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A MAXIMAL FUNCTION CHARACTERIZATION OF H^p ON THE SPACE OF HOMOGENEOUS TYPE

BY

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ABSTRACT. Let $\psi_0(x) \in \mathcal{S}(R^n)$ and let $\int_{R^n} \psi_0(y)\,dy \neq 0$. For $f \in \mathcal{S}'(R^n)$, $x \in R^n$ and $M > 0$, let

$$f^+(x) = \sup_{t>0} |f * \psi_{0t}(x)|$$

and let $f^{*M}(x) = \sup\{|f * \psi_t(x)|: t > 0, \psi(y) \in \mathcal{S}(R^n), \operatorname{supp} \psi \subset \{y \in R^n: |y| < 1\}, \|D^\alpha \psi\|_{L^\infty} < 1$ for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that $\sum_{i=1}^n \alpha_i < M\}$ where $\psi_t(y) = t^{-n}\psi(y/t)$.

Fefferman-Stein **[11]** showed

THEOREM A. *Let $p > 0$. Then there exists $M(p, n)$, depending only on p and n, such that if $M > M(p, n)$, then*

$$c\|f^+\|_{L^p} < \|f^{*M}\|_{L^p} < C\|f^+\|_{L^p}$$

for any $f \in \mathcal{S}'(R^n)$, where c and C are positive constants depending only on ψ_0, p, M and n.

We investigate this on the space of homogeneous type with certain assumptions.

1. Introduction. In this note, all functions are real valued and measurable. All numbers are real numbers.

In this section, we consider functions or distributions $\mathcal{S}'$ defined on R^n; the letter x denotes the vector $(x_1, \ldots, x_n) \in R^n$ and $|x|$ denotes $(\sum_{i=1}^n x_i^2)^{1/2}$.

First, we define $H^p(R^n)$ $(0 < p \leq 1)$ following Coifman-Weiss **[8]**.

A function $a(x)$ is called a p-atom $(0 < p \leq 1)$ if there exists a ball $B(x_0, r) = \{x: |x - x_0| < r\}$ such that

$$\operatorname{supp} a \subset B(x_0, r), \qquad \|a\|_{L^\infty} \leq |B(x_0, r)|^{-1/p}$$

and if $\int a(x)p(x)\,dx = 0$ for any polynomial $p(x)$ of degree $\leq [n/p - n]$, where $|B(x_0, r)|$ denotes the Lebesgue measure of $B(x_0, r)$ and $[t]$ denotes the integral part of t. For $f \in \mathcal{S}'(R^n)$ let

$$\|f\|_{H^p} = \inf\left\{\left(\sum_{i=1}^{\infty} |\lambda_i|^p\right)^{1/p} : \text{there exists a sequence of } p\text{-atoms } \{a_i(x)\}_{i=1}^{\infty} \text{ such that } f = \sum_{i=1}^{\infty} \lambda_i a_i \text{ in } \mathcal{S}'\right\}.$$

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If such a sequence $\{\lambda_i\}_{i=1}^{\infty}$ does not exist, let $\|f\|_{H^p} = +\infty$. We define

$$H^p(R^n) = \{f \in \mathcal{S}'(R^n): \|f\|_{H^p} < +\infty\}.$$

Using the result of Fefferman-Rivière-Sagher [10] that refined the Calderón-Zygmund decomposition, Coifman [5] showed

THEOREM B. *If* $1 \geqslant p > 0$ *and if* $M \geqslant [n/p - n] + 1$, *then*

$$c\|f^{*M}\|_{L^p} \leqslant \|f\|_{H^p} \leqslant C\|f^{*M}\|_{L^p}$$

for any $f \in \mathcal{S}'(R^n)$, *where* c *and* C *are positive constants depending only on* p, M *and* n.

Coifman [5] showed this for $n = 1$ and this is extended to $n \geqslant 2$ by Latter [14].

As a result of Theorem A and Theorem B, the space $H^p(R^n)$, defined by p-atoms, can be characterized by $\|f^+\|_{L^p}$, that is,

$$c\|f^+\|_{L^p} \leqslant \|f\|_{H^p} \leqslant C\|f^+\|_{L^p} \tag{$*$}$$

for any $f \in \mathcal{S}'$, where C and c depend only on p, n and ψ_0.

For $p = 1$, L. Carleson [3] showed another proof of $(*)$. Extending Carleson's proof, R. Coifman, G. Weiss and Y. Meyer showed that if $p = 1$, then $(*)$ holds on the space of homogeneous type (see [8, p. 642]). This proof used the duality of H^1-BMO and the fact that $\|\cdot\|_{H^1}$ is a norm. For $p < 1$, $\|\cdot\|_{H^p}$ is not a norm and the argument of dual spaces is not so available.

In this note, we extend Theorem A to the L^1-functions defined on the space X, where X is a space of homogeneous type with certain assumptions. On the other hand, it has been shown by Macias-Segovia [16] that Theorem B holds on X. Thus, as a corollary of these results, we see that $(*)$ holds for $p > 1 - \varepsilon$ on X, where ε is a positive number depending only on X.

Lastly, I would like to thank Professor R. Coifman who suggested the problem to show $(*)$ for $p < 1$ on the space of homogeneous type in 1976. I would like to thank Mr. M. Satake for valuable information.

2. Definition. In this section, x, y and z denote the elements of a topological space X and X is endowed with a Borel measure μ and a quasi-distance d. The latter is a mapping d: $X \times X \to R^+ \cup \{0\} = [0, \infty)$ satisfying

(0) $d(x, y) = d(y, x)$ for any $x, y \in X$,

(1) $d(x, y) > 0$ iff $x \neq y$,

(2) $d(x, z) \leqslant A(d(x, y) + d(y, z))$ for any $x, y, z \in X$,

(3) $A^{-1}r \leqslant \mu(B(x, r)) \leqslant r$ for any $x \in X$ and any $r \in (0, \mu(X))$.

