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# HOLOMORPHIC ACTIONS OF Sp($n$, $R$) WITH NONCOMPACT ISOTROPY GROUPS[1]

BY

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ABSTRACT. $U(p, q)$ is a subgroup of $S_p(n, R)$, for $p + q = n$. $B_q = S_p(n, r)/U(p, q)$ is realized as an open subset of the manifold of Lagrangian subspaces of $C^n \times C^n$. It is shown that $B_q$ carries a $(pq)$-pseudoconvex exhaustion function. $B_{pq} = S_p(n, r)/U(p) \times U(q)$ carries two distinct holomorphic structures making the projection to $B_q$, $B_0$ holomorphic respectively. The geometry of the correspondence between $B_q$ and $B_0$ via $B_{pq}$ is investigated.

**Introduction.** Throughout this article $p$, $q$, $n$ are nonnegative integers such that $p + q = n$. Let $S_n$ represent the space of symmetric $n \times n$ matrices $Z$ with complex entries. Let $Z = X + iY$ be the decomposition of $Z$ into real and imaginary parts. As is well known, there is a meromorphic action of Sp($n$, $R$) on $S_n$ given by $Z \to (AZ + B)(CZ + D)^{-1}$ where $G = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in$ Sp($n$, $R$) (see, for example, **[13]**). If $S_{n,q} = \{Z = X + iY \in S_n;\ Y$ is nonsingular and has $p$ positive and $q$ negative eigenvalues$\}$, then this action, where well defined, leaves the domains $S_{n,q}$ invariant. This action is well defined throughout $S_{n,0}$ and $S_{n,n}$, but may have poles on the other domains. However, each $S_{n,q}$ does correspond to a homogeneous space

$$B_q = \bar{S}_{n,q} = \mathrm{Sp}(n, R)/U(p, q),$$

since $U(p, q)$ is the isotropy of a suitable point in $S_{n,q}$. In this article we shall study some of the geometry of these spaces $B_q$. These domains have occurred implicitly in (at least) two other recent articles.[2] In **[15]**, Tolimieri shows that $\cup B_q$ parametrizes the set of complex polarizations of the Heisenberg algebra. In **[10]**, Matsushima considers complex tori polarized by hermitian line bundles with nonsingular, but not necessarily positive, definite curvature. Again $\cup B_q$ parametrizes the set of all such polarized complex tori.

In §1 we realize $UB_q$ as an open dense subset of the set $\mathfrak{L}$ of subspaces of $\mathbf{C}^{2n}$ which are Lagrangian for a fixed nondegenerate skew form $J$. These are the Sp($n$, $R$)-orbits under the standard action of $G(2n, \mathbf{C})$ on Gr($2n$, $n$, $\mathbf{C}$). In §2 we realize the $B_q$ as orbits of Sp($n$, $R$) on the dual of its Lie algebra. The structure inherited from $\mathfrak{L}$ provides these orbits with a polarization and associated hermitian

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[2]I wish to thank the referee for pointing out to me that much of the discussion of §1 is well known, and in fact Theorem 1.20 already appears in the following interesting paper: T. Maebashi, *A fixpoint theorem on $S_p(n)/U(n)$*, Math. J. Okayama Univ. **12** (1966), 91–105.

line bundle (a quantization). The representation associated to this orbit by the Blattner-Kostant-Sternberg theory **[9]**, **[14]** should be expected to lie in higher order cohomology. We have been unable to successfully pursue this line of reasoning (which was the main motivation of this article). In §3 we show that this complex polarization is unique to conjugation.

In §4 we show that $B_q$ has a $(pq)$-pseudoconvex exhaustion function. This pseudoconvexity is explained by the presence of compact subvarieties of dimension $pq$. Since $U(n) \cap U(p, q) = U(p) \times U(q)$, we can study the maps:

$$\begin{array}{ccc} & \mathrm{Sp}(n, R)/U(p) \times U(q) & \\ & \downarrow\pi_q \qquad\qquad \downarrow\pi_0 & \\ & B_q \qquad\qquad\quad B_0 & \end{array}$$

The homogeneous space $\mathrm{Sp}(n, R)/U(p) \times U(q)$ has two distinct $(q \neq 0, n)$ $\mathrm{Sp}(n, R)$-invariant complex structures $E_{pq}$, $E_{p0}$ making $\pi_q$, $\pi_0$ holomorphic respectively.[3] By means of this correspondence $B_0$ parametrizes a family of compact subvarieties of $B_q$ of dimension $pq$ which is a real form for the Kodaira space **[8]** of deformations of $\pi_q\pi_0^{-1}(p_0)$ $(= U(n)/U(p) \times U(q) = G(n, p))$ in $B_q$. Finally, in §5 we relate our work with the work of Tolimieri **[15]** on polarized complex tori and via that with the work of Matsushima **[10]**.

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**1. Hermitian forms on a complex vector space.** For the purposes of this section, a *hermitian vector space* shall consist in a complex vector space $V$ together with a nondegenerate hermitian form $H$:

(i) $H$ is real bilinear;
(ii) $H$ is complex linear in the first factor;
(iii) $H(u, v) = \overline{H(v, u)}$;
(iv) $H(u, v) = 0$ for all $v \in V$ implies $u = 0$.

We shall study such spaces from the point of view of the underlying real vector space $V_R$. The structure of complex vector space is given by the endomorphism $\beta \in \mathrm{GL}(V_R)$ representing multiplication by $i$. The only condition $\beta$ must satisfy to determine a complex structure is

$$\beta^2 = -I. \tag{1.1}$$

The hermitian form $H$ is a real bilinear complex-valued form. Let

$$H = S + iK \tag{1.2}$$

be its decomposition into real and imaginary parts.

The following result is standard and a proof can be found in **[11]**.

[3]Again, the referee has pointed out a reference for a more direct proof of Theorem 3.1: J. A. Wolf, *The geometry and structure of isotropy irreducible homogeneous spaces*, Acta Math. **120** (1968), 59–148.

1.3. THEOREM. *Let $\beta$ define a complex structure on the real vector space $V_R$, and let $H = S + iK$ be a nondegenerate hermitian form on $(V_R, \beta)$. Then*

(a) *$S$ is symmetric and $S(\beta v, \beta w) = S(v, w)$,*

(b) *$K$ is skew-symmetric and $K(\beta v, \beta w) = K(v, w)$,*

(c) *both $S$, $K$ are nondegenerate, and*

$$S(v, w) = K(\beta v, w), \qquad K(v, w) = -S(\beta v, w). \tag{1.4}$$

*Finally, either $S$ or $K$ satisfying* (a), (b) *respectively, determines the hermitian form $H$ via equations* (1.4) *and* (1.2).

Given the hermitian form $H$, we shall call $S$ its *Riemannian part* and $K$ its *Kähler* part.

Let

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \tag{1.5}$$

where $I_n$ is the $n \times n$ identity matrix. Thinking of $R^{2n}$ as the space of $n \times 1$ vectors, $J$ defines a skew form on $R^{2n}$ by $(v, w) \to {}^t wJv$.

1.6. DEFINITION. We shall let $B$ denote the collection of structures of hermitian vector space on $R^{2n}$ whose Kähler form is $J$. Then $B$ is given by complex structures $\beta$ which leave $J$ invariant:

$$B = \{ \beta \in \mathrm{GL}(2n, R);\ \beta^2 = -I, {}^t\beta J\beta = J \}. \tag{1.7}$$

The corresponding hermitian form is given by the matrix

$$H_\beta = J\beta + iJ. \tag{1.8}$$

There is another way to look at complex structures $\beta$ on a real vector space $V$ which we shall have to exploit. Extend $\beta$ complex linearly to the complexification $V^{\mathbf{C}}$. Since $\beta^2 + I = 0$, $V^{\mathbf{C}}$ splits into the direct sum of two eigenspaces $V_{\pm i}(\beta)$ of eigenvalue $\pm i$ respectively. Since $\beta$ is real, $V_{-i}(\beta) = \overline{V_i}(\beta)$. Conversely, given any direct sum decomposition $V^{\mathbf{C}} = W \oplus \overline{W}$, the linear transformation $\beta$ defined by $B | W = +i$, $B | \overline{W} = -i$ defines a real transformation $\beta$ on $V$ such that $\beta^2 = -I$, i.e., a complex structure. Finally, we observe that the map $V \to W$ defined by $v \to v - i\beta v$ defines a complex linear isomorphism of $(R^{2n}, \beta)$ with $W$. The following result is easily calculated.

1.9. PROPOSITION. *Let $\beta \in B$. Under the identification of $(R^{2n}, \beta)$ with $V_i(\beta)$, $H_\beta$ is given by*

$$H_\beta(v, w) = (i/2){}^t\overline{w}Jv.$$

For $\beta \in B$, $H_\beta$ is a nondegenerate hermitian form so (as a hermitian symmetric matrix) has $p$ positive eigenvalues and $q$ negative eigenvalues, with $p + q = n$. Let

$$B_q = \{ \beta \in B;\ H_\beta \text{ has } q \text{ negative eigenvalues} \}. \tag{1.10}$$

Let

$$I_{pq} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}, \qquad J_q = \begin{pmatrix} 0 & I_{pq} \\ -I_{pq} & 0 \end{pmatrix}. \tag{1.11}$$

Then $J_q^2 = -I$, so defines a complex structure on $R^{2n}$. Up to complex isomorphism, there is only one $n$-dimensional complex vector space, so there must be a complex linear transformation from $(R^{2n}, J)$ to $(R^{2n}, J_q)$; i.e., a $D_q \in \mathrm{GL}(2n, R)$ such that $D_qJ = J_qD_q$. This is given by the matrix

$$D_q = \begin{bmatrix} I_p & & & 0 \\ & I_q & & \\ & & I_p & \\ 0 & & & -I_q \end{bmatrix} \tag{1.12}$$

since

$${}^tD_q = D_q = D_q^{-1}, \qquad D_qJ_qD_q = J. \tag{1.13}$$

We also have

$$J_q^2 = -I, \qquad J_qJ = JJ_q, \qquad {}^tJ_qJJ_q = J, \tag{1.14}$$

so all the $\pm J_q$ are in $B$. In fact, it is easily verified that $J_p, {}^tJ_q \in B_q$.

The *symplectic group* is the subgroup of $\mathrm{GL}(2n, R)$ leaving the skew-form $J$ invariant:

$$\mathrm{Sp}(n, R) = \{ g \in \mathrm{GL}(2n, R), {}^tgJg = J \}. \tag{1.15}$$

Thus $B = \{ \beta \in \mathrm{Sp}(n, R); \beta^2 = -I \}$.

It is easily checked that $B$ is invariant under the adjoint action of $\mathrm{Sp}(n, R)$: for $\beta \in B$, and $g \in \mathrm{Sp}(n, R)$, $g^{-1}\beta g \in B$ also. We have the following theorem, proved by Tolimieri in **[15]**; we give a simpler proof.

1.16. THEOREM. (a) $B = \mathrm{Sp}(n, R) \cap \mathfrak{sp}(n, R) = \{ \beta \in \mathrm{Sp}(n, R);$ *$J\beta$ is symmetric*$\}$.

(b) *The orbits of $B$ under the adjoint action of* $\mathrm{Sp}(n, R)$ *are precisely the* $B_q$.

PROOF. (a) $\mathfrak{sp}(n, R)$, the Lie algebra of $\mathrm{Sp}(n, R)$, is determined by the condition

$$\mathfrak{sp}(n, R) = \{ X \in \mathfrak{gl}(2n, R), {}^tXJ + JX = 0 \},$$

where $\mathfrak{gl}(2n, R)$ is the space of $2n \times 2n$ matrices, the Lie algebra of $\mathrm{GL}(2n, R)$.

