cot\theta_j$, where t is the angle between $\bar{x} - \bar{y}$ and the axis. As $\bar{x}$ and $\bar{y}$ vary, the difference $\cot t - \cot\theta_j$ is easily maximized, and is bounded by $5(\gamma N_k)^{-1}$; then

$$|\phi(\bar{x}-\bar{y})| \leq 10\gamma^{-1} \quad \text{and} \quad \exp i\phi = 1 + O(\gamma^{-1}).$$

Then $K_t^j * f_j(x) = M + E$ where the main term M is

$$t^{-1}\int_{R_j}(x_2 - y_2)^{-1/2}\hat{\varphi}((x_2 - y_2)/t)\,d\bar{y}$$

and the error term E is

$$O(t^{-1}\gamma^{-1})\int_{R_j}(x_2 - y_2)^{-1/2}|\hat{\varphi}((x_2 - y_2)/t)|\,d\bar{y}.$$

For $\bar{x} \in \bar{R}_j, \bar{y} \in R_j$,

$$|x_2 - y_2| \leq |x - y| \leq (2+\gamma)N_k^2 \leq 4\gamma N_k^2, \qquad |x_2 - y_2|/t \leq 2\gamma N_k^2/\gamma\delta N_k^2 = 2/\delta$$

when $\gamma > 2$; hence $|x_2 - y_2|^{-1/2} \geq (4\gamma N_k^2)^{-1/2}$. As $\hat{\varphi}$ is continuous, we may choose δ sufficiently large such that $\operatorname{Re}\hat{\varphi}((x_2 - y_2)/t) \geq \hat{\varphi}(0)/2 = 1/2$; hence

$$\operatorname{Re} M \geq t^{-1}(4\gamma N_k^2)^{-1/2}|R_j| \cdot \tfrac{1}{2} = (4\gamma^{3/2}\delta)^{-1}.$$

To estimate E from above, we have

$$|x_2 - y_2| = |(\bar{x}-\bar{y})\cdot\bar{e}_2| = |\bar{x}-\bar{y}|\cdot\sin\psi \geq \tfrac{1}{4}|\bar{x}-\bar{y}| \geq \tfrac{1}{4}\gamma N_k^2,$$

and

$$|\hat{\varphi}((x_2 - y_2)/t)| \leq \|\hat{\varphi}\|_\infty \leq \|\varphi\|_1 = 1.$$

Thus

$$|E| \leq (100/t\gamma)\left(\tfrac{1}{4}\gamma N_k^2\right)^{-1/2}|R_j| \leq 200(\gamma^{5/2}\delta)^{-1}.$$

If we choose $\gamma > 1600$, $|M| > 2|E|$ and $|K_t^j * f_j(\bar{x})| \geq (8\gamma^{3/2}\delta)^{-1} = c_{\gamma\delta}$. This completes the proof.

III. Hyperbolic invariance. Let $r^2 = \sum_{i=1}^n x_i^2$, $s^2 = \sum_{j=1}^m x_{j+n}^2$. We shall establish negative multiplier results for $SO(n, m)$ invariant multipliers $\varphi(r^2 - s^2)$, when $\max(n, m) > 1$. (If $\max(n, m) = 1$, the Marcinkiewicz multiplier theorem shows there are many φ which yield L^p multipliers.)

Lemma 4(a) allows us to restrict attention to the case $n = 2$, $m = 1$. Our results in this case are weaker than those for $n = 3$, $m = 1$, so that we state two separate cases. Lemma 4 allows us to use case (a) if $\min(n, m) > 2$, and case (b) if $\min(n, m) = 2$.

THEOREM B. (a) *Let* φ *be in* $L^q \cap L^\infty(\mathbf{R}')$ *for some* q, $0 < q < \infty$, *and* $\varphi \not\equiv 0$. *Then* $m(\bar{x}) = \varphi(r^2 - s^2)$ *is* $L^p(\mathbf{R}^{m+n})$ *bounded iff* $p = 2$, *when* $\min(n, m) > 2$.

(b) *Let* φ *be as above, with* $\varphi(t) = O(t^{-\alpha})$ *for some* $\alpha > 0$. *Then* $m(\bar{x}) = \varphi(r^2 - s^2)$ *is* $L^p(\mathbf{R}^{n+m})$ *bounded iff* $p = 2$ *when* $\min(n, m) \geqslant 2$.