The balls $B(x, r) = \{y \in X: d(x, y) < r\}$ $(r > 0)$ form a basis of open neighbourhoods of the point x.

Further we assume that X is endowed with a nonnegative continuous function $K(r, x, y)$ defined on $R^+ \times X \times X$ satisfying

(4) $K(r, x, y) = 0$ if $d(x, y) > r$,

(5) $K(r, x, x) > A^{-1} > 0$,

(6) $K(r, x, y) \leqslant 1$,

(7) $|K(r, x, y) - K(r, x, z)| \leqslant (d(y, z)/r)^{\gamma}$

for any $x, y, z \in X$ and any $r \in R^+$, where γ (> 0) is independent of x, y, z and r. These definitions are due to [**8**]. Notice that there exist $C_1 > 0$ and $C_2 > 0$ such that

$$C_1 K(r, x, y) > 1 \tag{8}$$

for any $x \in X, y \in X$ and $r > 0$ satisfying $d(x, y) < C_2 r$.

For any $f(x) \in L^1_{\text{loc}}(X) = \{f: f \text{ is integrable on any bounded set}\}$, let

$$F(r, x, f) = \int_X K(r, x, y) f(y)\, d\mu(y)/r, \qquad f^+(x) = \sup_{r>0} |F(r, x, f)|.$$

For $f(x)$ and $\infty > p > 0$ let

$$M_p(f)(x) = \sup_{r>0} F(r, x, |f|^p)^{1/p}.$$

The following definition of $H^p(X)$ is also almost due to [**8**].

For $f(x) \in L^1_{\text{loc}}(X)$, let

$$L(f, 0) = \sup_{x \in X,\, r>0} \inf_{c \in R} \int_{B(x,r)} |f(y) - c|\, d\mu(y)/r,$$

$$L(f, \alpha) = \sup_{x \in X,\, y \in X,\, x \neq y} |f(x) - f(y)|/d(x, y)^\alpha \quad \text{for } \alpha > 0.$$

For $\alpha \geqslant 0$, let

$$\|f\|^{(\alpha)} = L(f, \alpha) \quad \text{if } \mu(X) = \infty,$$

$$\|f\|^{(\alpha)} = L(f, \alpha) + \left| \int_X f(y)\, d\mu(y) \right| \mu(X)^{-(\alpha+1)} \quad \text{if } \mu(X) < \infty,$$

$$\mathfrak{L}_\alpha(X) = \{f \in L^\infty(X): \|f\|^{(\alpha)} < \infty\}.$$

Then, $\|\cdot\|^{(\alpha)}$ is a norm. When $\alpha = 0$, it is a BMO norm. When $\alpha > 0$, it is a Lipschitz norm. If $\mu(X) = \infty$, then we consider the set of equivalence classes of functions defined by the relation "$f_1(x)$ and $f_2(x)$ in $\mathfrak{L}_\alpha$ are equivalent iff $f_1 - f_2$ is constant".

We say $a(x)$ is a p-atom if $\int a(y)\, d\mu(y) = 0$ and if there exists a ball $B(x_0, r_0)$ such that

$$\operatorname{supp} a(x) \subset B(x_0, r_0), \qquad \|a\|_\infty \leqslant r_0^{-1/p}.$$

In case $\mu(X) < \infty$ the constant function having $\mu(X)^{-1/p}$ is also considered to be a p-atom. It is clear that

$$\|a\|_{\mathfrak{L}^*_{1/p-1}} \leqslant 1$$

where $\mathfrak{L}^*_\alpha$ is the dual space of $\mathfrak{L}_\alpha$.

For $0 < p \leqslant 1$ and $f \in \mathfrak{L}^*_{1/p-1}$, let

$$\|f\|_{H^p} = \inf\left\{ \left(\sum_{i=1}^\infty |\lambda_i|^p \right)^{1/p} : \text{there exists a sequence of } p\text{-atoms } \{a_i(x)\} \text{ such that } f = \sum \lambda_i a_i \text{ in } \mathfrak{L}^*_{1/p-1} \right\}.$$

If such a sequence $\{\lambda_i\}$ does not exist, let $\|f\|_{H^p} = +\infty$. We define

$$H^p(X) = \{f \in \mathfrak{L}^*_{1/p-1}: \|f\|_{H^p} < +\infty\}.$$

Lastly, for $f \in L^1_{\mathrm{loc}}(X)$ we define

$$f^*(x) = \sup\left\{\left|\int f(y)\varphi(y)\,d\mu(y)\right|/r: r > 0, \operatorname{supp}\varphi \subset B(x, r),\right.$$
$$\left. L(\varphi, \gamma) \leq r^{-\gamma}, \|\varphi\|_{L^\infty} \leq 1\right\}.$$

3. The main theory. Our result is the following

THEOREM 1. *There exists $p_1 < 1$, only depending on X, such that for any $f \in L^1(X)$ and any $p > p_1$*

$$\|f^*\|_{L^p} \leq c_1\|f^+\|_{L^p},$$

where c_1 is a positive constant depending only on p and X.

REMARK. For $p > 1$, this is clear from the Hardy-Littlewood maximal theorem. For $p = 1$, this is shown by **[8]**.

Macias-Segovia **[16]** showed

THEOREM C. *If $f \in L^1(X)$ and if $1 \geq p > 1/(1+\gamma)$, then*

$$c_2\|f^*\|_{L^p} \leq \|f\|_{H^p} \leq c_3\|f^*\|_{L^p},$$

where c_2 and c_3 are positive constants depending only on p and X.

REMARK. This can also be proved by exactly the same way as **[15]**. **[16]** showed this theorem more generally for a "distribution" f.

As a corollary of Theorem 1 and Theorem C, we get

COROLLARY 1. *There exists $p_2 < 1$, only depending on X, such that for any $f \in L^1(X)$ and any $1 \geq p > p_2$*

$$\|f^+\|_{L^p} \leq c_4\|f\|_{H^p} \leq c_5\|f^*\|_{L^p} \leq c_6\|f^+\|_{L^p},$$

where c_4, c_5 and c_6 are positive constants depending only on p and X.