Let $\beta \in B$. Then $\beta^{-1} = -\beta$ and ${}^t\beta J = J\beta^{-1} = -J\beta$ so ${}^t\beta J + J\beta = 0$ and $\beta \in \mathrm{sp}(n, R)$. Clearly also $J\beta$ is symmetric: ${}^t(J\beta) = {}^t\beta {}^tJ = -{}^t\beta J = J\beta$.

The argument reverses: if $\beta \in \mathrm{Sp}(n, R)$ with $J\beta$ symmetric, then $\beta \in \mathfrak{sp}(n, R)$. Finally, we suppose that $X \in \mathrm{Sp}(n, R) \cap \mathfrak{sp}(n, R)$. Then ${}^tXJ + JX = 0$, ${}^tXJ = JX^{-1}$, so $JX^{-1} + JX = 0$. Multiply on the left by $XJ$, obtaining $-I - X^2 = 0$, or $X \in B$.

(b) We now consider the $\mathrm{Sp}(n, R)$ action on $B$. Let us represent that action by $g \cdot \beta = g^{-1}\beta g$. Notice that

$$J(g \cdot \beta) = Jg^{-1}\beta g = {}^tgJ\beta g,$$

so $H_{g \cdot \beta} = {}^tgH_\beta g$, and then the induced action on the hermitian form is the natural action. Since the signature is preserved under the natural action, the $\mathrm{Sp}(n, R)$ orbits of $B$ are contained in the $B_q$.

Let $\beta \in B_q$. The form $J\beta + iJ$ is a $\beta$-hermitian form and has signature $(p, q)$. Now, up to complex isomorphism, there is only one such form. Thus, there is a complex isomorphism $D$ of $(R^{2n}, \beta)$ with $(R^{2n}, {}^tJ_q)$ which carries $H_\beta$ to $Ht_{J_q}$. Explicitly,

$$\beta D = D{}^tJ_q, \qquad {}^tD(J_\beta + iJ)D = J{}^tJ_q + iJ.$$

The imaginary part of the second equation tells us that $D \in \mathrm{Sp}(n, R)$, and the first equation tells us $D \cdot \beta = {}^tJ_q$. Thus $B_q$ is precisely the Sp($n$, $R$)-orbit of ${}^tJ_q$.

Finally, we can easily compute the isotropy group of ${}^tJ_q$.

1.17. THEOREM. *The isotropy of ${}^tJ_q$ under the* Sp($n$, $R$)*-action is $U(p, q)$. Thus, as homogeneous spaces,*

$$B_q = \mathrm{Sp}(n, R)/U(p, q) \qquad (p + q = n). \tag{1.18}$$

PROOF. First of all we explicitly realize $U(p, q)$ as a subgroup of Sp($n$, $R$) as follows. $R^{2n}$ admits (up to isomorphism) only one nondegenerate skew form. Thus we could have defined Sp($n$, $R$) as well by

$$\mathrm{Sp}^q(n, R) = \left\{ g \in \mathrm{GL}(2n, R); {}^tgJ_qg = J_q \right\}, \tag{1.19}$$

since $J_q$ also defines a nondegenerate skew form on $R^{2n}$. In fact, $\mathrm{Sp}^q(n, R)$ is conjugate to Sp($n$, $R$) via the matrix $D_q$:

$$\mathrm{Sp}^q(n, R) = \left\{ D_qgD_q; g \in \mathrm{Sp}(n, R) \right\}.$$

Now $(R^{2n}, J)$ is a complex vector space, and $H_q = J_qJ + iJ_q$ is a hermitian form on $(R^{2n}, J)$ with $p$ positive and $q$ negative eigenvalues. By definition $U(p, q)$ is the group of complex-linear transformations of $(R^{2n}, J)$ leaving $H_q$ invariant:

$$U(p, q) = \left\{ g \in \mathrm{GL}(2n, R), gJ = Jg, {}^tgJ_qg = J_q \right\},$$

so $U(p, q)$ is a subgroup of $\mathrm{Sp}^q(n, R)$. Finally we verify that conjugation by $D_q$ carries $U(p, q)$ onto the isotropy group of ${}^tJ_q = -J_q$. The isotropy of ${}^tJ_q$ is

$$K(p, q) = \left\{ \beta \in \mathrm{GL}(2n, R); \beta J_q = J_q\beta, {}^t\beta J\beta = J \right\}.$$

Let $\beta \in K(p, q)$ and take $\alpha = D_q\beta D_q$. Then

$$\alpha J = D_q\beta D_qJ = D_q\beta J_qD_q = D_qJ_q\beta D_q = JD_q\beta D_q = J\alpha.$$

$${}^t\alpha J_q\alpha = D_q^t\beta D_qJD_q\beta D_q = D_q^t\beta J\beta D_q = D_qJD_q = J_q.$$

Thus $\alpha \in U(p, q)$. The argument is reversible for $\alpha \in U(p, q)$, $D_q\alpha D_q \in K(p, q)$.

We shall see in the next section that the homogeneous spaces $B_q$ have an (essentially) unique Sp($n$, $R$)-invariant structure as a complex manifold. Here we shall construct that structure geometrically, exhibiting the $B_q$ as a kind of flag domain, generalizing the Siegel upper half plane.

1.20. THEOREM. *$B$ is an open dense subset of an algebraic variety $\mathfrak{L}$.* Sp($n$, $R$) *acts on $B$ by holomorphic transformations.*

PROOF. Let $G(2n, n)$ be the Grassmannian of $n$-dimensional subspaces of $\mathbf{C}^{2n}$. The correspondence $\beta \to V_i(\beta)$ defines a one-one map (in fact a diffeomorphism) of $B$ into $G(2n, n)$. If $g \in \mathrm{Sp}(n, R)$, and $\beta' = g \cdot \beta$, we must also have $V_i(\beta') = g \cdot V_i(\beta)$ (as is easy to compute), where this last action is defined as the natural action of $\mathrm{GL}(2n, \mathbf{C})$ on $G(2n, n)$ restricted to $\mathrm{Sp}(n, R)$. Thus $\mathrm{Sp}(n, R)$ acts by holomorphic transformations on $G(2n, n)$ so as to preserve the image of $B$.

We now calculate the image of $B$. As we have seen, $V \in G(2n, n)$ defines a complex structure on $R^{2n}$ if and only if $V \cap \overline{V} = 0$; in this case the complex structure is given by $\beta | R^{2n}$ where $\beta | V = +i$, $\beta | \overline{V} = -i$. The condition that $\beta \in B$ is just the condition that $V$ is isotropic for the complex skew form $J$ on $\mathbf{C}^{2n}$: for $\beta$ a complex structure on $R^{2n}$, we have

$$J(v - i\beta v, w - i\beta w) = J(v, w) - J(\beta v, \beta w) - i[J(v, \beta w) + J(\beta v, w)].$$

Thus $J$ is the Kähler form of a hermitian form on $(R^{2n}, \beta)$ if and only if $J | V_i(\beta) \equiv 0$, i.e., $V_i(\beta)$ is $J$-isotropic. Thus, if we call $\mathfrak{L}$ the set of Lagrangian subspaces of $G(2n, n)$ (maximally isotropic for $J$), we have $B = \{V \in \mathfrak{L}; V \cap \overline{V} = \{0\}\}$ so $B$ is given by an open condition on a closed subvariety, $\mathfrak{L}$, of $G(2n, n)$.

Let us calculate the $\mathrm{Sp}(n, R)$-action in a local coordinatization of $G(2n, n)$. Choose $V_1$ and $V_2$ as two complementary subspaces of $C^{2n}$ of dimension $n$. Relative to the splitting $\mathbf{C}^{2n} = V_1 \oplus V_2$, write $v \in \mathbf{C}^{2n}$ as a row vector, and $g \in \mathrm{GL}(2n, \mathbf{C})$ as a $2 \times 2$ matrix:

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \qquad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Now, for $\tau \in L(V_2, V_1)$, the graph of $\tau$ is in $G(2n, n)$, and the correspondence $\tau \to$ graph of $\tau$ provides a local chart (for a Zariski open subset) of $G(2n, n)$. Suppose $V_i(\beta)$ is in this coordinate neighborhood.

$$V_i(\beta) = \left\{\begin{pmatrix} \tau v \\ v \end{pmatrix}; v \in V_2\right\}.$$

For $g \in \mathrm{Sp}(n, R)$, $V_i(g \cdot \beta) = g \cdot V_i(\beta)$, so

$$V_i(g \cdot \beta) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \tau v \\ v \end{pmatrix} = \begin{pmatrix} (A + B)v \\ (C + D)v \end{pmatrix}.$$

$V_i(g \cdot \beta)$ is also in this coordinate neighborhood if and only if there is a $\tau' \in L(V_2, V_1)$ with

$$V_i(g \cdot \beta) = \left\{\begin{pmatrix} \tau' v \\ v \end{pmatrix}; v \in V_2\right\}.$$

In particular, we must have $C\tau + D$ invertible, and the coordinate $\tau'$ of $V_i(g \cdot \beta)$ is given by

$$\tau' = (A\tau + B)(C\tau + D)^{-1}. \tag{1.21}$$

That is, the action of $\mathrm{Sp}(n, R)$ is the usual fractional linear action in any coordinate chart.

REMARK. Of course (1.21) is valid throughout the given coordinatization of $G(2n, n)$ and for all $g \in \mathrm{GL}(2n, R)$. The question remains: what $\tau \in L(V_2, V_1)$ correspond to $\mathfrak{L}$? There is an easy answer if $V_1$ and $V_2$ are themselves Lagrangian. In this case we can find bases $E_1^j$, $E_2^j$ of $V_j$, $j = 1, 2$, so that

$$J(E_r^i, E_s^j) = \delta_r^s \cdot \begin{cases} 0 & \text{if } i = j, \\ 1 & \text{if } i = 1, j = 2, \\ -1 & \text{if } i = 2, j = 1. \end{cases}$$

Relative to these bases, $\tau \in L(V_2, V_1)$ is given an $n \times n$ matrix. As in [6, p. 118], we compute that $\tau$ corresponds to a Lagrangian subspace if and only if it is symmetric. (By the way, this also verifies that $\mathfrak{L}$ is defined by algebraic equations.)

Finally, we make explicit the relationship of $B$ with the usual Siegel upper half plane. Our choice of $J$ introduces a direct sum decomposition of $R^{2n}$ into isotropic subspaces

$$V_1 = \left\{ \begin{pmatrix} v \\ 0 \end{pmatrix}; v \in R^n \right\}, \qquad V_2 = \left\{ \begin{pmatrix} 0 \\ v \end{pmatrix}; v \in R \right\},$$

so that the $V_j^{\mathbf{C}}$ are Lagrangian in $\mathbf{C}^{2n}$, and the standard basis gives the bases of the above remark.

1.22. Definition. $\beta$ is *regular* if there is a $\tau \in LV_2^{\mathbf{C}}, V_1^{\mathbf{C}}$ such that

$$V_i(\beta) = \{ w + \tau w; w \in V_2^{\mathbf{C}} \}.$$

$S_q = \{ \beta \in B_q; \beta \text{ is regular} \}$.

As already observed, it follows from [6] that $\tau$ corresponds to a complex structure $\beta$ with a Lagrangian $V_i(\beta)$ if and only if $\tau$ is symmetric. Furthermore, the Sp($n$, $R$)-action is given by fractional linear transformations in the $\tau$ coordinate. In the following we give a direct proof.

