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A PARTITION THEOREM FOR THE INFINITE SUBTREES OF A TREE

BY

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ABSTRACT. We prove a generalization for infinite trees of Silver's partition theorem. This theorem implies a version for trees of the Nash-Williams partition theorem.

1. Introduction. First we establish some notation. An ordinal will be identified with the set of smaller ordinals, and a cardinal will be an initial ordinal. For example, $4 = \{0, 1, 2, 3\}$; and $\omega = \aleph_0$ is the set of all nonnegative integers as well as the cardinality of that set. If X is a set, then $|X|$ is the cardinaltiy of X. If κ is a cardinal, then $[X]^\kappa = \{Y \subseteq X: |Y| = \kappa\}$, $[X]^{<\kappa} = \{Y \subseteq X: |Y| < \kappa\}$, and $[X]^{\leqslant\kappa} = [X]^{<\kappa} \cup [X]^\kappa$.

In [2], Erdös and Rado made the following definition: a family of sets $\mathfrak{F} \subseteq [\omega]^{\aleph_0}$ is Ramsey provided there exists $X \in [\omega]^{\aleph_0}$ with either $[X]^{\aleph_0} \subseteq \mathfrak{F}$ or $[X]^{\aleph_0} \cap \mathfrak{F} = \varnothing$. Erdös and Rado also proved that the axiom of choice implies that there exists $\mathfrak{F} \subseteq [\omega]^{\aleph_0}$ that is not Ramsey.

However, $[\omega]^{\aleph_0}$ is naturally embedded in $2^\omega = \{f: f \text{ is a function from } \omega \text{ into } 2\}$, and so we can consider $[\omega]^{\aleph_0}$ with the induced topology, where 2^ω has the Tychonoff product topology. In this topology, the work of Nash-Williams [8] and of Galvin and Prikry [3] shows that each Borel set is Ramsey. Silver [10] extended these results to show that every analytic set is Ramsey (see Corollary 1.12 below). And recently, Ellentuck [1] and others (see [5] and [11]) have found simpler proofs of Silver's result.

The primary result of this paper (Theorem 1.9 below) is a version for trees of Silver's theorem. This result for trees implies Silver's theorem. Also, just as Silver's theorem implies the Nash-Williams partition theorem (Theorem 3.1 below) and Ramsey's theorem, so our result implies a version for trees of the Nash-Williams theorem (Theorem 3.3 below) and a version for trees of Ramsey's theorem (Corollary 3.4 below). This last mentioned Ramsey's theorem for trees was originally proved in [6].

In order to work with trees, we need several definitions. These are listed together here for convenient reference.

Suppose $P = \langle P, \leqslant \rangle$ is a partially ordered set. (We use a single symbol both for a structure and for its underlying set.) If $p \in P$, we write

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$$\mathrm{Pred}(p, P) = \{q \in P: q \leqslant p\}, \qquad \mathrm{Pred}^*(p, P) = \mathrm{Pred}(p, P) - \{p\},$$
$$\mathrm{Succ}(p, P) = \{q \in P: q \geqslant p\}, \qquad \mathrm{Succ}^*(p, P) = \mathrm{Succ}(p, P) - \{p\}.$$

We shall be primarily concerned with rooted trees of finite height or of height ω, so the following definition of a tree will be used.

DEFINITION 1.1. A *tree* $t = \langle T, \leqslant \rangle$ is a partially ordered set satisfying:

(1) T has a unique least element, called the root of T and denoted $\mathrm{Root}(T)$, and

(2) for each $t \in T$, $\mathrm{Pred}(t, T)$ is a finite chain, i.e., $\mathrm{Pred}(t, T)$ is a finite, linearly ordered set in $\langle T, \leqslant \rangle$.

The elements of a tree T will sometimes be called *nodes*. If $t \in T$, then the *level* of t in T, denoted $\mathrm{Lev}(t, T)$, is the cardinality of $\mathrm{Pred}^*(t, T)$. If $n \in \omega$, $T(n) = \{t \in T\colon \mathrm{Lev}(t, T) = n\}$, i.e., $T(n)$ is the set of nodes on the nth level of T. The height of T is $\mathrm{Height}(T) = \sup\{|\mathrm{Pred}(t, T)|\colon t \in T\}$. For example, if $n \in \omega$ implies $T(n) \neq \varnothing$, then T must have height ω. A *branch* of T is a maximal chain in $\langle T, \leqslant \rangle$. We call T an *α-tree* (where $\alpha \leqslant \omega$) provided each branch of T has cardinality α. Thus each α-tree has height α, but a tree with height α need not be an α-tree.

If s and t are nodes of T, we say s is an *immediate successor* of t when s is minimal in $\mathrm{Succ}^*(t, T)$, or equivalently, when $t = \max\{\mathrm{Pred}^*(s, T)\}$. We write $\mathrm{IS}(t, T)$ for the collection of all immediate successors of t in T.

If κ is a cardinal (finite or infinite), and if $\alpha \leqslant \omega$, an *(α, κ)-tree* is an α-tree with each nonmaximal node having exactly κ immediate successors. An $(\alpha, <\kappa)$-tree is an α-tree with each nonmaximal node having fewer than κ immediate successors, and an *$(\alpha, \leqslant \kappa)$-tree* is an α-tree with each nonmaximal node having at most κ immediate successors.

If $0 \leqslant \alpha \leqslant \beta \leqslant \omega$, we write $\mathrm{Incr}(\alpha, \beta)$ for the set of all strictly increasing functions from α into β.

Below is a formal definition of when a tree S is strongly embedded in another tree T. Intuitively, for S to be strongly embedded in T, S must be a subset of T with the induced partial order. S must preserve the branching structure of T, i.e. given a (nonmaximal) node of S, if that node has k immediate successors in T, then that node must have k corresponding immediate successors in S. Also, S must preserve the level structure of T, i.e. all nodes of S on a common level (of S) must be from a common level in T.

DEFINITION 1.2. Suppose S is an α-tree and T is a β-tree with $0 \leqslant \alpha \leqslant \beta \leqslant \omega$. S is *strongly embedded* in T provided the following hold.

(1) $S \subseteq T$, and the partial order on S is induced from T.

(2) If $s \in S$ is nonmaximal in S and $t \in \mathrm{IS}(s, T)$ then $\mathrm{Succ}(t, T) \cap \mathrm{IS}(s, S)$ is a singleton.