For the proof of Theorem 1, we need the following four lemmas.

In the following, N and Z mean $\{1, 2, 3, \ldots\}$ and $\{0, \pm 1, \pm 2, \ldots\}$ respectively. The letters C and C_i $(i = 3, 4, \ldots)$ denote the positive constants that depend only on A and γ. The various uses of C do not all denote the same constant.

LEMMA 1. *Let $d\nu$ be a positive measure over $X \times R^+$ such that*

$$\nu(B(x, r) \times (0, r)) < r^{1+\delta} \tag{10}$$

for any $x \in X$ and any $r \in R^+$, where $\delta \geq 0$ is independent of r and x. Then

$$\left(\int\int_{X \times R^+} |F(r, y, f)|^{p(1+\delta)}\,d\nu(y, r)\right)^{1/(p(1+\delta))} \leq C_{p,\delta}\|f\|_{L^p_{d\mu}(X)}$$

for any $p > 1$ and any $f \in L^p(X)$, where $C_{p,\delta}$ is independent of f.

REMARK. This lemma is essentially known. For the case $\delta = 0$, see [**18**, p. 236]. For the case $\delta > 0$, see Duren [**23**].

PROOF. Let $f \in L^p(X)$. Let $\lambda > 0$,

$$V_\lambda = \{(x, r) \in X \times R^+ : |F(r, x, f)| > \lambda\}, \qquad q = 2A. \tag{11}$$

Let $W_{n,\lambda} = \{x \in X : \sup_{q^{n-1} < r \leqslant q^n} |F(r, x, f)| > \lambda\}$; then there exists $M_{f,\lambda}$ such that $W_{n,\lambda} = \varnothing$ for any $n > M$. For each $n \leqslant M$, there exist disjoint balls $\{B(y_{nj}, q^n)\}_j$ such that

$$y_{nj} \in W_{n,\lambda}, \qquad B(y_{nj}, q^n) \cap \left(\bigcup_{m=n+1}^{M} \bigcup_i B(y_{mi}, q^m) \right) = \varnothing \tag{12}$$

and that for any $x \in W_{n,\lambda}$

$$B(x, q^n) \cap \left(\bigcup_{m=n}^{M} \bigcup_i B(y_{mi}, q^m) \right) \neq \varnothing.$$

By (2) and (11)

$$V_\lambda \subset \bigcup_n \bigcup_j \left(B(y_{nj}, q^{n+1}) \times (0, q^n) \right).$$

Thus

$$\begin{aligned}
\lambda^{p(1+\delta)} \nu(V_\lambda) &\leqslant \sum_n \sum_j \nu\left(B(y_{nj}, q^{n+1}) \times (0, q^n)\right) \lambda^{p(1+\delta)} \\
&\leqslant \sum_n \sum_j q^{(n+1)(1+\delta)} \left(\int_{B(y_{nj}, q^n)} |f(y)|\, d\mu(y) / q^{n-1} \right)^{p(1+\delta)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by (10), (12)} \\
&\leqslant \sum_n \sum_j q^{(n+1)(1+\delta)} q^{p(1+\delta)} \left(\int_{B(y_{nj}, q^n)} |f(y)|^p\, d\mu(y) / q^n \right)^{1+\delta} \\
&\leqslant C_{p,\delta} \left(\sum_n \sum_j \int_{B(y_{nj}, q^n)} |f(y)|^p\, d\mu(y) \right)^{1+\delta} \\
&\leqslant C_{p,\delta} \left(\int_X |f(y)|^p\, d\mu(y) \right)^{1+\delta}.
\end{aligned}$$

Then, Lemma 1 follows from the Marcinkiewicz interpolation theorem.

LEMMA 2. *Let $g(x)$ be a nonnegative function defined on X. Then for each $t > 0$ there exist $\{x(g, t, j)\}_{j=1,2,\ldots} \subset X$ such that*

$$1 \leqslant C_1 \sum_j K(t, x(g, t, j), y) \leqslant C_3 \quad \textit{for any } y \in X, \tag{20}$$

$$g(x(g, t, j)) \leqslant C_4 F(t, x(g, t, j), g^{1/2})^2 \quad \textit{for any } j. \tag{21}$$

PROOF. First, we can select $\{y(t, j)\}_{j=1,2,\ldots}$ such that

$$d(y(t, i), y(t, j)) \geqslant (2A)^{-1} C_2 t \qquad (i \neq j), \tag{22}$$

$$\sum_j \chi_{B(y(t,j),(2A)^{-1}C_2t)}(x) \geqslant 1 \quad \text{for any } x \in X. \tag{23}$$

For each $y(t,j)$, we select $x(g,t,j)$ such that

$$d(x(g,t,j),y(t,j)) \leqslant (2A)^{-1}C_2t, \tag{24}$$

$$g(x(g,t,j)) \leqslant \left(\int_{B(y(t,j),(2A)^{-1}C_2t)} g(y)^{1/2}\,d\mu(y)/\left((2A)^{-2}C_2t\right)\right)^2. \tag{25}$$

Then, (20) and (21) follow from (8), (22), (23), (24) and (25).

LEMMA 3. *There exist $p_1 < 1$ and C_5, only depending on X, such that*

$$\left|\int f(y)\varphi(y)\,d\mu(y)\right|/r_0 \leqslant C_5\left(\int_{B(x_0,r_0)} f^+(y)^{p_1}\,d\mu(y)/r_0\right)^{1/p_1}$$

for any $f \in L^1_{\mathrm{loc}}(X)$ and any φ, x_0, r_0 satisfying

$$\operatorname{supp}\varphi \subset B(x_0,r_0), \qquad L(\varphi,\gamma) \leqslant r_0^{-\gamma}, \qquad \|\varphi\|_{L^\infty} \leqslant 1.$$

REMARK. I borrowed the idea of this proof from Carleson-Garnett **[4]** and Jones **[13]**.

