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THE GENUS OF A MAP

BY

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ABSTRACT. The elements $[f']$ $(f'\colon X' \to Y')$ of the genus $-$ $G(f)$ of a map $f\colon X \to Y$ are equivalence classes of homotopy classes of maps f' which satisfy: For every prime p there exist homotopy equivalences $h_p\colon X'_p \to X_p$ and $k_p\colon Y'_p \to Y_p$ so that $f_p h_p \sim k_p f'_p$. The genus of f under X $-$ $G^X(f)$ and the genus of f over Y $-$ $G_Y(f)$ are defined similarly.

In this paper we prove that under certain conditions on f, the sets $G(f)$, $G^X(f)$ and $G_Y(f)$ are finite and admit an abelian group structure. We also compare the genus of f to those of X and Y, calculate it for some principal fibrations of the form $K(G, n-1) \to X \to Y$, and deal with the noncancellation phenomenon.

1. Introduction. In this paper we use the structure of the genus of an H_0-space, which was investigated by Zabrodsky [8], to study the structure of the genus of a map $f\colon X \to Y$. In some cases we calculate the genus and compare it with those of X and Y.

All spaces considered are pointed and of the homotopy type of simply connected CW-complexes of finite type and either with a finite number of nonzero homology groups or with a finite number of nonzero homotopy groups.

Throughout this paper we work in the homotopy category. We recall that for a CW-complex X, the genus of X is the set $G(X)$ of homotopy types of spaces Y with $Y_p \approx X_p$ for every prime p (where $(\ )_p$ denotes the p-localization operation). We define analogously the genus $G(f)$ of f, $G^X(f)$–the genus of maps under X and $G_Y(f)$ the genus of maps over Y.

1.1. DEFINITION. Let $f\colon X \to Y$ be a map. The elements $[f'] \in G(f)$ $(f'\colon X' \to Y')$ are equivalence classes of homotopy classes of f' which satisfy: For every prime p there exist homotopy equivalences $h_p\colon X'_p \to X_p$ and $k_p\colon Y'_p \to Y_p$ so that $f_p h_p \sim k_p f'_p$. (We denote the genus of f either by $G(f)$ or by $G(X, Y, f)$.)

The elements $[f'] \in G^X(f)$ $(f'\colon X \to Y')$ are equivalence classes of homotopy classes of maps f' which satisfy: For every prime p there exists a homotopy equivalence $k_p\colon Y'_p \to Y_p$ so that $k_p f'_p \sim f_p$. (Two maps $f_i\colon X \to Y_i$, $i = 1, 2$, are equivalent under X if there exists a homotopy equivalence $k\colon Y_1 \to Y_2$ with $kf_1 \sim f_2$.)

The elements $[f'] \in G_Y(f)$ $(f'\colon X' \to Y)$ are equivalence classes of homotopy classes of maps f' so that for every prime p there exists a homotopy equivalence $h_p\colon X'_p \to X_p$ so that $f_p h_p \sim f'_p$.

To state the main results of this study we need the following notations: Let t be an integer, X a space and $f\colon X \to Y$ a map. Denote by Z_t the group Z/tZ, by Z^*_t

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the units in Z_t, by $l(X)$ the number of integers n with $QH^n(X, Q) \neq 0$ and by $[f, f]_t$ the set of pairs (h, k) of t-equivalences $h: X \to X$, $k: Y \to Y$ satisfying $kf \sim fh$. (A t-equivalence of X is a map $f: X \to X$ so that $H^*(f, Z) \otimes Z_p$ is an isomorphism for every prime p which divides t.)

THEOREM I. *Let $f: X \to Y$ be a map which satisfies one of the following conditions*:

(a) *$f: X \to Y$ is an H-map, $H^*(X, Q)$ and $H^*(Y, Q)$ are primitively generated and $H^*(f, Q)$ is either a monomorphism, an epimorphism, an isomorphism or zero.*

(b) *$X = S^{2n-1}$, Y is an H-space and $H^*(Y, Q)$ is primitively generated.*

(c) *$X = S^{2n-1}$, $Y = S^{2m-1}$.*

Then $G(f)$ admits an abelian group structure and there exist integers k and $\hat{t}$ (depending on X, Y and f) and an exact sequence

$$[f, f]_{\hat{t}} \xrightarrow{\alpha'} [(Z_{\hat{t}}^*)/\pm 1]^k \xrightarrow{\hat{\xi}} G(f) \to 0$$

where α' is the composition

$$[f, f]_{\hat{t}} \to \operatorname{aut}(QH^*(X, Z)/torsion \otimes Z_{\hat{t}}) \times \operatorname{aut}(QH^*(Y, Z)/torsion \otimes Z_{\hat{t}})$$
$$\xrightarrow{|\det| \times |\det|} [(Z_{\hat{t}}^*)/\pm 1]^{l(X)+l(Y)} \to [(Z_{\hat{t}}^*)/\pm 1]^k.$$

THEOREM II. *If $f: X \to Y$ is a map satisfying the conditions of Theorem* I *then $G_Y(f)$ admits an abelian group structure and there exist an integer $\hat{t}$ depending on X, Y and f and an exact sequence*

$$[f, f]'_{\hat{t}} \xrightarrow{\alpha'} [(Z_{\hat{t}}^*)/\pm 1]^{l(X)} \xrightarrow{\hat{\xi}} G_Y(f) \to 0$$

where $[f, f]'_{\hat{t}} = \{(h, 1) \in [f, f]_{\hat{t}}\}$.

THEOREM III. *Let $f: X \to Y$ be a map satisfying one of the following conditions*:

(a) *X, Y are H-spaces; $H^*(X, Q)$ and $H^*(Y, Q)$ are primitively generated, and $H^*(f, Q)$ is either a monomorphism, an epimorphism, an isomorphism or zero.*

(b) *$X = S^{2n-1}$, Y is an H-space, and $H^*(Y, Q)$ is primitively generated.*

Then $G^X(f)$ admits an abelian group structure and there exist an integer $\hat{t}$ depending on X, Y and f and an exact sequence

$$[f, f]''_{\hat{t}} \xrightarrow{\alpha'} [(Z_{\hat{t}}^*)/\pm 1]^{l(Y)} \xrightarrow{\xi} G^X(f) \to 0$$

where $[f, f]''_{\hat{t}} = \{(1, k) \in [f, f]_{\hat{t}}\}$.

The proof of Theorem I relies heavily on the fact that for maps which satisfy the conditions of Theorem I, a map $f': X' \to Y'$ belongs to the genus of the map f: $X \to Y$ iff for every prime p there exist p-equivalences $h: X' \to X$, $k: Y' \to Y$ so that $fh \sim kf'$. The proofs of Theorems II and III rely on similar facts. These facts are proved in §2. The main theorems are proved in §3. In §4 some simple conclusions are derived. §5 deals with the kernel of the obvious map $G(X, Y, f) \to G(X) \times G(Y)$ and §6 applies this map and the main theorems to calculate the genus of some principal fibrations of the form $K(G, n-1) \to X \to Y$. The last section, §7, deals with the noncancellation phenomenon.

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2. Localization and p-equivalences. Let (X, μ) be an H-space and let (Y, ψ) be a co-H-space. We shall denote by $+$ the operation on $[Z, X]$ and $[Y, Z]$ induced by μ and ψ, respectively, by ϕ_n the n-power map

$$\phi_n = \underbrace{\mu(\mu \times 1) \cdots (\mu \times 1 \times \cdots \times 1)}_{n-1} \circ \underbrace{(\Delta \times 1 \times \cdots \times 1) \cdots \Delta}_{n-1}$$

(Δ–the diagonal)

and by η_n the map

$$\eta_n = \underbrace{\mathcal{F} \cdots (\mathcal{F} \vee 1 \vee \cdots \vee 1)}_{n-1} \circ \underbrace{(\psi \vee 1 \vee \cdots \vee 1) \cdots (\psi \vee 1)\psi}_{n-1}$$

($\mathcal{F}$–the folding map).

2.1. THEOREMS (HILTON, MISLIN, ROITBERG, SEE [3, II,6]). *Let X be a connected H-space and W a space with a finite number of homology groups. For every map f: $W_p \to X_p$ there exist an integer n, $(n, p) = 1$, and a function g: $W \to X$ so that $g_p \sim \phi_n f$.*

Moreover, given two functions f, g: $W \to X$ so that $f_p \sim g_p$, there exists an integer m, $(m, p) = 1$, so that $\phi_m f \sim \phi_m g$.

2.2. THEOREM. *Let f: $X \to Y$ be a map and let l be an integer. Suppose $l = p_1^{w_1} \cdots p_s^{w_s}$.*

(a) *If f is an H-map then*:

(1) *Given two spaces X', Y', a function f': $X' \to Y'$ and homotopy equivalences h_{p_i}: $X'_{p_i} \to X_{p_i}$, k_{p_i}: $Y'_{p_i} \to Y_{p_i}$ satisfying $f_{p_i}h_{p_i} \sim k_{p_i}f'_{p_i}$ $(r = 1, \ldots, s)$, there exist l-equivalences h: $X' \to X$, k: $Y' \to Y$ so that $fh \sim kf'$.*

(2) *Given a map f': $X' \to Y$ in $G_Y(f)$, there exists an l-equivalence h: $X' \to X$ so that $fh \sim f'$.*

(3) *Given a map f': $X \to Y'$ in $G^X(f)$, there exists an l-equivalence k: $Y' \to Y$ so that $kf' \sim f$.*

(b) *If Y is an H-space and $X = SX''$ then*:

(1) *Given two spaces X', Y', a function f': $SX' \to Y'$ and homotopy equivalences h_{p_i}: $(SX')_{p_i} \to X_{p_i}$, k_{p_i}: $Y'_{p_i} \to Y_{p_i}$ satisfying $f_{p_i}h_{p_i} \sim k_{p_i}f'_{p_i}$ $(i = 1, \ldots, s)$, there exist l-equivalences h: $SX' \to X$, k: $Y' \to Y$ so that $fh \sim kf'$.*

(2) *Given a map f: $SX' \to Y$ in $G_Y(f)$, there exists an l-equivalence h: $SX' \to X$ so that $fh \sim f'$.*

(3) *Given a map f': $X \to Y'$ in $G^X(f)$, there exists an l-equivalence k: $Y' \to Y$ so that $kf' \sim f$.*

(c) *If $X = S^n$ and $Y = S^{2m-1}$ then*:

(1) *Given a function f': $X \to Y$ and homotopy equivalences h_{p_i}: $X_{p_i} \to X_{p_i}$, k_{p_i}: $Y_{p_i} \to Y_{p_i}$ satisfying $f_{p_i}h_{p_i} \sim k_{p_i}f'_{p_i}$ $(i = 1, \ldots, s)$ there exist l-equivalences h: $X \to X$, k: $Y \to Y$ so that $fh \sim kf'$.*

(2) *Given a map f': $X \to Y'$ in $G_Y(f)$ there exists an l-equivalence h: $X \to X$ so that $fh \sim f'$.*

Proof. (a) (1) Since X and Y are H-spaces, by 2.1, for every i there exist integers $n_{1,i}$, $n_{2,i}$, $(n_{1,i}, p_i) = (n_{2,i}, p_i) = 1$, so that $\phi_{n_{1,i}} \circ h_{p_i}$ and $\phi_{n_{2,i}} \circ k_{p_i}$ are induced by functions $h'_i\colon X' \to X$ and $k'_i\colon Y' \to Y$.

As f is an H-map, for every i, the p-localization of $f(\phi_{n_{2,i}} \circ h'_i)$ and $(\phi_{n_{1,i}} \circ k'_i)f'$ are homotopic:

$$\begin{tikzcd}
X'_{p_i} \arrow[r, "h_{p_i}"] \arrow[d, "f'_{p_i}"] \arrow[rr, bend left, "\phi_{n_{2,i}} \circ (h'_i)_{p_i}"] & X_{p_i} \arrow[r, "\phi_{n_{2,i}} \circ \phi_{n_{1,i}}"] \arrow[d, "f_{p_i}"] & X_{p_i} \arrow[d, "f_{p_i}"] \\
Y'_{p_i} \arrow[r, "k_{p_i}"] \arrow[rr, bend right, "\phi_{n_{1,i}} \circ (k'_i)_{p_i}"'] & Y_{p_i} \arrow[r, "\phi_{n_{1,i}} \circ \phi_{n_{2,i}}"] & Y_{p_i}
\end{tikzcd}$$

Hence, there exist integers n_i, $(n_i, p_i) = 1$, so that

$$(\phi_{n_i}\phi_{n_{1,i}}k'_i)f' \sim \phi_{n_i}(\phi_{n_{1,i}}k'_i)f' \sim \phi_{n_i}f(\phi_{n_{2,i}}h'_i) \sim f(\phi_{n_i}\phi_{n_{2,i}}h'_i).$$

Define $m = \prod_{i=1}^{s} p_i$, $h''_i = \phi_{m/p_i}\phi_{n_in_{2,i}}h'_i$, $k''_i = \phi_{n/p_i}\phi_{n_in_{1,i}}k'_i$. Then $h = \sum_{i=1}^{s} h''_i$ and $k = \sum_{i=1}^{s} k''_i$ are the desired maps. Indeed since $\pi_*(h) \otimes Z_{p_{i_0}} = \sum_i \pi_*(h''_i) \otimes Z_{p_{i_0}} = \pi_*(h''_{i_0}) \otimes Z_{p_{i_0}}$ and h''_{i_0} is a p_{i_0}-equivalence, h is an l-equivalence. Similarly one gets that k is an l-equivalence. It is clear that $fh \sim kf'$.