(3) There exists $f \in \mathrm{Incr}(\alpha, \beta)$ such that $S(n) \subseteq T(f(n))$ for each $n \in \alpha$.

The function f in (3) is called the *level assignment function* for S in T, and we write $f = \mathrm{LAF}(S, T)$.

Given $f \in \mathrm{Incr}(\alpha, \beta)$, we write $\mathrm{Str}_f(T)$ for the collection of all α-trees strongly embedded in the β-tree T that have f as level assignment function in T. Also, we write

$$\mathrm{Str}^{\alpha}(T) = \bigcup_{f \in \mathrm{Incr}(\alpha,\beta)} \mathrm{Str}_f(T),$$

$$\mathrm{Str}^{<\alpha}(T) = \bigcup_{n \in \alpha} \mathrm{Str}^n(T),$$

$$\mathrm{Str}^{\leqslant\alpha}(T) = \mathrm{Str}^{\alpha}(T) \cup \mathrm{Str}^{<\alpha}(T).$$

The proof we give of our main theorem involves consideration of finite sequences of trees. So we shall extend the above notation to finite sequences of trees. Suppose d is a positive integer and $\langle T_i \colon i \in d\rangle$ is a sequence of β-trees for some $0 \leqslant \beta \leqslant \omega$. If $0 \leqslant \alpha \leqslant \beta$ and $f \in \mathrm{Incr}(\alpha, \beta)$, then we write

$$\begin{aligned}\mathrm{Str}_f(T_i \colon i \in d) &= \{\langle S_i \colon i \in d\rangle \colon S_i \in \mathrm{Str}_f(T_i) \text{ for each } i \in d\} \\ &= \prod_{i \in d} \mathrm{Str}_f(T_i),\end{aligned}$$

$$\mathrm{Str}^{\alpha}(T_i \colon i \in d) = \bigcup_{f \in \mathrm{Incr}(\alpha,\beta)} \mathrm{Str}_f(T_i \colon i \in d),$$

$$\mathrm{Str}^{<\alpha}(T_i \colon i \in d) = \bigcup_{n \in \alpha} \mathrm{Str}^n(T_i \colon i \in d).$$

$\mathrm{Str}^{\leqslant\alpha}(T_i \colon i \in d)$ is defined similarly.

It should be noted that if S, R and T are ω-trees with $S \in \mathrm{Str}_f(T)$ and $R \in \mathrm{Str}_g(S)$, then $R \in \mathrm{Str}_h(T)$ where $h(n) = f(g(n))$ for each $n \in \omega$.

Definition 1.3. We write Id for the identity function on ω, i.e., Id: $\omega \to \omega$ with $\mathrm{Id}(n) = n$ for each $n \in \omega$. Thus $\mathrm{Id}|n$, the restriction of Id to n, is the identity function on n.

Definition 1.4. Suppose s is an α-tree and T is a β-tree for some $0 \leqslant \alpha \leqslant \beta \leqslant \omega$. Then S is a *strong initial segment* of T (denoted $S <{}^* T$) provided S is the unique tree satisfying $S \in \mathrm{Str}_{\mathrm{Id}|\alpha}(T)$.

Definition 1.5. Suppose T is an ω-tree and $A \in \mathrm{Str}^{<\omega}(T)$. Then we shall write $\mathrm{Str}(A, T) = \{R \in \mathrm{Str}^{\omega}(T) \colon A <{}^* R\}$. So, in particular, $\mathrm{Str}(\phi, T) = \mathrm{Str}^{\omega}(T)$. Also, we shall write $\mathrm{Dmt}(A, T)$ for the maximal tree of $\mathrm{Str}(A, T)$ and call $\mathrm{Dmt}(A, T)$ the *dominating* tree of A in T, i.e.,

$$\mathrm{Dmt}(A, T) = A \cup \{\mathrm{Succ}(t, T) \colon t \text{ is a maximal node of } A\}$$

where $\mathrm{Dmt}(A, T)$ has the partial order induced from T.

Definition 1.6. Suppose that d is a positive integer, that $\langle T_i \colon i \in d\rangle$ is a sequence of ω-trees, $n \in \omega$, $f \in \mathrm{Incr}(n, \omega)$, and that $A_i \in \mathrm{Str}_f(T_i)$ for each $i \in d$. We shall write

$$\mathrm{Str}(A_i, T_i \colon i \in d) = \bigcup_{\substack{g \in \mathrm{Incr}(\omega,\omega) \\ g|n = \mathrm{Id}|n}} \left(\prod_{i \in d} \mathrm{Str}_g(\mathrm{Dmt}(A_i, T_i)) \right).$$

Intuitively, $\mathrm{Str}(A_i, T_i \colon i \in d)$ consists of all sequences $\langle S_i \colon i \in d\rangle$ in $\mathrm{Str}^{\omega}(T_i \colon i \in d)$ that (for each $i \in d$) have A_i being a strong initial segment of S_i.

Definition 1.7. Suppose T is an ω-tree and $R \subseteq \mathrm{Str}^{\omega}(T)$. We say R is T-Ramsey provided there exits $T' \in \mathrm{Str}^{\omega}(T)$ with either $\mathrm{Str}^{\omega}(T') \subseteq R$ or $\mathrm{Str}^{\omega}(T') \cap R = \varnothing$.

Considering ω with its usual ordering as the trivial $(\omega, 1)$-tree, then ω-Ramsey means just Ramsey in the traditional sense mentioned above.

DEFINITION 1.8. Suppose d is a positive integer and $\langle T_i: i \in d\rangle$ is a sequence of ω-trees. We say that a set $R \subseteq \mathrm{Str}^\omega(T_i: i \in d)$ is *completely* $\langle T_i: i \in d\rangle$*-Ramsey* provided the following holds. If $\langle S_i: i \in d\rangle \in \mathrm{Str}^\omega(T_i: i \in d)$, and if $\langle A_i: i \in d\rangle \in \mathrm{Str}^{<\omega}(S_i: i \in d)$, then there exists $\langle R_i: i \in d\rangle \in \mathrm{Str}(A_i, S_i: i \in d)$ such that either $\mathrm{Str}(A_i, R_i: i \in d) \subseteq R$ or $\mathrm{Str}(A_i, R_i: i \in d) \cap R = \varnothing$.