PROOF. We may assume that $r_0 = 1$ and that $\varphi \geqslant 0$. Let

$$\varepsilon = 1/(4C_3) \tag{30}$$

and let η be a sufficiently small positive number, only depending on X. We inductively construct $\{x_{sj}\}_{s=1,2,\ldots,j=1,2,\ldots,j(s)} \subset B(x_0,1)$ satisfying the following.

(31) $\|\sum_{j=1}^{j(s)} \chi_{B_{sj}}\|_\infty \leqslant C_3$ for any $s \in N$, where $B_{sj} = B(x_{sj}, C_2\eta^s)$,

(32) $f^+(x_{sj}) \leqslant C_4 F(\eta^s, x_{sj}, f^{+1/2})^2$,

(33) $0 \leqslant \varphi_s(x) \leqslant (1-\varepsilon)^s \chi_{B(x_0,1)}(x)$, where

(34) $\varphi_s(x) = \varphi(x) - \sum_{i=1}^s \varepsilon(1-\varepsilon)^{i-1}\sum_{j=1}^{j(i)} C_1 K(\eta^i, x_{ij}, x)$.

Let $\varphi_0(x) = \varphi(x)$. Assume that $\{x_{ij}\}_{i=1,\ldots,s-1,j=1,\ldots,j(i)}$ have been constructed and that $\varphi_{s-1}(x)$ is defined by (34). Then, by (31) and (7),

$$\begin{aligned}
|\varphi_{s-1}(x) - \varphi_{s-1}(y)| &\leqslant |\varphi(x) - \varphi(y)| \\
&\quad + \sum_{i=1}^{s-1} \varepsilon(1-\varepsilon)^{i-1}\sum_j C_1\left|K(\eta^i, x_{ij}, x) - K(\eta^i, x_{ij}, y)\right| \\
&\leqslant d(x,y)^\gamma + \sum_{i=1}^{s-1} \varepsilon(1-\varepsilon)^{i-1} C_1 2C_3 (d(x,y)/\eta^i)^\gamma \\
&\leqslant d(x,y)^\gamma\left\{1 + \varepsilon(1-\varepsilon)^{-1}2C_1C_3((1-\varepsilon)/\eta^\gamma)^{s-1}(1-\eta^\gamma/(1-\varepsilon))^{-1}\right\} \\
&\leqslant C((1-\varepsilon)/\eta^\gamma)^{s-1} d(x,y)^\gamma.
\end{aligned} \tag{35}$$

Let $\Omega_{s,\lambda} = \{x \in X: \varphi_{s-1}(x) > \lambda(1-\varepsilon)^{s-1}\}$. Applying Lemma 2 to $g(x) = f^+(x)$ and $t = \eta^s$, we get $\{x(f^+,\eta^s,j)\}_{j=1,2,\ldots}$ such that (20) and (21). Let $\{x_{sj}\}_{j=1}^{j(s)}$ be a subset of $\{x(f^+,\eta^s,j)\}_j$ which is contained in $\Omega_{s,2/3}$. Then (31) and (32) are satisfied. By (20),

$$\varepsilon(1-\varepsilon)^{s-1}C_1\sum_{j=1}^{j(s)} K(\eta^s, x_{sj}, y) \leqslant C_3\varepsilon(1-\varepsilon)^{s-1} \quad \text{for any } y \in X. \tag{36}$$

If $\operatorname{supp} K(\eta^s, x, \cdot) \cap \Omega_{s,1-\varepsilon} \neq \varnothing$, then by (35) $x \in \Omega_{s,2/3}$ because η is small. Thus by (20)

$$\varepsilon(1-\varepsilon)^{s-1} < \varepsilon(1-\varepsilon)^{s-1} C_1 \sum_{j=1}^{j(s)} K(\eta^s, x_{sj}, y) \quad \text{for any } y \in \Omega_{s,1-\varepsilon}. \tag{37}$$

Similarly, if $\operatorname{supp} K(\eta^s, x, \cdot) \cap \Omega^c_{s,1/2} \neq \varnothing$, then $x \notin \Omega_{s,2/3}$ by (35). So,

$$\sum_j K(\eta^s, x_{sj}, y) = 0 \quad \text{for any } y \in \Omega^c_{s,1/2} \tag{38}$$

and (33) follows from (30), (36), (37) and (38).

Thus

$$\varphi(x) = \sum_{s \in N} \sum_{j=1}^{j(s)} \varepsilon(1-\varepsilon)^{s-1} C_1 K(\eta^s, x_{sj}, x)$$

and

$$\begin{aligned}\int f(y)\varphi(y)\, d\mu(y) &= \sum_{s \in N} \varepsilon(1-\varepsilon)^{s-1} \sum_j C_1 \int f(y) K(\eta^s, x_{sj}, y)\, d\mu(y) \\ &= C_1 \varepsilon (1-\varepsilon)^{-1} \sum_s \sum_j (1-\varepsilon)^s \eta^s F(\eta^s, x_{sj}, f).\end{aligned}$$

By (32),

$$\begin{aligned}\left| \sum_s \sum_j (1-\varepsilon)^s \eta^s F(\eta^s, x_{sj}, f) \right| &< \sum \sum C_4 (1-\varepsilon)^s \eta^s F\left(\eta^s, x_{sj}, f^{+1/2}\right)^2 \\ &= C_4 \int\int_{X \times R^+} F(r, x, f^{+1/2})^2\, d\nu(x, r),\end{aligned}$$

where $\nu = \sum_s \sum_j (1-\varepsilon)^s \eta^s \delta_{(x_{sj}, \eta^s)}$ and $\delta_{(x,r)}$ is the Dirac measure of the point $(x, r) \in X \times R^+$. Note that

$$\nu(B(x, r) \times (0, r)) \leqslant Cr(1-\varepsilon)^{\log r/\log \eta} = Cr^{1+\log(1-\varepsilon)/\log \eta}$$

and that

$$F(r, x, f^{+1/2}) = F\left(r, x, f^{+1/2}\chi_{B(x_0,1)}\right) \quad \text{on } \operatorname{supp} \nu.$$