(2) Since for every p_i there exists a homotopy equivalence $h_{p_i}\colon X'_{p_i} \to X_{p_i}$ satisfying $f_{p_i}h_{p_i} \sim f'_{p_i} = 1_{Y_{p_i}}f'_{p_i}$, it follows from (1) that there exists an integer n, $(n, l) = 1$, and an l-equivalence $h'\colon X' \to X$ so that $fh' \sim \phi_n f'$.

Assume that $n = q_1^{r_1} \cdot \ldots \cdot q_v^{r_v}$ where every q_i is a prime. Since for every q_i there exists a homotopy equivalence $h_{q_i}\colon X'_{q_i} \to X_{q_i}$ satisfying $f_{q_i}h_{q_i} \sim f'_{q_i}$, it follows from (1) that there exist an integer m, $(m, n) = 1$, and an l-equivalence $h''\colon X' \to X$ so that $fh'' \sim \phi_m f'$.

Let a and b be integers satisfying $an + blm = 1$. Define $h\colon X' \to X$ by $h = \phi_a h' + \phi_{bl}h''$. Since $\pi_*(h) \otimes Z_l = \pi_*(\phi_a h') \otimes Z_l$ and h' is an l-equivalence, h is an l-equivalence. But $fh = f(\phi_a h' + \phi_{bl}h'') \sim \phi_a fh' + \phi_{bl}fh'' \sim \phi_a\phi_n f' + \phi_{bl}\phi_m f' \sim \phi_{an+blm}f' \sim f'$; hence h is the desired map.

(3) is proved similarly.

(b) (1) Since Y is an H-space and $(SX')_{p_i} = SX'_{p_i}$, there exist integers $n_{1,i}$, $n_{2,i}$, $(n_{1,i}, p_i) = (n_{2,i}, p_i) = 1$, so that the maps $h_{p_i}\eta_{n_{1,i}}$ and $\phi_{n_{2,i}}k_{p_i}$ are induced by maps $h'_i\colon SX' \to X$ and $k'_i\colon Y' \sim Y$.

Consider the following diagram:

$$\begin{tikzcd}
SX'_{p_i} \arrow[r, "\eta_{n_{1,i}} \circ \eta_{n_{2,i}}"] \arrow[d, "f'_{p_i}"] \arrow[rr, bend left, "(h'_i)_{p_i} \circ \eta_{n_{2,i}}"] & SX'_{p_i} \arrow[r, "h_{p_i}"] & X_{p_i} \arrow[d, "f_{p_i}"] \\
Y'_{p_i} \arrow[r, "k_{p_i}"] \arrow[rr, bend right, "\phi_{n_{1,i}}(k'_i)_{p_i}"'] & Y_{p_i} \arrow[r, "\phi_{n_{1,i}} \circ \phi_{n_{2,i}}"] & Y_{p_i}
\end{tikzcd}$$

Since $f_{p_i}h_{p_i}\eta_{n_{1,i}}\eta_{n_{2,i}} \sim \phi_{n_{1,i}}\phi_{n_{2,i}}k_{p_i}f'_{p_i}$, for every i the p_i-localization of $f(h'_i\eta_{n_{2,i}})$ and $(\phi_{n_{1,i}}k'_i)f'$ are homotopic. Therefore there exist integers n_i, $(n_i, p_i) = 1$, so that

$$(\phi_{n_i}\phi_{n_{1,i}}k'_i)f' \sim \phi_{n_i}(\phi_{n_{1,i}}k'_i)f' \sim \phi_{n_i}(fh'_i\eta_{n_{2,i}}) \sim f(h'_i\eta_{n_{2,i}}\eta_{n_i}).$$

Define $m = \prod_{i=1}^{s} p_i$,

$$h''_i = h'_i\eta_{n_{2,i}n_i}\eta_{m/p_i}, \quad k'_i = \phi_{m/p_i}\phi_{n_in_{1,i}}k'_i.$$

Then $h = \sum_i h''_i$ and $k = \sum_i k''_i$ are the desired maps: h and k are obviously l-equivalences and $fh \sim kf'$.

(2) and (3) follow from (1) in the same way that (2) and (3) of part (a) follow from (1) of part (a).

(c) (1) Choose localizations φ_i: $S^n \to S^n_{p_i}$, ψ_i: $S^{2m-1} \to S^{2m-1}_{p_i}$ so that the following diagram is commutative:

$$\begin{array}{ccccc}
S_n & \overset{\varphi_i}{\to} & S^n_{p_i} & \overset{h_{p_i}}{\to} & S^n_{p_i} \\
\downarrow f' & & \downarrow f'_{p_i} & & \downarrow f_{p_i} \\
S^{2m-1} & \overset{\psi_i}{\to} & S^{2m-1}_{p_i} & \overset{k_{p_i}}{\to} & S^{2m-1}_{p_i}
\end{array}$$

Since $h_{p_i}\varphi_i \in (\pi_n S^n)_{p_i}$ and $k_{p_i}\psi_i \in (\pi_{2m-1}S^{2m-1})_{p_i}$, for every i, there exists an integer v_i, $(v_i, p_i) = 1$, so that $h_{p_i}\varphi_i\eta_{v_i}$ and $k_{p_i}\psi_i\eta_{v_i}$ are induced by maps $\tilde{h}_i$: $S^n \to S^n$ and $\tilde{k}_i$: $S^{2m-1} \to S^{2m-1}$.

Consider the following diagram:

$$\begin{tikzcd}
S^n \arrow[rr, "\tilde{h}_i"] \arrow[d, "\eta_{v_i}"] \arrow[ddd, bend right=60, "f'"'] & & S^n \arrow[d, "\varphi_i"] \arrow[ddd, bend left=60, "f"] \\
S^n \arrow[r, "\varphi_i"] \arrow[d, "f'"] & S^n_{p_i} \arrow[r, "h_{p_i}"] \arrow[d, "f'_{p_i}"] & S^n_{p_i} \arrow[d, "f_{p_i}"] \\
S^{2m-1} \arrow[r, "\psi_i"] & S^{2m-1}_{p_i} \arrow[r, "k_{p_i}"] & S^{2m-1}_{p_i} \\
S^{2m-1} \arrow[u, "\eta_{v_i}"] \arrow[rr, "\tilde{k}_i"] & & S^{2m-1} \arrow[u, "\psi_i"']
\end{tikzcd}$$

For $n = 2m - 1$ the diagram commutes; hence in this case $f_{p_i}(\tilde{h}_i)_{p_i} \sim (\tilde{k}_i)_{p_i} f'_{p_i}$. For $n > 2m - 1$ the two squares and all the rectangles except the left one are commutative. Therefore in order to prove the existence of p_i-equivalences, $h'_i\colon S^n \to S^n$ and $k'_i\colon S^{2m-1} \to S^{2m-1}$ so that $f_{p_i}(h'_i)_{p_i} \sim (k'_i) f'_{p_i}$, it suffices to prove

2.1.1. Lemma. *For every map $f'\colon S^n \to S^{2m-1}$ and every integer v, there exists an integer t so that $\eta_{v^t} f' \sim f' \eta_{v^t}$.*

Proof of 2.1.1. Let $f_1, f_2\colon S^n \to S^{2m-1}$ be arbitrary maps. Consider the diagram:

$$\begin{tikzcd}
S^n \arrow[r, "\eta_v"] \arrow[d, "\psi"] & S^n \arrow[d, "\psi"] & & \\
S^n \vee S^n \arrow[r, "\eta_v \vee \eta_v"] & S^n \vee S^n \arrow[r, "f_1 \vee f_2"] & S^{2m-1} \vee S^{2m-1} \arrow[r, "\mathfrak{F}"] \arrow[d, "\eta_v \vee \eta_v"] & S^{2m-1} \arrow[d, "\eta_v"] \\
& & S^{2m-1} \vee S^{2m-1} \arrow[r, "\mathfrak{F}"] & S^{2m-1}
\end{tikzcd}$$

Obviously the right-hand square is commutative. Considering the homology homomorphisms one can easily see that the left-hand square is commutative as well. Hence $\eta_v(f_1 + f_2) \sim \eta_v f_1 + \eta_v f_2$ and $(f_1 + f_2)\eta_v \sim f_1\eta_v + f_2\eta_v$.

Since the order of $f'\colon S^n \to S^{2m-1}$ is finite there exist $f_1, f_2\colon S^n \to S^{2m-1}$ satisfying $f' = f_1 + f_2$, $(|f_1|, v) = 1$ and $|f_2|/v^a$ (for some integer a). Hence $f_2\eta_{v^a} \sim *$. As for every p_j, p_j/v, $H^*(\eta_v, Z_{p_j}) = 0$, there exists an integer b so that, for every p_j/v, $\pi_*(\eta_{v^b}, Z_{p_j}) = 0$ (Zabrodsky [9, proof of Theorem 4.1.4]). Hence for every p_j/v, $0 = (\eta_{v^b})_*\colon \pi_* S^{2m-1} \otimes Z_{p_j} \to \pi_* S^{2m-1} \otimes Z_{p_j}$. Consequently there exists an integer c so that $0 = (\eta_{v^{bc}})\colon {}^{|f_2|}\pi_* S^{2m-1} \to {}^{|f_2|}\pi_* S^{2m-1}$. Define $d = abc$. Obviously for every $d' > d$, $(f_1 + f_2)\eta_{v^{d'}} \sim f_1\eta_{v^{d'}}$ and $\eta_{v^{d'}}(f_1 + f_2) \sim \eta_{v^{d'}} f_1$.

Since $(|f_1|, v^d) = 1$ there exists an integer $\tilde{d} > d$ so that $v^{\tilde{d}} \equiv 1(|f_1|)$. Consider the map $\eta_{v^{\tilde{d}}}\colon S^{2m-1} \to S^{2m-1}$. This map is an l-equivalence; hence there exists an integer e so that $1 = \eta_{(v^{\tilde{d}})^e}\colon {}^{|f_1|}\pi_n S^{2m-1} \to {}^{|f_1|}\pi_n S^{2m-1}$. Define $t = (\tilde{d})^e$. Since

$$f'\eta_{v^t} = (f_1 + f_2)\eta_{v^t} \sim f_1\eta_{v^t} \sim f_1 \sim \eta_{v^t} f_1 \sim \eta_{v^t}(f_1 + f_2) = \eta_{v^t} f',$$

t is the desired integer which completes the proof of 2.1.1.

Let h_i': $S^n \to S^n$, k_i': $S^{2m-1} \to S^{2m-1}$ be p_i-equivalences satisfying $f_{p_i}(h_i')_{p_i} \sim (k_i')_{p_i} f'_{p_i}$. As $k_i' f'$, $fh_i' \in \pi_n S^{2m-1}$ there exist integers u_i, $(u_i, p_i) = 1$, so that $fh_i'\eta_{u_i} \sim k_i' f' \eta_{u_i}$ (η_{u_i}: $S^n \to S^n$). Consequently we obtain from 2.1.1 that there exist integers w_i satisfying

$$fh_i'\eta_{u_i^{w_i}} \sim k_i' f' \eta_{u_i^{w_i}} \sim (k_i'\eta_{u_i^{w_i}})f'.$$

Suppose that $\eta_{p_i^{r_i}} f' \sim k_i' f' \eta_{p_i^{r_i}}$, $\eta_{2^t} f' \sim f'\eta_{2^t}$ and $p_i \neq 2$ for $i > 1$. Define

$$v = 2^t \prod p_i^{r_i},$$

$$h_i'' = \begin{cases} h_1'\eta_{u_1^{w_1}}\eta_{v/(2^t p_1^{r_1})}, & i = 1, \\ h_i'\eta_{u_i^{w_i}}\eta_{v/p_i^{r_i}}, & i \neq 1; \end{cases}$$

$$k_i'' = \begin{cases} k_1'\eta_{u_1^{w_1}}\eta_{v/(2^t p_1^{r_1})}, & i = 1, \\ k_i'\eta_{u_i^{w_i}}\eta_{v/p_i^{r_i}}, & i \neq 1; \end{cases}$$

$$h = \sum_i h_i'', \qquad k = \sum_i k_i''.$$

Obviously h and k are l-equivalences. In order to prove that $fh \sim kf'$ it is enough to prove that $(\sum_i k_i'')f' \sim \sum(k_i'' f')$.