1.23. Theorem. *Let*

$$\beta = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in B.$$

(a) *If $\beta \in B_q$ is regular, then $\tau$ is symmetric and* $\operatorname{Im} \tau$ *has signature $(p, q)$.*

(b) *$\beta$ is regular if and only if $(D + iI)$ is invertible. In this case $\tau = B(D + iI)^{-1}$.*

(c) *If $B$ is invertible, $\beta$ is regular. In particular, if $\beta \in B_0$ or $B_n$, then $\beta$ is regular.*

(d) *If $\tau = \tau_1 + i\tau_2$ is a symmetric $n \times n$ matrix with nondegenerate imaginary part, then $\tau$ comes from a $\beta \in B$. $\beta$ is given by*

$$\beta = \begin{pmatrix} \tau_1\tau_2^{-1} - \tau_1\tau_2^{-1} & \tau_1 - \tau_2 \\ -\tau_2^{-1} - \tau_2^{-1} & \tau_1 \end{pmatrix}. \tag{1.24}$$

Proof. (a) Let $v = v_0 - i\beta v_0$, $w = w_0 - i\beta w_0$, $v_0, w_0 \in R^{2n}$. As already observed (Proposition 1.9)

(i) ${}^t wJv = 0$,

(ii) ${}^t\overline{w}Jv = -2iH_\beta(v_0, w_0)$.

For $\beta$ regular, we calculate (i), (ii) directly using $\tau$. Let $v, w \in V_2^{\mathbf{C}}$.

$$0 = ({}^t w {}^t\tau, {}^t w)\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} \tau v \\ v \end{pmatrix} = ({}^t w {}^t\tau, {}^t w)\begin{pmatrix} v \\ -\tau v \end{pmatrix} = {}^t w {}^t\tau v - {}^t w\tau v,$$

for all $v, w$. Thus ${}^t\tau - \tau = 0$.

$$({}^t\overline{w}{}^t\overline{\tau}, \overline{w})\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} \tau v \\ v \end{pmatrix} = {}^t\overline{w}({}^t\overline{\tau} - \tau)v = -2i{}^t\overline{w}(\operatorname{Im} \tau)v.$$

Thus, for $v = v_0 - i\beta v_0$, $w = w_0 - i\beta w_0$, $v_0, w_0 \in R^{2n}$,

$$ {}^t\overline{w}(\operatorname{Im} \tau)v = H_\beta(v_0, w_0), $$

so Im $\tau$ has the same signature as $H_\beta$.

(b) Let $X = \binom{0}{v} \in V_2^{\mathbf{C}}$. Then $X - i\beta X \in V_i(\beta)$, and

$$ X - i\beta X = \begin{pmatrix} 0 \\ v \end{pmatrix} - i\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 0 \\ v \end{pmatrix} = -i\begin{pmatrix} Bv \\ (D + iI)v \end{pmatrix}. \tag{1.25} $$

Thus, $\beta$ is regular if and only if there is a $\tau$ such that

$$ \left\{ \begin{pmatrix} Bv \\ (D + iI)v \end{pmatrix}; v \in V_2^{\mathbf{C}} \right\} = \left\{ \begin{pmatrix} \tau w \\ w \end{pmatrix}; w \in V_2^{\mathbf{C}} \right\}, $$

i.e., if and only if $D + iI$ is invertible, in which case $\tau = B(D + iI)^{-1}$.

(c) Note that, for $v, w \in V_2$, $S_\beta(v, w) = -{}^twBv$. Thus, $B$ is symmetric, so that if $B$ is invertible, $B^2$ is positive definite. Suppose then that $B^{-1}$ exists; we will show that $D + iI$ is invertible as well. Suppose that $(D + iI)v = 0$ for some $v \in V_2^{\mathbf{C}}$. Then by (1.25)

$$ X = \begin{pmatrix} Bv \\ 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} B\bar{v} \\ (D + iI)\bar{v} \end{pmatrix} $$

are both in $V_i(\beta)$, so ${}^tYJX = 0$. But ${}^tYJX = {}^t\bar{v}B^2v$. Thus $v = 0$, so that $D + iI$ is invertible and $\beta$ is regular. In particular, if $S_\beta$ is definite, then $B$ must also be definite, i.e., invertible.

(d) Finally, suppose $\tau = \tau_1 + i\tau_2$ is symmetric, and $\tau_2$ is nondegenerate. We solve $\tau = B(D + iI)^{-1}$ for $B$, $D$, obtaining $D = -\tau_2^{-1}\tau_1$, $B = -\tau_1\tau_2^{-1}\tau_1 - \tau_2$. Using $A = -{}^tD$ and $CB + D^2 = I$ (which follows from $\beta \in \mathrm{Sp}(n, R) \cap \mathrm{sp}(n, R)$), we find $A$, $C$: $A = \tau_1\tau_2^{-1}$, $C = -\tau_2^{-1}$. Thus (1.25) implies that $\beta$ has the form (1.24). Now, one calculates directly that the matrix given by (1.24) is indeed in $B$.

**2. Relations with the orbit theory.** Since $B = \mathrm{Sp}(n, R) \cap \mathfrak{sp}(n, R)$, and $\mathfrak{sp}(n, R) \cong \mathfrak{sp}(n, R)^*$ in a natural way, we may identify each $B_q$ with a coadjoint orbit. In this section we shall show that the complex structure $B_q$ inherits from $G(2n, n)$ is a polarization, and the determinant bundle $L_q$ of the tautological bundle of $G(2n, n)$ restricted to $B_q$ provides a quantization, i.e., it has an $\mathrm{Sp}(n, R)$-invariant hermitian metric whose curvature is the symplectic form of $B_q$ as coadjoint orbit.

Let $V = B \times R^{2n}$. $\mathrm{Sp}(n, R)$ acts on $V$ by the product action. We shall show by direct calculation that $V$ has the structure of a homogeneous holomorphic vector bundle over $B$. Since $V$ is a product space, its tangent bundle $T(V)$ has the canonical splitting $T(V) = T(B) \oplus R^{2n}$. Let $H^{1,0}(B)$ be the subbundle of $T(B)^{\mathbf{C}}$ of holomorphic tangent vectors (the $+i$ eigenspace of the complex structure inherited from $G(2n, n)$). Then we take the (1, 0)-space on $B$ to be

$$ H^{1,0}(V)_{(\beta,v)} = H^{1,0}(B) \oplus V_i(\beta). $$

Since $T(V)^{\mathbf{C}} = H^{1,0}(V) \oplus \overline{H^{1,0}(V)}$, this defines an almost complex structure which is obviously $\mathrm{Sp}(n, R)$-invariant $(g \cdot V_i(\beta) - V_i(g \cdot \beta))$. This structure is integrable.

2.1. PROPOSITION. *$V \to B$ has the structure of a homogeneous holomorphic vector bundle over $B$. The restriction $V_q = V|B_q$ can be described as*

$$V_q = \mathrm{Sp}(n, R) \times_{\rho(p,q)} \mathbf{C}^n,$$

*where $\rho(p, q)$: $U(p, q) \to \mathrm{GL}(n, \mathbf{C})$ is the standard representation.*

PROOF. Let $\beta_0 \in B$, and let $U$ be a coordinate neighborhood of $\beta_0$ with complex coordinates $z_1, \ldots, z_N$ ($N = n(n+1)/2$). Write $R^{2n} = V_1 \oplus V_2$ so that $V_i(\beta_0)$ is complementary to $V_2^{\mathbf{C}}$ (as in the preceding section, this splitting provides a coordinatization of a neighborhood $U$ of $V_i(\beta_0)$ by $L(V_2^{\mathbf{C}}, V_1^{\mathbf{C}})$). Let $l_1, \ldots, l_n$ be a basis for $V_2^{\mathbf{C}*}$, and define over $U$:

$$\xi_j(\beta, v) = l_j(v) - il_j(\beta v), \qquad 1 \leqslant j \leqslant n. \tag{2.2}$$

Then $z_1, \ldots, z_N, \xi_1, \ldots, \xi_n$ have differentials vanishing on $H^{0,1}(V)$, so give a coordinatization of $V|U$. By (2.2) the change of coordinates on the fiber is complex linear, so this makes $V$ a homogeneous holomorphic vector bundle over $B$.

As for the last statement, we have seen that $U(p, q)$ is the isotropy of ${}^tJ_q$. It is easily verified that $U(p, q)$ acts on the fiber over ${}^tJ_q$ via the standard representation of $U(p, q)$ on $\mathbf{C}^n$.

2.3. REMARK. Letting $\mathcal{V} \to G(2n, n)$ be the "tautological bundle", it is clear that $V$, as holomorphic Sp($n$, $R$)-homogeneous vector bundle, is just $\mathcal{V}|B$. We wanted to exhibit the local coordinates explicitly.

Now, fix $q$ and consider $V_q \to B_q$. The form $H_\beta = J\beta + iJ$ defines a nondegenerate hermitian form on the fibers of $V_q$ which is Sp($n$, $R$)-covariant. Then $L_q = \Lambda^n V_q \to B_q$ is a homogeneous hermitian line bundle over $B_q$ with metric $(-1)^q \Lambda^n H$. We show directly that this bundle gives the quantization and polarization associated to $B_q$ as coadjoint orbit.

First, let us identify the complex structure on $B_q$ in Lie algebra terms. What we mean is this. The set of symmetric matrices

$$S_q = \{X + iY, X, Y \text{ are real, } Y \text{ has signature } (p, q)\}$$

is an open dense subset of $B_q$, and Sp($n$, $R$) acts on $S_q$ by

$$\tau \xrightarrow{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)} (A\tau + B)(C\tau + D)^{-1}.$$

In this coordinatization $U(p, q)$ is the isotropy group if $iI_{pq}$. We can identify

$$T_{iI_{pq}}(B_q) = \mathrm{sp}(n, R/\mathfrak{u}(p, q)), \tag{2.4}$$

and by the Sp($n$, $R$)-homogeneity we can fully describe the complex structure by identifying multiplication by $i$ in the realization (2.4).

2.5. PROPOSITION. *Multiplication by $i$ on $T_{iI_{pq}}(B)$ is given, via the realization* (2.4), *by* ad $J_q|2$.

PROOF. Let $X = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathfrak{sp}(n, R)$, and consider the curve $\gamma(t) = \exp tX \circ (iI_{pq})$. We compute up to first order:

$$\begin{aligned}\gamma(t) &= \left[(I + tA)(iI_{pq}) + tB\right]\left[tC(iI_{pq}) + I + tD\right]^{-1} + o(t)\\ &= \left[(I + tA)(iI_{pq}) + tB\right]\left[I - tC(iI_{pq}) - tD\right] + o(t)\\ &= iI_{pq} + t\left(B + I_{pq}CI_{pq} + i(AI_{pq} - I_{pq}D)\right) + o(t).\end{aligned}$$

Thus the differential of $\pi$: $\mathrm{Sp}(n, R) \to B_q$ at $e$ is

$$d\pi_e(X) = (B + I_{pq}CI_{pq}) + i(AI_{pq} - I_{pq}D). \tag{2.6}$$

Now we compute ad $J_q$:

$$[J_q, X] = \begin{pmatrix} I_{pq}C + BI_{pq} & -AI_{pq} + I_{pq}D \\ -I_{pq}A + DI_{pq} & -(I_{pq}D + CI_{pq}) \end{pmatrix}.$$

Putting this in (2.6) we readily compute

$$d\pi_e(\mathrm{ad}\, J_q \cdot X) = 2id\pi_e(x),$$

so multiplication by $i$ is given by ad $\frac{1}{2}J_q$.

