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PLANE MODELS FOR RIEMANN SURFACES ADMITTING CERTAIN HALF-CANONICAL LINEAR SERIES. II

BY

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ABSTRACT. For $r > 2$, closed Riemann surfaces of genus $3r + 2$ admitting two simple half-canonical linear series g^r_{3r+1}, h^r_{3r+1} are characterized by the existence of certain plane models of degree $2r + 3$ where the linear series are apparent. The plane curves have $r - 2$ 3-fold singularities, one $(2r - 1)$-fold singularity Q, and two other double points (typically tacnodes) whose tangents pass through Q. The lines through Q cut out a g^1_4 which is unique. The case where the g^1_4 is the set of orbits of a noncyclic group of automorphisms of order four is characterized by the existence of $3r + 3$ pairs of half-canonical linear series of dimension $r - 1$, where the sum of the two linear series in any pair is linearly equivalent to $g^r_{3r+1} + h^r_{3r+1}$.

1. Introduction. By Torelli's theorem the conformal type of a closed Riemann surface is determined by the symplectic equivalence class of its period matrices. Consequently, special properties that a Riemann surface might possess should be reflected in its period matrix. An attractive approach to this problem is to examine the properties of the theta function especially when evaluated at points of finite order on the Jacobian since quite often these properties are independent of the particular period matrix at hand. Thus Riemann characterized hyperelliptic Riemann surfaces of genus three by the theta function vanishing to order two at one half-period [**13**], and Weber analogously characterized hyperelliptic Riemann surfaces of genus four [**15**]. Martens completed the program of characterizing hyperelliptic Riemann surfaces in terms of vanishing properties of theta functions at half-periods [**12**]. It seems clear that further hyperelliptic automorphism groups can be characterized by the vanishing of certain homogeneous polynomial expressions in theta-nulls, although the author does not know if this program has been carried out.

An extension of this work was begun by Farkas when he discovered extraordinary vanishing properties at half-periods for the theta functions for Riemann surfaces admitting fixed-point-free automorphisms of order two [**10**]. Following this work of Martens and Farkas the present author was able to characterize Riemann surfaces admitting other automorphisms of period two (involutions) in terms of vanishing properties of the theta function [**4**].

In another direction, it has long been known that a Riemann surface of genus six

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whose theta function vanishes at a single half-period to order three admits a plane model as a nonsingular quintic. In Part I of this paper the author extended this type of result to Riemann surfaces of genus $3r$ where the theta function vanishes to order $r + 1$ at one half-period [**6**]. The present work is an extension of this latter kind of result.

Kraus seems to be the first author to investigate plane curves where the corresponding theta function vanishes at half-periods [**11**]. Here he used Riemann's solution to the Jacobi inversion problem which asserts, among many other things, a one-to-one correspondence between the vanishing of the theta function at half-periods to order $r + 1$ and the existence of complete half-canonical linear series of dimensions r, g_{p-1}^r's. Kraus constructed plane curves with fairly obvious half-canonical linear series. The present author continued this work (no intermediate work is known) and constructed plane curves of genus $3r$, $3r + 2$, and $3r + 3$ admitting respectively one, two and four simple half-canonical linear series of dimension r. (In the last case the sum of the four linear series is bicanonical [**2**].)

These two kinds of investigations concerning theta functions, characterizing automorphism groups and certain plane models, are not unrelated. For if a Riemann surface has genus less than $3r$ and the theta-function vanishes at a half-period to order $r + 1$ then this implies the existence of an involution. Similarly, surfaces of genus less than $3r + 2$ or $3r + 3$ admitting respectively two or four half-canonical series of dimension r also admit involutions [**4**].

To the author's surprise and pleasure the indicated vanishing properties for the theta function for Riemann surfaces of genus precisely $3r$, $3r + 2$, and $3r + 3$ usually characterize the existence of the plane models discussed in [**2**]. This allows one to compute the dimension in Teichmüller space of surfaces admitting such vanishing properties since by classical methods it is easy to compute the dimensions of these families of plane curves.

Basic to these investigations is the classical work of Castelnuovo on the inequality that bears his name [**7**], [**8**]. In almost all these cases the simple half-canonical g_{p-1}^r determines a curve in P^r that lies on a rational normal scroll whose rulings cut out a g_4^1 on the curve. Thus we almost always have a unique g_4^1 which imposes two linear conditions on g_{p-1}^r. From this the existence of the various plane models follows easily.

As a bonus it turns out that one can often characterize the case when this g_4^1 is itself the orbit space of a noncyclic group of automorphisms of order four. This characterization is again in terms of the vanishing properties of the theta function at half-periods. So the investigation ends where it began with characterizing certain automorphism groups in terms of vanishing properties of the theta function.

In Part I of this paper we considered Riemann surfaces of genus $3r$ admitting a simple half-canonical linear series of dimension r. In Part II we now consider Riemann surfaces of genus $3r + 2$ admitting two simple half-canonical linear series of dimension r. In Part III we hope to consider the case of genus $3r + 3$ where the Riemann surface admits four simple half-canonical linear series of dimension r whose sum is bicanonical.

In §2 of this paper we will summarize the necessary preliminary material. In §3 we will consider the general case $r \geqslant 4$, and we leave the cases $r = 1, 2$, and 3 to §4 where more special techniques appear to be necessary. In §5 we consider when the g_4^1 is the set of orbits for a noncyclic group of automorphisms of order four. Finally, we have included an appendix which completes the discussion of automorphisms in Part I.

All results will be stated in terms of half-canonical linear series. We will omit the translation via Riemann's vanishing theorem to statements about the vanishing of the theta function of half-periods. An example of this type of translation will be found in [3].

2. Notation, definitions, and preliminary results. Let W_p be a closed Riemann surface of genus p. A divisor $D = n_1z_1 + n_2z_2 + \cdots + n_sz_s$ is a zero chain on W_p where the n_i are integers and the z_i are points on W_p. A divisor will be called *integral* ($D \geqslant 0$) if each $n_i \geqslant 0$. If f is a meromorphic function on W_p then its divisor, (f), can be written: $(f) = D_1 - D_2$; where D_1 is the divisor of zeros of f and D_2 is the divisor of poles. We say two divisors D_1 and D_2 (not necessarily integral) are (linearly) equivalent if there is a (meromorphic) function f so that $(f) = D_1 - D_2$. If D is a divisor we say that a function f is a *multiple* of D if $(f) + D \geqslant 0$. The set of multiples of D is a finite dimensional vector space. A linear series of degree n and dimension r, written g_n^r, is a set of integral divisors $\{(f) + D\}$ where D is an integral divisor of degree n and f ranges over an $r + 1$ dimensional subspace of the multiples of D. Such a linear series will be called complete if the multiples of D have dimension $r + 1$. In such a case we write $|D| = g_n^r$; that is, $g_n^r = \{D' \geqslant 0\colon D' \equiv D\}$. For example, the canonical series g_{2p-2}^{p-1} is equal to $|K|$ where K is the set of zeros of an abelian differential of the first kind.

The Riemann-Roch theorem asserts that for a complete g_n^r, $r = n - p + i$ where i is the index of specialty[2] of any divisor in g_n^r. An integral divisor will be called *special* if it is part of an integral canonical divisor. If $g_n^r + g_{n'}^{r'}$ is the canonical series then the Brill-Noether form of the Riemann-Roch theorem asserts that $n - 2r = n' - 2r'$. Clifford's theorem asserts that in this case $n - 2r$ is nonnegative and is zero if and only if $n = 0$, $n = 2p - 2$, or W_p is hyperelliptic.

For two integral divisors D_1 and D_2, (D_1, D_2) will denote their greatest common divisor. We will say that D_1 and D_2 are *disjoint* if $(D_1, D_2) = 0$.