Then, by Lemma 1,

$$\begin{aligned}&\int\int_{X \times R^+} F\left(r, x, f^{+1/2}\chi_{B(x_0,1)}\right)^2 d\nu(x, r) \\ &\qquad \leqslant C\left(\int_X (f^+(y)^{1/2}\chi(y))^{2/(1+\delta)}\, d\mu(y)\right)^{1+\delta} \\ &\qquad\qquad\qquad \text{where } \delta = \log(1-\varepsilon)/\log \eta \\ &\qquad \leqslant C\|f^+\chi\|_{L^{1/(1+\delta)}} \\ &\qquad = C\left(\int_{B(x_0,1)} f^+(y)^{1/(1+\delta)}\, d\mu(y)\right)^{1+\delta}.\end{aligned}$$

This completes the proof of Lemma 3.

LEMMA 4. *If $f \in L^p(X)$, with $1 < p \leqslant \infty$, then*

$$\|M_1(f)\|_{L^p} \leqslant C_p\|f\|_{L^p}$$

where C_p is independent of f.

This is the Hardy-Littlewood maximal theorem. We omit the proof.

PROOF OF THEOREM 1. By Lemma 3, $f^*(x) \leqslant CM_{p_1}(f^+)(x)$. Thus, by Lemma 4,

$$\|f^*\|_{L^p} \leqslant C\|M_{p_1}(f^+)\|_{L^p} = C\|M_1(f^{+p_1})\|_{L^{p/p_1}}^{1/p_1} \leqslant C_{p,p_1}\|f^+\|_{L^p}$$

if $p > p_1$.

4. The kernel whose support is not compact. In this section, we relax the restriction (4). Let $K_1(r, x, y)$ be a nonnegative continuous function defined on $R^+ \times X \times X$ such that

(40) $K_1(r, x, y) \leqslant (1 + d(x, y)/r)^{-1-\gamma}$,

(41) $K_1(r, x, x) > A^{-1} > 0$,

(42) $|K_1(r, x, y) - K_1(r, x, z)| \leqslant (d(y, z)/r)^\gamma(1 + d(x, y)/r)^{-1-2\gamma}$ if $d(y, z) < (r + d(x, y))/(4A)$

for any $x, y, z \in X$ and any $r \in R^+$. In this case (8) holds; i.e.

(43) $C_1K_1(r, x, y) > 1$

for any $x \in X, y \in X$ and $r > 0$ satisfying $d(x, y) < C_2r$.

For any $f \in L^1(X)$, let

$$F_1(r, x, f) = \int_X K_1(r, x, y)f(y)\, d\mu(y)/r,$$

$$f^{(+)}(x) = \sup_{r>0}|F_1(r, x, f)|.$$

Extending Theorem 1, we get

THEOREM 1′. *There exists $p_3 < 1$, only depending on X, such that for any $f \in L^1(X)$ and any $p > p_3$*

$$\|f^*\|_{L^p} \leqslant c_7\|f^{(+)}\|_{L^p},$$

where c_7 is a positive constant depending only on p and X.

As a corollary of Theorem 1′ and Theorem C, we get

COROLLARY 1′. *There exists $p_4 < 1$, only depending on X, such that for any $f \in L^1(X)$ and any $1 \geqslant p > p_4$*

$$\|f^{(+)}\|_{L^p} \leqslant c_8\|f\|_{H^p} \leqslant c_9\|f^*\|_{L^p} \leqslant c_{10}\|f^{(+)}\|_{L^p},$$

where c_8, c_9 and c_{10} are positive constants depending only on p and X.

REMARK. The inequality $\|f^{(+)}\|_{L^p} \leqslant c_8\|f\|_{H^p}$ follows easily from (42).

For the proof of Theorem 1′, it suffices to prove the following.

LEMMA 3′. *There exist $p_3 < 1$ and C_5', only depending on X, such that*

$$\left|\int f(y)\varphi(y)\, d\mu(y)\right|/r_0 \leqslant C_5'M_{p_3}(f^{(+)})(x_0)$$

for any $f \in L^1(X)$ and any φ, x_0, r_0 satisfying

$$\operatorname{supp} \varphi \subset B(x_0, r_0), \qquad L(\varphi, \gamma) \leqslant r_0^{-\gamma}, \qquad \|\varphi\|_{L^\infty} \leqslant 1.$$

Theorem 1′ can be proved in exactly the same way as Theorem 1, replacing Lemma 3 by Lemma 3′. For the proof of Lemma 3′, we need the following three lemmas.

In the following, let x_0 be fixed and let $d(y) = 1 + d(x_0, y)$.

LEMMA 5. *If $d(x, y) \leqslant d(y)/(2A)$, then $d(y)/(2A) \leqslant d(x) \leqslant 2Ad(y)$.*

We omit the proof.

LEMMA 2′. *Let $g(x)$ be a nonnegative function defined on X. Then for each $0 < t < (4A)^{-5}$, there exist $\{x'(g, t, j)\}_{j=1,2,\dots}$ such that*

$$1 \leqslant \sum_j \chi_{B(x'(g,t,j), C_2 t d(x'(g,t,j)))}(x) \leqslant C_3' \quad \text{for any } x \in X, \tag{50}$$

$$g(x'(g, t, j)) \leqslant C_4' F\left(td(x'(g, t, j)), x'(g, t, j), g^{1/2}\right)^2. \tag{51}$$

In particular,

$$(2A)^{1+\gamma/2} C_1 \sum_j d(x'(g, t, j))^{-1-\gamma/2} K_1(td(x'(g, t, j)), x'(g, t, j), x) \cdot \chi_{B(x'(g,t,j), C_2 t d(x'(g,t,j)))}(x) \geqslant d(x)^{-1-\gamma/2} \tag{52}$$

for any $x \in X$.