Now that we have this fact, we shall go through the Kostant machinery **[9]** to verify our claims on $B_q$.

Since $\mathrm{Sp}(n, R)$ is simple its Lie algebra is naturally self-dual, the duality being given by $\langle X, Y\rangle = \mathrm{tr}\, {}^tXY$.

Now $B_q$ is the orbit of ${}^tJ_q$ under the adjoint action of $\mathrm{Sp}(n, R)$ on $\mathfrak{sp}(n, R)$. Moving to $\mathfrak{sp}(n, R)^*$ via the given duality, we identify $B_q$ as the coadjoint orbit of the functional $f_0$: $f_0(X) = \mathrm{tr}\, J_qX$. Let $\mathfrak{h}^+$ be the union of the 0 and $+i$ eigenspaces of ad $\frac{1}{2}J_q$. Then $\mathfrak{h}^+$ defines the complex structure on $B_q$ so must be a Lie algebra.

2.7. Proposition. *$\mathfrak{h}^+$ is a polarization at $f_0$.*

Proof. To show $\mathfrak{h}^+$ is a polarization, we need only show that it is isotropic for the symmetric form

$$\Omega(X, Y) = f_0([X, Y]) = \mathrm{tr}\, J_q[X, Y]. \tag{2.8}$$

That amounts to showing

$$\Omega\left(\mathrm{ad}\, \tfrac{1}{2}J_q \cdot X, Y\right) + \Omega\left(X, \mathrm{ad}\, \tfrac{1}{2}J_q \cdot Y\right) = 0. \tag{2.9}$$

But, by the Jacobi identity

$$\left[\left[\tfrac{1}{2}J_q, X\right], Y\right] + \left[X, \left[\tfrac{1}{2}J_q, Y\right]\right] = \left[\tfrac{1}{2}J_q, [X, Y]\right]. \tag{2.10}$$

Since $\frac{1}{2}J_q$ is in the Lie algebra of the isotropy group at $f_0$, $\Omega(\frac{1}{2}J_q, Z) = 0$ for all $Z$, so the right-hand side of (2.10) vanishes, verifying (2.9).

2.11. Remark. In the next section we shall see that $\mathfrak{h}^+$, $\bar{\mathfrak{h}}^+$ are the only complex polarizations at $f_0$.

Now, according to the orbit theory, $f_0$ is a character on its isotropy algebra $\mathfrak{u}(p, q)$, and if $e^{2\pi i f_0}$ defines a character on $U(p, q)$, that character determines a holomorphic hermitian line bundle on $B_q$ with $\Omega$ as the curvature of the holomorphic connection. Indeed $e^{2\pi i f_0}$ must be a power of the determinant and since

$$f_0({}^tJ_q) = \mathrm{tr}\, J_q{}^tJ_q = \mathrm{tr}\, I = n$$

and ${}^tJ_q = \exp((\pi i/2){}^tJ_q)$ is multiplication by $i$ on $V_{{}^tJ_q}$, that power is one. Thus, the line bundle predicted by the orbit theory is just that introduced previously: $L_q \to B_q$.

Finally, we compute explicitly the curvature $\Omega$. To do this, as well as the computations in the next section, we need the following explicit description of the Lie algebra of Sp($n$, $R$).

Throughout this discussion $i$ will appear both as an index and as $\sqrt{-1}$. We are certain there will be no confusion. $E_{ij}$ is the matrix with a 1 in the $(i, j)$th entry; all other entries are 0. Note $E_{ij}E_{kl} = \delta_{jk}E_{il}$. $\mathfrak{u}(n) = \{X \in \mathfrak{gl}(n, \mathbf{C}); {}^t\overline{X} + X = 0\}$ has the basis

$$H_i = iE_{ii}, \qquad U_{ij} = E_{ij} - E_{ji}, \qquad V_{ij} = i(E_{ij} + E_{ji}) \qquad (i < j).$$

$$[H_i, U_{jk}] = \begin{cases} 0, & i \neq j \text{ or } k, \\ V_{jk}, & i = j, \\ -V_{jk}, & i = k, \end{cases} \qquad [H_i, V_{jk}] = \begin{cases} 0, & i \neq j \text{ or } k, \\ -U_{jk}, & i = j, \\ U_{jk}, & i = k, \end{cases}$$

$H_1, \ldots, H_n$ span the Cartan subalgebra $\mathfrak{t}$. The root spaces determined by $\mathfrak{t}$ are spanned by the

$$Z_{jk} = U_{jk} - iV_{jk}, \qquad \overline{Z}_{jk} = U_{jk} + iV_{jk} \qquad (j < k)$$

with the corresponding roots

$$\lambda^{jk} = i(h^j - h^k)\bar{\lambda}^{jk} = -i(h^j - h^k) \tag{2.12}$$

where $h^1, \ldots, h^n$ span $\mathfrak{t}^*$ and are dual to the basis $H_1, \ldots, H_n$ of $\mathfrak{t}$. We will need

$$[Z_{ij}, \overline{Z}_{kl}] = 4i(H_i - H_j), \qquad i = k, j = l, \tag{2.13}$$

and otherwise is 0 or another root vector.

Now

$$\mathfrak{sp}(n, R) = \{X \in \mathfrak{gl}(2n, R); {}^tXJ + JX = 0\}.$$

The given basis of $\mathfrak{u}(n)$ appears in $\mathfrak{sp}(n, R)$ as

$$H_i = \begin{pmatrix} 0 & E_{ii} \\ -E_{ii} & 0 \end{pmatrix}, \qquad U_{ij} = \begin{pmatrix} E_{ij} - E_{ji} & 0 \\ 0 & E_{ij} - E_{ji} \end{pmatrix},$$

$$V_{ij} = \begin{pmatrix} 0 & E_{ij} + E_{ji} \\ -E_{ij} - E_{ji} & 0 \end{pmatrix}.$$

The rest of $\mathfrak{sp}(n, R)$ is spanned by these subalgebras:

$$\mathcal{K}_0 = \left\{\begin{pmatrix} \Delta & 0 \\ 0 & -{}^t\Delta \end{pmatrix}; \Delta \text{ upper triangular}\right\}, \qquad \mathcal{K}_1 = \left\{\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}; {}^tX = X\right\}.$$

Since ${}^t\mathfrak{sp}(n, R) = \mathfrak{sp}(n, R)$ we can replace the subalgebras $\mathcal{K}_0$, $\mathcal{K}_1$ by the following subspaces which makes computations easier:

$$\mathcal{K}_0' = \left\{\begin{pmatrix} \Delta & 0 \\ 0 & -\Delta \end{pmatrix}; {}^t\Delta = \Delta\right\}, \qquad \mathcal{K}_1' = \left\{\begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}; {}^tX = X\right\}.$$

Note: $[\mathcal{K}_1', \mathcal{K}_1'] \subset \mathfrak{u}(n)$, and is *not* an algebra. Either $\mathcal{K}_0$ or $\mathcal{K}_0'$ together with the $\{U_{ij}\}$ span the same space. The rest of our basis of $\mathfrak{sp}(n, R)$ consists of the vectors

$$G_{ii} = \begin{pmatrix} 0 & E_{ii} \\ E_{ii} & 0 \end{pmatrix}, \qquad G_{ij} = \begin{pmatrix} 0 & E_{ij} + E_{ji} \\ E_{ij} + E_{ji} & 0 \end{pmatrix}, \qquad i < j\ (\mathcal{H}_1'),$$

$$H_{ii} = \begin{pmatrix} E_{ii} & 0 \\ 0 & -E_{ii} \end{pmatrix}, \qquad H_{ij} = \begin{pmatrix} E_{ij} + E_{ji} & \\ 0 & -(E_{ij} + E_{ji}) \end{pmatrix}, \qquad i < j\ (\mathcal{H}_0').$$

Now

$$[H_i, G_{jj}] = 2\delta_{ij} H_{jj}, \qquad [H_i, H_{jj}] = -2\delta_{ij} G_{jj},$$

$$[H_i, G_{jk}] = \begin{Bmatrix} H_{jk}, & i = j \text{ or } k & -G_{jk} \\ 0, & \text{otherwise} & 0 \end{Bmatrix} = [H_i, H_{jk}].$$

Thus the noncompact root vectors are the

$$W_{ij} = G_{ij} - iH_{ij}, \qquad \overline{W}_{ij} = G_{ij} + iH_{ij} \qquad (i \leqslant j)$$

with corresponding roots

$$\mu^{ij} = i(h^i + h^j), \qquad \bar{\mu}^{ij} = -i(h^i + h^j). \tag{2.14}$$

Finally,

$$[W_{ij}, \overline{W}_{kl}] = -4i(H_i + H_j), \qquad i = k, j = l, \tag{2.15}$$

and otherwise is 0 or another root vector.

Since $\mathfrak{h}^+$ is ad $t$-invariant, it is a sum of root spaces. Let us calculate which roots vectors are in $\mathfrak{h}^+$. We compute the 0, $\pm 2i$ eigenspaces of ad $J_q$.

$$J_q = \sum_{i \leqslant p} H_i - \sum_{i > p} H_i.$$

$$\begin{aligned} [J_q, Z_{jk}] &= \sum_{i \leqslant p} [H_i, Z_{jk}] - \sum_{i > p} [H_i, Z_{jk}] \\ &= \sum_{i \leqslant p} i(\delta_{ij} - \delta_{ik}) Z_{jk} - \sum_{i > p} i(\delta_{ij} - \delta_{ik}) Z_{jk} \\ &= \begin{cases} 0, & j < k \leqslant p \text{ or } p < j < k, \\ {}_i Z_{jk}, & j \leqslant p < k. \end{cases} \end{aligned}$$

Thus the $Z_{jk}$ are in $\mathfrak{h}^+$ if $j \leqslant p < k$, the corresponding $\overline{Z}_{jk}$ are in $\bar{\mathfrak{h}}^+$, and all other compact root vectors are isotropic (in $\mathfrak{u}(p, q)$).

$$\begin{aligned} &= \sum_{i \leqslant p} [H_i, W_{jk}] - \sum_{i > p} [H_i, W_{jk}] \\ &= \sum_{i \leqslant p} i(\delta_{ij} + \delta_{ik}) W_{jk} - \sum_{i > p} i(\delta_{ij} + \delta_{ik}) W_{jk} \\ &= \begin{cases} 2iW_{jk}, & j \leqslant k \leqslant p, \\ -2iw_{jk}, & p < j \leqslant k, \\ 0, & j \leqslant p < k. \end{cases} \end{aligned}$$

Thus the $W_{jk}$ are in $\mathfrak{h}^+$ if $j \leqslant k \leqslant p$, $\overline{W}_{jk} \in \mathfrak{h}^+$ if $p < j \leqslant k$; their conjugates are in $\bar{\mathfrak{h}}^+$, and all other noncompact roots are isotropic. Now $f_0 = -\Sigma_{i \leqslant p} h^i + \Sigma_{i > p} h^i$, and the curvature form at $f_0$ is given by $f_0([X, \overline{Y}])$ for $X, Y \in \mathfrak{h}^+$ mod $\mathfrak{u}(p, q)$. We have provided previously a root vector basis for $\mathfrak{h}^+ \bmod \mathfrak{u}(p, q)$ such that $f_0([X, \overline{Y}]) = 0$ unless $X = Y$; i.e., this basis diagonalizes the curvature form. We compute the eigenvalues directly:

$$\begin{cases} f_0\left(\left[Z_{jk}, \overline{Z}_{jk}\right]\right) = -8i & (j \leqslant p < k); \\ f_0\left(\left[W_{jk}, \overline{W}_{jk}\right]\right) = 8i & (j \leqslant k \leqslant p); \\ f_0\left(\left[\overline{W}_{jk}, W_{jk}\right]\right) = 8i & (p < j \leqslant k). \end{cases} \tag{2.16}$$

In summary, we have

2.17. THEOREM. *The curvature form of the hermitian line bundle $\Lambda^n V \to B_q$ has $pq$ negative eigenvalues, and all the rest are positive. The negative eigenvalues occur only in the compact directions.*

**3. Uniqueness of the complex structure.** The object of this section is to show that $\mathfrak{h}^+$ and $\overline{\mathfrak{h}}^+$ are the only proper complex Lie subalgebras of $\mathfrak{sp}(n, R)^{\mathbf{C}}$ containing $\mathfrak{u}(p, q)$. It follows that these are the unique polarizations at $f_0$, and, in particular, that the complex structure of $B_q$ described in §1, and its conjugate, are the only $\mathrm{Sp}(n, R)$-invariant complex structures on $\mathrm{Sp}(n, R)/U(p, q)$.