Let ι_j: $S^{2m-1} \to S^{2m-1} \times (S^{2m-1})^{s-1}$ be the inclusion into the jth factor. There exists a map α: $S^{2m-1} \times (S^{2m-1})^{s-1} \to S^{2m-1}$ so that for every $j > 1$ the diagram

$$\begin{tikzcd}
S^{2m-1} \arrow[d, "\iota_j"] \arrow[dr, "\eta_{2^t}"] & \\
S^{2m-1} \times (S^{2m-1})^{s-1} \arrow[r, "\alpha"] & S^{2m-1} \\
S^{2m-1} \arrow[u, "\iota_1"'] \arrow[ur, "1"'] &
\end{tikzcd}$$

is commutative. Therefore the following diagram is commutative:

$$\begin{tikzcd}
 & S^{2m-1} \times (S^{2m-1})^{s-1} \arrow[r, "\alpha"] & S^{2m-1} \\
S^n \times (S^n)^{s-1} \arrow[ur, "k_1''f' \times \prod_{i=2}^{s} \frac{k_i''f'}{2^t}"] & S^{2m-1} \vee \cdots \vee S^{2m-1} \arrow[u, "U"'] \arrow[r, "1 \vee 2^t \vee \cdots \vee 2^t"] & S^{2m-1} \vee \cdots \vee S^{2m-1} \arrow[u, "\mathfrak{F}(\mathfrak{F} \vee 1) \cdots (\mathfrak{F} \vee 1 \vee \cdots \vee 1)"'] \\
S^n \arrow[u, "\Delta^s"'] \arrow[r, "(\psi \vee 1 \vee \cdots \vee 1) \circ \cdots \circ (\psi \vee 1)\psi"'] \arrow[uurr, bend right=60, "\sum_i (k_i''f')"'] & S^n \vee \cdots \vee S^n \arrow[u, "k_1''f' \vee \frac{k_2'f'}{2^t} \vee \cdots \vee \frac{k_s''f'}{2^t}"'] \arrow[ur, "k_1f' \vee \cdots \vee k_sf'"'] &
\end{tikzcd}$$

$$(\Delta^s(x) = (x, \ldots, x)), \quad \text{namely} \quad \sum_i (k_i''f') = \alpha \circ \left(k_1''f' \times \prod k_i''f'/2^t\right) \circ \Delta^s.$$

Consequently the commutativity of the diagram

$$\begin{tikzcd}
S^n \arrow[r, "\Delta^s"] \arrow[d, "f'"] & S^n \times (S^n)^{s-1} \arrow[d, "f' \times (f')^{s-1}"] \arrow[dr, "k_1 f' \times \prod_{i>1} \frac{k_i'' f'}{2^t}"] & \\
S^{2m-1} \arrow[r] \arrow[drr] & S^{2m-1} \times (S^{2m-1})^{s-1} \arrow[r, "k_1'' \times \prod_{i>1} \frac{k_i''}{2^t}"'] & S^{2m-1} \times (S^{2m-1})^{s-1} \arrow[d, "\alpha"] \\
 & & S^{2m-1}
\end{tikzcd}$$

implies that $(\Sigma_i\, k_i'')f' \sim \Sigma_i(k_i'' f')$.

Since for any two maps $g_1, g_2\colon S^n \to S^n$, $f(g_1 + g_2) \sim fg_1 + fg_2$, (2) follows from (1) in the same way that (2) of part (a) follows from (1) of the same part.

Notation. Let X be an H_0-space, denote by $N(X)$ the least integer satisfying either, for every $n > N(X)$, $\pi_n X = 0$ or, for every $n > N(X)$, $H_n X = 0$. (Recall that we consider only spaces with $N(X) < \infty$.)

2.3. Corollary. (a) *Given an H-fibration $F \to X \xrightarrow{f} Y$; if $H^*(f, Q)$ is surjective then $G_Y(f) = 0$.*

(b) *Given a fibration $F \to X \xrightarrow{f} Y$ so that Y is an H-space and X is an H_0-space: If $H^*(f, Q)$ is injective then $G^X(f) = 0$.*

(c) (a) *and* (b) *hold also for a fibration of the form $F \to SX \to Y$ where Y is an H-space.*

Proof. (a) Let $F \to X \xrightarrow{f} Y$ be an H-fibration and let $l = \Pi_{n<N}|\text{torsion } \pi_n X| \cdot |\text{torsion } \pi_n F|$ where $N = \max\{N(X), N(Y)\}$. By Theorem 2.2 there exist l-equivalences $h\colon X' \to X$ and $h'\colon F \to F$ so that the following diagram commutes

$$\begin{tikzcd}
F' \arrow[r] \arrow[d, "h'"] & X' \arrow[d, "h"] \arrow[dr, "f'"] & \\
F \arrow[r] & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

We shall prove that h is a homotopy equivalence.

As $h_\#\colon {}^t\pi_* X' \to {}^t\pi_* X$ and $h'_\#\colon {}^t\pi_* F' \to {}^t\pi_* F$ (${}^t\pi_* X = \text{torsion}(\pi_* X)$) are isomorphisms and as h, h' are 0-equivalences, $h_\#\colon \pi_* X'/\text{torsion} \to \pi_* X/\text{torsion}$ and $h'_\#\colon \pi_* F'/\text{torsion} \to \pi_* F/\text{torsion}$ are monomorphisms, so are $h_\#\colon \pi_* X' \to \pi_* X$ and $h'_\#\colon \pi_* F' \to \pi_* F$ and h is a homotopy equivalence if and only if $h_\#$ is a surjection.

Consider the following diagram:

$$\begin{tikzcd}
\pi_n X' \arrow[r, "f'_\#"] \arrow[d, "h_\#"] & \pi_n Y \arrow[r, "\partial'"] \arrow[d, equal] & \pi_{n-1} F' \arrow[d, "h'_\#"] \\
\pi_n X \arrow[r, "f_\#"] & \pi_n Y \arrow[r, "\partial"] & \pi_{n-1} F
\end{tikzcd}$$

Let $v \in \pi_n X$ be of infinite order. As $H^*(f, Q)$ is surjective, $H_*(f, Q)$ and $\pi_*(f) \otimes Q$ are injective and so is $\pi_*(f)/\text{torsion}$; hence $w = f_\# v$ is of infinite order as well. $0 = \partial w = h'_\# \partial' w$; hence $\partial'(w) = 0$ and there exists $v' \in \pi_n X'$ so that $f'_\# v' = w$. Hence, $v - h_\# v' \in \ker f_\# \subseteq \text{torsion } \pi_n X \subseteq \text{im } h_\#$, $v \in \text{im } h_\#$ and $h_\#$ is surjective.

(b) and (c) are proved similarly.

3. The structure of $G(f)$, $G^X(f)$ **and** $G_Y(f)$. In this section we use Zabrodsky's method of constructing the genus of an H_0-space (with a finite number of homotopy or homology groups–Zabrodsky **[8]**) to obtain elements in the genus of a map $f: X \to Y$ where X and Y are H_0-spaces. We go on to prove that every element in the genus of a map which satisfies the conditions of Theorem I is obtained in this way. The same method is also good for constructing $G^X(f)$, $(G_Y(f))$ for maps which satisfy the conditions of Theorem III (I).

3.1. Definitions and notations. Let P be the set of all primes. For any integer t denote by P_t the set of all primes which divide t and by $\bar{t}$ the set $P - P_t$.

Let X be an H_0-space, i.e. $H^*(X, Q)$ is a free commutative graded algebra. Denote by $[X, X]_t$ the set of homotopy classes of t-equivalences $f: X \to X$. Denote by $t(X)$ the number $\Pi_{n < N(X)} |{}^t H^n(X)|$, by $K(X)$ the space $K(QH^*(X, Z)/\text{torsion})$ and by $l(x)$ the number of integers n for which $QH^n(X, Q) \neq 0$. Let Γ be a splitting $\text{Hom}°(QH^*(X, Z)/\text{torsion}, \ QH^*(X, Z)/\text{torsion}) \to [K(X), K(X)]$: $QH^*(\Gamma(f), Z)/\text{torsion} = f$.

Let A be an $n \times n$ matrix. We shall say that A is diagonal if

$$A_{ij} = \begin{cases} \lambda_i, & i = j, i \leqslant \min(m, n), \\ 0, & \text{otherwise} \end{cases}$$

(some of the λ_i's may be zero).

Suppose f: $X \to Y$ is a map and X and Y are H_0-spaces. Let $B_X = \{x_{m'_1}, x_{m'_2}, \dots, x_{m'_r}\}$, $\dim x_{m'_i} = m'_i$, $m'_i \leqslant m'_{i+1}$ and $B_Y = \{y_{n'_1}, y_{n'_2}, \dots, y_{n'_s}\}$, $\dim y_{n'_i} = n'_i$, $n'_i \leqslant n'_{i+1}$ be bases for $QH^*(X, Z)/\text{torsion}$ and $QH^*(Y, Z)/\text{torsion}$ in which $QH^*(f, Z)/\text{torsion}$ is represented by a diagonal matrix A.

Assume that $QH^{m_j}(Y, Q) \neq 0$ for $j = 1, \dots, l(X)$, $m_1 < m_2 < \dots < m_{l(X)}$, and that $QH^{n_k}(Y, Q) \neq 0$ for $k = 1, \dots, l(Y)$, $n_1 < n_2 < \dots < n_{l(Y)}$. Obviously for every $1 \leqslant i \leqslant r$ there exists a j $(1 \leqslant j \leqslant l(X))$ so that $m'_i = m_j$, and for every $1 \leqslant i \leqslant s$ there exists a K $(1 \leqslant k \leqslant l(Y))$ so that $n'_i = n_k$.

Let $\tilde{t} = \Pi_{n < N} |{}^t H^n(X, Z)|$ $(N = \max\{N(X), N(Y)\})$ and let ψ: $X \to K(X)$, φ: $Y \to K(Y)$ be rational equivalences realizing $\{\tilde{t}x_{m_1}, \dots, \tilde{t}x_{m_r}\}$ and $\{\tilde{t}y_{n'_1}, \dots, \tilde{t}y_{n'_s}\}$, respectively. Denote by t the least common multiple of $\tilde{t}$, $\Pi_{n < N} |{}^t H^n(Y, Z)|$, $\Pi_{n < N} |\pi_n(\text{fiber } \varphi)|$, $\Pi_{n < N} |\pi_n(\text{fiber } \psi)|$ and the nonzero elements of A.

Denote by $D \subset Z^{l(X) + l(Y)}$ the set of pairs (d, d'), $d = (d_{m_1}, \dots, d_{m_{l(X)}}) \in Z^{l(X)}$, $d' = (d'_{n_1}, \dots, d'_{n_{l(Y)}}) \in Z^{l(Y)}$ satisfying the following conditions:

(a) For every i, $(d_{n_i}, t) = (d_{m_i}, t) = 1$.

(b) If $QH^{n_i}(f, Q)$ is a monomorphism and $QH^{n_i}(Y, Q) \neq 0$ then d'_{n_i} / d_{n_i}.

(c) If $QH^{n_i}(f, Q)$ is an epimorphism and $QH^{n_i}(X, Q) \neq 0$ then d_{n_i} / d'_{n_i}.

(d) If $QH^{n_i}(f, Q)$ is an isomorphism then $d_{n_i} = d'_{n_i}$.

3.2. THEOREM (ZABRODSKY [8]). *Let X be an H_0-space with $QH^{m_i}(X, Q) \neq 0$ for $i = 1, \dots, l(X)$ and let ψ: $X \to K(X)$ be a rational equivalence. Suppose $t(X, \psi)$ is an integer divisible by $\prod_{n < N(X)} |{}^t\pi_n(\text{fiber } \psi)|$.*

Then $G(X)$ admits an abelian group structure and there exists an exact sequence

$$[X, X]_{t(X,\psi)} \xrightarrow{\alpha} \left[(Z^*_{t(X,\psi)})/\pm 1\right]^{l(X)} \xrightarrow{\xi} G(X) \to 0$$

where α is the composition

$$[X, X]_{t(X,\psi)} \to \operatorname{aut}(QH^*(X, Z)/\mathit{torsion} \otimes Z_{t(X,\psi)}) \xrightarrow{|\det|} \left[(Z^*_{t(X,\psi)})/\pm 1\right]^{l(X)}$$

and ξ is given as follows: Let $d_1, \dots, d_{l(X)}$ be integers satisfying $(d_i, t(X, \psi)) = 1$ for every i and let $I_{d_1, \dots, d_{l(X)}}$: $QH^*(X, Z)/\mathit{torsion} \to QH^*(X, Z)/\mathit{torsion}$ satisfy $\det(I_{d_1, \dots, d_{l(X)}} | QH^{m_i}(X, Z)/\mathit{torsion}) = d_i$. *Consider the following pull-back diagram*

$$\begin{tikzcd}
\tilde{X} \arrow[r] \arrow[d] & X \arrow[d, "\psi"] \\
K(X) \arrow[r, "\Gamma(I_{d_1, \dots, d_{l(X)}})"] & K(X)
\end{tikzcd}$$

If X has a finite number of homotopy groups define $\xi(d_1, \dots, d_{l(X)}) = \tilde{X}$ and if X is finite dimensional define $\xi(d_1, \dots, d_{l(X)}) = HL_{\dim X}(\tilde{X})$.

3.3. DEFINITION. Let X be an H_0-space so that $QH^{m_i}(X, Q) \neq 0$ for $i = 1, \dots, l(X)$ and let f: $X \to X$ be a map. Suppose $d = (d_1, \dots, d_{l(X)}) \in Z^{l(X)}$. We say that f *realizes* d if, for every i, $\det(QH^{m_i}(f, Z)/\text{torsion}) = d_i$.