PROOF. First, we can select $\{y'(t, j)\}_{j=1,2,\dots}$ such that

$$d(y'(t, i), y'(t, j)) \geqslant (2A)^{-2} C_2 t \min(d(y'(t, i)), d(y'(t, j))) \qquad (i \neq j), \tag{53}$$

$$\sum_j \chi_{B(y'(t,j)(2A)^{-2} C_2 t d(y'(t,j)))}(x) \geqslant 1. \tag{54}$$

For each $y'(t, j)$, we select $x'(g, t, j)$ such that

$$d(x'(g, t, j), y'(t, j)) < (2A)^{-2} C_2 t d(y'(t, j)), \tag{55}$$

$$g(x'(g, t, j))^{1/2} \leqslant \int_{B\left(y'(t,j), (2A)^{-2} C_2 t d(y'(t,j))\right)} g(y)^{1/2}\, d\mu(y) \Big/ \left((2A)^{-3} C_2 t d(y'(t, j))\right). \tag{56}$$

The first inequality of (50) follows from (54), (55) and Lemma 5. The second inequality of (50) follows from (53) and (55). (51) follows from (55) and (56). If $x \in B(y, C_2 t d(y))$, then

$$d(x) \geqslant d(y)/(2A) \tag{57}$$

by Lemma 5. Thus (52) follows from (57), (50) and (43).

LEMMA 6. *Let $0 < r < 1$ and let $\{x_j\}_{j=1,2,\dots}$ be such that*

$$\sum_j \chi_{B(x_j, C_2 r d(x_j))}(x) \leqslant C_3' \quad \text{for any } x \in X. \tag{60}$$

Let $0 \leqslant a, a + \gamma/2 \leqslant b \leqslant 2\gamma, 0 \leqslant M$ *and let*

$$u_j(x) = d(x_j)^{-1-a}\big(1 + d(x_j, x)/\big(rd(x_j)\big)\big)^{-1-b}\chi_M\big(d(x_j, x)/\big(rd(x_j)\big)\big)$$

where $\chi_M(\cdot)$ *is the characteristic function of* $[M, \infty)$. *Then*

$$\sum_j u_j(x) \leqslant C_6 d(x)^{-1-a}\max\big(r^b, (1 + M)^{-b}\big).$$

PROOF. For each $t \in N$, let $v_t(x) = \Sigma_j'' u_j(x)$, where Σ_j'' means $\Sigma_{j:2^{t-1} \leqslant d(x_j) < 2^t}$. First,

$$\begin{aligned} v_t(x) &\leqslant 2^{-(t-1)(1+a)}\sum_j{}''\big(1 + d(x_j, x)/(r2^t)\big)^{-1-b}\chi_M\big(d(x_j, x)/(r2^{t-1})\big) \\ &\leqslant C2^{-(t-1)(1+a)}(r2^t)^{-1} \\ &\quad \cdot\int\big(1 + d(y, x)/(r2^t)\big)^{-1-b}\chi_M\big(d(y, x)/(r2^{t-1})\big)\,d\mu(y) \quad \text{by (60)} \\ &\leqslant C2^{-(t-1)(1+a)}(1 + M)^{-b}/b. \end{aligned} \tag{61}$$

If $2^{t-1} > 2Ad(x)$, then $d(x_j, x) \geqslant Cd(x_j)$. Thus,

$$\begin{aligned} v_t(x) &\leqslant C2^{-(t-1)(1+a)}r^{1+b}\sum_j{}''1 \\ &\leqslant C2^{-(t-1)(1+a)}r^{1+b}r^{-1} \quad \text{by (60)}. \end{aligned} \tag{62}$$

If $2^t < d(x)/(2A)$, then $d(x_j, x) \geqslant Cd(x)$. Thus,

$$\begin{aligned} v_t(x) &\leqslant C2^{-(t-1)(1+a)}\big(1 + d(x)/(r2^t)\big)^{-1-b}\sum_j{}''1 \\ &\leqslant C2^{-(t-1)(1+a)}d(x)^{-1-b}r^{1+b}2^{t(1+b)}r^{-1} \quad \text{by (60)}. \end{aligned} \tag{63}$$

Summing up (61)–(63), we get the desired estimate.

PROOF OF LEMMA 3′. We may assume $r_0 = 1$ and $\|\varphi\|_{L^\infty} < 2^{-1-\gamma/2}$. Let

$$\varepsilon = \min\big(1/C_8, (2A)^{-1-\gamma/2}/2\big), \tag{70}$$

where

$$C_7 = 2(2A)^{1+\gamma/2}C_1, \quad C_8 = 4C_6C_7. \tag{71}$$

Let η be a sufficiently small positive number to be determined later.

We inductively construct $\{x_{sj}\}_{s\in N, 1 \leqslant j < j(s)} \subset X$, and $\{\varepsilon_{sj}\}_{s\in N, 1 \leqslant j < j(s)} \subset \{-1, 0, 1\}$, where $j(s)$ can be ∞, satisfying

(72) $\|\Sigma_j \chi_{B_{sj}}(x)\|_{L^\infty} \leqslant C_3'$ for any $s \in N$, where $B_{sj} = B(x_{sj}, C_2\eta^s d(x_{sj}))$,

(73) $f^{(+)}(x_{sj}) \leqslant C_4' F(\eta^s d(x_{sj}), x_{sj}, f^{(+)1/2})^2$,

(74) $|\varphi_s(x)| \leqslant (1 - \varepsilon)^s d(x)^{-1-\gamma/2}$, *where*

$$\begin{aligned} \varphi_s(x) = \varphi(x) - \sum_{i=1}^{s} C_7\varepsilon(1 - \varepsilon)^{i-1} \\ \cdot \sum_{1 \leqslant j < j(i)} \varepsilon_{ij} d(x_{ij})^{-1-\gamma/2}K_1\big(\eta^i d(x_{ij}), x_{ij}, x\big). \end{aligned} \tag{75}$$

Let $\varphi_0(x) = \varphi(x)$. Assume that $\{x_{ij}\}$, $\{\varepsilon_{ij}\}$ $(1 \leqslant i \leqslant s-1,\ 1 \leqslant j < j(i))$ have been constructed and that $\varphi_{s-1}(x)$ is defined by (75).