3.1. THEOREM. *Let $\mathfrak{h}$ be a complex Lie subalgebra of $\mathfrak{sp}(n, \mathbf{C})$ which contains $\mathfrak{u}(p, q)$. There are only four possibilities*: $\mathfrak{u}(p, q)^{\mathbf{C}}$, $\mathfrak{h}^+$, $\overline{\mathfrak{h}}^+$, $\mathfrak{sp}(n, \mathbf{C})$.

PROOF. Since $\mathfrak{h}$ contains the Cartan algebra $\mathfrak{t}$, $\mathfrak{h}$ must be a sum of root spaces. Let $\Delta$ be the set of roots corresponding to root vectors in $\mathfrak{h}$, but not in $\mathfrak{u}(p, q)^{\mathbf{C}}$, and let $\Delta^+$ be the corresponding set of roots for $\mathfrak{h}^+$. We will show that if $\Delta \cap \Delta^+$ is nonempty, then $\Delta \supset \Delta^+$. A similar argument will show that $\Delta \cap \overline{\Delta}^+ \neq \varnothing$ implies $\Delta \supset \overline{\Delta}^+$, and that will conclude the proof.

We introduce the following notation:

$$\begin{aligned} \Delta_p^+ &= \{\lambda^{jk}; j < k \leqslant p\}, & \Delta_q^+ &= \{\lambda^{jk}; p < j < k\}, \\ \Delta_{oc}^+ &= \{\lambda^{jk}; j \leqslant p < k\}, & \Delta_{nc}^+ &= \{\mu^{jk}; j \leqslant p < k\}, \\ \Delta_{np}^+ &= \{\mu^{jk}; j \leqslant k \leqslant p\}, & \Delta_{nq}^+ &= \{\mu^{jk}; p < j \leqslant k\}. \end{aligned}$$

Similarly we define $\Delta_p^-$, $\Delta_q^-$, etc., using the negative roots. Thus $\Delta^+ = \Delta_{oc}^+ \cup \Delta_{np}^+ \cup \Delta_{nq}^-$, and $\Delta_p^\pm \cup \Delta_q^\pm \cup \Delta_{nc}^\pm$ correspond to root vectors in $\mathfrak{u}(p, q)$. Since $\mathfrak{h}$ is a Lie algebra, we have: if $\alpha \in \Delta$ and $\beta \in \Delta$ or corresponds to a root vector in $\mathfrak{u}(p, q)$, and $\alpha + \beta$ is a root, then $\alpha + \beta \in \Delta$. We shall need the following root computations:

$$\lambda^{ij} + \lambda^{rs} = \begin{cases} \lambda^{is}, & j = r, \\ \lambda^{rj}, & i = s, \\ \text{not a root otherwise,} \end{cases} \tag{1}$$

$$\lambda^{ij} + \overline{\lambda}^{rs} = \begin{cases} \lambda^{sj}, & i = r, \\ \lambda^{ir}, & j = s, \\ \text{not a root otherwise,} \end{cases} \tag{2}$$

$$\lambda^{ij} + \mu^{rs} = \begin{cases} \mu^{is}, & j = r, \\ \mu^{ir}, & j = s, \\ \text{not a root otherwise,} \end{cases} \tag{3}$$

$$\lambda^{ij} + \bar{\mu}^{rs} = \begin{cases} \bar{\mu}^{js}, & i = r, \\ \bar{\mu}^{jr}, & i = s, \\ \text{not a root otherwise,} \end{cases} \tag{4}$$

$$\bar{\lambda}^{ij} + \mu^{rs} = \begin{cases} \mu^{js}, & i = r, \\ \mu^{jr}, & i = s, \\ \text{not a root otherwise,} \end{cases} \tag{5}$$

$$\mu^{ij} + \mu^{rs} \text{ is never a root } (p^+ \text{ is abelian}) \tag{6}$$

$$\mu^{ij} + \bar{\mu}^{rs} = \begin{cases} \lambda^{ir}, & j = s, \\ \lambda^{is}, & j = r, \\ \lambda^{jr}, & i = s, \\ \lambda^{js}, & i = r, \\ \text{not a root otherwise.} \end{cases} \tag{7}$$

(We use the convention that (2.12), (2.14) define $\lambda^{ij}$, $\mu^{ij}$ for all indices $i, j$. We also note for further use that if the sum of two roots is a root, the corresponding root vector is the bracket of the original root vectors.) We now return to the proof of Theorem 3.1.

(i) Suppose $\lambda^{ij} \in \Delta$ for some $i \leqslant p < j$. Then

(a) $\lambda^{rj} = \lambda^{ri} + \lambda^{ij} \in \Delta$, $r \leqslant p$, $\lambda^{rs} = \lambda^{rj} + \lambda^{js} \in \Delta$, $r \leqslant p \leqslant s$. Thus $\Delta \supset \Delta^+_{oc}$.

(b) Let $j \leqslant k \leqslant p$. Take $s > p$: $\mu^{jk} = \mu^{js} + \lambda^{ks} \in \Delta$. Thus $\Delta \supset \Delta^+_{np}$.

(c) Let $p < j \leqslant k$. Take $s \leqslant p$: $\bar{\mu}^{jk} = \bar{\mu}^{sk} + \lambda^{sj} \in \Delta$. Thus $\Delta \supset \Delta^+_{nq}$.

We conclude that if $\Delta$ contains any $\lambda^{ij}$, $i \leqslant p < j$, then $\Delta \supset \Delta^+$.

(ii) Suppose $\mu^{ik} \in \Delta$, $i \leqslant k \leqslant p$. Let $j > p$, $\lambda^{ij} = \mu^{ik} + \bar{\mu}^{kj} \in \Delta$. Thus again $\Delta \supset \Delta^+$.

(iii) Suppose $\mu^{jk} \in \Delta$, $p < j \leqslant k$. Let $i \leqslant p$. $\lambda^{ij} = \mu^{ik} + \bar{\mu}^{jk} \in \Delta$, so once again $\Delta \supset \Delta^+$. The theorem is proved.

3.2. Remark. Part (i)(a) alone proves that $U(n)/U(p) \times U(q)$ has precisely two distinct invariant complex structures. In the same way, replacing $\lambda^{ij}$ by $\mu^{ij}$, we can prove that $U(p, q)/U(p) \times U(q)$ also has only two distinct invariant structures.

3.3. Remark. More generally, suppose that $G_0$ is any subgroup of $\mathrm{Sp}(n, R)$ containing the compact Cartan algebra. An invariant complex structure $\mathrm{Sp}(n, R)/G_0$ is determined by a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{sp}(n, \mathbf{C})$ such that $\mathfrak{h} + \bar{\mathfrak{h}} = \mathfrak{sp}(n, \mathbf{C})$, $\mathfrak{h} \cap \bar{\mathfrak{h}} = \mathfrak{g}_0^{\mathbf{C}}$. Since $\mathfrak{h} \supset \mathfrak{t}$, $\mathfrak{h}$ is ad $t$-invariant, so must be a sum of root spaces. Thus we conclude that $\mathrm{Sp}(n, R)/G_0$ has at most finitely many invariant complex structures. In particular, for $G_0 = U(p) \times U(q)$ we find eight possibilities: Either $\mathfrak{h}$ or $\bar{\mathfrak{h}}$ corresponds to any of the following four rootsets:

$$\Delta^+_{oc} \cup \Delta^+_{nc} \cup \Delta^+_{np} \cup \Delta^{\pm}_{nq}; \qquad \Delta^+_{oc} \cup \Delta^-_{nc} \cup \Delta^{\pm}_{np} \cup \Delta^-_{nq}.$$

Since the computation is not particularly enlightening, we suppress it.

**4. Pseudoconvexity of $B_q$.** We return now to the $\mathbf{C}^n$-vector bundle $V \to B_q$ whose fibers $V_\beta$, $\beta \in B_q$ are $n$-dimensional vector spaces endowed with a nondegenerate hermitian form $H_\beta = J\beta + iJ$. Let $V_\beta = V'_\beta \oplus V''_\beta$ so that $H_\beta | V' \gg 0$ and $H_\beta | V''_\beta \ll 0$. If we replace the complex structure on $V''_\beta$ by its conjugate, then the corresponding $V'_\beta \oplus \overline{V}''_\beta = V_{\beta'}$, with $\beta' \in B_0$. This "correspondence" between splittings of $V_\beta$ and points in $B_0$ was exploited by Matsushima in **[10]**; we shall discuss the relationship of our work with that article in the next section. In this section we shall explore the geometry of this correspondence, realizing it as a generalization of the "Penrose correspondence" as described by Wells in **[16]**. Much of our techniques are motivated by, and depend upon the work of, Griffiths and Schmid **[5]**.

4.1. THEOREM. *$B_q$ is a strongly pq-convex domain. In fact, there is a smooth real-valued function $\phi$ defined on $\mathfrak{L}$ such that $-\ln|\phi|$ is a pq-convex exhaustion function for each $B_q$.*

PROOF. As we have seen in §2, $m_{1,\beta} = (-1)^q \Lambda^n H_\beta$ is a metric on $L_q = \Lambda^n V_\beta \to B_q$ with curvature $i\partial\bar\partial(-\ln m_1) = \Omega$ as described in (2.16).

Now, fix a metric $E$ on $\mathbf{C}^{2n}$. Then every $V_\beta$, $\beta \in \mathfrak{L}$ inherits a positive-definite metric and the bundle $V \to B_q$ becomes a (nonhomogeneous) hermitian vector bundle. Taking $\Lambda^n$, $L_q \to B_q$ becomes a hermitian vector bundle in a new way, with metric $m_2$. As such $L_q \to B_q$ is the restriction of $\Lambda^n T \to G(2n, n)$ to $B_q$, where $T$ is the tautological bundle. Since $\Lambda^n T$ is negative over $G(2n, n)$ (see, e.g. **[5]**) this metric has negative definite curvature $\Omega_2 = i\partial\bar\partial(-\ln m_2)$. The quotient $\phi_q = m_1 m_2^{-1}$ is thus a well-defined function on $B_q$ and its Hessian $\mathcal{H}(\phi_q) = i\partial\bar\partial(-\ln \phi_q) = \Omega - \Omega_2$ has at most $pq$ nonpositive eigenvalues. We now show that $-\ln \phi_q$ is an exhaustion function by computing it another way.