PROOF. We present the full proof only of part (a); the proof of part (b) is similar, but there are more error terms. Assume then that m is an L^p multiplier; we may assume $p > 2$, and by Lemma 4(a) we may assume $n = 3$, $m = 1$. Applying Lemma 4(a), (b), we see that the multiplier $m_t(x, y, z) = \varphi(t\{(x^2 + y^2 + z^2) - 1\}) = \varphi(t(r^2 - 1))$ are uniformly bounded on $L^p(\mathbf{R}^3)$. As in the proof of Theorem A, we may assume φ is in $L^1(\mathbf{R}^1)$, $\varphi \geqslant 0$ and $\int \varphi = 1$. Let

$$T_t(R) = m_t(R) = \int_0^\infty \frac{J_{1/2}(2\pi rR)}{(rR)^{1/2}} \varphi(t(r^2 - 1)) r^2 \, dr$$
$$= \int_0^\infty \frac{\sin 2\pi rR}{rR} \varphi(t(r^2 - 1)) r^2 \, dr.$$

We shall first show that

$$T_t(R) = \frac{\operatorname{Im}}{2tR} e^{2\pi i R} \hat{\varphi}\left(\frac{R}{2t}\right) + E_0(t, R) \tag{5'}$$

where $E(t, r) = o(t^{-1}R^{-1})$ as $t \sim R$ tend to infinity. For,

$$T_t(R) = \frac{\operatorname{Im}}{R} \int_0^\infty e^{2\pi i R r} \varphi(t(r^2 - 1)) r \, dr$$
$$= \frac{\operatorname{Im}}{2tR} \int_{-t}^\infty \exp\left\{2\pi i R\left(\frac{s}{t} + 1\right)^{1/2}\right\} \varphi(s) \, ds$$
$$= \frac{\operatorname{Im}}{2tR} \int_{-t^{1/4}}^{t^{1/4}} \exp\left\{2\pi i R\left(\frac{s}{t} + 1\right)^{1/2}\right\} \varphi(s) \, ds$$
$$+ o(t^{-1}R^{-1});$$

the o-estimate follows from the integrability of φ.

In the region $(-t^{1/4}; t^{1/4})$ Taylor series shows that

$$(s/t + 1)^{1/2} = 1 + s/2t + O((s/t)^2) = 1 + s/2t + O(t^{-3/2}),$$

hence

$$\exp(2\pi i R(s/t + 1)^{1/2}) = e^{2\pi i R} e^{2\pi i R s/2t} + O(Rt^{-3/2})$$

and

$$T_t(R) = \frac{\operatorname{Im}}{2tR} \int_{-t^{1/4}}^{t^{1/4}} e^{2\pi i R s/2t} \varphi(s) \, ds + O(t^{-9/2}) + o(t^{-1}R^{-1})$$
$$= \frac{\operatorname{Im}}{2tR} \int_{-\infty}^\infty e^{2\pi i R} e^{2\pi i R s/2t} \varphi(s) \, ds + o(t^{-1}R^{-1})$$

as desired, if $t \sim R$.

The proof of the remainder of the theorem is now similar to that of Theorem A, with obvious modifications to account for error terms and for the three-dimensional nature of the problem. We first construct three-dimensional analogues of the sets in Lemma 8. Let K, R_j, $\bar{R}_j$ be as in Lemma 8, and let $\delta > 0$ be given. Let

$$S_j^0 = \{\bar{y} \in R_j \mid |\bar{v}_j \cdot \bar{y} - n| < \delta \text{ for some } n\},$$

$$\overline{S_j^0} = \{\bar{x} \in \bar{R}_j \mid |\bar{v}_j \cdot \bar{x} - (n + \tfrac{1}{4})| < \delta \text{ for some } n\}.$$

Let $L = K \times [0, \alpha N_k]$, $S_j = S_j^0 \times [0, \alpha N_k]$, $\bar{S}_j$, etc.

Now let $\bar{v}_j$ denote the direction of the longest side of S_j, and let $\bar{w}_j$ be any perpendicular to $\bar{v}_j$. For the remainder of the proof, we shall assume $\bar{x} \in \bar{S}_j$ and $\bar{y} \in S_j$. We then have estimates analogous to those of Lemma 8.

$$\alpha^2\beta\delta N_k^4 \leqslant |S_j| \leqslant 2\alpha^2\beta\delta N_k^4. \tag{i}$$

$$(1/\delta)|S_j| \leqslant |\bar{S}_j \cap L|. \tag{ii}$$

$$|L| \leqslant \frac{6}{\delta}(c \log\log N_k)^{-1} \sum |S_j|. \tag{iii}$$

$$\gamma N_k^2 \leqslant |\bar{x} - \bar{y}| \leqslant ((\beta + \gamma)^2 + 2\alpha^2)^{1/2} N_k^2. \tag{iv}$$

$$\gamma N_k^2 \leqslant |(\bar{x} - \bar{y}) \cdot \bar{v}_j| \leqslant (2\beta + \gamma) N_k^2. \tag{v}$$

$$(\bar{x} - \bar{y}) \cdot \bar{w}_j \leqslant \sqrt{2}\, \alpha N_k. \tag{vi}$$

(i), (iii), (iv), (v), (vi) are obvious, while (ii) follows from the proof of Lemma 8, as given in **[2]**.