If $d = 1$ so $\langle T_i: i \in d\rangle = \langle T_0\rangle$, we shall say R is "completely T_0-Ramsey" instead of saying R is "completely $\langle T_0\rangle$-Ramsey." So R is completely T-Ramsey means that for each $S \in \mathrm{Str}^\omega(T)$ and each $A \in \mathrm{Str}^{<\omega}(S)$, there exists $S' \in \mathrm{Str}(A, S)$ with either $\mathrm{Str}(A, S') \subseteq R$ or $\mathrm{Str}(A, S') \cap R = \varnothing$. Clearly, if R is complete T-Ramsey, then R is T-Ramsey.

Given a sequence of ω-trees $\langle T_i: i \in d\rangle$ where d is a positive integer, we shall define a topology on $\mathrm{Str}^\omega(T_i: i \in d)$ by taking $\{\mathrm{Str}(A_i, S_i: i \in d): \langle S_i: i \in d\rangle \in \mathrm{Str}^\omega(T_i: i \in d)$ and $\langle A_i: i \in d\rangle \in \mathrm{Str}^{<\omega}(S_i: i \in d)\}$ as a basis. This topology will be called the *tree topology* on $\mathrm{Str}^\omega(T_i: i \in d)$. If t is a single ω-tree, then the tree topology on $\mathrm{Str}^\omega(T)$ has $\{\mathrm{Str}(A, S): S \in \mathrm{Str}^\omega(T)$ and $A \in \mathrm{Str}^{<\omega}(S)\}$ as a basis.

For completeness, we also define the analytic sets in a topology. Suppose $\tau = \langle X, \mathbf{G}\rangle$ is a topological space, i.e., X is a set and $\mathbf{G}$ is the family of open subsets X. Write $\mathbf{F}$ for the family of closed subsets of X. Suppose T is an arbitrary $(\omega, \aleph_0)$-tree, and $\mathbf{B}$ is the set of *all* branches of T. Then $A \subseteq X$ is analytic in τ if there exists a function $f: T \to \mathbf{F}$ such that

$$A = \bigcup_{B \in \mathbf{B}} \Big(\bigcap_{t \in B} f(t) \Big).$$

It is well known that every Borel set is analytic.

Using these definitions, we can state our main theorem.

THEOREM 1.9. *Suppose T is an $(\omega, < \aleph_0)$-tree and $R \subseteq \mathrm{Str}^\omega(T)$ is an analytic set in the tree topology on* $\mathrm{Str}^\omega(T)$. *Then R is completely T-Ramsey; hence R is T-Ramsey.*

First let us see how Theorem 1.9 implies Silver's partition theorem. If A and B are subset of ω, we write $A < B$ to mean: for each $a \in A$ and $b \in B$, we have $a < b$.

DEFINITION 1.10. If $A \in [\omega]^{<\aleph_0}$ and $X \subseteq \omega$, then we say A is an initial segment of X and write $A \ll X$ provided there exists $Y \subseteq \omega$ with $A < Y$ and $A \cup Y = X$.

If we consider $[\omega]^{\aleph_0}$ to be embedded in 2^ω has the Tychonoff product topology, then we shall call the induced topology on $[\omega]^{\aleph_0}$ the *classical* topology. If for each $A \in [\omega]^{<\aleph_0}$ we write

$$I_A = \{ Y \in [\omega]^{\aleph_0}: A \ll Y \}$$

then $\{I_A: A \in [\omega]^{<\aleph_0}\}$ is a basis for the classical topology on $[\omega]^{\aleph_0}$.

If we instead consider ω with the usual ordering to be the trivial $(\omega, 1)$-tree, then the tree topology on $\mathrm{Str}^\omega(\omega) = [\omega]^{\aleph_0}$ is finer (has more open sets) than the classical topology. A typical basic open set for the tree topology on $\mathrm{Str}^\omega(\omega) = [\omega]^{\aleph_0}$ is of the form

$$J_{A,X} = \{ Y \in [X]^{\aleph_0} : A \ll Y \}$$

where $X \in [\omega]^{\aleph_0}$ and $A \in [X]^{<\aleph_0}$. We shall call the tree topology on $\mathrm{Str}^{\omega}(\omega) = [\omega]^{\aleph_0}$ the *Ellentuck topology* since it is identical to the topology on $[\omega]^{\aleph_0}$ introduced by Ellentuck in **[1]**.

Since we have noted that ω is just a particular $(\omega, < \aleph_0)$-tree, we have the following corollary to Theorem 1.9.

COROLLARY 1.11 (ELLENTUCK **[1]**). *If $R \subseteq [\omega]^{\aleph_0}$ is analytic in the Ellentuck topology on $[\omega]^{\aleph_0}$, then R is Ramsey.*

Since the Ellentuck topology is finer than the classical topology, (1.11) implies Silver's partition theorem.

COROLLARY 1.12 (SILVER **[10]**). *If $R \subseteq [\omega]^{\aleph_0}$ is analytic in the classical topology on $[\omega]^{\aleph_0}$, then R is Ramsey.*

2. Proof of the main theorem. In this section, we shall give a proof of Theorem 1.9. In fact, we shall prove the stronger Theorem 2.1 below.

Suppose $\tau = \langle X, \mathbf{G} \rangle$ is a topological space, i.e., $\mathbf{G}$ is the family of open subsets of the set X. Remember that $N \subseteq X$ is *nowhere dense* provided the closure of N contains no nonempty open sets. A set $M \subseteq X$ is *meager* if it is a countable union of nowhere dense sets. And a set $B \subseteq X$ has the *Baire property* provided there exists an open set $U \in \mathbf{G}$ such that $B \triangle U = (B - U) \cup (U - B)$ is meager.

THEOREM 2.1. *Suppose d is a positive integer and $\langle T_i : i \in d \rangle$ is a sequence of $(\omega, < \aleph_0)$-trees. Then a set $R \subseteq \mathrm{Str}^{\omega}(T_i : i \in d)$ is completely $\langle T_i : i \in d \rangle$-Ramsey if and only if R has the Baire property in the tree topology on $\mathrm{Str}^{\omega}(T_i : i \in d)$.*

It is well known (see Kuratowski **[4**, p. 94**]**) that each analytic set in a topology has the Baire property in that topology. Using this fact and taking $d = 1$ in Theorem 2.1, we obtain Theorem 1.9. So we turn to the proof of Theorem 2.1. Our proof of (2.1) combines the ideas of Ellentuck **[1]**, of Galvin and Prikry **[3]**, of Nash-Williams **[8]** and of this author **[6]**.