In this paper we will use the following terms. A *pair* will be an integral divisor of degree two and a *triple* will be an integral divisor of degree three. The points of a pair or triple need not be distinct. Finally, we will often write "$D \equiv g_n^r$" for "$|D| = g_n^r$" where there is no danger of confusion.

Linear series may be complete or incomplete, with or without fixed points, and simple or composite. If g_n^r is a linear series on W_p then $g_n^r - x$ will denote the linear

[2]The index of specialty of a divisor D is the dimension of the space of holomorphic differentials whose divisors contain D.

series of degree $n - 1$ of divisors of g_n^r passing through x not counting x. If x is not a fixed point of g_n^r then $g_n^r - x = g_{n-1}^{r-1}$.

A linear series g_n^r ($r \geqslant 2$) is simple if, for a general choice of x on W_p, $g_n^r - x$ has the same fixed points as g_n^r. In this situation for $r \geqslant 3$ and a general choice of x, $g_n^r - x$ is also simple. If g_n^r is simple and without fixed points then W_p can be realized as a curve C_n in P^r, and the hyperplane sections of the curve cut out the g_n^r. By a *k-point* of g_n^r we mean an integral divisor of degree k, $x_1 + \cdots + x_k$, so that whenever a divisor of g_n^r contains one of the x's, it contains all k of them, but not necessarily any further point of W_p. On the corresponding projective curve C_n this corresponds to a singularity of multiplicity k, which we will also call a *k-point of* C_n.

A linear series g_n^r is composite if, for every choice of x on W_p, $g_n^r - x$ has more fixed points than g_n^r. In this case W_p is a t-sheeted covering of a surface W_q, and a divisor of nonfixed points of g_n^r is a union of fibers of the map ϕ: $W_p \to W_q$. In such a case W_q admits a $g_{(n-f)/t}^r$ which lifts via ϕ to g_{n-f}^r, the nonfixed points of g_n^r. If x is not a fixed point of g_n^r then $g_n^r - x$ has $t - 1$ additional fixed points, the other points in the fiber of ϕ containing x. If g_n^r is complete on W_p, so is $g_{(n-f)/t}^r$ on W_q.

If g_n^r is a linear series an integral divisor $E = x_1 + \cdots + x_m$ is said to impose t (linear) conditions on g_n^r if there are t points of E, say $x_1 + \cdots + x_t$ (not necessarily distinct) so that $g_n^r - (x_1 + \cdots + x_t)$ has dimension $r - t$ and has $x_{t+1} + \cdots + x_m$ among its fixed points. It follows that if the first t x's are distinct points of W_p then for each k from 1 to t there is a divisor of g_n^r containing all the x_i, $i = 1, 2, \ldots, t$, except x_k.

If g_n^r is a linear series and g_m^s is another we say that g_m^s imposes t (linear) conditions on g_n^r if one (and therefore every) divisor in g_m^s imposes t conditions on g_n^r. If g_n^r is complete this means that there is a complete g_{n-m}^{r-t} so that $g_n^r \equiv g_m^s + g_{n-m}^{r-t}$. In particular, if a g_m^1 imposes one condition on g_n^r then g_n^r is composite and $g_n^r \equiv rg_m^1 + D_{n-rm}$ where D_{n-rm} is the divisor of fixed points of g_n^r.

We will use the following corollary of the Riemann-Roch theorem. A g_n^1 ($n \leqslant p$) without fixed points imposes $n - 1$ conditions on the canonical series and so imposes at most $n - 1$ conditions on any special linear series. A simple special g_m^s without fixed points imposes at most $m - s$ conditions on any other special linear series whose dimension is at least $m - s$.

If W_p admits a simple g_n^2 without fixed points then W_p admits a plane model of degree n, C_n. If d is the number of double points suitably counted then

$$p = (n - 1)(n - 2)/2 - d. \tag{2.1}$$

To compute the dimension R of all plane curves of degree n with s fixed ordinary singularities of multiplicities $k_1, k_2, \ldots, k_s$ we use the formula

$$R = n(n + 3)/2 - \sum_{k=1}^{s} k_j(k_j + 1)/2 + \varepsilon \tag{2.2}$$

where $\varepsilon \geqslant 0$. If we allow the singularities to vary in P^2 then the variety of such curves has dimension at least $R + 2s$. If one wishes to consider the Riemann surfaces corresponding to such a family then in computing the dimension of such Riemann surfaces in Teichmüller space one must subtract, at least, eight for the plane collineations.

For plane curves we shall have occasion to consider standard quadratic transformations with three fundamental points. If these points are (1, 0, 0), (0, 1, 0) and (0, 0, 1) then $x_i \to x_j x_k$, $\{i, j, k\} = \{1, 2, 3\}$ gives the transformation [**14**, p. 74].

If W_p admits a simple g_n^2 then formula (2.1) assures us that

$$p \leqslant (n-1)(n-2)/2.$$

The following inequality, due to Castelnuovo, generalizes this result.

2.1. THEOREM ([7], [**8**], [**9**]). *Let W_p admit a simple g_n^r, $r \geqslant 3$. Then*

$$p \leqslant (n - r + \varepsilon)(n - 1 - \varepsilon)/2(r - 1) \tag{2.3}$$

where $0 \leqslant \varepsilon \leqslant r - 1$ and $n - r + \varepsilon \equiv 0 \pmod{r-1}$. If we have equality in formula (2.3), $n > 2r$, and $r \neq 5$, then C_n, the corresponding curve in P^r, lies on a rational normal scroll whose rulings cut out a g_m^1 on W_p which imposes two conditions on g_n^r.

The case of most interest in this paper is when W_p admits a simple g_{3r-1}^r and $p = 3r$. In that case $2g_{3r-1}^r$ is canonical, we have equality in formula (2.3), and W_{3r} admits a g_4^1 which imposes two conditions on g_{3r-1}^r [**6**, Lemma 3.1].

As a consequence of the methods used by Castelnuovo the following results can be derived [**5**, Lemma 4.2]. They give lower bounds on the dimensions of the sums of certain linear series.

2.2. LEMMA. *Suppose $p \neq 0$.*

(i) *Suppose g_n^r and g_m^s are different simple linear series without fixed points, $n \geqslant r + s$, and $r \geqslant s$. Then $g_n^r + g_m^s = g_{n+m}^{r+2s+\varepsilon}$ where $\varepsilon \geqslant 0$.*

(ii) *Suppose g_n^r is simple. Then $g_n^r + tg_m^1 = g_{n+tm}^{r+2t+\varepsilon}$ where $\varepsilon \geqslant 0$.*

(iii) *Suppose for g_n^1 and g_m^1 the two coverings of P^1 determined by these linear series do not admit a common factoring. Suppose $n \geqslant t + 1$. Then $sg_n^1 + tg_m^1 = g_{sn+tm}^{t+ts+s+\varepsilon}$ where $\varepsilon \geqslant 0$.*

On W_p a linear series g_{p-1}^r is said to be *half-canonical* if the sum of any two divisors in g_{p-1}^r is canonical; that is, $2g_{p-1}^r \equiv g_{2p-2}^{p-1}$. The following results are basic for half-canonical linear series.

2.3. LEMMA [**6**, LEMMA 2.4]. *A g_m^1 without fixed points imposes at most $[m/2]$ conditions on a half-canonical g_{p-1}^r $(m < 2r + 1)$.*

2.4. THEOREM [**6**, THEOREM 2.5]. *Let W_p admit a simple g_m^s without fixed points and a half-canonical g_{p-1}^r where $m - s \leqslant 2r$. Then g_m^s imposes at most $[(m - s + 1)/2]$ conditions on g_{p-1}^r. Also g_{p-1}^r must be simple.*

2.5. COROLLARY. *A g_3^1 imposes one condition on a half-canonical g_{p-1}^r. A g_4^1 imposes at most two conditions on a half-canonical g_{p-1}^r.*

DEFINITION. A Riemann surface admitting a g_3^1 without fixed points will be called *trigonal*.