Let $f_j(\bar{y}) = \chi_{S_j}(\bar{y}) \cos[2\pi(\bar{v}_j \cdot \bar{y})]$. As in the proof of Theorem A, we shall show that when $t = dN_k^2$,

$$|T_t f_j(\bar{x})| \geqslant C(\alpha, \beta, \gamma, \delta, d) > 0 \tag{6}$$

independently of N_k; it then follows that

$$\|T_t\|_{(p,p)} \geqslant C(\alpha, \beta, \gamma, \delta, d)(C \log\log N_k)^{1/2 - 1/p},$$

which will yield a contradiction, when $N_k \to \infty$.

The T_t commute with rotations, so that it suffices to prove (6) when $\bar{v}_j = (1, 0, 0)$. Let $\bar{x} = (x_1, x_2, x_3)$, and similarly for $\bar{y}$. Then

$$T_t f_j(\bar{x}) = \frac{\mathrm{Im}}{2t} \int_{S_j} e^{2\pi i|\bar{x} - \bar{y}|} \varphi\left(\frac{|\bar{x} - \bar{y}|}{2t}\right) |\bar{x} - \bar{y}|^{-1} \cos 2\pi \bar{v}_j \cdot \bar{y}\, d\bar{y}$$

$$+ \int_{S_j} E_0(t, |\bar{x} - \bar{y}|) \cos 2\pi(\bar{v}_j \cdot \bar{y})\, d\bar{y} = M_1 + E_1.$$

Recall that $E_0(t, R) = o(t^{-1}R^{-1})$ and

$$|\bar{x} - \bar{y}| \geqslant \gamma N_k^2, \qquad |S_j| \leqslant 2\alpha^2\beta\delta N_k^4, \qquad t = dN_k^2,$$

then $E_1 = o(1)$. To estimate M_1, let $\hat{\varphi} = F + iG$; then

$$\begin{aligned} M_1 &= \frac{1}{2t}\int_{S_j} \sin 2\pi|\bar{x} - \bar{y}| F\left(\frac{|\bar{x}-\bar{y}|}{2t}\right)|\bar{x}-\bar{y}|^{-1} \cos 2\pi\bar{v}_j\cdot\bar{y}\, d\bar{y} \\ &\quad + \frac{1}{2t}\int_{S_j} \cos 2\pi|\bar{x} - \bar{y}| G\left(\frac{|\bar{x}-\bar{y}|}{2t}\right)|\bar{x}-\bar{y}|^{-1} \cos 2\pi\bar{v}_j\cdot\bar{y}\, d\bar{y} \\ &= M_2 + E_2. \end{aligned}$$

To estimate E_2, note that as $\bar{x}$ and $\bar{y}$ vary,

$$\frac{|\bar{x}-\bar{y}|}{2t} \text{ varies from 0 to } \frac{((\beta+\gamma)^2 + 2\alpha^2)^{1/2}}{2d} = \varepsilon.$$

Let $\tilde{G} = \sup_{0\leq s\leq\varepsilon} |G(s)|$. Then

$$|E_2| \leq (1/2dN_k^2)|S_j|(\gamma N_k^2)^{-1}\tilde{G} \leq (\alpha^2\beta\delta/\gamma d)\tilde{G}.$$

To estimate M_2, note that

$$\begin{aligned} |\bar{x}-\bar{y}| &= |x_1 - y_1|\left[1 + \frac{(x_2-y_2)^2 + (x_3-y_3)^2}{(x_1-y_1)^2}\right]^{1/2} \\ &= |x_1 - y_1| + D(\bar{x},\bar{y}) \end{aligned}$$

where

$$|D(\bar{x},\bar{y})| \leq \alpha^2/\gamma + o(N_k^{-1}).$$

Then

$$\begin{aligned} M_2 &= \frac{1}{2t}\int_{S_j} \sin 2\pi|x_1 - y_1|(\cos 2\pi D)\cdot|\bar{x}-\bar{y}|^{-1} F\left(\frac{|\bar{x}-\bar{y}|}{2t}\right)\cos 2\pi\bar{v}_j\cdot\bar{y}\, d\bar{y} \\ &\quad + \frac{1}{2t}\int_{S_j} \cos 2\pi|x_1 - y_1|(\sin 2\pi D)\cdot|\bar{x}-\bar{y}|^{-1} F\left(\frac{|\bar{x}-\bar{y}|}{2t}\right)\cos 2\pi\bar{v}_j\cdot\bar{y}\, d\bar{y} \\ &= M_3 + E_3. \end{aligned}$$