We shall need the following "pigeon-hole principle for trees" in the proof of Theorem 2.1. A proof and the history of Theorem 2.2 can be found in §2 of **[6]**.

THEOREM 2.2 (HALPERN-LÄUCHLI-LAVER-PINCUS). *Suppose d is a positive integer and $\langle T_i : i \in d \rangle$ is a sequence $(\omega, < \aleph_0)$-trees. If F: $\mathrm{Str}^1(T_i : i \in d) \to 2$ then there must exist $k \in 2$ and $\langle S_i : i \in d \rangle \in \mathrm{Str}^{\omega}(T_i : i \in d)$ such that F has the constant value k on $\mathrm{Str}^1(S_i : i \in d)$.*

We shall also need the following strightforward lemma.

LEMMA 2.3. *If T is an $(\omega, < \aleph_0)$-tree, if $t \in T$, and if $f \in \mathrm{Incr}(\omega, \omega)$ with $f(0) = \mathrm{Lev}(t, T)$, then there must exist $S \in \mathrm{Str}_f(T)$ with Root$(S) = t$.*

DEFINITION 2.4. Suppose that d is a positive integer and $\langle T_i\colon i \in d\rangle$ is a sequence of $(\omega, <\aleph_0)$-trees, and that $R \subseteq \mathrm{Str}^{\omega}(T_i\colon i \in d)$. Also, suppose that $\langle S_i\colon i \in d\rangle \in \mathrm{Str}^{\omega}(T_i\colon i \in d)$ and $\langle A_i\colon i \in d\rangle \in \mathrm{Str}^{<\omega}(S_i\colon i \in d)$. Then $\langle S_i\colon i \in d\rangle$ *accepts* $\langle A_i\colon i \in d\rangle$ *with respect to* R provided $\mathrm{Str}(A_i, S_i\colon i \in d) \subseteq R$. We say $\langle S_i\colon i \in d\rangle$ *rejects* $\langle A_i\colon i \in d\rangle$ *with respect to* R provided that each $\langle R_i\colon i \in d\rangle \in \mathrm{Str}^{\omega}(S_i\colon i \in d)$ with $\langle A_i\colon i \in d\rangle \in \mathrm{Str}^{<\omega}(R_i\colon i \in d)$ does not accept $\langle A_i\colon i \in d\rangle$ with respect to R.

When it is clear which set R is being considered, we shall omit the phrase "with respect to R".

The following lemmas build up to a proof of Theorem 2.1. In Lemmas 2.5 through 2.14 we assume that $\langle T_i\colon i \in d\rangle$, R, $\langle A_i\colon i \in d\rangle$ and $\langle S_i\colon i \in d\rangle$ are as described in the hypothesis of Definition 2.4.

LEMMA 2.5. *If* $\langle S_i\colon i \in d\rangle$ *accepts* (*or rejects*) $\langle A_i\colon i \in d\rangle$, *then each*

$$\langle R_i\colon i \in d\rangle \in \mathrm{Str}^{\omega}(S_i\colon i \in d)$$

with $\langle A_i\colon i \in d\rangle \in \mathrm{Str}^{<\omega}(R_i\colon i \in d)$ *accepts* (*or rejects, respectively*) $\langle A_i\colon i \in d\rangle$.

LEMMA 2.6. $\langle S_i\colon i \in d\rangle$ *accepts* (*or rejects*) $\langle A_i\colon i \in d\rangle$, *if and only if*, $\langle \mathrm{Dmt}(A_i, S_i)\colon i \in d\rangle$ *accepts* (*or rejects, respectively*) $\langle A_i\colon i \in d\rangle$.

LEMMA 2.7. *There exists* $\langle R_i\colon i \in d\rangle \in \mathrm{Str}(A_i, S_i\colon i \in d)$ *such that* $\langle R_i\colon i \in d\rangle$ *either accepts or rejects* $\langle A_i\colon i \in d\rangle$

The above lemmas are all immediate from Definition 2.4. For the next lemma, we introduce an additional definition. If $\langle S_i\colon i \in d\rangle$ either accepts or rejects $\langle A_i\colon i \in d\rangle$, then we say that $\langle S_i\colon i \in d\rangle$ *decides* $\langle A_i\colon i \in d\rangle$.

LEMMA 2.8. *Given* $\langle T_i\colon i \in d\rangle$ *as in Definition* 2.4, *there exists*

$$\langle R_i\colon i \in d\rangle \in \mathrm{Str}^{\omega}(T_i\colon i \in d)$$

such that $\langle R_i\colon i \in d\rangle$ *decides each* $\langle B_i\colon i \in d\rangle \in \mathrm{Str}^{1}(R_i\colon i \in d)$.

The proof of Lemma 2.8 is not difficult. One recursively picks an array of trees $\langle T(i, n)\colon i \in d,\ n \in \omega\rangle$ such that for each $i \in d$, the sequence $\langle T(i, n)\colon n \in \omega\rangle$ decreases as a function of n, i.e., $T(i, n + 1) \subseteq T(i, n)$. Eventually it will be that

$$R_i = \bigcap_{n \in \omega} T(i, n).$$

One can assure that the R_i so defined are indeed $(\omega, <\aleph_0)$-trees (and are strongly embedded in the T_i) by choosing the $T(i, n)$ with $T(i, j)(n) = T(i, n)(n)$ for all $j \geqslant n$, i.e., the nth level of $T(i, n)$ determines the nth level of all $T(i, j)$ with $j \geqslant n$, and hence the nth level of R_i.

Because of Lemma 2.5, we can assure that $\langle R_i\colon i \in d\rangle$ decides each $\langle B_i\colon i \in d\rangle \in \mathrm{Str}^{1}(R_i\colon i \in d)$ by selecting the $T(i, n)$ so that $\langle T(i, n)\colon i \in d\rangle$ decides each $\langle B_i\colon i \in d\rangle \in \mathrm{Str}^{1}(T(i, n)\colon i \in d)$ with $B_i \subseteq T(i, n)(n)$ for each i. (Then $\langle T(i, n)\colon i \in d\rangle$ automatically decides all $\langle B_i\colon i \in d\rangle$ with $B_i \subseteq T(i, n)(j)$ for some $j < n$.) Such a selection of the $T(i, n)$ is easy to make using repeated applicatons of Lemma 2.7 (and of Lemma 2.3).