Before continuing with results on half-canonical linear series we must briefly discuss automorphisms. By $A(W_p)$ we shall mean the finite group of automorphisms (conformal self-maps) of W_p ($p \geqslant 2$). We shall be particularly interested in automorphisms of order two (involutions). A Riemann surface W_p will be called q-hyperelliptic if W_p admits an involution T and the genus of $W_p/\langle T \rangle$ is at most q. Thus P^1, tori, and hyperelliptic Riemann surfaces are 0-hyperelliptic. 1-hyperelliptic surfaces which are not 0-hyperelliptic are called elliptic-hyperelliptic. We have the following useful result concerning involutions and half-canonical linear series.

2.6. LEMMA [6, LEMMA 2.6]. *Suppose W_p is q-hyperelliptic and admits a simple half-canonical g^r_{p-1}. Then $r \leqslant q$.*

We will also be interested in noncyclic groups of order four. If W_p admits such a group and T_1, T_2, and T_3 are the involutions in the group G, let p_i be the genus of $W_p/\langle T_i \rangle$ for $i = 1, 2$, and 3, and let p_0 be the genus of W_p/G. Then $p + 2p_0 = p_1 + p_2 + p_3$ [1]. Such a group G on W_p will be denoted by the symbol $(p; p_1, p_2, p_3; p_0)$.

By using Lemma 2.6 we can obtain the following [6, Lemma 2.7].

2.7. LEMMA. *Let W_p admit a simple half-canonical g^r_{p-1}.*[3] *Suppose W_p admits a g^s_m and a g^1_4 without fixed points so that g^1_4 imposes at most two conditions on g^s_m. Suppose finally that $s \leqslant r$ and $m - 4s < 0$. Then g^s_m is simple.*

We will include a proof of the following lemma even though it is almost identical to that of the preceding lemma.

2.8. LEMMA. *Suppose W_p admits a simple half-canonical g^r_{p-1} and a g^1_4 without fixed points. Suppose g^1_4 imposes at most two conditions on another complete linear series g^s_m where $m - 2s < 2r$, $s \geqslant 2$. Suppose finally that g^s_m is composite. Then $g^s_m = sg^1_4 + D_{m-4s}$ where D_{m-4s} is the divisor of fixed points of g^s_m.*

PROOF. Since g^s_m is composite there is a t-sheeted covering ϕ: $W_p \to W_q$ and a complete $g^s_{(m-f)/t}$ on W_q which lifts to the nonfixed points of g^s_m. Since g^1_4 imposes at most two conditions on g^s_m we see that on W_q there is a $g^1_{4/t}$ which lifts to g^1_4 and so $t = 2$ or 4. If $t = 4$ then W_q is P^1 and g^1_1 imposes one condition on $g^s_{(m-f)/4}$. The result follows by lifting to W_p. If $t = 2$ and $g^s_{(m-f)/2}$ is special then g^1_2 imposes one condition on $g^s_{(m-f)/2}$ and the result follows. If $g^s_{(m-f)/2}$ is nonspecial then $q = (m - f)/2 - s$ and $q \geqslant r$ by Lemma 2.6. This leads to the contradiction $m - 2s \geqslant 2r$. Q.E.D.

The last lemma will be used constantly in the sequel.

2.9. LEMMA. *Suppose W_p admits a g^1_4 without fixed points so that $2g^1_4 = g^2_8$ is complete. Suppose h^0_4 is complete and $2h^0_4 \equiv 2g^1_4$. Then $h^0_4 = P + Q$ where P and Q are disjoint pairs satisfying $2P \equiv 2Q \equiv g^1_4$.*

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[3] In [6, Lemma 2.7] the assumption that $p = 3r$ is unnecessary.

PROOF. $2h_4^0$ contains points in at most only two divisors of g_4^1 and consequently, the same is true of h_4^0. But h_4^0 must contain points in at least two distinct divisors of g_4^1. Consequently, h_4^0 must contain points in precisely two distinct divisors of g_4^1. The result now follows (or [6, Lemma 2.8]). Q.E.D.

3. The cases $r \geqslant 4$. We now suppose that W_{3r+2} admits two half-canonical g_{3r+1}^r's. We will prove the existence of certain plane models for $r \geqslant 4$ in this section, but some of the preliminary lemmas will be valid for smaller r. We first show that the g_{3r+1}^r's must be simple.

3.1. LEMMA. *Suppose $r \geqslant 2$ and W_{3r+2} admits a composite half-canonical g_{3r+1}^r. Then either g_{3r+1}^r is unique or W_{3r+2} admits at least 16 half-canonical g_{3r+1}^r's. Thus if W_{3r+2} admits 2, 3, . . . , 14, or 15 half-canonical g_{3r+1}^r's they must all be simple.*

PROOF. Since g_{3r+1}^r is composite there is a t-sheeted covering ϕ: $W_p \to W_q$ and a complete simple $g_{(3r+1-f)/t}^r$ on W_q which lifts to the nonfixed points of g_{3r+1}^r. (f is the number of fixed points of g_{3r+1}^r.) If $t \geqslant 4$, since $(3r+1-f)/t \geqslant r$ we have $1-f \geqslant r$, a contradiction. If $t = 3$ then $1 - f \geqslant 0$; consequently, $f = 1$, W_q admits a g_r^r, and $q = 0$. W_p admits a g_3^1 and $g_{3r+1}^r = rg_3^1 + x$ where x is the fixed point. By Corollary 2.5 all half-canonical linear series are now composite. If h_{3r+1}^r is a second half-canonical linear series then $h_{3r+1}^r = rg_3^1 + y$. Since $2g_{3r+1}^r \equiv 2h_{3r+1}^r$ it follows that $2x \equiv 2y$, a contradiction. Consequently, if $t = 3$ then the half-canonical g_{3r+1}^r is unique.

Now suppose $t = 2$. If $g_{(3r+1-f)/2}^r$ is special then by Clifford's theorem we arrive at the contradiction $(3r+1-f)/2 \geqslant 2r$. Consequently $g_{(3r+1-f)/2}^r$ is not special and is complete, so that $q = (r+1-f)/2$ by the Riemann-Roch theorem. Thus W_p is q-hyperelliptic where $p = 3r+2$ and $q = (r+1-f)/2$. By the results of [4, p. 51] this implies that W_p has vanishing properties for the theta function like a hyperelliptic Riemann surface of genus $p - 2q = 2r + 1 + f$. Moreover W_p admits 2^{2q} sets of such vanishing properties, one for each half-period in the Jacobian of W_q. Thus if r is odd and $f = 0$ then W_p admits 2^{r+1} $(= 2^{2q})$ half-canonical g_{3r+1}^r's. In general it can be shown that W_p admits $2^{r-f+1}\binom{4r+2f+4}{f}$ half-canonical g_{3r+1}^r's. Since $f \geqslant 1$ when r is even, the smallest value of this expression is 16 ($r = 3$, $f = 0$).[4] Q.E.D.

3.2. LEMMA. *Suppose $r \geqslant 2$ and W_{3r+2} admits two simple half-canonical linear series g_{3r+1}^r and h_{3r+1}^r. Then neither has a fixed point.*

PROOF. If $g_{3r+1}^r = g_{3r-1}^r + x + y$ where x and y are fixed points then g_{3r-1}^r is composite by Castelnuovo's inequality, Theorem 2.1.