If $d(x,y) \leqslant \eta^{s-1}d(x)/(4A)^2$, then $d(x,y) \leqslant (\eta^{s-1}d(x_{ij}) + d(x_{ij},x))/(4A)$. Thus,

$$\begin{aligned}
|\varphi_{s-1}(x) - \varphi_{s-1}(y)| &\leqslant |\varphi(x) - \varphi(y)| + \sum_{i=1}^{s-1} C_7\varepsilon(1-\varepsilon)^{i-1} \\
&\quad \cdot \sum_j d(x_{ij})^{-1-\gamma/2}\left|K_1\big(\eta^i d(x_{ij}), x_{ij}, x\big) - K_1\big(\eta^i d(x_{ij}), x_{ij}, y\big)\right| \\
&\leqslant |\varphi(x) - \varphi(y)| + 2\sum_{i=1}^{s-1} C_7\varepsilon(1-\varepsilon)^i \\
&\quad \cdot \sum_j d(x_{ij})^{-1-\gamma/2}\big(d(x,y)/\big(\eta^i d(x_{ij})\big)\big)^\gamma\big(1 + d(x_{ij},x)/\big(\eta^i d(x_{ij})\big)\big)^{-1-2\gamma}
\end{aligned}$$

by (42). The second term is equal to

$$\begin{aligned}
2d(x,y)^\gamma C_7\varepsilon \sum_{i=1}^{s-1}(1-\varepsilon)^i\eta^{-i\gamma}\sum_j d(x_{ij})^{-1-3\gamma/2}\big(1 + d(x_{ij},x)/\big(\eta^i d(x_{ij})\big)\big)^{-1-2\gamma} \\
\leqslant 2d(x,y)^\gamma C_7\varepsilon\sum_{i=1}^{s-1}\big((1-\varepsilon)/\eta^\gamma\big)^i C_6 d(x)^{-1-3\gamma/2} \\
\leqslant d(x,y)^\gamma\big((1-\varepsilon)/\eta^\gamma\big)^{s-1}d(x)^{-1-3\gamma/2} \qquad (76)
\end{aligned}$$

by Lemma 6, (70) and (71).

Let

$$\Omega_{s,\lambda} = \big\{x \in X\colon \varphi_{s-1}(x) > \lambda(1-\varepsilon)^{s-1}d(x)^{-1-\gamma/2}\big\}.$$

Applying Lemma 2′ to $g(x) = f^{(+)}(x)$ and $t = \eta^s$, we get $\{x'(f^{(+)},\eta^s,j)\}_j$ such that (50) and (51). Let $x_{sj} = x'(f^{(+)},\eta^s,j)$. Then, (72) and (73) are satisfied. Let $\varepsilon_{sj} = \operatorname{sign}(\varphi_{s-1}(x_{sj}))$ and let

$$w_s(x) = C_7\varepsilon(1-\varepsilon)^{s-1}\sum_j \varepsilon_{sj}d(x_{sj})^{-1-\gamma/2}K_1\big(\eta^s d(x_{sj}), x_{sj}, x\big).$$

Note that

$$\begin{aligned}
|w_s(x)| &\leqslant C_7\varepsilon(1-\varepsilon)^{s-1}\sum_j d(x_{sj})^{-1-\gamma/2}\big(1 + d(x_{sj},x)/\big(\eta^s d(x_{sj})\big)\big)^{-1-\gamma} \\
&\leqslant 4^{-1}(1-\varepsilon)^{s-1}d(x)^{-1-\gamma/2} \qquad (77)
\end{aligned}$$

by Lemma 6, (70) and (71).

If $d(x,y) < C_9\eta^{s-1}d(y)$, where $C_9 = (\varepsilon(2A)^{-1-3\gamma/2}/2)^{1/\gamma}$, then

$$d(y)/(2A) \leqslant d(x) \leqslant 2Ad(y) \qquad (78)$$

by Lemma 5 and

$$\begin{aligned}
|\varphi_{s-1}(x) - \varphi_{s-1}(y)| &\leqslant |\varphi(x) - \varphi(y)| + 2^{-1}\varepsilon(1-\varepsilon)^{s-1}d(y)^{-1-\gamma/2} \quad \text{by (76)} \\
&\leqslant \varepsilon(1-\varepsilon)^{s-1}d(y)^{-1-\gamma/2} \qquad (79)
\end{aligned}$$

by supp $\varphi \subset B(x_0, 1)$, $L(\varphi, \gamma) \leqslant 1$. Thus, if $y \notin \Omega_{s,0}$ and if $d(x, y) \leqslant C_9\eta^{s-1}d(y)$, then by (79) and (78),

$$\varphi_{s-1}(x) < \varepsilon(1-\varepsilon)^{s-1}d(y)^{-1-\gamma/2} < (2A)^{1+\gamma/2}\varepsilon(1-\varepsilon)^{s-1}d(x)^{-1-\gamma/2}$$

and, by (70),

$$B\big(y, C_9\eta^{s-1}d(y)\big) \cap \Omega_{s,1/2} = \varnothing. \tag{80}$$

So, if $x \in \Omega_{s,1/2}$, then by (52), (71) and (80),

$$w_s(x) > 2\varepsilon(1-\varepsilon)^{s-1}d(x)^{-1-\gamma/2} - C_7\varepsilon(1-\varepsilon)^{s-1}\sum_j d(x_{sj})^{-1-\gamma/2}$$
$$\cdot\big|K_1\big(\eta^s d(x_{sj}), x_{sj}, x\big)\big|\chi_{C_9\eta^{-1}}\big(d(x, x_{sj})/\big(\eta^s d(x_{sj})\big)\big).$$