3.4. PROPOSITION. *Let f: $X \to Y$ be a map.*

(a) *If X, Y are H_0-spaces, then for every pair $(d, d') \in D$ there exist a map f': $X' \to Y'$, $G(f)$ and t-equivalences h: $X' \to X$ and k: $Y' \to Y$ so that h realizes d, k realizes d' and $fh \sim kf'$.*

(b) *Let f be an H-map and suppose $H^*(X, Q)$ and $H^*(Y, Q)$ are primitively generated. Then for every pair $(d, d') \in D$ there exist an H-map f': $X' \to Y'$ and H-maps h and k so that* (a) *is satisfied.*

PROOF. Let B_X, B_Y, φ and ψ be as in 3.1 and let f: $K(X) \to K(Y)$ satisfy $g\psi \sim \varphi f$.
Let α: $K(X) \to K(X)$ and β: $K(Y) \to K(Y)$ satisfy:
$\alpha = \prod \alpha_i$ (α_i: $K(Z, m_i') \to K(Z, m_i')$), $\det(QH^{m_i}(\alpha, Z)/\text{torsion}) = d_{m_i}$.
$\beta = \prod \beta_i$ (β_i: $K(Z, n_i') \to K(Z, n_i')$), $\det(QH^{n_i}(\beta, Z)/\text{torsion}) = d'_{n_i}$.
$g\alpha \sim \beta g$ (such α and β exist since, for every i, $g^*(\iota_{n_i'}) = \lambda_i \iota_{n_i'}$ or $g^*(\iota_{n_i}) = 0$).
Consider the diagram

$$\begin{tikzcd}
X' \arrow[rrr, "h"] \arrow[dr, "\theta"] \arrow[ddd, "f'"'] & & & X \arrow[dl, "\psi"'] \arrow[ddd, "f"] \\
& K(X) \arrow[r, "\alpha"] \arrow[d, "g"] & K(X) \arrow[d, "g"] & \\
& K(Y) \arrow[r, "\beta"] & K(Y) & \\
Y' \arrow[ur, "\phi"] \arrow[rrr, "k"] & & & Y \arrow[ul, "\varphi"']
\end{tikzcd} \tag{3.4.1}$$

where X' is the pull-back of $X \xrightarrow{\psi} K(X) \xleftarrow{\alpha} K(X)$ and Y' is the pull-back of $Y \xrightarrow{\varphi} K(Y) \xleftarrow{\beta} K(Y)$.

Since the lower trapezoid is a pull-back and $\varphi f h \sim \beta g \theta$, there exists a map f': $X' \to Y'$ so that $kf' \sim fh$ and $\phi f' \sim g\theta$. Consider the diagram:

$$\begin{array}{ccccc} X'_{t} & \xrightarrow{\theta_{\bar t}} & K(X)_{\bar t} & \xrightarrow{(\psi_{\bar t})^{-1}} & X_{\bar t} \\ \downarrow f'_{t} & & \downarrow g_{\bar t} & & \downarrow f_{\bar t} \\ Y'_{t} & \xrightarrow{\phi_{\bar t}} & K(Y)_{\bar t} & \xrightarrow{(\varphi_{\bar t})^{-1}} & Y_{\bar t} \end{array}$$

(If f: $X \to Y$ is a homotopy equivalence f^{-1} denotes the homotopy inverse of f.) Since this diagram is commutative and its horizontal rows are homotopy equivalences the map f': $X' \to Y'$ belongs to $G(f)$.

(b) Choose bases $\{x_{m'_1}, \dots, x_{m'_r}\}$, $\{y_{n'_1}, \dots, y_{n'_s}\}$ for $PH^*(X, Z)$/torsion and $PH^*(Y, Z)$/torsion, respectively, in which $PH^*(f, Z)$/torsion is represented by a diagonal matrix. (By Curjel **[2]** such bases exist.) Let ψ: $X \to K(X)$, φ: $Y \to K(Y)$ realize $\{\bar t x_{m'_1}, \dots, \bar t x_{m'_r}\}$ and $\{\bar t y_{n'_1}, \dots, \bar t y_{n'_s}\}$, respectively. Obviously ψ and φ are H-maps.

Let g, α and β be as in part (a). Consider diagram 3.4.1. Obviously h and k are H-maps and $f' \in G(f)$. We shall prove that f' is an H-map:

Since the maps

$$(\Omega\beta)_*: [X' \times X', \Omega K(Y)] \to [X' \times X', \Omega K(Y)],$$
$$(\Omega\varphi)_*: [X' \times X', \Omega Y] \to [X' \times X', \Omega K(Y)]$$

are t and $\bar t$ equivalences, respectively, the map

$$(\Omega\beta)_* + (\Omega\varphi)_*: [X' \times X', \Omega K(Y)] \oplus [X' \times X', \Omega Y] \to [X' \times X', \Omega K(Y)]$$
$$\big((\Omega\beta)_* + (\Omega\varphi)_*\big)(a, b) = (\Omega\beta)_*(a) + (\Omega\varphi)_*(b)$$

is an epimorphism (Arkowitz **[1]**, Proposition 4.3]). This together with the fact that $\phi f'$ and kf' are H-maps implies (Arkowitz **[1]**, Proposition 10.3]) that f' is an H-map.

3.5. Corollary. (a) *If* $(d, 1) \in D$ *then Proposition* 3.4 *is true for* $G_Y(f)$.

(b) *If* $(1, d') \in D$ *then Proposition* 3.4 *is true for* $G^X(f)$.

Proof. (a) Choose $\phi = \varphi$, $k = 1$, $\beta = 1$.

(b) Choose $\theta = \psi$, $h = 1$, $\alpha = 1$.

3.6. Proposition. *Suppose* X_1, X_2 *are* H*-spaces so that* $H^*(X_i, Q)$ $(i = 1, 2)$ *are primitively generated*; Y_1, Y_2 *are* H_0*-spaces*; f: $X_1 \to X_2$ *an* H*-map, and* g: $Y_1 \to Y_2$ *a map*.

Let $B_{X_i} = \{x_{i_1}, \dots, x_{im_i}\}$ *be bases for* $PH^*(X_i, Z)$/*torsion, and let* $B_{Y_i} = \{y_{i_1}, \dots, y_{in_i}\}$ *be bases for* $H^*(Y_i, Z)$/*torsion. Denote by* A *and* B *the matrices of* $PH^*(f, Z)$/*torsion and* $H^*(g, Z)$/*torsion in these bases.*

There exists an integer $t(f, g)$ *depending on* X_i, Y_i, f *and* g *so that*: *Given a pair of matrices* (*over* Z) (C_1, C_2) *satisfying* $C_1A = BC_2$, *there exist functions* h_i: $Y_i \to X_i$ $(i = 1, 2)$ *so that the following conditions are satisfied*:

(a) *The matrix of* $H^*(h_i, Z)/\mathit{torsion}|(PH^*(X_i, Z)/\mathit{torsion})$ *relative to the bases* B_{X_i} *and* B_{Y_i} *is* $t(f, g)C_i$.

(b) $fh_1 \sim h_2 g$.

PROOF. It is enough to prove the proposition in case that B_{X_i} and B_{Y_i} $(i = 1, 2)$ are bases in which the matrices A and B are diagonal.

Let λ be the multiple of the nonzero elements of A and B and let t be as in 3.1.

Let $\mathcal{C} = \{(C_1, C_2) \mid |(C_1)_{ij}| \leq t\lambda, |(C_2)_{ij}| \leq t\lambda \text{ for every } i \text{ and } j, C_1 A = BC_2\}$. To each pair $(C_1, C_2) \in \mathcal{C}$ correspond functions h_i: $Y_i \to X_i$ the matrices of which relative to B_{X_i} and B_{Y_i} are $\lambda t C_i$ (Zabrodsky [8, Proposition 1.8]). Since X_i and Y_i are $\bar{t}$-equivalent to $K(X_i)$ and $K(Y_i)$, respectively, the $\bar{t}$-localizations of fh_1 and $h_2 g$ coincide. Hence there exists an integer $s_{(C_1, C_2)}$ so that $(\phi_{s_{(C_1,C_2)}} h_2)g \sim f(\phi_{s_{(C_1,C_2)}} h_1)$. This together with the finiteness of the set $\mathcal{C}$ implies the existence of an integer s which is good for every pair $(C_1, C_2) \in \mathcal{C}$. We shall prove that $s = t(f, g)$.

As A and B are diagonal and $C_1 A = BC_2$, for every i and j, $(C_1)_{ij} a_{jj} = b_{ii}(C_2)_{ij}$. If $b_{ii} = b \cdot b'$ where $b/(C_1)_{ij}$ and b'/a_{jj} then

$$[(C_1)_{ij}/b] \cdot ba_{jj} = (C_1)_{ij} a_{jj} = b_{ii}(C_2)_{ij} = ba_{jj}[(C_2)_{ij}/(a_{jj}/b')].$$

Assume that $[(C_1)_{ij}/b] = tl_{ij} + c_{ij}$ where $|c_{ij}| < t$ or $c_{ij} = 0$. For every $1 \leq k \leq \max\{l_{ij}\} + 1$ define matrices C_1^k, C_2^k as follows:

$$(C_1^k)_{ij} = \begin{cases} c_{ij} b, & k = 1, \\ tb, & 1 < k \leq l_{ij} + 1, \\ 0, & k > l_{ij} + 1, \end{cases} \qquad (C_2^k)_{ij} = \begin{cases} c_{ij}[a_{jj}/b'], & k = 1, \\ t \cdot [a_{jj}/b'], & 1 < k \leq l_{ij} + 1, \\ 0, & k > l_{ij} + 1. \end{cases}$$

Obviously, for every k, the pair $(C_1^k, C_2^k) \in \mathcal{C}$ and $(C_1, C_2) = \Sigma_k (C_1^k, C_2^k)$.

Let h_i^k: $Y_i \to X_i$ $(i = 1, 2)$ be maps the matrices of which are sC_i^k and which satisfy $fh_1^k \sim h_2^k g$. Since B_{X_i} are bases for $PH^*(X_i, Z)/\text{torsion}$ and f is an H-map, the matrices of $\Sigma_k h_1^k$ (relative to B_{X_i} and B_{Y_i}) are sC_i and $f(\Sigma_k h_1^k) \sim (\Sigma h_2^k)g$. Consequently $t(f, g) = s$.

3.7. REMARK. If $C_1 A = BC_2$ and $C_1 = 0$ $(C_2 = 0)$ we obtain that

$$* \sim h_2 g (fh_1 \sim *).$$

3.8. COROLLARY. *Proposition* 3.6 *remains true if we substitute* $X_1 = Y_1 = \bigvee_{\mathit{finite}} S^{m_j}$, f: $X_1 \to X_2$ *a function and replace* B_{X_1} *and* B_{Y_1} *by bases* B'_{X_1} *and* B'_{Y_1} *of* $H^*(\bigvee S^{m_j})$.

PROOF. Analogous to the proof of Theorem 3.6, since if k_1, k_2: $Y_1 \to X_1$ are functions the matrices of which (relative to the bases B'_{X_1} and B'_{Y_1}) are C_1 and C_2 and l_1, l_2: $Y_2 \to X_2$ are functions which satisfy $fk_1 \sim l_1 g$, $fk_2 \sim l_2 g$. Then the matrix of $k_1 + k_2$ is $C_1 + C_2$ and $f(k_1 + k_2) \sim (l_1 + l_2)g$.

3.9. COROLLARY. *If in Proposition* 3.6 $Y_2 = X_2$, $B_{Y_2} = B_{X_2}$ *we obtain that there exists an integer* $t(f, g)$ *depending on* X_1, Y_1, X_2, f *and* g, *so that for every matrix* C *which satisfies* $CA = B$, *there exists a function* h: $Y_1 \to X_1$ *so that*:

(a) *The matrix of* $H^*(h, Z)/\mathit{torsion}|(PH^*(X_1, Z)/\mathit{torsion})$ *relative to the bases* B_{X_1} *and* B_{Y_1} *is* $t(f, g)C$.

(b) $fh \sim \phi_{t(f,g)} g$.

Proof. It is enough to prove the assertion in the case that B_{X_i} $(i = 1, 2)$ are bases in which the matrix A of $PH^*(f, Z)/\text{torsion}$ is diagonal.

Since $CA = B$, the matrix C is of the form

$$C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}$$

where C_1 is completely determined by A and B. The corollary follows from the fact that every matrix D of the form

$$D = \begin{pmatrix} 0 & C_2 \\ C_3 & C_4 \end{pmatrix}$$

can be written as $D = \sum_{k \text{ finite}} D_k$, where $|(D_k)_{ij}| < t$ for every i and j and $D_k A = 0$ for every k.

3.10. **Corollary.** *If in Proposition* 3.6 $Y_1 = X_1$, $B_{X_1} = B_{Y_1} =$ *a basis for* $H^*(Y_1, Z)/$*torsion, we obtain that there exists an integer* $t(f, g)$ *so that for every matrix* C *which satisfies* $A = BC$ *there exists a map* $k\colon Y_2 \to X_2$ *so that*:

(a) *The matrix of* $H^*(k, Z)/$*torsion*$|(PH^*(X, Z)/$*torsion*) *relative to the bases* B_{X_2} *and* B_{Y_2} *is* $t(f, g)C$.

(b) $kg \sim \phi_{t(f,g)} f$.

Proof. Similar to the proof of 3.9.

Remark. The corollary remains true if one replaces the conditions that X_1 is an H-space and f is an H-map by the conditions that X_1 is an H_0-space and f is a map.

If A and B are matrices denote by $A * B$ the matrix $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$.