If $g_{3r+1}^r = g_{3r}^r + x$ then by Lemma 2.2(i) we see that $g_{3r}^r + h_{3r+1}^r = g_{6r+1}^{3r+\varepsilon}$ where $\varepsilon \geqslant 0$. By the Riemann-Roch theorem it follows that this latter series is special and so there is a point y so that $g_{3r}^r + h_{3r+1}^r + y \equiv K$, a canonical divisor. Since h_{3r+1}^r is half-canonical it follows that $h_{3r+1}^r = g_{3r}^r + y$. Thus $K \equiv 2(g_{3r}^r + x) \equiv 2(g_{3r}^r + y)$, or $2x \equiv 2y$. This contradiction completes the proof. Q.E.D.

[4] In fact, if $f \geqslant 2$ then W_p admits half-canonical g_{p-1}^{r+1}'s.

3.3. LEMMA. *If $r \geqslant 3$ and W_{3r+2} admits two simple half-canonical linear series of dimension r then W_{3r+2} admits a g_4^1 without fixed points.*

PROOF. By the methods of [4, pp. 67, 68, 74] there is a smooth, two-sheeted cover of W_{3r+2} of genus $6r+3$ and g_{3r+1}^r and h_{3r+1}^r lift to this cover to become equivalent. On W_{6r+3} they determine a half-canonical g_{6r+2}^{2r+1}. By the discussion following Theorem 2.1, W_{6r+3} admits a g_4^1 and consequently W_{3r+1} does also. That it is without fixed points follows from Corollary 2.5. Q.E.D.

Until further notice we will assume that W_{3r+2} admits two simple half-canonical linear series g_{3r+1}^r and h_{3r+1}^r and also a g_4^1 without fixed points. Since g_4^1 imposes two conditions on g_{3r+1}^r and h_{3r+1}^r we have

$$g_{3r+1}^r \equiv g_4^1 + g_{3r-3}^{r-2} \quad \text{and} \quad h_{3r+1}^r \equiv g_4^1 + h_{3r-3}^{r-2} \tag{3.1}$$

where these equations define g_{3r-3}^{r-2} and h_{3r-3}^{r-2}.

3.4. LEMMA. *If $r \geqslant 6$ then g_{3r-3}^{r-2} and h_{3r-3}^{r-2} are both simple.*

PROOF. We use Lemma 2.7 of §2. Since g_4^1 imposes at most two conditions on g_{3r-3}^{r-2} and $3r-3 < 4(r-2)$ we see that g_{3r-3}^{r-2} is simple. Q.E.D.

3.5. LEMMA. *If $r = 4$ or 5 then either g_{3r-3}^{r-2} or h_{3r-3}^{r-2} is simple.*

PROOF. If $r = 5$ then $g_{3r-3}^{r-2} = g_{12}^3$. By Lemma 2.8 we see that if g_{12}^3 is composite it must be $3g_4^1$. Since $h_{12}^3 \neq 3g_4^1$ in this case we see that h_{12}^3 must be simple.

If $r = 4$ and both are composite we see by Lemma 2.8 that $g_9^2 = 2g_4^1 + x$ and $h_9^2 = 2g_4^1 + y$. Since $2g_9^2 \equiv 2h_9^2$ we arrive at the contradiction $2x \equiv 2y$. Q.E.D.

3.6. LEMMA. *If $r \geqslant 4$ and g_{3r-3}^{r-2} is simple, then g_{3r-3}^{r-2} is without fixed points.*

PROOF. If $g_{3r-3}^{r-2} = g_{3r-4}^{r-2} + x$ then $g_{3r+1}^r \equiv g_4^1 + g_{3r-3}^{r-2} + x = g_{3r}^{r+\varepsilon} + x$ by Lemma 2.2(ii). We see that $\varepsilon = 0$ and that g_{3r+1}^r has a fixed point. This contradicts Lemma 3.2. Q.E.D.

DEFINITION. For a plane curve a (2, 4)-*singularity* or simply a (2, 4)-*point* will be a double point with a unique tangent line which has four intersections with the curve at the double point. This tangent will be called the (2, 4)-*tangent*.

Examples of (2, 4)-points are tacnodes and ramphoid cusps. [14, p. 56]. A (2, 4)-point contributes at least two to the double points of the plane curve suitably counted.

For a simple g_n^r a (2, 4)-point is a 2-point P which imposes one condition on g_n^r and $2P$ imposes two conditions on g_n^r.

3.7. THEOREM. *Suppose $r \geqslant 4$ and W_{3r+2} admits precisely two half-canonical linear series g_{3r+1}^r and h_{3r+1}^r. Then W_{3r+2} admits as a plane model a curve C_{2r+3} with $r-2$ 3-points, one $(2r-1)$-point, and two (2, 4)-points with both (2, 4)-tangents passing through the $(2r-1)$-point.*

PROOF. We know that both half-canonical series are simple and we can assume by Lemmas 3.4, 3.5, and 3.6 that g_{3r-3}^{r-2} is simple and without fixed points. By Theorem 2.4, g_{3r-3}^{r-2} imposes at most r conditions on h_{3r+1}^r. Thus there is a complete

h_4^0 so that $h_{3r+1}^r \equiv g_{3r-3}^{r-2} + h_4^0$. Since $h_{3r+1}^r \equiv h_{3r-3}^{r-2} + g_4^1$ we have $K \equiv 2h_{3r+1}^r \equiv h_{3r-3}^{r-2} + g_4^1 + g_{3r-3}^{r-2} + h_4^0$. Since we have half-canonical series we see that

$$h_{3r+1}^r \equiv h_4^0 + g_{3r-3}^{r-2} \equiv h_{3r-3}^{r-2} + g_4^1,$$

$$g_{3r+1}^r \equiv h_4^0 + h_{3r-3}^{r-2} \equiv g_{3r-3}^{r-2} + g_4^1. \tag{3.2}$$

It follows that $2h_4^0 \equiv 2g_4^1$. Since $p \geqslant 14$ Theorem 2.1 assures us that any g_8^3 must be composite, and this is impossible. Consequently $2g_4^1 = g_8^2$ is complete. By Lemma 2.9 there are two pairs P and Q so that $h_4^0 = P + Q$ and $2P \equiv 2Q \equiv g_4^1$ where $(P, Q) = 0$.

Now choose on W_p $r - 2$ points $x_1, \ldots, x_{r-2}$ generically so that they lie on different divisors, $x_i + T_i$, of g_4^1 and so that these divisors are disjoint from P and Q. Now

$$g_{3r+1}^r - (x_1 + \cdots + x_{r-2}) = g_{2r+3}^2 = g_4^1 + D_{2r-1}$$

since g_4^1 still imposes two conditions on the simple g_{2r+3}^2. g_{2r+3}^2 corresponds to a plane curve C_{2r+3}. Each triple T_i corresponds to a 3-point of C_{2r+3} and D_{2r-1} corresponds to a $(2r - 1)$-point.

We now show that P and Q determine (2, 4)-points on C_{2r+3}.

$$\begin{aligned} g_{2r+3}^2 &= g_{3r+1}^r - (x_1 + \cdots + x_{r-2}) \\ &= h_4^0 + h_{3r-3}^{r-2} - (x_1 + \cdots + x_{r-2}) \\ &= P + Q + E_{2r-1} = 2P + D_{2r-1} \qquad (2P \equiv g_4^1) \end{aligned}$$

where E_{2r-1} is some divisor on W_p of degree $2r - 1$. By the choice of the x's we can assume that $(E_{2r-1}, P) = 0$ so that the last two divisors in the above formulas are not the same. Thus the pair P lies on two distinct divisors of g_{2r+3}^2 and so is a double point. That it is a (2, 4)-point follows from the form of the last divisor above and we see that the (2, 4)-tangent cuts out $2P + D_{2r-1}$.