For $\beta \in \mathfrak{L}$, let $\phi(\beta)$ be the product of the eigenvalues of $H_\beta$ relative to the fixed metric $E | V_\beta$. Now, by Proposition 1.9 (any $\beta \in \mathfrak{L}$ is allowable, by continuity), $H_\beta(v, w) = (i/2)^t\bar{w}Jv$. Since $V_\beta$ is Lagrangian, we have $H_\beta(v, w) = 0$ for all $v \in V_\beta$ if and only if $\bar{w} \in V_\beta$. Thus $H_\beta$ is singular if and only if $V_\beta \cap \overline{V}_\beta \neq \{0\}$, i.e., precisely when $\beta \notin B$. Thus $\mathfrak{L} - B = \{\beta;\ \phi(\beta) = 0\}$, so for each $q$, $-\ln|\phi|$ is an exhaustion function for $B_q$. Now, if we trivialize the bundle $V$ (locally) on $B_q$, $H_\beta$ and $E | V_\beta$ are represented by hermitian matrices and $\phi(\beta)$ is the product of the eigenvalues of the transformation $H^0_\beta$ defined by $H_\beta(v, w) = E(H^0_\beta v, w)$. That is, as matrices, $H^0_\beta = E^{-1}H_\beta$ and

$$|\phi(\beta)| = \left|\det H^0_\beta\right| = (-1)^q \det H_\beta (\det E)^{-1} = m_1 m_2^{-1} = \phi_q(\beta).$$

We shall see later (in Proposition 4.12) that $-\ln|\phi|$ has a positive semidefinite Hessian: the $pq$ nonpositive eigenvalues are actually zero.

EXAMPLES. **(1)** $n = 1$. Let $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and let $\mathbf{C}^2$ be endowed with the Euclidean metric $\|v\|^2 = |v_1|^2 + |v_2|^2$. All one-dimensional subspaces of $\mathbf{C}^2$ are $J$-isotropic, so $\mathfrak{L}$ is all of $\mathbf{P}^1$. Fix $v = (v_1, v_2) \neq (0, 0)$, and consider the hermitian form $H(v, w) = \frac{1}{2} i^t\bar{w}Jv$ restricted to $\mathbf{C}v$:

$$H(\lambda v, \lambda v) = -\tfrac{1}{2} i\lambda\bar{\lambda}(v_1\bar{v}_2 - \bar{v}_1 v_2) = |\lambda|^2 \operatorname{Im}(v_1\bar{v}_2).$$

$$\|\lambda v\|^2 = |\lambda|^2(|v_1|^2 + |v_2|^2),$$

so

$$\phi(\mathbf{C}_v) = \frac{\operatorname{Im} v_1\bar{v}_2}{|v_1|^2 + |v_2|^2} = \frac{\operatorname{Im} z}{1 + |z|^2}$$

using the inhomogeneous coordinate $z = v_1/v_2$. Identifying $P^1$ with the sphere in $\mathbf{R}^3$, $\phi$ is just the cosine of the polar angle.

**(2)** $n = 2$. For ease in computation we take $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. Let $e_i$, $1 < i < 4$ be the standard basis for $\mathbf{C}^4$, and $\{e^i\}$ the dual basis. The two form $(v, w) \to {}^t wJv$ associated to $J$ is

$$\omega = e^1 \wedge e^3 + e^2 \wedge e^4. \tag{4.2}$$

This is the Kähler form associated to the hermitian form $(v, w) \to i\,{}^t\bar{w}Jv$, given by

$$\Omega = i(e^1 \wedge \bar{e}^3 - e^3 \wedge \bar{e}^1 + e^2 \wedge \bar{e}^4 - e^4 \wedge \bar{e}^2). \tag{4.3}$$

$G(4, 2)$ is the subvariety of $\mathbf{P}(\Lambda^2\mathbf{C}^4)$ defined by $v \wedge v = 0$ $(v \in \Lambda^2\mathbf{C}^4)$. In coordinates

$$\Lambda^2\mathbf{C}^4 = \left\{ \sum_{1<i<j<4} v^{ij} e_{ij} \right\}$$

$(e_{ij} = e_i \wedge e_j)$; the defining equation becomes

$$v^{12}v^{34} - v^{13}v^{24} + v^{14}v^{23} = 0. \tag{4.4}$$

Now $v \in \Lambda^2\mathbf{C}^4$ represents a Lagrangian 2-plane if and only if $\langle \omega, v \rangle = 0$, which in coordinates is the linear space given by the equation

$$v^{13} + v^{24} = 0. \tag{4.5}$$

Thus

$$\mathfrak{L} = \{v \in \mathbf{P}(\Lambda^2 C^4);\ (4.4) \text{ and } (4.5) \text{ hold}\}$$

can be realized as a hypersurface in $P^4$ with coordinates $[x_0, \ldots, x_4]$ via the substitution

$$x_0 = v^{13} = -v^{24}, \qquad x_1 = v^{12}, \qquad x_2 = v^{34}, \qquad x_3 = v^{14}, \qquad x_4 = v^{23}.$$

Then

$$\mathfrak{L} = \{x \in \mathbf{P}^4;\ x_0^2 + x_1x_2 + x_3x_4 = 0\}. \tag{4.6}$$

Now, we compute the metric. For $\beta \in \mathfrak{L}$ with coordinates $v \in \Lambda^2\mathbf{C}^4$, the metric induced by (4.3) on $\Lambda^2 V_\beta$ (with coordinate $v \wedge v$) is given by the linear form

$$\Omega \wedge \Omega = e^{13} \wedge \bar{e}^{13} + e^{24} \wedge \bar{e}^{24} + e^{12} \wedge \bar{e}^{34} + e^{34} \wedge \bar{e}^{12} + e^{14} \wedge \bar{e}^{23} + e^{23} \wedge \bar{e}^{14}.$$

In $\{v\}$-coordinates the formula is

$$|v^{13}|^2 + |v^{24}|^2 + v^{12}\bar{v}^{34} + \bar{v}^{12}v^{34} + v^{14}\bar{v}^{23} + \bar{v}^{14}v^{23},$$

or in the coordinates of (4.6), $2(|x_0|^2 + \mathrm{Re}(x_1\bar{x}_2 + x_3\bar{x}_4))$. Thus, in these coordinates

$$\phi(x) = \frac{|x_0|^2 + \mathrm{Re}(x_1\bar{x}_2 + x_3\bar{x}_4)}{\Sigma|x_1|^2} \tag{4.7}$$

up to a constant.

Now, by Theorem 4.1, $B_q$ is a $pq$-complete manifold in the terminology of Andreotti-Grauert [1], and Andreotti-Norguet [2]. In general, in a $Q$-complete manifold every compact analytic cycle has dimension at most $Q$. We now want to describe the geometry of a family of compact linear subspaces of $B_q$ of dimension $pq$.

Let $G(2n, n, p)$ be the manifold of pairs $(V, W)$ of subspaces of $\mathbf{C}^{2n}$ with $V \supset W$, $\dim V = n$, $\dim W = p$. Let $\pi$: $G(2n, n, p) \to G(2n, n)$ be the projection onto the first factor. For $s = 0$ or $q$, let

$$E_{ps} = \{(V, W) \in G(2n, n, p);\ V \in B_s,\ H_V|W \gg 0\}.$$

$E_{ps}$ is an open subset of $\pi^{-1}(B_s)$, and thus is a complex manifold. We shall let $\pi_s = \pi|E_{ps}$. We restrict to $s = 0$ or $q$ for only in those cases is $H_V|W'$ definite, so long as $W'$ is complementary to $W$.

4.8. THEOREM. (i) $\mathrm{Sp}(n, R)$ *acts on* $E_{ps}$ *by holomorphic transformations.*

(ii) $E_{ps} = \mathrm{Sp}(n, R)/U(p) \times U(q)$ *as* $\mathrm{Sp}(n, R)$*-spaces.*

(iii) *The complex structure on* $E_{ps}$ *is given by taking as holomorphic vectors the sum of the root spaces corresponding to the roots*

$$\Delta_{ps} = \Delta_{oc}^+ \cup \Delta_{nc}^+ \cup \Delta_{n(n-s)}^+ \cup \Delta_{ns}^- \qquad (s = 0 \text{ or } q). \tag{4.9}$$

PROOF. (i) is obvious. We now show that $\mathrm{Sp}(n, R)$ acts transitively on $E_{ps}$. Let $(V_0, W_0)$, $(V, W) \in E_{ps}$. Since $\mathrm{Sp}(n, R)$ acts transitively on $B_q$, we may assume $V = V_0$. Then $W_0$, $W$ are two subspaces of $V$ on which $H_V$ is positive definite. For $X$ a subspace of $V$, let $X = \{\omega \in V;\ H_V(\omega, x) = 0 \text{ for all } x \in X\}$. Then $W^\perp$, $W_0^\perp$ are complementary to $W$, $W_0$ respectively, and since $s = 0$ or $q$, $H_V$ is definite on these spaces. A linear transformation taking orthonormal bases of $W_0$, $W_0^\perp$ to $W$, $W^\perp$ respectively is unitary with respect to $H_V$, so in particular is in $\mathrm{Sp}(n, R)$. Thus $(V, W_0)$, $(V, W)$ are in the same $\mathrm{Sp}(n, R)$-orbit.

Suppose $g \in \mathrm{Sp}(n, R)$ fixes $(V, W)$, where $V$ corresponds to ${}^tJ_q$, and $W$ is the span of the first $p$ basis vectors. Then, necessarily, $g \in U(p, q)$. But $g$ fixes both $W$ and $W^\perp$, and is unitary on each of these spaces, so $g \in U(p) \times U(q)$. Thus (ii) is proved.

(iii) Let $s = 0$. Since $\pi_0$: $E_{p0} \to B_0$ is holomorphic, (4.9) is right as far as the noncompact roots are concerned. For $V \in B_0$, $\pi_0^{-1}(V)$ is $G(V, p)$, the set of $p$-dimensional subspaces of $V$. For $V_0$ corresponding to ${}^tJ$, $G(V_0, p) \cong U(n)/U(p) \times U(q)$, and the holomorphic structure is given by the root spaces corresponding to $\Delta_{oc}^+$. Thus (4.9) is proved for $s = 0$.

Now take $s = q$. Since $\pi_q: E_{pq} \to B_q$ is holomorphic, $\Delta_{pq}$ must contain $\Delta^+_{oc} \cup \Delta^+_{np} \cup \Delta^-_{nq}$. As before, since the fiber is holomorphically embedded, and is biholomorphic to $\{W \subset V;\ \dim W = p,\ H_V|W \gg 0\} = U(p, q)/U(p) \times U(q)$ with holomorphic structure corresponding to $\Delta^+_{nc}$, we have $\Delta_{pc} \supset \Delta^+_{nc}$. Theorem 4.8 is proved.

Because of (ii) we may think of $E_{p0}$ and $E_{pq}$ as the same $\mathrm{Sp}(n, R)$-homogeneous space with two different complex structures. Because of (4.9) these complex structures coincide along with fibers of both $\pi_q$ and $\pi_0$. Thus $\pi_q$ and $\pi_0$ are holomorphic along the fibers of $\pi_0$ and $\pi_q$ respectively.