If X, Y are H_0-spaces and for every $1 \leq i \leq l$ either $H^{s_i}(X, Q) \neq 0$ or $H^{s_i}(Y, Q) \neq 0$ we can identify

$\mathrm{Hom}^0_Z(QH^*(Y, Z)/\text{torsion}, QH^*(X, Z)/\text{torsion})$ with the set of matrices $A^Q = A^Q_{s_1} * A^Q_{s_2} * \cdots * A^Q_{s_l}$,

$\mathrm{Hom}^0_Z(DH^*(Y, Z)/\text{torsion}, DH^*(X, Z)/\text{torsion})$ with the set of matrices $A^D = A^D_{s_1} * A^D_{s_2} * \cdots * A^D_{s_l}$ and

$\mathrm{Hom}^0_Z(QH^*(Y, Z)/\text{torsion}, H^*(X, Z)/\text{torsion})$ with the set of matrices $A = A_{s_1} * A_{s_2} * \cdots * A_{s_l}$ where

$$A_{s_i} = \begin{bmatrix} A^Q_{s_i} \\ A^D_{s_i} \end{bmatrix}$$

and the order of the matrices $A^Q_{s_i}$, $A^D_{s_i}$ is completely determined by

$$H^*(X, Z)/\text{torsion} \quad \text{and} \quad H^*(Y, Z)/\text{torsion}.$$

We denote

$$A = \begin{pmatrix} A^Q \\ A^D \end{pmatrix}.$$

3.11. PROPOSITION. *Suppose $f\colon X \to Y$ is an H-map, $H^*(X, Q)$ and $H^*(Y, Q)$ are primitively generated and $H^*(f, Q)$ is either a monomorphism, an epimorphism, an isomorphism or zero.*

Given a fibration $F'' \to X'' \xrightarrow{f''} Y''$ in $G(f)$ and a commutative diagram

$$\begin{array}{ccc} X'' & \xrightarrow{h''} & X \\ \downarrow f'' & & \downarrow f \\ Y'' & \xrightarrow{k''} & X \end{array} \tag{3.11.1}$$

where h'' and k'' realize $\tilde{d} = (\pm d_{m_1}, \ldots, \pm d_{m_{t(X)}})$ and $\tilde{d}' = (\pm d'_{n_1}, \ldots, \pm d'_{n_{t(Y)}})$, respectively. Then the map $f''\colon X'' \to Y''$ is homotopy equivalent to the map $f'\colon X' \to Y'$, which corresponds to the pair (d, d') ($d = (d_{m_1}, \ldots, d_{m_{t(X)}})$, $d' = (d'_{n_1}, \ldots, d'_{n_{t(Y)}})$) in the construction of Proposition 3.4.

REMARKS. (1) The pair (C_1, C_2) which appears in Proposition 3.6 is equal to the matrix

$$\begin{pmatrix} C_1^Q & C_2^Q \\ C_1^D & C_2^D \end{pmatrix}.$$

(2) When we write in the proof functions which correspond to the matrix

$$\begin{pmatrix} t_1 C_1^Q & t_2 C_2^Q \\ t_3 C_1^D & t_4 C_4^D \end{pmatrix},$$

where $t(f', f'')/t_i$ for every i, we mean functions $h\colon X'' \to X'$, $k\colon Y'' \to Y'$ so that the matrices of

$$H^*(h, Z)/\text{torsion}|(PH^*(X', Z)/\text{torsion})$$

and

$$H^*(k, Z)/\text{torsion}|(PH^*(Y', Z)/\text{torsion})$$

are

$$\begin{pmatrix} t_1 & C_1^Q \\ t_3 & C_1^D \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} t_2 & C_2^Q \\ t_4 & C_2^D \end{pmatrix},$$

respectively, and which satisfy $f'h \sim kf''$. (Such functions exist by Proposition 3.6.)

PROOF OF PROPOSITION 3.11. We shall prove the proposition in the case $\tilde{d} = d$, $\tilde{d}' = d'$. The proof in the case $\tilde{d} = (\pm d_{m_1}, \ldots, \pm d_{m_{t(X)}}) \neq d$ or $\tilde{d}' = (\pm d'_{n_1}, \ldots, \pm d'_{n_{t(Y)}}) \neq d'$ is similar.

By Zabrodsky [8] $X'' \approx X'$, $Y'' \approx Y'$ and $F'' \approx F'$. After localization at t of diagram 3.11.1 and of the outer square of diagram 3.4.1 we obtain a commutative diagram:

$$\begin{tikzcd} X'_t \approx X''_t \arrow[rr, "h''_t"] \arrow[dr, "h_t^{-1} h''_t"] \arrow[dd, "f''_t"'] & & X_t \arrow[dd, "f_t"] \\ & X'_t \arrow[ur, "h_t"'] \arrow[d, "f'_t"] & \\ & Y'_t \arrow[dr, "k_t"] & \\ Y'_t \approx Y''_t \arrow[ur, "k_t^{-1} k''_t"] \arrow[rr, "k''_t"] & & Y_t \end{tikzcd}$$

The fact that the left trapezoid is commutative and f' is an H-map imply (Theorem 2.2) the existence of an integer n_1, $(n_1, t) = 1$, and maps $\tilde{h}: X'' \to X'$, $\tilde{k}: Y'' \to Y'$ so that $f'\tilde{h} \sim \tilde{k}f''$, $\tilde{h}_t \sim \phi_{n_1} h_t^{-1} h_t''$ and $\tilde{k}_t \sim \phi_{n_1} k_t^{-1} k_t''$. As h and k are H-maps $(h\tilde{h})_t \sim \phi_{n_1} h_t''$ and $(k\tilde{k})_t \sim \phi_{n_2} k_t''$. Therefore there exists an integer n_2, $(n_2, t) = 1$, so that $\phi_{n_2}(\phi_{n_1} h'') \sim \phi_{n_2}(h\tilde{h}) \sim h(\phi_{n_2}\tilde{h})$ and $\phi_{n_2}(\phi_{n_1} k'') \sim \phi_{n_2}(k\tilde{k}) \sim k(\phi_{n_2}\tilde{k})$. The maps $h' = \phi_{n_2}\tilde{h}$, $k' = \phi_{n_2}\tilde{k}$ satisfy $\phi_n h'' \sim hh'$ and $\phi_n k'' \sim kk'$ $(n = n_1 n_2)$. Consequently the fact that h and h'' realize d and k and k'' realize d' imply that h' and k' realize the same as ϕ_n.

Let χ: $ZH^*(X'', Z)/\text{torsion} \to H^*(X'', Z)$ be a rational splitting, i.e. the map $QH^*(X'', Z)/\text{torsion} \xrightarrow{\chi} H^*(X'', Z) \xrightarrow{\text{proj}} QH^*(X'', Z)/\text{torsion}$ is a monomorphism of maximal rank. We shall identify $QH^*(X'', Z)/\text{torsion}$ with

$$\chi(QH^*(X'', Z)/\text{torsion}).$$

Choose bases for $PH^*(X', Z)/\text{torsion}$ and $PH^*(Y', Z)/\text{torsion}$ in which $PH^*(f', Z)/\text{torsion}$ is represented by a diagonal matrix M, and bases for $QH^*(X'', Z)/\text{torsion}$ and $QH^*(Y, Z)/\text{torsion}$ in which $QH^*(f'', Z)/\text{torsion}$ is represented by a diagonal matrix N^Q. Denote by N the matrix of

$$H^*(f'', Z)/\text{torsion}|(QH^*(Y'', Z)/\text{torsion})$$

relative to these bases. Obviously

$$N = \begin{pmatrix} N^Q \\ N^D \end{pmatrix}.$$

Let r be the number of generators of $QH^*(X, Q)$ and let $\tilde{r}$ be the number of generators of $QH^*(Y, Q)$. Define $t' = \lambda^2 t(f', f'')$ where λ is the multiple of the nonzero elements of M and N. Consider the matrices A^Q and $\tilde{A}^Q$ of $QH^*(h', Z)/\text{torsion}$ and $QH^*(k', Z)/\text{torsion}$, respectively. Since h' and k' realize the same as ϕ_n and $(n, t') = 1$, the matrices $(nI)^{-1} \cdot (A^Q \otimes Z_{t'})$ and $(nI)^{-1} \cdot (\tilde{A}^Q \otimes Z_{t'})$ belong to $\mathrm{SL}(r, Z_{t'})$ and $\mathrm{SL}(\tilde{r}, Z_{t'})$, respectively. As for every n there exists an epimorphism β_m: $\mathrm{SL}(m, Z) \to \mathrm{SL}(m, Z_{t'})$, there exist matrices $E \in \mathrm{SL}(r, Z)$, $\tilde{E} \in \mathrm{SL}(\tilde{r}, Z)$ so that $\beta_r(E) = (nI)^{-1} \cdot (A^Q \otimes Z_{t'})$ and $\beta_r(\tilde{E}) = (nI)^{-1} \cdot (A^Q \otimes Z_{t'})$. Consequently there exist matrices B and $\tilde{B}$ (over Z) so that $A^Q = nE + t'B$ and $\tilde{A}^Q = n\tilde{E} + t'\tilde{B}$.

We shall use the conditions on A^Q, $\tilde{A}^Q$ and $H^*(f, Q)$ to construct homotopy equivalences $\bar{h}$: $X'' \to X'$ and $\bar{k}$: $Y'' \to Y'$ so that $f'\bar{h} \sim \bar{k}f''$. We shall discuss separately each condition on $H^*(f, Q)$.

(a) $H^*(f, Q) = 0$.

Let h_1: $X'' \to X'$, k_1: $Y'' \to Y'$ be functions which correspond to the matrix $\left(\begin{smallmatrix}-t'B\\0\end{smallmatrix}\ \begin{smallmatrix}-t'B\\0\end{smallmatrix}\right)$ and let h_2: $X'' \to X'$, k_2: $Y'' \to Y'$ be functions which correspond to the matrix $\left(\begin{smallmatrix}-t'B\\0\end{smallmatrix}\ \begin{smallmatrix}-t'B\\0\end{smallmatrix}\right)$. Define maps $\bar{h}$: $X'' \to X'$, $\bar{k}$: $Y'' \to Y'$ by $\bar{h} = a(h' + h_1) + bh_2$, $\bar{k} = a(k' + k_1) + bk_2$, where a and b are integers satisfying $an + bt' = 1$. Since the matrices of $QH^*(\bar{h}, Z)/\text{torsion}$ and $QH^*(\bar{k}, Z)/\text{torsion}$ are E and $\tilde{E}$, respectively, $\bar{h}$ and $\bar{k}$ are homotopy equivalences. Obviously $f'\bar{h} \sim \bar{k}f''$ (f' is an H-map); hence $\bar{h}$ and $\bar{k}$ are the desired maps.

(b) *$H^*(f, Q)$ is a monomorphism.*

Assume that $A^Q_{s_i}$ is a $v_i \times v_i$ matrix and that $\tilde{A}^Q_{s_i}$ is a $w_i \times w_i$ matrix. Assume also that for every i

$$M_{s_i} = \begin{bmatrix} m_{1,s_i} & & \\ & \ddots & \\ & & m_{w_i,s_i} \\ \hline & 0 & \end{bmatrix}\}\, v_i - w_i \qquad \text{and} \qquad N^Q_{s_i} = \begin{bmatrix} n_{1,s_i} & & \\ & \ddots & \\ & & n_{w_i,s_i} \\ \hline & 0 & \end{bmatrix}\}\, v_i - w_i .$$

As $A^Q M = N^Q \tilde{A}^Q$ and $(\det \tilde{A}^Q_{s_i}, \lambda) = 1$, for every i the matrix $A^Q_{s_i}$ is of the form

$$A^Q_{s_i} = \begin{array}{c} w_i \times w_i \\ \\ \end{array} \left[\begin{array}{c|c} C_{s_i} & * \\ \hline 0 & \tilde{C}_{s_i} \end{array}\right] \begin{array}{c} \\ (v_i - w_i) \times (v_i - w_i) \end{array}$$

where $|\det C_{s_i}| = n^{w_i}$ and $|\det \tilde{C}_{s_i}| = n^{v_i - w_i}$. Define

$$M'_{s_i} = \left[\begin{array}{c|c} \underbrace{\begin{matrix} \lambda/m_{1,s_i} & & 0 \\ & \ddots & \\ 0 & & \lambda/m_{w_i,s_i} \end{matrix}}_{w_i \times w_i} & \underbrace{0}_{w_i \times (v_i - w_i)} \end{array}\right],$$

$M' = M'_{s_1} * M'_{s_2} * \cdots * M'_{s_l}$ and $\bar{B} = \lambda N \tilde{B} M'$. As $\bar{B}$ satisfies

$$t(f', f'')\bar{B}M = \lambda t(f', f'')N\tilde{B}M'M = \lambda t(f', f'')N\tilde{B}(\lambda I) = \lambda^2 t(f', f'')N\tilde{B} = t'N\tilde{B},$$

there exist maps h_1: $X'' \to X'$, k_1: $Y'' \to Y'$ which correspond to the matrix

$$\begin{bmatrix} -t(f', f'')\bar{B}^Q & -t'\tilde{B} \\ -t(f', f'')\bar{B}^D & 0 \end{bmatrix}.$$

Define $h_2 = h' + h_1$, $k_2 = k' + k_1$. As f' is an H-map, $f'h_2 \sim k_2 f''$. From this homotopy and from the definition of h_2 and k_2 it follows that the matrices of $QH^*(h_2, Z)/\text{torsion}$ and $QH^*(k_2, Z)/\text{torsion}$ are $A' = A^Q - t(f', f'')\bar{B}^Q$ and $n\tilde{E}$, respectively, and that $A'M = nN^Q\tilde{E}$. The last equality together with the facts that $(n, \lambda) = 1$ and $(\bar{B}^Q_{s_i})_{kj} = 0$ for every k and j that satisfy either $k > w_i$ or $j > w_i$, imply that for every i the matrix A'_{s_i} is of the form

$$A'_{s_i} = \begin{array}{c} w_i \times w_i \\ \\ \end{array} \left[\begin{array}{c|c} nE''_{s_i} & * \\ \hline 0 & nE'_{s_i} + t'B'_{s_i} \end{array}\right] \begin{array}{c} \\ (v_i - w_i) \times (v_i - w_i) \end{array}$$

where $E'_{s_i} \in \mathrm{GL}(v_i - w_i, Z)$, $E''_{s_i} \in \mathrm{GL}(w_i, Z)$ and $nE' + t'B'_{s_i} = \tilde{C}_{s_i}$.