LEMMA 2.9. *Given $\langle T_i: i \in d\rangle$ as assumed in Definition 2.4, there exists*

$$\langle R_i: i \in d\rangle \in \mathrm{Str}^{\omega}(T_i: i \in d)$$

such that either $\langle R_i: i \in d\rangle$ accepts all $\langle B_i: i \in d\rangle \in \mathrm{Str}^1(R_i: i \in d)$ or $\langle R_i: i \in d\rangle$ rejects all $\langle B_i: i \in d\rangle \in \mathrm{Str}^1(R_i: i \in d)$.

The proof of Lemma 2.9 is easy. One need only apply Theorem 2.2 (Halpern-Lauchli-Laver-Pincus) to the result of Lemma 2.8.

LEMMA 2.10. *Given $\langle S_i: i \in d\rangle$ as assumed in Definition 2.4, if $\langle S_i: i \in d\rangle$ rejects $\langle \phi: i \in d\rangle$, then there exists $\langle R_i: i \in d\rangle \in \mathrm{Str}^{\omega}(S_i: i \in d)$ such that $\langle R_i: i \in d\rangle$ rejects all $\langle B_i: i \in d\rangle \in \mathrm{Str}^1(R_i: i \in d)$.*

The $\langle R_i: i \in d\rangle$ from Lemma 2.9 must satisfy Lemma 2.10; otherwise Lemma 2.9 yields that $\langle R_i: i \in d\rangle$ accepts all $\langle B_i: i \in d\rangle \in \mathrm{Str}^1(R_i: i \in d)$. Then $\mathrm{Str}^{\omega}(R_i: i \in d) \subseteq R$, and $\langle S_i: i \in d\rangle$ would not reject $\langle \phi: i \in d\rangle$.

LEMMA 2.11. *Given $\langle S_i: i \in d\rangle$ and $\langle A_i: i \in d\rangle$ as in the supposition of Definition 2.4, let $N = \mathit{Height}(A_i)$. If $\langle S_i: i \in d\rangle$ rejects $\langle A_i: i \in d\rangle$, then there exists*

$$\langle R_i: i \in d\rangle \in \mathrm{Str}(A_i, S_i: i \in d)$$

such that $\langle R_i: i \in d\rangle$ rejects all $\langle B_i: i \in d\rangle \in \mathrm{Str}^{N+1}(R_i: i \in d)$ with $A_i <{}^* B_i$ for each $i \in d$.

If $\langle S_i: i \in d\rangle$ and $\langle A_i: i \in d\rangle$ satisfy the hypothesis of Lemma 2.11, then we can assume $A_i <{}^* S_i$ for each $i \in d$. Letting $N = \mathrm{Height}(A_i)$, we can write each $S_i - A_i$ as a union of disjoint sets

$$S_i - A_i = \bigcup \{\mathrm{Succ}(a, S_i): \text{there exists } b \in A_i(N-1) \text{ with } a \in \mathrm{IS}(b, S_i)\}.$$

We shall concentrate on the array of trees

$$\langle \mathrm{Succ}(a, S_i): i \in d \text{ and there exists } b \in A_i(N-1) \text{ with } a \in \mathrm{IS}(b, S_i)\rangle. \quad (1)$$

(We consider $\mathrm{Succ}(a, S_i)$ a tree by giving it the induced partial order.) Since (1) is cumbersome to write, we shall make the notational convention that $M_i = \bigcup_{b \in A_i(N-1)} \mathrm{IS}(b, S_i)$, so (1) becomes

$$\langle \mathrm{Succ}(a, S_i): i \in d, a \in M_i\rangle. \quad (2)$$

We define

$$R' \subseteq \mathrm{Str}^{\omega}(\mathrm{Succ}(a, S_i): i \in d, a \in M_i)$$

by $\langle Q(a, i): i \in d, a \in M_i\rangle \in R'$ if and only if $\langle Q(a, i): i \in d, a \in M_i\rangle \in \mathrm{Str}^{\omega}(\mathrm{Succ}(a, S_i): i \in d, a \in M_i)$ and $\langle (\bigcup_{a \in M_i} Q(a, i)) \cup A_i: i \in d\rangle \in R$. Then to prove Lemma 2.11 one applies Lemma 2.10 to the sequence of trees (2) and the set R'.

LEMMA 2.12. *Given $\langle S_i: i \in d\rangle$ and $\langle A_i: i \in d\rangle$ as assumed in Definition 2.4, suppose $\langle S_i: i \in d\rangle$ rejects $\langle C_i: i \in d\rangle$, N is a positive integer, $\mathit{Height}(C_i) = N$,*

$$\langle C_i: i \in d\rangle \in \mathrm{Str}^N(A_i: i \in d),$$

and every maximal node of C_i is also maximal in the corresponding A_i. Then there must exist $\langle R_i\colon i \in d\rangle \in \mathrm{Str}(A_i, S_i\colon i \in d)$ which rejects all $\langle B_i\colon i \in d\rangle \in \mathrm{Str}^{N+1}(R_i\colon i \in d)$ with $C_i <^* B_i$ for each $i \in d$.

Lemma 2.12 is a straightforward generalization of Lemma 2.11. Using a recursive definition similar to the one in the proof of Lemma 2.8 along with repeated applications of Lemma 2.12, one can prove the following lemma.

LEMMA 2.13. *Given $\langle S_i\colon i \in d\rangle$ as in Definition 2.4, if $\langle S_i\colon i \in d\rangle$ rejects $\langle \phi\colon i \in d\rangle$, then there exists $\langle R_i\colon i \in d\rangle \in \mathrm{Str}^{\omega}(S_i\colon i \in d)$ such that $\langle R_i\colon i \in d\rangle$ rejects all $\langle B_i\colon i \in d\rangle \in \mathrm{Str}^{<\omega}(R_i\colon i \in d)$.*

Also, just as Lemma 2.10 was generalized to Lemma 2.11, so from Lemma 2.13 we obtain the following lemma.

LEMMA 2.14. *Given $\langle S_i\colon i \in d\rangle$ and $\langle A_i\colon i \in d\rangle$ as in Definition 2.4, if $\langle S_i\colon i \in d\rangle$ rejects $\langle A_i\colon i \in d\rangle$, then there exists*

$$\langle R_i\colon i \in d\rangle \in \mathrm{Str}(A_i, S_i\colon i \in d)$$

such that $\langle R_i\colon i \in d\rangle$ rejects all $\langle C_i\colon i \in d\rangle \in \mathrm{Str}^{<\omega}(R_i\colon i \in d)$ with $A_i <^* C_i$ for each $i \in d$.