For the double points of C_{2r+3} (suitably counted) the T_i's contribute at least $3(r - 2)$, D_{2r-1} contributes at least $(2r - 1)(2r - 2)/2$, and P and Q contribute at least 4. By formula (2.1) we have

$$\begin{aligned} p = 3r + 2 &\leqslant (2r + 2)(2r + 1)/2 - 3(r - 2) \\ &\quad - (2r - 1)(2r - 2)/2 - 4. \end{aligned} \tag{3.3}$$

But the right-hand side of this last formula is $3r + 2$ so that each of the $r + 1$ singularities contributes the minimum possible to the double points of C_{2r+3} suitably counted. Q.E.D.

If $r = 4$ or 5, h_{3r-3}^{r-2} can be composite. This will occur when the T's and P and Q are collinear in P^2.

3.8. Corollary. *If $r \geqslant 4$, W_{3r+2} can admit at most two simple half-canonical g_{3r+1}^r's.*

Proof. Suppose k_{3r+1}^r is a third simple half-canonical series. As above, there is a complete k_4^0 so that $k_{3r+1}^r \equiv g_{3r-3}^{r-2} + k_4^0$, $2k_4^0 \equiv 2g_4^1$ and $k_4^0 = R + S$ where R and S are pairs. Since $k_4^0 \neq h_4^0$ the proof of Theorem 3.7 shows that C_{2r+3} admits a third (2, 4)-point which will add another two to the double points suitably counted. Since we already have equality in formula (3.3) this is impossible. Thus the existence of k^r_{3r+1} is impossible. Q.E.D.

By the same argument we have the following corollary.

3.9. COROLLARY. *Let R be a pair so that R imposes one condition on g^r_{3r+1}. Then $R = P$ or $R = Q$.*

PROOF. R would add at least one to the double points of C_{2r+3} suitably counted. This is one too many unless $R = P$ or Q. Q.E.D.

If all the singularities are in general position in P^2 we can perform quadratic transformations to obtain the models of [**2**, p. 18]. First perform a standard quadratic transformation with fundamental points P, Q and one of the 3-points. This gives a curve C_{4r-1} with $r-3$ 3-points and six further singularities at the pairs of opposite vertices of a complete quadrilateral of multiplicities $2, 2, 2r-2, 2r-2, 2r-1, 2r-1$. Now perform a quadratic transformation with the two singularities of multiplicities $2r-1$ and one 3-point as fundamental points. One obtains a C_{4r-3} with singularities of order $2, 2, 2r-3, 2r-3, 2r-2, 2r-2$, and $r-4$ 3-points. One proceeds with $r-4$ similar quadratic transformations to obtain C_{2r+5} with precisely six singularities of multiplicities $2, 2, r+1, r+1, r+2$, and $r+2$ and no 3-points. Singularities of the same multiplicity occur at opposite vertices of a complete quadrilateral.

To obtain the models of [**2**, p. 21] for $V = 1$ we now perform a quadratic transformation on C_{2r+5} with fundamental points at singularities of multiplicities $r+2$, $r+2$, and $r+1$. We obtain a curve C_{r+5} with an $(r+1)$-point R and two $(2, 4)$-points with the $(2, 4)$ tangents passing through R. For r even we could obtain the same model by the formula

$$g^r_{3r+1} - ((r-2)/2)g^1_4 = g^2_{r+5}.$$

These latter plane models allow us to compute the dimension in Teichmüller space of the family of Riemann surfaces of genus $3r+2$ admitting two simple half-canonical g^r_{3r+1}'s, since the models are almost unique. The dimension is $5r+5$ which is equal to $(r+5)(r+8)/2 - (r+1)(r+2)/2 - 6 - 6 - 2$. We subtract 6 for each $(2, 4)$-point (which is a tacnode in general) since the direction of the tangent is prescribed. We also subtract two for the two dimensional family of collineations of P^2 which leave the three singularities fixed.

4. The cases $r = 3, 2$, and 1. We now consider the cases not covered in §3.

$r = 3$. W_{11} admits two half-canonical linear series g^3_{10} and h^3_{10} which are simple and without fixed points (Lemmas 3.1, 3.2), and W_{11} also admits a unique g^1_4 without fixed points (Lemma 3.3). Consequently we can write

$$g^3_{10} \equiv g^1_4 + g^1_6, \qquad h^3_{10} \equiv g^1_4 + h^1_6.$$

4.1. LEMMA. *If g^1_6 has a fixed point it must have two fixed points.*

PROOF. If $g^1_6 = g^1_5 + x$ then $g^3_{10} \equiv g^1_4 + g^1_5 + x \equiv g^{3+\varepsilon}_9 + x$ by Lemma 2.2(iii). Consequently $\varepsilon = 0$ and g^3_{10} has a fixed point. This is a contradiction. Q.E.D.

We distinguish three cases:

(i) g_6^1 and h_6^1 are both without fixed points.
(ii) g_6^1 is without fixed points, $h_6^1 = g_4^1 + R$ where R is a pair.
(iii) $g_6^1 = g_4^1 + P$ ($g_{10}^3 \equiv 2g_4^1 + P$), $h_6^1 = g_4^1 + Q$ ($h_{10}^3 \equiv 2g_4^1 + Q$)
where P and Q are pairs so that $2P \equiv 2Q \equiv g_4^1$ and $(P, Q) = 0$.

4.2. LEMMA. *There is a divisor* $h_4^0 = P + Q$ *where* $2P \equiv 2Q \equiv g_4^1$. *Also*

$$g_{10}^3 \equiv g_4^1 + g_6^1 \equiv h_4^0 + h_6^1, \qquad h_{10}^3 \equiv g_4^1 + h_6^1 \equiv h_4^0 + g_6^1. \tag{4.1}$$

PROOF. *Case* (iii). Define P and Q by the equations used in describing case (iii). Then

$$h_4^0 + h_6^1 = P + Q + Q + g_4^1 = P + g_4^1 + g_4^1 = g_{10}^3$$

and similarly for h_{10}^3.

Cases (i) *and* (ii). Since g_6^1 is without fixed points define h_4^0 by $h_4^0 = h_{10}^3 - g_6^1$. Then

$$K \equiv 2h_{10}^3 \equiv h_4^0 + g_6^1 + g_4^1 + h_6^1.$$

Since we are dealing with half-canonical divisors formula (4.1) follows. Since $2h_4^0 = 2g_4^1$ the proof is completed by Lemma 2.9. Q.E.D.

Now in all cases $g_{10}^3 \equiv 2P + g_6^1 \equiv 2Q + g_6^1 \equiv P + Q + h_6^1$, so that $g_{10}^3 - Q \equiv Q + g_6^1 \equiv P + h_6^1 \equiv h_{10}^3 - P \equiv g_8^s$ where $s = 1$ or 2. In cases (i) and (ii) it is easy to see that $s = 2$, but it is not so obvious in case (iii), so we proceed as follows.

Consider $g_{10}^3 + Q \equiv 2Q + g_8^s \equiv g_4^1 + g_8^s \equiv g_{12}^t$. Now $g_{10}^3 + Q \equiv 2P + g_8^s \equiv h_{10}^3 + P$ and $(P, Q) = 0$, so we see that g_{12}^t is without fixed points. Also $g_{12}^t + g_8^s \equiv K$ and so $t = s + 2$. But if $s = 1$ then $t = 3$ and g_{12}^t has fixed points; consequently $s = 2$. ($s = 3$ is impossible.) Moreover, since $g_4^1 + g_8^2 \equiv g_{12}^4$ we see that g_4^1 imposes at most two conditions on g_8^2. By Lemma 2.8 it follows that g_8^2 is simple since $g_8^2 \neq 2g_4^1$.