For every i denote by D_{s_i} the matrix

$$D_{s_i} = \begin{array}{c} w_i \times w_i \\ \\ \end{array} \left[\begin{array}{c|c} 0 & t'D'_{s_i} \\ \hline 0 & -t'B'_{s_i} \end{array}\right] \begin{array}{c} \\ (v_i - w_i) \times (v_i - w_i) \end{array}$$

where $(A'_{s_i})_{kj} + t'(D'_{s_i})_{kj} \equiv 0 \pmod{n}$ for every k and j that satisfy either $k < w_i$ or $j > w_i$. Define $D = D_{s_1} * D_{s_2} * \cdots * D_{s_l}$. Since $DM = 0$ there exists a function h_3: $X'' \to X'$ so that $f'h_3 \sim *$.

Define functions h_4: $X'' \to X'$, k_4: $Y'' \to Y'$ by $h_4 = h_2 + h_3$, $k_4 = k_2$. It is clear that $f'h_4 \sim k_4 f''$, that the matrix of $QH^*(k_4, Z)/\text{torsion}$ is $n\tilde{E}$, that there exists a matrix $\bar{E} \in \mathrm{GL}(r, Z)$ so that the matrix of $QH^*(h_4, Z)/\text{torsion}$ is $n\bar{E}$ and that $\bar{E}M = N^Q\tilde{E}$.

Let $h_5: X'' \to X'$, $k_5: Y'' \to Y'$ be functions which correspond to the matrix

$$\begin{bmatrix} t'\bar{E} & t'\tilde{E} \\ t(f', f'')\bar{\bar{E}} & 0 \end{bmatrix}$$

where $\bar{\bar{E}} = \lambda N^D\tilde{E}M'$ (such functions exist since $t'N^Q\tilde{E} = t'\bar{E}M$ and $t'N^D\tilde{E} = t(f', f'')\bar{\bar{E}}M$). Define $\bar{h} = ah_4 + bh_5$, $\bar{k} = ak_4 + bk_5$, where a and b are integers satisfying $an + bt' = 1$. As the matrices of $\bar{h}$ and $\bar{k}$ are $\bar{E}$ and $\tilde{E}$, respectively, and as f' is an H-map, $\bar{h}$ and $\bar{k}$ are homotopy equivalences and $f'\bar{h} \sim \bar{k}f''$.

(c) *$H^*(f, Q)$ is an epimorphism.*

Assume that $A^Q_{s_i}$ is a $v_i \times v_i$ matrix, that $\tilde{A}^Q_{s_i}$ is a $w_i \times w_i$ matrix and for every i

$$N_{s_i} = \Bigg[\,\underbrace{\begin{matrix} n_{1,s_i} & & \\ & \ddots & \\ & & n_{v_i,s_i} \end{matrix}}_{v_i \times v_i} \;\Bigg|\; \underbrace{\begin{matrix} \\ 0 \\ \\ \end{matrix}}_{v_i \times (w_i - v_i)}\,\Bigg]$$

and

$$M_{s_i} = \Bigg[\,\underbrace{\begin{matrix} m_{1,s_i} & & \\ & \ddots & \\ & & m_{v_i,s_i} \end{matrix}}_{v_i \times v_i} \;\Bigg|\; \underbrace{\begin{matrix} \\ 0 \\ \\ \end{matrix}}_{v_i \times (w_i - v_i)}\,\Bigg].$$

Since $A^QM = N^Q\tilde{A}^Q$ and $(\det A^Q_{s_i}, \lambda) = 1$, for every i the matrix $\tilde{A}^Q_{s_i}$ is of the form

$$\tilde{A}^Q_{s_i} = \begin{array}{r} v_i \times v_i \\ \\ \end{array}\left[\begin{array}{c|c} C_{s_i} & 0 \\ \hline * & \tilde{C}_{s_i} \end{array}\right]\begin{array}{l} \\ (w_i - v_i) \times (w_i - v_i) \end{array}$$

where $|\det C_{s_i}| = n^{v_i}$ and $|\det \tilde{C}_{s_i}| = n^{w_i - v_i}$. Define

$$N'_{s_i} = \begin{array}{r} v_i \times v_i \\ \\ \\ (w_i - v_i) \times v_i \end{array}\left[\begin{array}{ccc} \lambda/n_{1,s_i} & & 0 \\ & \ddots & \\ 0 & & \lambda/n_{v_i,s_i} \\ \hline & 0 & \end{array}\right],$$

$N' = N'_{s_1} * N'_{s_2} * \cdots * N'_{s_l}$ and $B' = N'BM$.

Let $\bar{N}$ and $\tilde{N}$ be the matrices of $DH^*(f'', Z)/\text{torsion}$ and $H^*(f'', Z)/\text{torsion}$, respectively. As for every decomposable element $d \in H^*(X'', Z)/\text{torsion}$ there exists a decomposable element $d' \in H^*(Y'', Z)$ satisfying $(f'') * d' = \lambda d$, there exists a matrix B'' so that $-\lambda N^DB' = \bar{N}B''$. Define $\bar{B} = \binom{\lambda B'}{B''}$, $\bar{\bar{B}} = \binom{\bar{B}}{0}$.

As B' satisfies

$$\begin{aligned} \lambda t(f', f'')N^QB' &= \lambda t(f', f'')N^QN'BM \\ &= \lambda t(f', f'')(\lambda I)BM = \lambda^2 t(f', f'')BM = t'BM, \end{aligned}$$

$\bar{B}$ and $\bar{\bar{B}}$ satisfy $t(f', f'')\tilde{N}\bar{B} = t'\bar{\bar{B}}M$. Consequently there exist functions $h_1\colon X'' \to X'$, $k_1\colon Y'' \to Y'$ which correspond to the matrix

$$\begin{pmatrix} -t'B & -\lambda t(f', f'')B' \\ 0 & -t(f', f'')B'' \end{pmatrix}.$$

Define $h_2 = h' + h_1$, $k_2 = k' + k_1$. The matrices of $QH^*(h_2, Z)/\text{torsion}$ and $QH^*(k_2, Z)/\text{torsion}$ are nE and $A' = \tilde{A}^Q - \lambda t(f', f'')B$. As $f'h_2 \sim k_2 f''$, E and A' satisfy $nEM = N^Q A'$. This together with the fact that $(n, \lambda) = 1$ and $(B'_{s_i})_{kj} = 0$ for every pair (k, j) satisfying either $k > v_i$ or $j > v_i$, imply that for every i the matrix A'_{s_i} is of the form

$$A'_{s_i} = \begin{array}{c} v_i \times v_i \\ \\ \end{array}\left[\begin{array}{c|c} nE''_{s_i} & 0 \\ \hline 0 & nE'_{s_i} + t'B'_{s_i} \end{array}\right] \begin{array}{c} \\ (w_i - v_i) \times (w_i - v_i) \end{array}$$

where $E'_{s_i} \in \mathrm{GL}(w_i - v_i, Z)$, $E''_{s_i} \in \mathrm{GL}(v_i, Z)$ and $nE'_{s_i} + t'B'_{s_i} = \tilde{C}_{s_i}$.

For every i denote by D_{s_i} the matrix

$$D_{s_i} = \begin{array}{c} v_i \times v_i \\ \\ \end{array}\left(\begin{array}{c|c} 0 & 0 \\ \hline t'D'_{s_i} & -t'B'_{s_i} \end{array}\right) \begin{array}{c} \\ (w_i - v_i) \times (w_i - v_i) \end{array}$$

where $(A'_{s_i})_{ij} + t'(D'_{s_i})_{ij} \equiv 0 \pmod{n}$ for every i and j satisfying either $i > v_i$ or $j < v_i$. Define $D = D_{s_1} * D_{s_2} * \cdots * D_{s_l}$. Since, for every i and j, D_{ij} is divisible by λ, there exists a matrix $\bar{D}$ so that $-N^D D = \bar{N}\bar{D}$. Denote $D' = (\frac{D}{\bar{D}})$. D' satisfies $\tilde{N}D' = 0$; therefore there exists a function $k_3\colon Y'' \to Y'$ so that $k_3 f'' \sim *$.

Define $h_4\colon X'' \to X'$, and $k_4\colon Y'' \to Y'$ by $h_4 = h_2$, $k_4 = k_2 + k_3$. It is obvious that $f'h_4 \sim k_4 f''$, that the matrix of $QH^*(h_4, Z)/\text{torsion}$ is nE, that there exists a matrix $\bar{E}$ so that the matrix of $QH^*(k_4, Z)/\text{torsion}$ is $n\bar{E}$ and that $EM = N^Q\bar{E}$.

Let $h_5\colon X'' \to X'$, $k_5\colon Y'' \to Y'$ be functions which correspond to the matrix

$$\begin{bmatrix} t'E & t'\bar{E} \\ 0 & t(f', f'')\bar{\bar{E}} \end{bmatrix}$$

where the matrix $\bar{\bar{E}}$ satisfies $-t'N^D\bar{E} = t(f', f'')\bar{N}\bar{\bar{E}}$. Define $\bar{h} = ah_4 + bh_5$, $\bar{k} = ak_4 + bk_5$ where a and b are integers which satisfy $an + bt' = 1$. As the matrices of $QH^*(\bar{h}, Z)/\text{torsion}$ and $QH^*(\bar{k}, Z)/\text{torsion}$ are E and $\bar{E}$, respectively, $\bar{h}$ and $\bar{k}$ are homotopy equivalences. Obviously $f'\bar{h} \sim \bar{k}f''$; hence $\bar{h}$ and $\bar{k}$ are the desired maps.

3.12. Corollary. *Proposition* 3.11 *is also true for a map* $S^{2n-1} \to X$ *where* X *is an* H*-space so that* $H^*(X, Q)$ *is primitively generated.*

Proof. The assertion follows from Corollary 3.8 in the same way that Proposition 3.11 follows from Proposition 3.6.

3.13. Corollary. *Let* $f\colon X \to Y$ *be a map which satisfies conditions* (a) *or* (b) *of Theorem* I. *Given a map* $f''\colon X'' \to Y$ *in* $G_Y(f)$ *and a* t*-equivalence* $h''\colon X'' \to X'$ *realizing* $\tilde{d} = (\pm d_{m_1}, \ldots, \pm d_{m_{l(X)}})$. *Then the map* $f''\colon X'' \to Y$ *is homotopy equivalent to the map* $f'\colon X' \to Y$ *which corresponds to the pair* $(d, 1)$ $(d = (d_{m_1}, \ldots, d_{m_{l(X)}}))$ *by the construction of Proposition* 3.4.

PROOF. The corollary is obviously true if f satisfies condition (b), namely if $X = S^{2n-1}$.

Suppose f satisfies condition (a). By Corollary 2.3, $G_Y(f) = 0$ if $H^*(f, Q)$ is either an isomorphism or an epimorphism. Therefore we have only to check the cases $H^*(f, Q) = 0$ and $H^*(f, Q)$ is a monomorphism.

Choose bases for $PH^*(X', Z)/\text{torsion}$ and $PH^*(Y', Z)/\text{torsion}$ in which the matrix A of $PH^*(f', Z)/\text{torsion}$ is diagonal. Define $t' = \lambda t(f', f'')$, where λ is the multiple of the nonzero elements of A. Assume $\tilde{d} = d$. Using the considerations of Proposition 3.11 one obtains that there exists a map $h'\colon X'' \to X'$ which realizes the same as ϕ_n and satisfies $f'h' \sim \phi_n f''$, and that the map $f''\colon X'' \to Y$ is homotopy equivalent (over Y) to the map $f'\colon X' \to Y$.

(If $\tilde{d} = (\pm d_{m_1}, \ldots, \pm d_{m_{l(X)}}) \neq d$ the proof is similar.)

3.14. COROLLARY. *Let $f\colon X \to Y$ be a map which satisfies the conditions of Theorem* III. *Given a map $f''\colon X \to Y''$ in $G^X(f)$ and a t-equivalence $k''\colon Y'' \to Y$ realizing $\tilde{d}' = (\pm d_{n_1}, \ldots, \pm d_{n_{l(Y)}})$. Then the map $f''\colon X \to Y''$ is homotopy equivalent* (*under* X) *to the map $f'\colon X \to Y'$ which corresponds to the pair* $(1, d')$ ($d' = (d'_{n_1}, \ldots, d'_{n_{l(X)}})$) *by the construction of Proposition* 3.4.