We shall present more detailed proofs of the following lemmas.

LEMMA 2.15. *Suppose d is a positive integer, $\langle T_i\colon i \in d\rangle$ is a sequence of $(\omega, < \aleph_0)$-trees, and that $R \subseteq \mathrm{Str}^{\omega}(T_i\colon i \in d)$ is an open set in the tree topology on $\mathrm{Str}^{\omega}(T_i\colon i \in d)$. Then R is completely $\langle T_i\colon i \in d\rangle$-Ramsey.*

PROOF. Suppose that R and $\langle T_i\colon i \in d\rangle$ satisfy the hypothesis. Also, suppose $\langle S_i\colon i \in d\rangle \in \mathrm{Str}^{\omega}(T_i\colon i \in d)$ and $\langle A_i\colon i \in d\rangle \in \mathrm{Str}^{<\omega}(S_i\colon i \in d)$.

If some $\langle R_i\colon i \in d\rangle \in \mathrm{Str}(A_i, S_i\colon i \in d)$ accepts $\langle A_i\colon i \in d\rangle$, then

$$\mathrm{Str}(A_i, R_i\colon i \in d) \subseteq R,$$

and we are done.

Otherwise $\langle S_i\colon i \in d\rangle$ rejects $\langle A_i\colon i \in d\rangle$. So apply Lemma 2.14 to obtain $\langle R_i\colon i \in d\rangle \in \mathrm{Str}(A_i, S_i\colon i \in d)$ such that $\langle R_i\colon i \in d\rangle$ rejects each $\langle C_i\colon i \in d\rangle \in \mathrm{Str}^{<\omega}(R_i\colon i \in d)$ with $A_i <^* C_i$ for each $i \in d$. We claim $\mathrm{Str}(A_i, R_i\colon i \in d) \cap R = \varnothing$.

Suppose not and pick $\langle Q_i\colon i \in d\rangle \in \mathrm{Str}(A_i, R_i\colon i \in d) \cap R$. Since

$$\mathrm{Str}(A_i, R_i\colon i \in d) \cap R$$

is open, we can find a basic open set $\mathrm{Str}(B_i, P_i\colon i \in d)$ with

$$\langle Q_i\colon i \in d\rangle \in \mathrm{Str}(B_i, P_i\colon i \in d) \subseteq \mathrm{Str}(A_i, R_i\colon i \in d) \cap R.$$

In fact, we can assume $A_i <^* B_i <^* P_i$, for each $i \in d$, and $\langle P_i\colon i \in d\rangle \in \mathrm{Str}(A_i, R_i\colon i \in d)$. Then $\langle P_i\colon i \in d\rangle$ accepts $\langle B_i\colon i \in d\rangle$, but this contradicts the requirement that $\langle R_i\colon i \in d\rangle$ rejects $\langle B_i\colon i \in d\rangle$. The contradiction proves the lemma.

Lemma 2.16. *Suppose* $\langle T_i\colon i \in d\rangle$ *is a finite sequence of* $(\omega, <\aleph_0)$*-trees, and* $N \subseteq \mathrm{Str}^\omega(T_i\colon i \in d)$ *is nowhere dense in the tree topology. Then for each* $\langle S_i\colon i \in d\rangle \in \mathrm{Str}^\omega(T_i\colon i \in d)$ *and each* $\langle A_i\colon i \in d\rangle \in \mathrm{Str}^{<\omega}(S_i\colon i \in d)$*, there must exist* $\langle R_i\colon i \in d\rangle \in \mathrm{Str}(A_i, S_i\colon i \in d)$ *with* $\mathrm{Str}(A_i, R_i\colon i \in d) \cap N = \varnothing$.

This is immediate from Lemma 2.15 applied to the complement of the closure of N.

Lemma 2.17. *Suppose* $\langle T_i\colon i \in d\rangle$ *is a finite sequence of* $(\omega, <\aleph_0)$*-trees; then* $M \subseteq \mathrm{Str}^\omega(T_i\colon i \in d)$ *is meager in the tree topology if and only if* M *is nowhere dense in the tree topology.*

Proof. If $M \subseteq \mathrm{Str}^\omega(T_i\colon i \in d)$ is nowhere dense, then M is trivially meager.

So suppose $M = \bigcup_{n\in\omega} N_n$ where each $N_n \subseteq \mathrm{Str}^\omega(T_i\colon i \in d)$ is nowhere dense. In order to conclude that M is nowhere dense, it suffices to show that for each nonempty, open $R \subseteq \mathrm{Str}^\omega(T_i\colon i \in d)$, there exists a basic open neighborhood $\mathrm{Str}(A_i, R_i\colon i \in d)$ with $\mathrm{Str}(A_i, R_i\colon i \in d) \subseteq R - M$.

So assume such R is given, and pick $\langle S_i\colon i \in d\rangle \in \mathrm{Str}^\omega(T_i\colon i \in d)$ and $\langle A_i\colon i \in d\rangle \in \mathrm{Str}^{<\omega}(S_i\colon i \in d)$ so that $\mathrm{Str}(A_i, S_i\colon i \in d) \subseteq R$. Let $\mathrm{Height}(A_i) = H$, for each $i \in d$.

By induction on n, $n \in \omega$, we shall define two arrays of trees, $\langle T(i, n)\colon i \in d, n \in \omega\rangle$ and $\langle P(i, n)\colon i \in d, n \in \omega\rangle$, such that the following conditions hold for each $n \in \omega$.

(a) $\langle T(i, n)\colon i \in d\rangle \in \mathrm{Str}(A_i, S_i\colon i \in d)$.

(b) $\langle P(i, 0)\colon i \in d\rangle = \langle A_i\colon i \in d\rangle$, and if $n \geqslant 1$, then for each $i \in d$, $P(i, n) = \bigcup_{k\in n+H} T(i, n-1)(k)$, and $P(i, n)$ has the induced partial order.

(c) If $n \geqslant 1$, then

$$\langle T(i, n)\colon i \in d\rangle \in \mathrm{Str}(P(i, n), T(i, n-1)\colon i \in d).$$

(d) Suppose $H \leqslant k \leqslant n + H$ and $\langle B_i\colon i \in d\rangle \in \mathrm{Str}^k(P(i, n)\colon i \in d)$ with $A_i <^* B_i$ for each $i \in d$. Then for every $\langle Q_i\colon i \in d\rangle \in \mathrm{Str}(B_i, T(i, n)\colon i \in d)$ with $Q_i \cap P(i, n) = B_i$ for each $i \in d$, we have $\langle Q_i\colon i \in d\rangle \notin N_n$.