This can be seen directly as follows. Let $\tau: E_{p0} \to E_{pq}$ be the $\mathrm{Sp}(n, R)$-covariant correspondence given by the identifications (ii). $\tau$ can be described this way: for $(V, W) \in E_{p0}$, $\tau(V, W) = (W \oplus \overline{W}^\perp, W)$. Now, the fiber of $\pi_0$ through $(V, W)$ is $G(V, p)$. The map $W \to W^\perp$ of $G(V, p) \to G(V, q)$ is antiholomorphic. The map $G(V, q) \to G(2n, q)$: $X \to \overline{X}$ is also antiholomorphic. Thus the correspondence $\tau$ is the sum of a holomorphic map and the composition of two antiholomorphic maps, so is holomorphic.

We now have the following picture:

$$\begin{array}{ccc} E_{pq} & \xrightarrow{\tau} & E_{p0} \\ \downarrow \pi_q & & \downarrow \pi_0 \\ B_q & & B_0 \end{array}$$

The fibers of $\pi_0$ are biholomorphic to $G(n, p) \cong U(n)/U(p) \times U(q)$, and the fibers of $\pi_q$ are biholomorphic to the bounded symmetric domain $U(p, q)/U(p) \times U(q)$. $\tau$ is not holomorphic, but is holomorphic on the fibers of $\pi_q$ and $\pi_0$. For $p \in B_0$, $\pi_q|\tau\pi_0^{-1}(p)$ is proper and holomorphic (in fact, biholomorphic) and carries $\tau\pi_0^{-1}(p)$ onto a linear subvariety of $B_q$ of dimension $pq$. In this way $B_0$ parametrizes a collection of subvarieties of $B_q$ which are positively embedded, as we shall now show.

4.10. Proposition. *The varieties $\pi_q\tau\pi_0^{-1}(p)$ for $p \in B_0$ have positive normal bundle in $B_q$.*

Proof. Because of the homogeneity, we need only check this for one variety, the most convenient being $K = \pi_q\tau\pi_0^{-1}({}^tJ)$. The embedding of $K$ in $B_q$ is given by the commutative diagram

$$\begin{array}{ccc} U(n) & \to & \mathrm{Sp}(n, R) \\ \downarrow & & \downarrow \\ K = U(n)/U(p) \times U(q) & \to & \mathrm{Sp}(n, R)/U(p, q) = B_q \end{array} \tag{4.11}$$

(since $U(n) \cap U(p, q) = U(p) \times U(q)$). At the point ${}^tJ_q \in B_q$ corresponding to the identity we have already computed at the end of §2, the holomorphic tangent space $\mathfrak{h}^+ = \mathfrak{U}_0 \oplus \nu_p + \nu_q$ where

$$\mathfrak{U}_0 = \text{span of } \{Z_{jk}; j \leqslant p < k\},$$

$$\nu_p = \text{span of } \{W_{jk}; j \leqslant k \leqslant p\}, \qquad \nu_q = \text{span of } \{\overline{W}_{jk}; p < j \leqslant k\}.$$

Now all the vectors in $\mathfrak{U}_0$ are in $\mathfrak{U}(n)^C$, so are tangent to $K$; in fact they span the holomorphic tangent space to $K$. Thus the normal bundle sequence of the embedding (4.5) is given by the vector bundle sequence associated to

$$0 \to \mathfrak{U}_0 \to \mathfrak{h}^+ \to \nu_p \oplus \nu_q \to 0$$

as $U(p) \times U(q)$-modules. That is, the normal bundle to $K$ is the bundle on $U(n)/U(p) \times U(q)$ associated to the adjoint representation (in $\mathrm{Sp}(n, R)$) of $U(p) \times U(q)$ on $\nu_p \oplus \nu_q$. We identify that representation by looking at it infinitesimally: the adjoint action of $\mathfrak{U}(p) \oplus \mathfrak{U}(q)$ or $\nu_p \oplus \nu_q$ where

$$\mathfrak{U}(p) = \text{span of } \left\{H_j, j \leqslant p;\ Z_{jk}, \bar{Z}_{jk}, j < k \leqslant p\right\},$$

$$\mathfrak{U}(q) = \text{span of } \left\{H_j, j > p;\ Z_{jk}, \bar{Z}_{jk}, p < j < k\right\}.$$

Using the root conjugation of Theorem 3.1, we see that for $r = p$ or $q$, $\mathfrak{U}(r)$ acts trivially on $\nu_{n-r}$ so our representation is a direct sum $\rho_p \oplus \rho_q$, where $\rho_r$ is the adjoint action of $\mathfrak{U}(r)$ on $\nu_r$.

Now, by (2.14) each $W_{jk} \in \nu_p$ spans an eigenspace of $\rho_p|t$ with weight $\mu^{ij} = i(h^i + h^j)$. The representation $\rho^{(2)}$ of $U(p)$ on the space of $S_p^{(2)}$ of symmetric $p \times p$ matrices given by $\rho^{(2)}(g)(S) = {}^t gSg$ has the differential $d\rho^{(2)}(X)(S) = {}^t XS + SX$, and a direct computation shows that $\rho^{(2)}$ has the same decomposition by weights as $\rho_p$. Thus $\rho_p$ and $\rho^{(2)}$ are unitarily equivalent. We conclude that $\nu_p$ is the bundle on $U(n)/U(p) \times U(q) = G(n, p)$ associated to the representation of $U(p)$ on the space $S^2(C^p)$ of homogeneous functions of degree 2. Thus $\nu_p$ is $S^2(T_p^*)$, where $T_p$ is the tautological bundle of $p$-planes on $G(n, p)$. Similarly $\nu_q$ is $S^2(T_q^*)$, where $T_q$ is the tautological bundle of $q$-planes on $G(n, q)$ $(= G(n, p))$. Since $T_p^*$, $T_q^*$ are positive, so is $\nu_p \oplus \nu_q$.

Now $H^0(K, \nu_p)$ includes $S^2(\mathbf{C}^n)$, the space of quadratic homogeneous polynomials on $\mathbf{C}^n$; since the representation of $U(n)$ on $H^0(K, \nu_p)$ is irreducible, these spaces coincide. Thus $\dim H^0(K, \nu_p) = \frac{1}{2}n(n+1) = \dim H^0(K, \nu_q)$. Since higher cohomology of these bundles vanishes (again by the Borel-Weil theorem), the theorem of Kodaira [8] applies: $H^0(K, \nu_p \oplus \nu_q)$ is the tangent space at $K$ to the manifold $M$ of all deformations of the embedding $K \to B_q$. Note that $M$ is of dimension $n(n+1)$, which is exactly twice the dimension of $B_0$, which parametrizes some deformations of $B_q$. In fact, it is easily checked that a direction $(V_p, V_q) \in H^0(K, \nu_p \oplus \nu_q)$ corresponds to a tangent to $B_0$ if and only if $V_q = \bar{V}_p$; then $B_0$ is a totally real submanifold of $M$.

Finally, we compute that the exhaustion function of Theorem 4.1 is actually semidefinite.

4.12. PROPOSITION. *On $B_q$, $-\ln|\phi|$ has a positive semidefinite Hessian with pq zero eigenvalues, and all the rest positive.*

PROOF. Let $\mathbf{C}^{2n} = V_{t_J} \oplus \bar{V}_{t_J}$ and choose the metric $E$ of Theorem 4.1 so that the standard basis of $\mathbf{C}^{2n}$ is orthonormal. As in Proposition 4.10, let $K$ be the $U(n)$-orbit of $\beta_0 = {}^t J_q$; also $K = \pi_q \tau \pi_0^{-1}({}^t J)$. We shall compute the Hessian of $-\ln|\phi|$ at $\beta_0$.

Let $V_{t_J} = V_1 \oplus V_2$, where $V_1$ is the span of the first basis vectors and $V_2$ is the span of the last $q$ basis vectors. Then $V_{iJ_q} = V_1 \oplus \overline{V}_2$, and in terms of the given basis

$$H_{\beta_0} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}, \qquad E|_{V_{\beta_0}} = I. \tag{4.13}$$

Let $g \in U(n)$, $\beta = g^t J_q$. Then $g$ takes $V_j$ into a space $V_j'$ and the standard basis into bases for the vector spaces $V_j'$, $\overline{V}_j'$. Since $g$ is unitary, these transformed bases remain orthonormal for $E$, and by the prior description of the correspondence $\tau$ we have $H_\beta$ also given by (4.13) in terms of the transformed basis. Then $\det H_\beta = \pm 1$, so $\phi(\beta) = \phi(\beta_0) = 1$. Since $\phi$ is constant along $K$, its Hessian vanishes on its tangent space $\mathfrak{U}_0$ at $\beta_0$.

Now, in §2 we computed the curvature $\Omega$ of $m_1$ at the tangent space to $B_q$ at $\beta_0$. We note that the basis of root vectors diagonalizes $\Omega$. The discussion of **[5, §4]** applies to $\mathfrak{L}$ as a Kähler $C$-space, so that $(4.4)_X$ **[5**, p. 269] holds for the curvature $\Omega_2$ of the bundle $\Lambda^n T \to \mathfrak{L}$. Here the $e_\alpha$ are the same root vectors, $\omega^\alpha$ then duals, and $\pi$ is the representation of $\mathrm{Sp}(n, \mathbf{C})$ defining $\Lambda^n T$. Since $\pi$ is a character it vanishes except on the Cartan algebra, so in $(4.4)_X$, the coefficient of $\omega^\alpha \wedge \overline{\omega}^\beta$ vanishes unless $\alpha = \beta$ (see e.g. $(4.24)_X$). Then the root vectors diagonalize *both* $\Omega$ *and* $\Omega_2$, so also diagonalize $\Omega - \Omega_2$ which is the Hessian of $-\ln|\phi|$. Thus $\nu_p \oplus \nu_q$ and $\mathfrak{U}_0$ are both sums of eigenspaces of the Hessian of $-\ln|\phi|$, which is positive definite on the first space, and identically zero on the second. Thus it is semidefinite.

**5. Relations with complex tori.** Fix a hermitian metric $E$ on $\mathbf{C}^n$. For $\beta \in B$, let $H_\beta$ represent the hermitian form on $\mathbf{C}^n$ determined by $\beta$: $H_\beta = J\beta + iJ$ on $(R^{2n}, \beta)$. We consider $\mathbf{C}^n$ as a hermitian manifold with hermitian metric

$$H_\beta = e^{-H_\beta(z,\bar{z})} E(dz, d\bar{z}).$$

We consider the square-integrable Dolbeault cohomology groups on $\mathbf{C}^n$. Let $\Omega^r$ be the collection of smooth global $r$-forms on $\mathbf{C}^n$, and $\bar{\partial}$: $\Omega^r \to \Omega^{r+1}$ the Dolbeault resolution. Let $\mathfrak{L}^r$ be the $L^2$-closure of $\overline{\Omega}^r$ and consider $\bar{\partial}$: $\mathfrak{L}^r \to \mathfrak{L}^{r+1}$ as a densely defined closed operator. Let $\delta_\beta$: $\mathfrak{L}^{r+1} \to \mathfrak{L}^r$ be the adjoint of $\bar{\partial}$. Then the cohomology group is defined by

$$H_\beta^r = (\ker \bar{\partial} \colon \mathfrak{L}^r \to \mathfrak{L}^{r+1})/\text{closure } \bar{\partial}\mathfrak{L}^{r-1}).$$

Since any two metrics on $\mathbf{C}^n$ are comparable, $H_\beta^r$ is independent of the chosen metric $E$. The following lemma is easy to prove as in Hörmander **[7]**, since we are on Euclidean space.