PROOF. If $H^*(f, Q)$ is either a monomorphism or an isomorphism then $G_X(f) = 0$ by Corollary 2.3. If $H^*(f, Q)$ is either an epimorphism or zero, one chooses bases for $QH^*(X'', Z)/\text{torsion}$ and $QH^*(Y'', Z)/\text{torsion}$ in which the matrix B of $QH^*(f'', Z)/\text{torsion}$ is diagonal, then one defines $t' = \lambda t(f', f'')$ where λ is the multiple of the nonzero elements of B. Using the Corollary 3.14 follows from Corollary 3.10 in the same way that Proposition 3.11 follows from Proposition 3.6.

In Theorems I, II and III we referred to an integer $\hat{t}$. We shall define it now:

3.15. DEFINITION. (a) If $f\colon X \to Y$ is a map satisfying the conditions of Theorems I or III and $Y \neq S^{2n-1}$, we define $\hat{t} = t(f, f)$ (of 3.6).

(b) If $X = S^{2n-1}$, $Y = S^{2m-1}$ $(n > m)$ and the order of f is odd we define $\hat{t} = \text{order}(f)$.

(c) If $X = S^{2n-1}$, $Y = S^{2m-1}$ $(n > m)$ and the order of f is even, we define $\hat{t} = |f|^v$, where v is an integer satisfying $\eta_{|f|^v} f \sim *$.

(By the proof of Theorem 2.2 such an integer exists.)

3.16. PROPOSITION. (a) *Let $f\colon X \to Y$ be a map which satisfies the conditions of Theorem* I. *There exists a surjection $\xi' = \xi'_f\colon D \to G(f)$ satisfying the following conditions*:

(1) $\xi'(d, d') = \xi'(d + \hat{t}s, d' + \hat{t}s')$ *whenever* (d, d') *and* $(d + \hat{t}s, d' + \hat{t}s')$ *belong to* D.

(2) *If f is an H-map, then for every pair $(d, d') \in D$, $\xi'(d, d')$ is an H-map.*

(3) *If $D' = \{(d, 1) \in D\}$ then $\xi'|D'$ is on $G_Y(f)$ and for any two pairs* $(d, 1)$, $(d + \hat{t}s, 1)$ *in* D', $\xi'(d, 1) = \xi'(d + \hat{t}s, 1)$ *in* $G_Y(f)$.

(b) *If the map $f\colon X \to Y$ satisfies the conditions of Theorem* III *and if* $D'' = \{(1, d') \in D\}$ *then the map* $\xi'|D''$ (ξ' *from* (a)) *is on* $G^X(f)$ *and*

$$\xi'(1, d') = \xi'(1, d' + \hat{t}s')$$

in $G_X(f)$, *whenever* $(1, d')$ *and* $(1, d' + \hat{t}s')$ *belong to* D''.

Proof. Propositions 3.4 and 3.11 imply that there exists a surjection $\xi'\colon D \to G(f)$, that $\xi'|D'$ and $\xi'|D''$ are on $G_Y(f)$ and on $G^X(f)$, respectively, and that if f is an H-map, then $\xi'(d, d')$ is an H-map for every pair $(d, d') \in D$.

We shall prove part (a)(1) (parts (a)(3) and (b) are proved similarly). We shall distinguish two cases:

(a) f satisfies conditions (a) or (b) of Theorem I.

(b) $f\colon S^{2n-1} \to S^{2m-1}$ $(n > m)$.

The proof of case (a). It follows from Propositions 3.4 and 3.11 that for every pair $(1 + \hat{t}s, 1 + \hat{t}s') \in D$, $\xi'_f(1 + \hat{t}s, 1 + \hat{t}s') = \xi'_f(1, 1) = (f\colon X \to Y)$.

Let $(d_1, d_1') \in D$ be a pair which satisfies $(d_1 d, d_1' d') = (1 + \hat{t}a, 1 + \hat{t}a')$ where $(a, a') \in Z^{l(Z)} \times Z^{l(Y)}$. Assume that $\xi'_f(d, d') = (f'\colon X' \to Y')$ and that

$$\xi'_f(d + \hat{t}s, d' + \hat{t}s') = (f''\colon X'' \to Y'').$$

Since $(d_1(d + \hat{t}s), d_1'(d + \hat{t}s')) = (1 + \hat{t}(a + d_1 s), 1 + \hat{t}(a' + d_1' s'))$, it follows from Proposition 3.4 that $\xi'_{f'}(d_1, d_1') = \xi'_{f''}(d_1, d_1') = f$. Consequently there exists a commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "h_2"] \arrow[dr, "h_1"] \arrow[ddd, "f"'] & & X'' \arrow[ddd, "f''"] \\
& X' \arrow[d, "f'"] & \\
& Y' & \\
Y \arrow[ur, "k_1"] \arrow[rr, "k_2"] & & Y''
\end{tikzcd}$$

where h_1 and h_2 realize d_1, and k_1 and k_2 realize d_1'. Using this diagram we obtain (in the same way that we proved Proposition 3.11) that

$$\xi'(d, d') = \xi'(d + \hat{t}s, d' + \hat{t}s').$$

The proof of (b). Assume that $\xi'(d, d') = (f'\colon S^{2n-1} \to S^{2m-1})$ and that $\xi'(d + \hat{t}s, d' + \hat{t}s') = (f''\colon S^{2n-1} \to S^{2m-1})$. As for every $f' \in G(f)$ the order of f' is equal to the order of f, we obtain from the choice of $\hat{t}$ that $\eta_{d' + \hat{t}s'} f' \sim f\eta_{d + \hat{t}s}$; hence $\xi'(d, d') = \xi'(d + \hat{t}s, d' + \hat{t}s')$.

3.17. The proof of Theorems I, II, and III. We shall prove Theorem I. Theorems II and III are proved similarly.

Let $\beta\colon D \to [(Z_t^*)/\pm 1]^{l(X) + l(Y)}$ be the map

$$\beta\big(d_{m_1}, \ldots, d_{m_{l(X)}}, d_{n_1}, \ldots, d_{n_{l(Y)}}\big) = \big(d_{m_1}(\mathrm{mod}\ \hat{t}), \ldots, d_{m_{l(X)}}(\mathrm{mod}\ \hat{t}), d_{n_1}(\mathrm{mod}\ \hat{t}), \ldots, d_{n_{l(Y)}}(\mathrm{mod}\ \hat{t})\big).$$

By Proposition 3.16, ξ': $D \to G(f)$ factors through Im β. We shall calculate Im β. To this end we shall distinguish among four cases:

(1) $H^*(f, Q) = 0$. If $(d, d') \in D$ there is no relation between d and d'; consequently Im $\beta = [Z_{\hat{t}}^*/\pm 1]^{l(X)+l(Y)}$.

(2) *$H^*(f, Q)$ is an isomorphism.* $(d, d') \in D$ iff $d = d'$; consequently

$$\operatorname{Im} \beta = \left\{(d, d) \in \left[(Z_{\hat{t}}^*)/\pm 1\right]^{l(X)+l(Y)} \cong \left[Z_{\hat{t}}^*/\pm 1\right]^{l(X)}\right\}.$$

(3) *$H^*(f, Q)$ is a monomorphism.* Suppose $d = (d_{m_1}, \ldots, d_{m_{l(X)}}) \in [(Z_{\hat{t}}^*)/\pm 1]^{l(X)}$ and $d' = (d'_{n_1}, \ldots, d'_{n_{l(Y)}}) \in [(Z_{\hat{t}}^*)/\pm 1]^{l(Y)}$. Define $\tilde{d} = (\tilde{d}_{m_1}, \ldots, \tilde{d}_{m_{l(X)}}) \in Z^{l(X)}$ by

$$\tilde{d}_{m_i} = \begin{cases} d_{m_i}, & m_i \neq n_j \text{ for every } j, \\ c_i d'_{n_j}, & m_i = n_j, \end{cases}$$

where $0 < c_1 < \hat{t}$ is an integer satisfying $d_{m_j} \equiv c_i d'_{n_j} \pmod{\hat{t}}$. Obviously $(\tilde{d}, d') \in D$ and $\beta(\tilde{d}, d') = (d, d')$. Consequently Im $\beta = [(Z_{\hat{t}}^*)/\pm 1]^{l(X)+l(Y)}$.

(4) *$H^*(f, Q)$ is an epimorphism.* Suppose d and d' are as in (3). Define $\tilde{d} = (\tilde{d}_{n_1}, \ldots, \tilde{d}_{n_{l(Y)}}) \in Z^{l(Y)}$ by

$$\tilde{d}_{n_i} = \begin{cases} d'_{n_i}, & n_i \neq m_n \text{ for every } j, \\ c_i d_{m_j}, & n_i = m_j, \end{cases}$$

where $0 < c_i < \hat{t}$ is an integer satisfying $d'_{n_j} \equiv c_i d_{m_j} \pmod{\hat{t}}$. Obviously $(d, \tilde{d}) \in D$ and $\beta(d, \tilde{d}) = (d, d')$. Consequently Im $\beta = [(Z_{\hat{t}}^*)/\pm 1]^{l(X)+l(Y)}$.

Define an integer k as follows: If $H^*(f, Q)$ is either a monomorphism, an epimorphism or zero put $k = l(X) + l(Y)$ and if $H^*(f, Q)$ is an isomorphism put $k = l(X) = l(Y)$. By Proposition 3.16 and the calculation of Im β, the surjection ξ': $D \to G(f)$ induces a surjection ξ: $[(Z_{\hat{t}}^*)/\pm 1]^k \to G(f)$. Define an action on $G(f)$ by $\hat{\xi}(d, d') \cdot \hat{\xi}(d_1, d_1') = \hat{\xi}(dd_1, d'd_1')$. Propositions 3.4, 3.11 and 3.16 imply that the action is well defined, that $G(f)$ with this action is an abelian group and that the sequence

$$[f, f]_{\hat{t}} \xrightarrow{\alpha'} \left[(Z_{\hat{t}}^*)/\pm 1\right]^k \xrightarrow{\hat{\xi}} G(f) \to 0$$

is exact.

4. Some consequences of Theorems I, II, and III. We assume that all the maps satisfy the conditions of Theorem I or of Theorem III (Theorem I when we speak of $G(f)$ or $G_Y(f)$, Theorem III when we speak of $G^X(f)$).

4.1. LEMMA. *Let f: $S^{2n-1} \to S^{2m-1}$ $(n > m)$ be a map. If the order of f is odd then $G(f) = G_{S^{2m-1}}(f) = [(Z_{|f|}^*)/\pm 1]$.*

PROOF. $(d, d') \in \operatorname{Im} \alpha' \subset [(Z_{|f|}^*)/\pm 1]^2$ if and only if $d \equiv d' \pmod{|f|}$.

REMARK. It is clear that for any map of the form f: $S^n \to S^n$, $G(f) = 0$.

4.2. LEMMA. *If f: $X \to Y$ is a map and f is a rational equivalence then each map f': $X' \to Y'$ in $G(f)$ is obtained as the pull-back of $X \xrightarrow{f} Y \xleftarrow{k} Y'$ where k is a $\hat{t}$-equivalence. In particular for every $f' \in G(f)$, $F' \approx F$.*

PROOF. Follows from the construction that appears in Proposition 3.4.

4.3. LEMMA. *Let* $f_1: X_1 \to Y_1$ *and* $f_2: X_2 \xrightarrow{f_2} Y_2$ *be maps.*

(a) *Each map in* $G(f_1 \times f_2)$ $(G_{Y_1 \times Y_2}(f_1 \times f_2),\ G^{X_1 \times X_2}(f_1 \times f_2))$ *is of the form* $g_1 \times g_2$ *where* $g_i \in G(f_i)$ $(g_i \in G_{Y_i}(f_i),\ g_i \in G^{X_i}(f_i))$.

(b) *If* (1) $QH^n(X_1, Q) \neq 0$ *whenever* $QH^n(X_2, Q) \neq 0$,

(2) $QH^n(Y_1, Q) \neq 0$ *whenever* $QH^n(Y_2, Q) \neq 0$,

then

(1′) $G(f_1 \times f_2) = f_1 \times G(f_2)$.

(2′) $G^X(f_1 \times f_2) = f_1 \times G^X(f_2)$.

(3′) $G_Y(f_1 \times f_2) = f_1 \times G_Y(f_2)$.

(4′) *If* $Y_2 = K(Y_1)$, $X_2 = K(X_1)$ *and* $f_2 \sim *$ *then* $G(f_1 \times f_2) = G^X(f_1 \times f_2) = G_Y(f_1 \times f_2) = 0$.

4.4. COROLLARY. *Let* $f: X \to Y$ *be a map. There exists an integer n so that* $G(f^n: X^n \to Y^n) = 0$.

PROOF. If $f' \in G(f)$ and $f' = \hat{\xi}(d, d')$ then $(f')^n \in G(f^n)$ satisfies $(f')^n = \hat{\xi}(d^n, d'^n)$. Consequently $(f')^{\varphi(t)/2} \approx f^{\varphi(t)/2}$. ($\varphi(\hat{t})$ = the Euler number of $\hat{t}$ = the order of Z_t^*.)

REMARK. It is obvious that the corollary is also true for $G^X(f)$ and $G_Y(f)$.

4.5. LEMMA. *Every map in* $G(\text{proj}: X \times Y \to Y)$ *is of the form* $\text{proj}: X' \times Y' \to Y'$ *where* $X' \in G(X)$ *and* $Y' \in G(Y)$.