If $n = 0$, then condition (b) defines $\langle P(i, 0)\colon i \in d\rangle = \langle A_i\colon i \in d\rangle$. So we can apply Lemma 2.16 to get $\langle T(i, 0)\colon i \in d\rangle \in \mathrm{Str}(A_i, S_i\colon i \in d)$ such that $\mathrm{Str}(A_i, T(i, 0)\colon i \in d) \cap N_0 = \varnothing$.

Given $n \geqslant 1$ and the trees $T(i, k)$ and $P(i, k)$ for each $i \in d$ and $k \in n$, we want to select $T(i, n)$ and $P(i, n)$ for each $i \in d$. Now condition (b) determines $\langle P(i, n)\colon i \in d\rangle$ and hence $T(i, n)(j)$ for each $j \in n + H$ because of condition (c). So it remains to select $T(i, n)(j)$ for $j \geqslant n + H$.

Let $P'(i, n) = \bigcup_{k\in n+H+1} T(i, n-1)(k)$ for each $i \in d$, and let

$$\mathcal{C}(n) = \big\{\langle C(i)\colon i \in d\rangle \in \mathrm{Str}^{<n+H+1}(P'(i, n)\colon i \in d)\colon \text{for each } i \in d,$$
$$A_i <^* C(i) \text{ and } C(i)(\mathrm{Height}(C(i) - 1)) \subseteq P'(i, n)(n + H)\big\}.$$

Let $K = |\mathcal{C}(n)|$ and enumerate $\mathcal{C}(n)$ as $\{\mathbf{C}(p)\colon 1 \leqslant p \leqslant K\}$ where $\mathbf{C}(p) = \langle C(p, i)\colon i \in d\rangle$.

By induction on p, $p \in K+1$, we shall define trees $T(i, n, p)$ such that the following conditions hold for each $p \in K+1$.

(e) $T(i, n, 0) = T(i, n-1)$ for each $i \in d$.

(f) If $p \geqslant 1$, then $\langle T(i, n, p): i \in d\rangle \in \mathrm{Str}\langle(P(i, n), T(i, n, p-1): i \in d)\rangle$.

(g) Write $B_i = C(p, i) \cap P(i, n)$, and $H(p) = \mathrm{Height}(C(p, i))$, and $I(i) = C(p, i)(H(p)-1)$ for each $i \in d$.

If

$$V(i) = B_i \cup \left(\bigcup \{\mathrm{Succ}(a, S_i) \cap T(i, n, p): a \in I(i)\}\right) \tag{1}$$

has the induced partial order, then

$$\mathrm{Str}(B_i, V(i): i \in d) \cap N_n = \varnothing.$$

Condition (e) defines $T(i, n, 0)$. So suppose $p > 1$ and the trees $T(i, n, q)$ have been defined for $i \in d$ and $q \in p$. We shall use the notational conventions made in the first sentence of conditions (g). Let

$$U(i) = B_i \cup \left(\bigcup \{\mathrm{Succ}(a, S_i) \cap T(i, n, p-1): a \in I(i)\}\right).$$

Then apply Lemma 2.16 to $\langle U(i): i \in d\rangle$ and obtain $\langle V(i): i \in d\rangle \in \mathrm{Str}(B_i, U(i): i \in d)$ so that

$$\mathrm{Str}(B_i, V(i): i \in d) \cap N_n = \varnothing.$$

But then we can use Lemma 2.3 to find

$$\langle T(i, n, p): i \in d\rangle \in \mathrm{Str}(P(i, n), T(i, n, p-1): i \in d)$$

such that for each $i \in d$ and each $a \in I(i)$,

$$\mathrm{Succ}(a, S_i) \cap T(i, n, p) = \mathrm{Succ}(a, S_i) \cap V(i).$$

This assures that equation (1) holds, so the conditions (f) and (g) hold.

When the induction on $p \in K+1$ is complete, we set $T(i, n) = T(i, n, K)$, so the conditions (a) – (c) follow immediately. And condition (d) follows from condition (g) after a moment of thought. So we have completed our induction on $n \in \omega$.

By conditions (a)–(c) we can set

$$R_i = \bigcap_{n \in \omega} (T(i, n)) = A_i \cup \left(\bigcup_{n \in \omega} T(i, n)(n + H - 1)\right) = \bigcup_{n \in \omega} P(i, n)$$

for each $i \in d$, and get $\langle R_i: i \in d\rangle \in \mathrm{Str}(A_i, S_i: i \in d)$.

Now, it is clear that $\mathrm{Str}(A_i, R: i \in d) \subseteq \mathrm{Str}(A_i, S_i: i \in d) \subseteq R$, and we claim $\mathrm{Str}(A_i, R_i: i \in d) \cap M = \varnothing$ (which, if true, proves the lemma). Indeed, suppose $\langle Q_i: i \in d\rangle \in \mathrm{Str}(A_i, R_i: i \in d) \cap N_n$ for some $n \in \omega$. Let $B_i = Q_i \cap P(i, n)$ for each $i \in d$. Then $\langle Q_i: i \in d\rangle$ and $\langle B_i: i \in d\rangle$ satisfy the hypothesis of condition (d), and we conclude $\langle Q_i: i \in d\rangle \notin N_n$. This contradiction proves the lemma.

Lemmas 2.16–2.17 enable us to prove Theorem 2.1.

Proof of Theorem 2.1. Suppose $\langle T_i: i \in d\rangle$ is a finite sequence of $(\omega, < \aleph_0)$-trees.