5.1. Lemma. (i) $\bar{\partial}\mathfrak{L}^{r-1}$ *is closed in* $\mathfrak{L}^r$.
(ii) $H_\beta^r \cong \{\varphi \in \mathfrak{L}^r;\ \bar{\partial}\varphi = 0,\ \delta_\beta\varphi = 0\}$.

(i) implies (ii) as a standard Hilbert space fact. Now, in Rossi-Vergne **[12**, §3], the right-hand side of (ii) is explicitly identified.

5.2. THEOREM. *Let $\beta \in B_q$. There is a complex analytic change of coordinates $z \to u$ so that*

$$E = \sum du_i \otimes d\bar{u}_i, \qquad H_\beta = \sum_{i<p} \mu_i^2\, du_i \otimes d\bar{u}_i - \sum_{i>p} \mu_i^2\, du_i \otimes d\bar{u}_i,$$

*with $\mu_i > 0$, $1 \leqslant i \leqslant n$.*

(i) *$H_\beta^r = 0$ if $r \neq q$.*

(ii) *For $v_1 = \mu_1 u_1, \ldots, v_p = \mu_p u_p$, $v_{p+1} = \mu_{p+1}\bar{u}_{p+1}, \ldots, v_n = \mu_n \bar{u}_n$, we have*

$$\{\varphi;\ \bar{\partial}\varphi = 0,\ \delta_\beta \varphi = 0\} = \Bigg\{ f(v)\exp - \Big( - \sum_{i=p+1}^{n} |v_i|^2 \Big) d\bar{u}_{p+1} \wedge \cdots \wedge d\bar{u}_n;$$

*$f$ is holomorphic and*

$$\|\varphi\|^2 = \frac{1}{(\mu_1 \cdots \mu_n)^2} \int |f|^2 \exp\Big( - \sum_{i=1}^{n} |v_i|^2 \Big)\, dV < \infty \Bigg\}.$$

The coordinate change just described is any set of orthonormal coordinates diagonalizing $H$ relative to $E$. As $\beta$ varies in $B$ we can (locally) make this coordinate change vary holomorphically in $\beta$. Thus

5.3. THEOREM. *$\mathcal{H}^q = \bigcup_{\beta \in B_q} H_\beta^q \to B_q$ can be made into a holomorphic fiber with Hilbert space fibers.* Sp$(n, R)$ *acts on $\mathcal{H}^q$, but not unitarily.*

Actually, for $q = 0$ or $n$, we may redefine the Hilbert space structure by taking $E = H_\beta$ so that Sp$(n, R)$ does act unitarily.

We now make use of our geometric picture

$$\begin{array}{ccc} E_{pq} & \overset{\tau}{\leftrightarrow} & E_{p0} \\ \downarrow\pi_q & & \downarrow\pi_0 \\ B_q & & B_0 \end{array}$$

to better understand the bundle $\mathcal{H}^q$. Let $p \in \pi_q^{-1}(\beta)$, and let $\beta' = \pi_0\tau(p)$. We consider Theorem 5.2 where $E$ is taken as the metric $H_{\beta'}$. Then if $p = (V, W)$, we have realized $\mathcal{H}_\beta^q$ as the spaces of square-integrable functions holomorphic on $W$, and conjugate holomorphic on $W^\perp$, i.e., precisely as $\mathcal{H}_{\beta'}^0$. We may state this as

5.4. THEOREM. (i) *We may change the norm on the fibers of $\pi_q^*\mathcal{H}^q$ so that* Sp$(n, R)$ *acts unitarily and holomorphically on this fiber bundle over $E_{pq}$.*

(ii) *$\tau$ induces a natural* Sp$(n, R)$*-covariant* (*nonholomorphic*) *isomorphism $\Phi_q$: $\pi_q^*\mathcal{H}^q \cong \pi_0^*\mathcal{H}^0$.*

PROOF. For $p = (V, W) \in E_{pq}$, take the metric $E$ to be $H_{\pi_0\tau(p)}$. Then in Theorem 5.2, the $\mu_i = 1$, $1 \leqslant i \leqslant n$ for $u_1, \ldots, u_q$ any orthonormal basis of $W$, $u_{q+1}, \ldots, u_n$ any orthonormal basis of $W^\perp$. $\Phi_q$ is just the identification expressed in Theorem 5.2.

In particular, restricting to the fibers of $\pi_q$, $\Phi_q$ is an isomorphism on those fibers and $\pi_q^*(\mathcal{H}^q)$ is a trivial bundle. For $\beta \in B_q$, let $M(\beta) = \pi_0\tau\pi_q^{-1}(\beta)$. Then

$\mathcal{H}^0|M(\beta) \to M(\beta)$ is a holomorphic Hilbert space bundle on the complex manifold $M(\beta)$ (biholomorphic to the hermitian symmetric space $U(p, q)/U(p) \times U(q)$), and $\Phi_q$ induces isomorphisms of $\mathcal{H}^q_\beta$ with the fibers of $\mathcal{H}^0|M(\beta)$ varying smoothly along $M(\beta)$. This is, in the context of complex tori, exactly the situation discovered by Matsushima in **[10]**. Now we briefly indicate the relation of our work with that of Matsushima. Our exposition relies heavily on that of Tolimieri **[15]**.

Let $N$ be the $(2n + 1)$-dimensional Heisenberg group: topologically $N = R^{2n} \times R$ with the group law

$$(\xi, s)(\eta, t) = \left(\xi + \eta, s + t + \tfrac{1}{2}{}^t\eta J \xi\right).$$

As Tolimieri has shown, $B$ can be identified as the set of automorphisms $\beta$ of $N$, leaving the center $Z$ pointwise fixed, such that $\beta^2 = -I$. These he called CR-structures on $N$, which they are in the following sense. Let $z = (z_1, \ldots, z_n)$ be complex coordinates on $(R^{2n}, \beta)$. $N$ acts as a group of holomorphic transformations on $\mathbf{C}^n \times \mathbf{C}$ as follows:

$$(z, w) \xrightarrow{(\xi, s)} \left(z + \xi, w + s + \tfrac{1}{2}H_\beta(z, \xi) + H_\beta(\xi, \xi)\right). \tag{5.5}$$

$N$ is diffeomorphic to the orbit

$$\Sigma_\beta\colon \left\{(z, w);\ \operatorname{Im} w = \tfrac{1}{2}H_\beta(z, z)\right\}$$

which is a real hypersurface in $\mathbf{C}^{n+1}$, and as such inherits an $N$-invariant CR-structure. This CR-structure is determined by a Lie subalgebra $\mathfrak{h}_\beta$ of $\mathfrak{n}^{\mathbf{C}}$ such that $\mathfrak{h}_\beta \cap \bar{\mathfrak{h}}_\beta = \{0\}$, and here $\mathfrak{h}_\beta$ is given by the $-i$ eigenspace of the automorphism $\beta$.

Now let $N^0$ be the Heisenberg group with compact center: $N^0 = N/\{(0, n);\ n$ an integer$\}$. Replacing $N$ by $N^0$ and $w$ by $e^{2\pi i w}$, we realize $N^0$ as the hypersurface

$$\Sigma^0_\beta = \left\{(z, w);\ |w|^2 = e^{-H_\beta(z,z)}\right\}. \tag{5.6}$$

We shall consider (5.6) as the unit circle bundle of a hermitian line bundle $L^0_\beta \to \mathbf{C}^n$. We can describe this in terms of the orbit theory (cf. Tolimieri).

5.7. THEOREM. *Let $\mathcal{O}_1$ be the coadjoint orbit of* $(0, 1) \in n^*$. *Then for $\beta \in B$, $h_\beta$ defines a complex polarization at* $(0, 1)$, *realizing $\mathcal{O}_1$ as $\mathbf{C}^n$. $L^0_\beta \to \mathbf{C}^n$ is the line bundle of Kostant providing the quantization. $B$ parametrizes all complex polarizations at* $(0, 1)$.

Now let $\Lambda$ be the lattice of integral points in $R^{2n}$. We can identify $\Lambda$ with a closed subgroup $\Lambda^0$ of $N^0$: $\Lambda^0 = \{(n, 0);\ n \in \Lambda\}$. Then $N^0/\Lambda^0 \to R^{2n}/\Lambda$ is a circle bundle over a torus. Each $\beta \in B$ realizes $N^0$ as a circle bundle in a complex line bundle over $C^n$; this structure is $\Lambda^0$-invariant, so descends to $\Lambda^0/\Lambda^0 \to R^{2n}/\Lambda$. Let $T_\beta$ represent the complex torus $(R^{2n}/\Lambda, \beta)$, and $L_\beta \to T_\beta$ the hermitian line bundle defined by descending (5.6) mod $\Lambda^0$. Let $\Sigma_\beta$ represent the unit circle bundle in $L_\beta$. We shall call the pair $(L_\beta, T_\beta)$ a *polarized complex torus* (if $\beta \in B_0$, this is precisely a polarized abelian variety, and $B_0$ parametrizes the space of polarized abelian varieties).

5.8. THEOREM. *Let $\beta \in B_0$. Let $H^2(\Sigma^0_\beta)$ be the space of square-integrable CR-functions on $\Sigma^0_\beta$, and $H^2(\Sigma_\beta)$ the same on $\Sigma_\beta$. Then*

$$\Psi(f)(z, w) = \sum_{n \in \Lambda^0} f(T_{(n,0)}(z, w)) \tag{5.9}$$

*defines a continuous map of $H^2(\Sigma^0_\beta)$ onto $H^2(\Sigma_\beta)$. Any $f \in H^2(\Sigma^0_\beta)$ has a Taylor series expansion*

$$f(z, w) = \sum f^{(n)}(z) w^n,$$

*where $f^{(n)} \in H^0_{n\beta}$ (recall Theorem 5.2). The map (5.9) induces continuous surjective maps*

$$H^0_{n\beta} \to H^0(T_\beta, L_\beta^{-n}).$$

Finally, we state the result of Matsushima. For $\beta \in B_q$, we have $H^r(T_\beta, L_\beta^{-1}) = 0$, $r \neq q$ and $H^0(T_\beta, L_\beta^{-1}) \neq 0$ (this is originally a theorem of Mumford). Matsushima gives an analytic proof along the lines of **[12]** which takes place on the torus $T_\beta$. For each $p \in \pi_q^{-1}(\beta)$, there is an isomorphism $\phi_p$: $H^q(T_\beta, L_\beta^{-1})$ with $H^0(T_{\pi_0\tau(p)}, L^{-1}_{\pi_0\tau(p)})$ varying smoothly with $p$. Thus, we associate to $\beta \in B_q$ the vector bundle

$$\bigcup_{\beta' \in M(\beta)} H^0(T_{\beta'}, L_{\beta'}^{-1}) \to M(\beta).$$

Combining our work with Matsushima's we may generalize Theorem 5.8 to all $B_q$:

5.10. THEOREM. *Let $\beta \in B_q$. Let $H^q(\Sigma^0_\beta)$ be the space of square-integrable CR-cohomology in degree $q$ on $\Sigma^0_\beta$, and $H^2(\Sigma_\beta)$ the same on $\Sigma_\beta$. Then*

$$\Psi(\varphi) = \sum_{n \in \Lambda^0} \Psi(T_n) \tag{5.11}$$

*defines a continuous map of $H^2(\Sigma^0_\beta)$ onto $H^2(\Sigma_\beta)$. If we expand $\varphi$ in a Taylor series along the fiber coordinate this specializes to continuous surjections $\mathcal{H}^q_{n\beta} \to H^q(T_\beta, L_\beta^{-n})$.*

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