Note that $|E_3| \leq (2dN_k^2)^{-1}|S_j|(2\pi\alpha^2/2\gamma)(\gamma N_k^2)^{-1}\|F\|_\infty + o(1)$. But $\|F\|_\infty \leq \|F + iG\|_\infty \leq \|\varphi\|_1 = 1$, so that

$$|E_3| \leq (\alpha^2\beta\delta/\gamma d)(\pi\alpha^2/\gamma) + o(1).$$

To estimate M_3, note that S_j and $\bar{S}_j$ may all be oriented so that $x_1 - y_1 > 0$. As $\bar{v}_j\cdot\bar{y} = y_1$,

$$\begin{aligned} M_3 &= \frac{\sin 2\pi x_1}{2t}\int_{S_j} \cos^2 2\pi y_1 \cos 2\pi D|\bar{x}-\bar{y}|^{-1} F\left(\frac{|\bar{x}-\bar{y}|}{2t}\right) d\bar{y} \\ &\quad - \frac{\cos 2\pi x_1}{2t}\int_{S_j} \sin 2\pi y_1 \cos 2\pi y_1 \cos 2\pi D|\bar{x}-\bar{y}|^{-1} F\left(\frac{|\bar{x}-\bar{y}|}{2t}\right) d\bar{y} \\ &= M_4 + E_4. \end{aligned}$$

To estimate E_4, recall that $|x_1 - (n + \frac{1}{4})| \leqslant \delta$ and $|y_1 - n| \leqslant \delta$. If δ is small, $|\cos 2\pi x_1| \leqslant 2\pi\delta$ and $|\sin 2\pi y_1| \leqslant 2\pi\delta$, and therefore

$$|E_4| \leqslant 2\pi\delta(2dN_k)^{-2}|S_j| 2\pi\delta(\gamma N_k^2)^{-1}\|F\|_\infty$$

or $|E_4| \leqslant (\alpha^2\beta\delta/\gamma d)(4\pi^2\delta^2)$.

We shall now compute a lower bound for M_4. If we require $\delta \leqslant 10^{-3}$, $\sin 2\pi x_1 > \frac{1}{2}$ and $\cos^2 2\pi y_1 \geqslant \frac{1}{2}$. We shall also require $\alpha^2/2\gamma < 10^{-2}$, so that $\cos 2\pi D \geqslant \frac{1}{2}$. Then if $\tilde{F} = \inf_{0 \leqslant s \leqslant \varepsilon} F(s)$,

$$M_4 \geqslant (4dN_k^2)^{-1}|S_j|\frac{1}{2}\cdot\left[(\beta+\gamma)^2 + 2\alpha^2\right]^{-1/2}N_k^{-2}\tilde{F}$$

$$\geqslant \frac{\alpha^2\beta\delta}{\gamma d}\left(\frac{\gamma\tilde{F}}{8((\beta+\gamma)^2 + 2\alpha^2)^{1/2}}\right).$$

In all, $T_t f_j(x) = M_4 + E_2 + E_3 + E_4 + o(1)$ where $|E_2 + E_3 + E_4| \leqslant (\alpha^2\beta\delta/\gamma d)[\tilde{G} + \pi\alpha^2/\gamma + 4\pi^2\delta^2]$. We shall now choose $\beta = \gamma = \frac{1}{2}$; we still must require $\delta < 10^{-3}$ and now $\alpha^2 < 10^{-2}$. Then $\gamma((\beta+\gamma)^2 + 2\alpha^2)^{-1/2} \geqslant 1/4$ and $M_4 \geqslant (\alpha^2\delta/d)(\frac{1}{32})\tilde{F}$. Since $\varepsilon = ((\beta^2 + \gamma)^2 + 2\alpha^2)^{1/2}/2d \leqslant d^{-1}$, ε is small if d is large. We shall use the continuity of $\hat{\varphi} = F + iG$ to estimate $\tilde{F}$ and $\tilde{G}$. $1 = \hat{\varphi}(0)$, so that if d is large, $\inf_{0 \leqslant s \leqslant \varepsilon} F(s) \geqslant \frac{1}{2}$ and $\sup_{0 \leqslant s \leqslant \varepsilon} |G(s)| \leqslant 1/6.128$. Then $M_4 \geqslant \alpha^2\delta/d \cdot 1/64$. To insure that the error terms are only a fraction of M_4, choose δ so small that $4\pi^2\delta^2 \leqslant 1/6.128$ (still requiring $\delta \leqslant 10^{-3}$) and α so small that $\pi\alpha^2/\gamma \leqslant 1/6.128$ (still requiring $\alpha^2 < 10^{-2}$, $\gamma = \frac{1}{2}$).