The plane curve C_8 corresponding to g_8^2 has a 4-point D_4 since g_4^1 imposes two conditions on g_8^2. Consequently, we have

$$g_8^2 \equiv Q + g_6^1 \equiv P + h_6^1 \equiv D_4 + g_4^1.$$

In case (i) the divisors P, Q, and D_4 are pairwise disjoint. In case (ii) $D_4 = R + Q$ and in case (iii) $D_4 = P + Q$. In any case P and Q contribute at least two and D_4 contributes at least six to the double points suitably counted. Since ten is the maximum for the double points suitably counted, we see that P, Q, and D_4 contribute precisely the amounts indicated. Thus for C_8 cases (i), (ii), and (iii) correspond to none, one, or both of P and Q being in the first neighborhood of D_4. In all cases the (2, 4)-tangents at P and Q pass through D_4.

4.3. THEOREM. *Let* W_{11} *admit two simple half-canonical linear series. Then* W_{11} *admits a plane model* C_8 *with a* 4-*point and two* (2, 4)-*points with the* (2, 4)-*tangents passing through the* 4-*point. One or both of the* (2, 4)-*points may be in the first neighborhood of the* 4-*point.*

Again by a careful examination of the proof we see that Corollary 3.8 follows for $r = 3$.

4.4. COROLLARY. *W_{11} cannot admit three simple half-canonical g_{10}^3's.*

$r = 2$. Suppose W_8 admits two simple half-canonical linear series g_7^2 and h_7^2. We know that neither has a fixed point but an added argument is needed to show the existence of a g_4^1.

4.5. LEMMA. *W_8 admits a g_4^1.*

PROOF. W_8 admits a g_5^1 or a g_4^1. If W_8 admits a g_5^1 then by Lemma 2.3 we have $g_7^2 \equiv g_5^1 + P$ and $h_7^2 \equiv g_5^1 + Q$ where $(P, Q) = 0$. Then $2P \equiv 2Q$ and this gives a g_4^1.

4.6. LEMMA. *g_4^1 is unique and $2g_4^1 \equiv g_8^2$ is complete.*

PROOF. First we show that W_8 does not admit a g_8^3. If it does then $K - g_8^3 \equiv g_6^2$. If g_6^2 is simple then by Theorem 2.4 $g_7^2 = g_6^2 + x$ and $h_7^2 = h_6^2 + y$ and $2x \equiv 2y$, a contradiction. If g_6^2 is composite then W_7 is trigonal or 1-hyperelliptic, both possibilities being contradictions.

If W_p admits a g_4^1 and an h_4^1 then $g_4^1 + h_4^1 \equiv g_8^3$, a contradiction. Moreover $2g_4^1 \equiv g_8^{2+\varepsilon}$ and ε must be 0. Q.E.D.

Now let $g_7^2 \equiv g_4^1 + g_3^0$ and $h_7^2 \equiv g_4^1 + h_3^0$. Then $g_7^2 + h_3^0 \equiv g_3^0 + g_4^1 + h_3^0 \equiv g_3^0 + h_7^2 \equiv g_{10}^s$ where $s \geqslant 3$ since $s = 2$ is clearly impossible. Thus g_{10}^s is special and there is a divisor h_4^0 so that

$$K \equiv g_3^0 + g_4^1 + h_3^0 + h_4^0.$$

It follows that $g_7^2 = h_3^0 + h_4^0$ and $h_7^2 = g_3^0 + h_4^0$ and so h_4^0 has dimension zero. Moreover $2h_4^0 \equiv 2g_4^1$. By Lemma 2.9 $h_4^0 = P + Q$, $2P \equiv 2Q \equiv g_4^1$, and $(P, Q) = 0$. Thus

$$g_7^2 \equiv g_4^1 + g_3^0 \equiv 2P + g_3^0 \equiv 2Q + g_3^0 \equiv P + Q + h_3^0,$$

P and Q are (2, 4)-points for g_7^2 and g_3^0 is a 3-point. Since $p = 8 < 6 \cdot 5/2 - 3 - 2 - 2 = 8$ this accounts for all the singularities of g_7^2. It is possible for (P, g_3^0) or (Q, g_3^0) to be nonzero but not both since $(P, Q) = 0$. In this latter case one of the (2, 4)-points is in the first neighborhood of g_3^0 on C_7, the plane curve corresponding to g_7^2.

4.7. THEOREM. *Suppose W_8 admits two simple half-canonical linear series. Then W_8 admits a plane model, C_7, with a 3-point and two (2, 4)-points whose tangents pass through the 3-point. One but not both of the (2, 4)-points may be in the first neighborhood of the 3-point.*

Again we have the corollary:

4.8. COROLLARY. *W_8 cannot admit three simple half-canonical g_7^2's.*

$r = 1$. This case is covered in [3, p. 15]. However, there is an oversight in that discussion which should be mentioned. It is quite possible for the four double-points on the curve of degree seven to coalesce by pairs into two (2, 4)-points and so the four lines in Figure 1 degenerate into three lines, one line being counted twice.

5. Automorphisms. In this section we will disucss some possible automorphism groups of W_{3r+2}'s admitting two simple half-canonical g_{3r+1}^r's, $r > 2$.

Since g_4^1 is unique any automorphism must permute the divisors in g_4^1. If N is the subgroup of $A(W_{3r+2})$ which leaves each divisor in g_4^1 fixed then N is normal and $A(W_{3r+2})/N$ is isomorphic to a finite group of P^1. The order of N is one, two, or four. Another normal subgroup is that one which leaves each half-canonical g_{3r+1}^r fixed. This subgroup will have index one or two since any automorphism permutes the two g_{3r+1}^r's.

We shall devote the remainder of this section to considering the case where g_4^1 is the set of orbits of a noncyclic group of order four. As in §4 of Part I, it turns out that such a four-group of automorphisms can be characterized by certain vanishing properties of the theta function: that is, by the existence of certain half-canonical g_{3r+1}^{r-1}'s.

To see this suppose that W_{3r+2} admits a noncyclic group of automorphisms of order four, G, whose orbits are g_4^1. Then each branched orbit is of the form $2R$ where R is a pair of two distinct points. By the Riemann-Hurwitz formula there are $3r + 5$ such pairs of which two are the (2, 4)-points P and Q. Thus there are $3r + 3$ other pairs $R_1, R_2, \ldots, R_{3r+3}$. By formula (3.2) and the fact that $h_4^0 = P + Q$ and $2P \equiv 2Q \equiv g_4^1$ we see that $g_{3r+1}^r - P \equiv h_{3r+1}^r - Q \equiv g_{3r-1}^{r-1}$ and $g_{3r+1}^r - Q \equiv h_{3r+1}^r - P \equiv h_{3r-1}^{r-1}$. Now $g_{3r-1}^{r-1} + R_i$ is half-canonical and complete as is $h_{3r-1}^{r-1} + R_i$. Let

$$ {}_ig_{3r+1}^{r-1} = g_{3r-1}^{r-1} + R_i \quad \text{and} \quad {}_ih_{3r+1}^{r-1} = h_{3r-1}^{r-1} + R_i. $$

Then there are $3r + 3$ pairs $\{{}_ig_{3r+1}^{r-1}, {}_ih_{3r+1}^{r-1}\}$.

We will now show that the existence of $3r + 3$ such pairs of half-canonical g_{3r+1}^{r-1}'s will insure that every branched divisor in g_4^1 is of the form $2R$ where R is a pair with two distinct points, at least for $r \geqslant 10$. The following lemma is the key.