By Lemma 6, the second term is less than

$$C_7\varepsilon(1-\varepsilon)^{s-1}C_6 d(x)^{-1-\gamma/2}\big(C_9\eta^{-1}\big)^{-\gamma}.$$

Since η is sufficiently small, we see that

$$w_s(x) > \varepsilon(1-\varepsilon)^{s-1}d(x)^{-1-\gamma/2} \quad \text{on } \Omega_{s,1/2}. \tag{81}$$

Similarly,

$$w_s(x) < -\varepsilon(1-\varepsilon)^{s-1}d(x)^{-1-\gamma/2} \quad \text{on } (\Omega_{s,-1/2})^c. \tag{82}$$

In this way, by (77), (81) and (82), we see that $\varphi_s(x)$ defined by (75) satisfies (74).

Thus,

$$\varphi(x) = \sum_{s\in N}\sum_j C_7\varepsilon(1-\varepsilon)^{s-1}\varepsilon_{sj}d(x_{sj})^{-1-\gamma/2}K_1\big(\eta^s d(x_{sj}), x_{sj}, x\big).$$

So,

$$\left|\int f(y)\varphi(y)\,d\mu(y)\right| \leqslant C_7\sum_s\sum_j \varepsilon(1-\varepsilon)^{s-1}\eta^s d(x_{sj})^{-\gamma/2}f^{(+)}(x_{sj})$$
$$\leqslant C\int\int_{X\times R^+} F(r, x, f^{(+)1/2})^2\,d\nu(x, r)$$

by (73), where

$$\nu = \sum_s\sum_j \varepsilon(1-\varepsilon)^s\eta^s d(x_{sj})^{-\gamma/2}\delta_{(x_{sj},\eta^s d(x_{sj}))}$$
$$\leqslant C\sum_{t\in N}\varepsilon 2^{-t\gamma/2}\sum_{s,j:2^{t-1}\leqslant d(x_{sj})<2^t}(1-\varepsilon)^s\eta^s\delta_{(x_{sj},\eta^s d(x_{sj}))}$$
$$= \sum_{t\in N}2^{-t\gamma/2}\nu_t. \tag{83}$$

Note that $\nu_t(B(x, r)\times(0, r)) \leqslant (2^{-t}r)^{1+\log(1-\varepsilon)/\log\eta}$ and that

$$F(r, x, f^{(+)1/2}) = F\big(r, x, f^{(+)1/2}\chi_{B(x_0, C2^t)}\big) \quad \text{on supp } \nu_t.$$

Let $\delta = \log(1 - \varepsilon)/\log \eta$. Then, by Lemma 1,

$$\int\int_{X \times R^+} F(r, x, f^{(+)1/2}\chi_{B(x_0, C2^t)})^2 \, d\nu_t(x, r)$$
$$\leqslant C2^{-t(1+\delta)}\left(\int_{B(x_0, C2^t)} f^{(+)1/(1+\delta)} \, d\mu\right)^{1+\delta}$$
$$\leqslant CM_{1/(1+\delta)}(f^{(+)})(x_0)$$

for each $t \in N$. Thus, by (83), we get

$$\left|\int f(y)\varphi(y) \, d\mu(y)\right| \leqslant CM_{1/(1+\delta)}(f^{(+)})(x_0).$$

5. Examples.

EXAMPLE 1. If we set $X = R^n$, $d(x, y) = |x - y|^n$ and

$$K(r, x, y) = \psi_0((x - y)/r^{1/n}),$$

(where $\psi_0 \in \mathfrak{D}(R^n)$, $\operatorname{supp} \psi_0 \subset \{x \in R^n: |x| < 1\}$, $|\psi_0(x) - \psi_0(y)| \leqslant |x - y|$, $\psi_0(x) \geqslant 0$, $\psi_0(0) > 0$), then (0)–(7) are satisfied with $\gamma = 1/n$. In this case, the definitions of H^p in §§1 and 2 coincide for $p > n/(n + 1)$. Since $\mathfrak{L}_{1/p-1}(R^n) = \{0\}$ for $p < n/(n + 1)$, the definition in §2 is not valid for $p < n/(n + 1)$.

$$K_1(r, x, y) = \left(1 + |x - y|^2/r^{2/n}\right)^{-(n+1)/2}$$

satisfies (40)–(42) and $K_1(r, x, y)/r$ is the Poisson kernel.

EXAMPLE 2. If we set $X = \Sigma_{2n-1} = \{z \in C^n: z \cdot \bar{z} = \sum_{j=1}^n z_j\bar{z}_j = 1\}$ and $d(z, w) = |1 - z \cdot \bar{w}|^n$, then Σ_{2n-1} is a space of homogeneous type by using the Lebesgue surface measure. Let $\varphi_0(t) \in C^\infty(0, \infty)$ be a function such that $\varphi_0(t) = 1$ on $(0, 1/2)$, $\varphi_0(t) = 0$ on $(1, \infty)$ and $\varphi_0(t) \geqslant 0$. Then, $K(r, z, w) = \varphi_0(d(z, w)/r)$ satisfies (0)–(7) with $\gamma = 1/(2n)$.

$$K_1(r, z, w) = |1 - tz \cdot w|^{-2n}(1 - t^2)^n r,$$

where $t = 1 - r^{1/n}$ $(0 < r \leqslant 1)$, satisfies (40)–(42) and $K_1(r, z, w)/r$ is the Poisson-Szegö kernel. ($H^p(\Sigma_{2n-1})$ has been investigated by many mathematicians. For example, see [7], [8], [12] and [19].)

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