If $R \subseteq \mathrm{Str}^{\omega}(T_i\colon i \in d)$ has the Baire property (i.e., $R \triangle U = (R - U) \cup (U - R)$ is meager for some open set U), then we want to show R is completely $\langle T_i\colon i \in d\rangle$-Ramsey. Now Lemma 2.17 states that $R \triangle U$ is in fact nowhere dense (in the tree topology). So suppose $\langle S_i\colon i \in d\rangle \in \mathrm{Str}^{\omega}(T_i\colon i \in d)$ and $\langle A_i\colon i \in d\rangle \in \mathrm{Str}^{<\omega}(T_i\colon i \in d)$. Since U is open, Lemma 2.15 implies there exists $\langle R_i\colon i \in d\rangle \in \mathrm{Str}(A_i, S_i\colon i \in d)$ with either $\mathrm{Str}(A_i, R_i\colon i \in d) \subseteq U$ or $\mathrm{Str}(A_i, R_i\colon i \in d) \cap U = \varnothing$. But the fact that $R \triangle U$ is nowhere dense and Lemma 2.16 yield $\langle Q_i\colon i \in d\rangle \in \mathrm{Str}(A_i, R_i\colon i \in d)$ such that

$$\mathrm{Str}(A_i, Q_i\colon i \in d) \cap (R \triangle U) = \varnothing.$$

Thus $\mathrm{Str}(A_i, R_i\colon i \in d) \subseteq \cup$ implies $\mathrm{Str}(A_i, Q_i\colon i \in d) \subseteq R$, while

$$\mathrm{Str}(A_i, R_i\colon i \in d) \cap U = \varnothing$$

implies $\mathrm{Str}(A_i, Q_i\colon i \in d) \cap R = \varnothing$.

Conversely, suppose R is completely $\langle T_i\colon i \in d\rangle$-Ramsey. Let $\mathrm{int}(R)$ be the interior of R. We shall show that $R - \mathrm{int}(R)$ is nowhere dense. To show this, it suffices to show that for each nonempty, open set U, there exists a basic open set $\mathrm{Str}(A_i, R_i\colon i \in d) \subseteq U - (R - \mathrm{int}(R))$.

Indeed, given nonempty open U, pick $\langle S_i\colon i \in d\rangle \in \mathrm{Str}^{\omega}(T_i\colon i \in d)$ and $\langle A_i\colon i \in d\rangle \in \mathrm{Str}^{<\omega}(S_i\colon i \in d)$ such that $\mathrm{Str}(A_i, S_i\colon i \in d) \subseteq U$. Since R is completely $\langle T_i\colon i \in d\rangle$-Ramsey, there must exist $\langle R_i\colon i \in d\rangle \in \mathrm{Str}(A_i, S_i\colon i \in d)$ with either $\mathrm{Str}(A_i, R_i\colon i \in d) \subseteq R$ or $\mathrm{Str}(A_i, R_i\colon i \in d) \cap R = \varnothing$. In the first case, $\mathrm{Str}(A_i, R_i\colon i \in d)$ is open, so $\mathrm{Str}(A_i, R_i\colon i \in d) \subseteq \mathrm{int}(R)$. So in either case, $\mathrm{Str}(A_i, R_i\colon i \in d) \subseteq U - (R - \mathrm{int}(R))$. This complete the proof of Theorem 2.1.

3. A Nash-Williams partition theorem for trees. A family of finite sets $\mathcal{Q} \subseteq [\omega]^{<\aleph_0}$ is said to be *thin* provided it is not the case that there exist distinct sets $A, B \in \mathcal{Q}$ with $A \ll B$. In [8], Nash-Williams proved the following generalization of Ramsey's theorem.

THEOREM 3.1 (NASH-WILLIAMS). *Suppose that $\mathcal{Q} \subseteq [\omega]^{<\aleph_0}$ is thin, that r is a positive integer, and that $\mathcal{Q} = \bigcup_{i \in r} C_i$. Then there must exist $X \in [\omega]^{\aleph_0}$ and $k \in r$ such that $\mathcal{Q} \cap [X]^{<\aleph_0} \subseteq C_k$.*

We shall show that Theorem 1.9 implies a generalization for trees of Theorem 3.1.

DEFINITION 3.2. Suppose that T is an ω-tree. A family of subtrees $\mathcal{B} \subseteq \mathrm{Str}^{<\omega}(T)$ is said to be *thin* provided that it is not the case that there exist distinct trees A, $B \in \mathcal{B}$ with $A <{}^*B$.

THEOREM 3.3. *Suppose that T is an $(\omega, < \aleph_0)$-tree, that $\mathcal{B} \subseteq \mathrm{Str}^{<\omega}(T)$ is thin, that r is a positive integer, and that $\mathcal{B} = \bigcup_{i \in r} C_i$. Then there must exist $S \in \mathrm{Str}^{\omega}(T)$ and $k \in r$ such that $\mathcal{B} \cap \mathrm{Str}^{<\omega}(S) \subseteq C_k$.*

Theorem 3.3 becomes Theorem 3.1 if we take T to be the trivial $(\omega, 1)$-tree, i.e., $T = \omega$. Also, note that for each $n \in \omega$, it is clear that $\mathrm{Str}^{n}(T)$ is a thin family of subtrees whenever T is an ω-tree. Hence, we have the following generalization for trees of Ramsey's theorem.

COROLLARY 3.4. *Suppose that T is an $(\omega, < \aleph_0)$-tree, that n and r are positive integers, and that* $\mathrm{Str}^n(T) \subseteq \bigcup_{i \in r} C_i$. *Then there must exist $k \in r$ and $S \in \mathrm{Str}^\omega(T)$ with* $\mathrm{Str}^n(S) \subseteq C_k$.

A finitary version of (3.4) and related results can be found in **[6]**.

PROOF OF THEOREM 3.3. Suppose that T and $\mathfrak{B}$ satisfy the hypothesis. By a standard argument, we may assume that $r = 2$. So suppose $\mathfrak{B} = C_0 \cup C_1$. Define

$$P = \{ R \in \mathrm{Str}^\omega(T) \colon \text{there exists } A \in C_0 \text{ with } A <^* R \}.$$

Since $C_0 \subseteq \mathrm{Str}^{<\omega}(T)$, it must be that P is an open set in the tree topology on $\mathrm{Str}^\omega(T)$. Thus Theorem 1.9 (or Lemma 2.15) implies that there exists $S \in \mathrm{Str}^\omega(T)$ with either $\mathrm{Str}^\omega(S) \subseteq P$ or $\mathrm{Str}^\omega(S) \cap P = \varnothing$.

If $\mathrm{Str}^\omega(S) \subseteq P$, then the fact that $\mathfrak{B}$ is thin requires $\mathfrak{B} \cap \mathrm{Str}^{<\omega}(S) \subseteq C_0$. Similarly, if $\mathrm{Str}^\omega(S) \cap P = \varnothing$, then $C_0 \cap \mathrm{Str}^{<\omega}(S) = \varnothing$, so $\mathfrak{B} \cap \mathrm{Str}^{<\omega}(S) \subseteq C_1$. This proves Theorem 3.3.

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