5.1. LEMMA. *Suppose W_{3r+2} admits two half-canonical g_{3r+1}^r's, $r > 10$. If W_{3r+2} admits a half-canonical g_{3r+1}^{r-1} then $g_{3r+1}^{r-1} \equiv g_{3r+1}^r - P + R$ (or $g_{3r+1}^r - Q + R$) where $2R \equiv g_4^1$.*

PROOF. We prove the lemma by assuming g_{3r+1}^{r-1} has precisely t fixed points, $t = 0, 1, 2, \ldots,$ and showing that t must equal 2. This involves several cases.

Case (0). Assume that g_{3r+1}^{r-1} has no fixed points. By Lemma 2.7, g_{3r+1}^{r-1} is simple and $g_{3r+1}^{r-1} \equiv g_4^1 + g_{3r-3}^{r-3}$ where again by Lemma 2.7, g_{3r-3}^{r-3} is simple. g_{3r-3}^{r-3} is without fixed points and by Theorem 2.4 imposes at most r conditions on g_{3r+1}^r, so we have

$$ g_{3r+1}^r = g_{3r-3}^{r-3} + G_4^0 \tag{5.1} $$

where $2G_4^0 \equiv 2g_4^1$, $G_4^0 = R_1 + R_2$, $2R_1 \equiv 2R_2 \equiv g_4^1$, and $(R_1, R_2) = 0$. Now $R_1 + R_2$ imposes three conditions on g_{3r+1}^r by formula (5.1) so one of the pairs, say R_1, imposes two conditions on g_{3r+1}^r. But $2R_1 \equiv g_4^1$ which also imposes two conditions on g_{3r+1}^r. Since $g_{3r+1}^r - R_1 = g_{3r-3}^{r-3} + R_2$ we see that R_1 must be a fixed divisor of g_{3r-3}^{r-3}, a contradiction.

Case (1). Assume that g_{3r+1}^{r-1} admits precisely one fixed point. $g_{3r+1}^{r-1} = g_{3r}^{r-1} + x$. Also $g_{3r+1}^{r-1} \equiv g_{3r-4}^{r-3} + g_4^1 + x$ and g_{3r}^{r-1} and g_{3r-4}^{r-3} are simple by Lemma 2.7. Again

$g^r_{3r+1} \equiv g^{r-3}_{3r-4} + G^0_5$ where $2G^0_5 \equiv 2g^1_4 + 2x \equiv g^s_{10}$, $s > 2$.

If $s \geqslant 3$ then g^s_{10} is simple and $p \leqslant 16$, a contradiction. Consequently $G^0_5 = G^0_4 + x$, $2G^0_4 \equiv 2g^1_4$, $G^0_4 = R_1 + R_2$, $2R_1 \equiv 2R_2 \equiv g^1_4$ and $(R_1, R_2) = 0$. Thus

$$g^r_{3r+1} \equiv g^{r-3}_{3r-4} + R_1 + R_2 + x.$$

If either R_1 or R_2 imposes two conditions on g^r_{3r+1} we can argue as in case (0) to obtain a further fixed point of g^{r-3}_{3r-4} which is a contradiction. Consequently R_1 and R_2 must each impose one condition on g^r_{3r+1}. By Corollary 3.9 this implies that $R_1 = P$ and $R_2 = Q$ (or vice versa). Consequently $g^r_{3r+1} \equiv g^{r-3}_{3r-4} + P + Q + x$. But exactly the same argument can be applied to h^r_{3r+1} and we arrive at the contradiction $g^r_{3r+1} = h^r_{3r+1}$.

Case (3). Assume that g^{r-1}_{3r+1} admits precisely three fixed points. $g^{r-1}_{3r+1} = g^{r-1}_{3r-2} + T_1$ and g^{r-1}_{3r-2} imposes at most r conditions on g^r_{3r+1}. Thus $g^r_{3r+1} = g^{r-1}_{3r-2} + T_2$. It follows that $2T_1 \equiv 2T_2 \equiv g^s_6$ where $s = 1$. Consequently $g^1_6 = g^1_4 + 2x$. Consequently $T_2 = R + x$ where $2R \equiv g^1_4$ and R imposes one condition on g^r_{3r+1}. This implies that $R = P$ or Q and therefore that g^r_{3r+1} has a fixed point x, a contradiction.

Case (4). Assume that g^{r-1}_{3r+1} has precisely 4 fixed points, $g^{r-1}_{3r+1} = g^{r-1}_{3r-3} + G^0_4$. g^{r-1}_{3r-3} imposes at most $r - 1$ conditions on g^r_{3r+1} so that $g^r_{3r+1} = g^{r-1}_{3r-3} + g^1_4$, contradicting the fact that g^1_4 imposes two conditions on g^r_{3r+1}.

Case (5). Suppose g^{r-1}_{3r+1} has $5 + k$ fixed points, $k \geqslant 0$. Then $g^{r-1}_{3r+1} = g^{r-1}_{3r-4-k} + F$ where the degree of F is $5 + k$. By Castelnuovo's inequality g^{r-1}_{3r-4-k} is composite which contradicts Lemma 3.4.

Case (2). g^{r-1}_{3r+1} must admit precisely two fixed points. If $g^{r-1}_{3r+1} = g^{r-1}_{3r-1} + R$ then g^{r-1}_{3r-1} imposes at most r conditions on g^r_{3r+1}, so $g^r_{3r+1} = g^{r-1}_{3r-1} + R'$ where $2R \equiv 2R' \equiv g^1_4$ and R' imposes one condition on g^r_{3r+1}. By Corollary 3.9 it follows that $R' = P$ or Q. If $g^r_{3r+1} \equiv g^{r-1}_{3r-1} + P$ and $h^r_{3r+1} \equiv h^{r-1}_{3r-1} + P$ then $h^{r-1}_{3r-1} + R$ is a second half-canonical series of dimension $r - 1$. Q.E.D.

We are now ready to complete the proof of the following theorem.

5.2. THEOREM. *Suppose W_{3r+2} ($r \geqslant 10$) admits two simple half-canonical linear series g^r_{3r+1} and h^r_{3r+1}. Then g^1_4 is the set of orbits of a noncyclic group of automorphisms of order four if and only if there are $3r + 3$ pairs of half-canonical linear series $\{{}_ig^{r-1}_{3r+1}, {}_ih^{r-1}_{3r+1}\}$ so that $g^r_{3r+1} + h^r_{3r+1} \equiv {}_ig^{r-1}_{3r+1} + {}_ih^{r-1}_{3r+1}$ for all i.*

PROOF. Assuming the existence of the $3r + 3$ pairs of half-canonical linear series we infer the existence of $3r + 3$ pairs R_i so that $2R_i \equiv g^1_4$. Consequently we have $3r + 5$ pairs, $P, Q, R_1, R_2, \ldots, R_{3r+3}$ whose doubles are in g^1_4. Each pair contributes at least two to the ramification of the four-sheeted covering of P^1 given by g^1_4. But the total ramification of this cover is $6r + 10$. Consequently each of the $3r + 5$ pairs must contribute precisely two to the ramification of this cover. This means that each pair must be two distinct points. Q.E.D.

Suppose W_{3r+2} admits such a four group G. Let G_1, G_2, and G_3 be the three subgroups of order two. By Lemma 2.6 the genus of W_{3r+2}/G_i must be r or more. Since the sum of these three genera must be $3r + 2$, we see that G is either a

$(3r + 2; r + 2, r, r; 0)$ or a $(3r + 2; r + 1, r + 1, r; 0)$. We shall presently give examples of the latter type.