Then $|E_2 + E_3 + E_4| \leqslant \alpha^2\delta/d \cdot 1/128$, whence $|T_t f_j(\bar{x})| \geqslant \alpha^2\delta/64d + o(1)$. If N_k is large, $|T_t f_j(\bar{x})| \geqslant \alpha^2\delta/128d = C(\alpha, \beta, \gamma, \delta, d) > 0$. This completes the proof.

IV. Applications and extensions. (1) A. P. Calderón observed that every bounded rational function of a real variable gives rise to an $L^p(\mathbf{R}^1)$ multiplier if $1 < p < \infty$. The work of Littman, McCarthy and Rivière [9] showed this cannot hold in higher dimensions, but Rivière [12] conjectured that bounded rational functions must be L^p multipliers for some $p \neq 2$. Theorem A shows this conjecture is false.

(2) Let G be a noncompact connected semisimple Lie group with finite center, Lie algebra g and Killing form B. Greenleaf, Moskowitz and Rothchild [5] showed that there are no nontrivial finite measures on g invariant under the action of Ad G. This means that there are no nontrivial multipliers of $L^1(g)$ which commute with the action of Ad G. R. Strichartz conjectured this holds for all $p \neq 2$. Theorem B provides evidence in favor of the conjecture. Defining a Fourier transform on $L^p(g)$ by $\hat{f}(y) = \int_g e^{iB(x,y)} f(x)\, dx$, a multiplier is seen to commute with Ad G if and only if m is Ad G invariant. If Ad G acted transitively on the level sets $B(x, x) =$ constant, it would follow that $m(x) = \varphi(B(x, x))$. Diagonalizing B and observing $\dim g \geqslant 3$, we see that Theorem B applies, and m is indeed not a multiplier of $L^p(g)$. In general, Ad G does not act transitively, but there is a three-dimensional subspace on which B has signature $(+, -, -)$ and on which Ad G acts transitively. We use Lemma 4(a) to restrict m to this subspace, and then apply Theorem B. This provides a partial solution to Strichartz' conjecture, with

restrictions on the growth of m. But $m(x) = |B(x, x)|^{it}$ is easily seen not to be a multiplier, but fails to satisfy our growth restrictions. The general case of Strichartz' conjecture therefore remains open.

(3) The results in this paper were adapted to the analysis of second order differential operators, or, more generally, multipliers constant on quadratic surfaces. The techniques may nonetheless be used to obtain unboundedness of multipliers of the form $\varphi(\xi - \phi(\eta))$ where φ has growth restrictions and ϕ curvature restrictions. As A. Cordoba has kindly pointed out, positive and negative results for such operators may be obtained using the methods of [**1**]. He has informed us that an analysis of such operators has been carried out in the doctoral dissertation of his student, A. Ruiz.

(4) The techniques of this paper have more general applicability for the study of the L^p spectrum of constant coefficient differential operators. We shall consider a class of "good" differential operators for which our techniques are applicable. Using Lemma 5, the central result of this paper may be phrased as a classification.

CLASSIFICATION. Let D be a "good" operator. Then either

(a) $\sigma(D_{0,p}) = \overline{\text{range } d}$ for all p, $1 < p < +\infty$, or

(b) $\sigma(D_{0,p}) = \overline{\text{range } d}$ when $p = 2$, while $\sigma(D_{0,p}) = \mathbf{C}$ for $1 < p < \infty, p \neq 2$.

The basic problem is to find the largest class of good operators for which the classification holds. It is clear that we cannot expect this for all operators; for example, the operator with symbol $(\zeta_2 - \zeta_1^2 + i)(\zeta_2 + \zeta_1^2 + i)$ is $L^p(\mathbf{R}^2)$ invertible for some $p \neq 2$, but not for all p (see [**7**] and [**13**]).

Using Theorems A and B, and some simple variants of the Marcinkiewicz and Hörmander multiplier theorems, one can see that if d is a polynomial of degree 2, with real coefficients, then D is a "good" operator. It seems plausible that this holds for arbitrary 2nd order operators. We have made some progress on this question, and shall return to the problem in a later paper.

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