But first let us heuristically count the dimension of the space of such W_{3r+2}'s. Since W_{3r+2}'s with two simple half-canonical linear series of dimension r have dimension $5r + 5$, and the g_4^1 has two pairs P and Q so that $2P \equiv 2Q \equiv g_4^1$, it appears that the existence of the $3r + 3$ additional pairs R_i imposes $3r + 3$ additional conditions. We would guess that W_{3r+2}'s with such a four group of automorphisms would have dimension $5r + 5 - (3r + 3) = 2r + 2$. In any event our examples will depend on $2r + 2$ parameters.

(Before proceeding to the examples the reader may wish to consult the Appendix where the analogous case of a W_{3r} admitting a simple g_{3r-1}^r is considered. Here the four group is a $(3r; r, r, r; 0)$. The construction of this latter example is easier than that of the $(3r + 2; r + 1, r + 1, r; 0)$.)

Now assume that r is odd. Let $P_1(x)$ and $P_2(x)$ be polynomials of degree $r + 1$ with $2r + 2$ distinct roots so that the Riemann surface W_r of $\sqrt{P_1 P_2}$ has genus r. Let $w = \sqrt{P_1/P_2}$. If H is the hyperelliptic involution of W_r then $H^* w = -w$. w has order $r + 1$ and is regular over ∞. Now fix a complex number A and let $u = (A - w)/(A + w)$. Choose A so that u has no multiple zeros and poles. u is a function of degree $r + 1$ and $H^* u = 1/u$. If $D_1 - D_2$ is the divisor of u then $HD_1 = D_2$. If $\phi: W_r \to P^1$ is the two-sheeted cover of W_r over P^1, then it is easy to see that $\phi(D_1)$ $(= \phi(D_2))$ is the set of zeros of $P_1(x) - A^2 P_2(x)$. Now let d_1 and d_2 be two different complex numbers which are not roots of P_1, P_2, or $P_1 - A^2 P_2$. Let $v = (x - d_1)u/(x - d_2)$, a function of order $r + 3$ on W_r. Let W_{3r+2} be the two-sheeted cover of W_r where $\sqrt{v}$ is singled valued. The covering $W_{3r+2} \to W_r$ is branched above the zeros and poles of w and also above $\phi^{-1}(d_1)$ and $\phi^{-1}(d_2)$. Thus the total ramification of this cover is $2r + 6$. On W_{3r+2} let P be the two points above $\phi^{-1}(d_1)$, let Q be the pair above $\phi^{-1}(d_2)$, let D_1' be the points above D_1 and let D_2' be the points above D_2. The divisor of $\sqrt{v}$ is $P + D_1' - (Q + D_2')$. On W_r the function $v_1 = (x - d_2)^2 v/(x - d_1)^2$ also has a square root on W_{3r+2} and $(\sqrt{v_1})$ is $(Q + D_1') - (P + D_2')$. Let $g_{r+3}^1 = |P + D_1'| = |Q + D_2'|$ and $h_{r+3}^1 = |Q + D_1'| = |P + D_2'|$.

Now $2P \equiv 2Q \equiv g_4^1$, where g_4^1 is the set of orbits for G, a noncyclic group of order four on W_{3r+2}. Consequently, $2g_{r+3}^1 \equiv 2h_{r+3}^1$. Also by Lemma 2.2(iii) we have

$$((r - 1)/2)g_4^1 + g_{r+3}^1 \equiv g_{3r+1}^{r+\varepsilon}$$

where $g_{3r+1}^{r+\varepsilon}$ is simple and so $\varepsilon = 0$ by Castelnuovo's inequality. Consequently g_{r+3}^1 is complete. Similarly $((r - 1)/2)g_4^1 + h_{r+3}^1 \equiv h_{3r+1}^r$. It follows that $2g_{3r+1}^r \equiv 2h_{3r+1}^r$.

It remains to show that $2g_{3r+1}^r \equiv (r - 1)g_4^1 + P + D_1' + Q + D_2'$ is canonical. On W_r let B be the $2r + 2$ branch points of $\phi: W_r \to P^1$. Then $B \equiv (r + 1)g_2^1$. If B' is the lift of B to W_{3r+2} then

$$B' \equiv (r + 1)g_4^1. \tag{5.2}$$

But on W_{3r+2} the branched locus is $2g_4^1 + K$ where K is the canonical divisor on W_{3r+2}. That is

$$K \equiv B' + P + D_1' + Q + D_2' - 2g_4^1.$$

This together with formula (5.2) completes the demonstration.

We now state without proof the following.

5.3. COROLLARY. *If W_{3r+2} admits two simple half-canonical linear series of dimension r and a four group G of automorphisms whose orbits are a g_4^1 then*

(i) *r is odd,*

(ii) *G is a $(3r + 2; r + 1, r + 1, r; 0)$, and*

(iii) *G arises exactly as in the above example.*

The above examples depend on $2r + 2$ parameters as follows: $2r - 1$ for W_r; one for A in the definition of u, and two for d_1 and d_2. The function field on W_{3r+2} is $(P_3 = P_1 - A^2P_2)$ $C(x, \sqrt{P_1P_2}, \sqrt{P_1(x - d_1)(x - d_2)P_3})$.

6. Appendix, an example with reference to Part I. For r odd, we shall construct a $2r$-dimensional family of W_{3r}'s admitting a $(3r; r, r, r; 0)$ and a g_{r+1}^1 without fixed points. By the formula $g_{3r-1}^r = g_{r+1}^1 + ((r - 1)/2)g_4^1$ ([6], or Lemma 2.2(iii)) this will insure that W_{3r} admits a g_{3r-1}^r which is simple and necessarily half-canonical.

We start with two polynomials $P_1(x)$ and $P_2(x)$ each of degree $r + 1$ and all of whose $2r + 2$ zeros are distinct. Thus the Riemann surface of $\sqrt{P_1P_2}$ is a hyperelliptic surface W_r of genus r. On this surface consider the function $w = \sqrt{P_1/P_2}$. If H is the hyperelliptic involution of W_r then $H^*w = -w$. Also w is of order $r + 1$ and is regular over ∞. Now fix a complex number A so that the function $u = (A - w)/(A + w)$, again a function of degree $r + 1$, has no multiple zeros or poles. Notice that $H^*u = 1/u$, so that if the divisor of u is $D_1 - D_2$ then $HD_1 = D_2$. If ϕ: $W_r \to P^1$ is the two-sheeted cover of W_r over P^1 then it is easy to see that $\phi(D_1)$ $(= \phi(D_2))$ is the set of zeros of $P_1 - A^2P_2$ (which we will call P_3). Now consider the two-sheeted cover of W_r with branch points over $D_1 + D_2$ where the function $\sqrt{u}$ is single valued. The ramification of this cover is $2r + 2$ so this cover, W_{3r}, has genus $3r$. D_1 and D_2 lift to divisors D_1' and D_2' and $D_1' - D_2'$ is the divisor of $\sqrt{u}$. Consequently, $|D_1'| = g_{r+1}^1$, and W_{3r} is the desired Riemann surface. It depends on $2r$ parameters, $2r - 1$ for the choice of W_r and one for the choice of A in the definition of u.

Since the main purpose of these examples is to show the characterization of Theorem 4.3, Part I, is not a vacuous statement, we will conclude with some corollaries without further proofs.

6.1. COROLLARY. *If W_{3r} admits a simple g_{3r-1}^r and a $(3r; r, r, r; 0)$ then*:

(i) *r is odd, and*

(ii) *the group arises exactly as in the previous examples.*

The function field for W_{3r} is $C(x, \sqrt{P_1P_2}, \sqrt{P_1P_3})$.

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