4.6. LEMMA. *Let* $f: X \xrightarrow{f} Y$ *be a map* $(Y \neq S^{2m-1})$.

(a) *Every map in* $G(\phi_n f)$ $(G^X(\phi_n f))$ *is of the form* $\phi_n f'$ *where* $f' \in G(f)$ $(f' \in G^X(f))$.

(b) *If X is an H-space then every map in* $G(f\phi_n)$ $(G_Y(f\phi_n))$ *is of the form* $f'\phi_n$ *where* $f' \in G(f)$ $(f' \in G_Y(f))$.

(c) *If* $X = S^{2n-1}$ *then every map in* $G(f\eta_n)$ $(G_Y(f\eta_n))$ *is of the form* $f'\eta_n$ *where* $f' \in G(f)$ $(f' \in G_Y(f))$.

PROOF. We shall prove (a). The proofs of (b) and (c) are similar.

Since f^* and $(\phi_n f)^*$ can be diagonalized simultaneously we can apply diagram 3.4.1 (with the same φ and ψ) to construct $G(f)$ and $G(\phi_n f)$. Suppose $\xi_f'(d, d') = (X' \xrightarrow{f'} Y')$ and $\xi_{\phi_n f} = (g': X' \to Y')$. From diagram 3.4.1 we obtain that there exist an H-map $k: Y' \to Y$ and a map $h: X' \to X$ (obviously if f is an H-map h is, also, an H-map) which realize d' and d, respectively, and which satisfy $kg \sim (\phi_n f)h$ and $kf' \sim fh$. The last homotopy together with the fact that k is an H-map imply that $k(\phi_n f') \sim (\phi_n f)h$. Therefore by Proposition 3.11 the map $\phi_n f'$ is homotopy equivalent to g.

5. The map $G(X, Y, f) \to G(X) \times G(Y)$. The map $G(X, Y, f) \to G(X) \times G(Y)$ exists for every map $f: X \to Y$. An immediate consequence of Theorem I is that for maps $f: X \to Y$ which satisfy the conditions of Theorem I the above map is a homomorphism and the compositions $G(X, Y, f) \to G(X) \times G(Y) \xrightarrow{\text{proj}} G(X)$ and $G(X, Y, f) \to G(X) \times G(Y) \xrightarrow{\text{proj}} G(Y)$ are epimorphisms.

In this section we deal with the kernel of the map $G(X, Y, f) \to G(X) \times G(Y)$ only for maps $f\colon X \to Y$ which satisfy the conditions of Theorem I. In case that this map is a monomorphism and $G(Y) = 0$ ($G(X) = 0$) we conclude (by the previous paragraph) that $G(X, Y, f) \cong G(X)$ ($G(X, Y, f) \cong G(Y)$).

All the notations in the next lemma, except the addition of indices to indicate the dependence in d and d', are taken from diagram 3.4.1.

5.1. LEMMA. *Let X and Y be H_0-spaces so that $H^*(X, Z)$ and $\pi_* Y$ are torsion free. If $f\colon X \to Y$ is a map which satisfies the conditions of Theorem* I *then*

$$|\ker(G(X, Y, f) \to G(X) \times G(Y))|$$
$$= \left|\left\{ [\gamma_{d'} g \delta_d \psi] \in [X, K(Y)] \,|\, (d, d') \in D, d_i, d_i' < \hat{t}^2, \forall i \right\}\right|$$

where $\gamma_{d'}\colon K(Y) \to K(Y)$ and $\delta_d\colon K(X) \to K(X)$ are homotopy equivalences satisfying $\gamma_{d'}\phi_{d'} \sim \varphi$ and $\delta_d \psi \sim \theta_d$.

PROOF. It follows from diagram 3.4.1 that if $\xi'(d, d') = f'$ then $\phi_{d'} f' \sim g \in \theta_d$. Suppose $\xi'(d, d')$ belongs to $\ker(G(X, Y, f) \to G(X) \times G(Y))$. The above homotopy together with the homotopies $\gamma_{d'}\phi_{d'} \sim \varphi$ and $\delta_d \psi \sim \theta_d$ imply that $\gamma_{d'}^{-1}\varphi f' \sim g\delta_d\psi$ (where $\gamma_{d'}^{-1}$ denotes the homotopy inverse of $\gamma_{d'}$) or equivalently that $\varphi f' \sim \gamma_{d'} g \delta_d \psi$. The truth of the lemma follows from the last homotopy and from the fact that the map $[X, Y] \to \mathrm{Hom}(H^*(Y, Q), H^*(X, Q))$ is one-to-one (Zabrodsky [**9**, Lemma 5.3.1]).

REMARK. By 3.17 it is enough to take pairs $(d, d') \in D$ which satisfy $d_i, d_i' < \hat{t}^2$ for every i.

5.2. *Examples of maps for which the map $G(X, Y, f) \to G(X) \times G(Y)$ is a monomorphism.* (All the maps considered are assumed to satisfy the conditions of Theorem I.)

EXAMPLE 1. $X \to K = \Pi_{\text{finite}}\, k(Z, n_i)$, $H^*(X, Z)$ is torsion free and $H^*(f, Z)$ is onto.

Every $d' \in Z^{l(K)}$ can be realized in K. Assume that $(d, d') \in D$ and that d can be realized in X. Let $h\colon X \to X$ be a map which realizes d. As $H^*(f, Z)$ is onto d_n / d_n' whenever $QH^n(X, Q) \neq 0$. Consequently there exists a map $k\colon K \to K$ which realizes d' and satisfies $fh \sim kf$. Therefore by Proposition 3.11, $\xi'(d, d') = f$ and the map $G(X, K, f) \to G(X) \times G(K) \to G(X)$ is an isomorphism.

EXAMPLE 2. $f\colon X \to K(X)$, f is a rational equivalence and $H^*(f, Z)$ is onto.

$G(f) \simeq G(X)$, since if $f'\colon X \to K(X)$ belong to $G(f)$ there exists a pull-back diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & X \\ \downarrow f' & & \downarrow f \\ K(X) & \xrightarrow{k} & K(X) \end{array}$$

where h and k are $\hat{t}$-equivalences (Lemma 4.2). Consequently $H^*(f', Z)$ is onto and there exists a homotopy equivalence $g\colon K(X) \to K(X)$ so that $gf' \sim f$.

To state the next two examples one needs the following notations: If X is a CW-complex denote by $h_n\colon X \to X_n$ the homotopy approximation of X in dim $< n$ (i.e. $\pi_k h_n$ is an isomorphism for $k \leqslant n$ and $\pi_k X_n = 0$ for $k > n$).

EXAMPLE 3. $h_n\colon X \to X_n$.

Assume that $(d, d') \in D$ and that d and d' can be realized in X and X_n, respectively. Since if $f\colon X \to X$ realizes d, $f_n\colon X_n \to X_n$ realizes d', one obtains from Proposition 3.11 that $\xi'(d, d') = h_n$ and the map $G(h_n) \to G(X) \times G(X_n)$ is a monomorphism. Moreover, since the map $H^k(h_n, Z)$ is an isomorphism in dim $\leq n$, the composition $G(h_n) \to G(X) \times G(X_n) \to G(X)$ is an isomorphism and $G(h_n) \cong G(X)$.

EXAMPLE 4. $f\colon \mathrm{SU}(m) \to \mathrm{SU}_{2n-1}$, $m \leq n$, and $H^*(f, Z)$ is onto.

Assume that $(d, d') \in D$, that d and d' can be realized in $\mathrm{SU}(m)$ and SU_{2n-1}, respectively, and that $\xi'(d, d') = g \in G(f)$. Since $f_\#\colon \pi_*\mathrm{SU}(m) \to \pi_*\mathrm{SU}_{2n-1}$ is an isomorphism in dim $\leq 2m - 1$ and an epimorphism in dim $2m$, $g_\#$ is also an isomorphism in dim $\leq 2m - 1$ and an epimorphism in dim $2m$. Therefore $H^*(g, Z)$ is an epimorphism and the fact that the map $G(f) \to G(\mathrm{SU}(m)) \times G(\mathrm{SU}_{2n-1})$ is a monomorphism follows from the next lemma:

5.3. LEMMA. *Given maps $f_1, f_2\colon \mathrm{SU}(m) \to \mathrm{SU}_{2n-1}$ so that $H^*(f_1, Z)$ and $H^*(f_2, Z)$ are surjections. There exists a homotopy equivalence $g\colon \mathrm{SU}_{2n-1} \to \mathrm{SU}_{2n-1}$ so that $gf_1 \sim f_2$.*

PROOF. By Lemma 1.5 in Zabrodsky [7], there exists a map $g\colon \mathrm{SU}_{2n-1} \to \mathrm{SU}_{2n-1}$ so that $gf_1 \sim f_2$. Obviously $H^k(g, Z)$ is an isomorphism for $k \leq 2m - 1$. Assume that g is not a homotopy equivalence and that k is the least integer for which $QH^{2k+1}(g, Z) \neq \pm 1$. Consider the diagram

$$\begin{tikzcd}
 & \mathrm{SU}_{2n-1} \arrow[dr, "h_{2k+1}"] & \\
\mathrm{SU}(m) \arrow[ur, "f_2"] \arrow[r, "h_{2k+1} \circ f_1"] & \mathrm{SU}_{2k+1} \arrow[r, "g'_{2k+1}"] \arrow[d, "h_{2k+1,2k-1}"] & \mathrm{SU}_{2k+1} \arrow[d, "h_{2k+1,2k-1}"] \\
 & \mathrm{SU}_{2k-1} \arrow[r, "g_{2k-1}"] & \mathrm{SU}_{2k-1}
\end{tikzcd}$$

where g'_{2k+1} is a homotopy equivalence which covers the homotopy equivalence g_{2k-1}. (By Zabrodsky [7, Corollary 1.4] such a homotopy equivalence exists.)

As $h_{2k-1}f_2 \sim g_{2k-1}h_{2k-1}f_1$ and the fibration $K(Z, 2k + 1) \to \mathrm{SU}_{2k+1} \to \mathrm{SU}_{2k-1}$ is principal, there exists $w \in [\mathrm{SU}(m), K(Z, 2k + 1)]$ so that $h_{2k+1}f_2 \sim w * (g'_{2k+1}h_{2k+1}f_1)$ where $*$ is the action of $[\mathrm{SU}(m), K(Z, 2k + 1)]$ on $[\mathrm{SU}(m), \mathrm{SU}_{2n-1}]$. Obviously w is decomposable. Since $H^*(f_1, Z)$ is onto there exists a decomposable element $\bar{w} \in [\mathrm{SU}_{2k+1}, K(Z, 2k + 1)]$ so that $w \sim \bar{w}h_{2k+1}f_1$. Define $g''_{2k+1} = \bar{w} * g'_{2k+1}$. Obviously g''_{2k+1} is a homotopy equivalence and $g''_{2k+1}h_{2k+1}f_1 \sim h_{2k+1}f_2$. Consequently g''_{2k+1} can be lifted to a homotopy equivalence $g''\colon \mathrm{SU}_{2n-1} \to \mathrm{SU}_{2n-1}$ so that $g''f_1 \sim f_2$.

6. Computation of $G(a)$, $G_Y(a)$ and $G_{K(G,n)}(a)$ for some fibrations $X \xrightarrow{f} Y \xrightarrow{a} K(G, n)$. We assume that all the fibrations in this section are Hopf fibrations which satisfy the conditions of Theorem I.

In order to calculate $G(a)$, $G_Y(a)$ and $G_{K(G,n)}(a)$ (in some of the cases) we need the following lemma:

6.1. LEMMA. *If the fibration $K(G, n-1) \to X \xrightarrow{f} Y$ is innduced by a: $Y \to K(G, n)$ then $G(f) \cong G(a)$ and $G_Y(f) \cong G_Y(a)$.*

PROOF. Suppose $X' \xrightarrow{f'} Y' \xrightarrow{a'} K(G, n)$ belongs to $G(a)$. Define maps h: $G(a) \to G(f)$ by $a' \to f'$ where f' is the fiber of a' and k: $G(f) \to G(a)$ by $f' \to a'$ where f' is induced by a'. As each of the maps h and k is the inverse of the other $G(f) \cong G(a)$. The fact that $G_Y(a) \cong G_Y(f)$ is proved similarly.

Case 1. $F \to K(Z, m) \xrightarrow{a} K(Z, n)$.

Obviously $G(a) = 0$ for $n < m$. Assume that $n > m$ and that $(d, d') \in D \subseteq Z^2$. Since $H^n(a, Q) = 0$ there is no relation between d and d'. In contrast with this, the existence of maps h: $K(Z, m) \to K(Z, m)$ and k: $K(Z, n) \to K(Z, n)$ satisfying $ah \sim ka$ implies that $d \equiv d' \pmod{|a|}$. Consequently $G(a) \cong [(Z^*_{|a|})/\pm 1]$.

For the same reason $G_{K(Z,n)}(a) \cong G_{K(Z,m)}(a) \cong [(Z^*_{|a|})/\pm 1]$.

Case 2. $F \to K(Z^m, n) \xrightarrow{a} K(Z^l, n)$, $k, m > 1$.

$G(a) = 0$, since one can choose bases for $H^n(K(Z^l, n),\ Z) = Z^l$ and $H^n(K(Z^m, n)Z) = Z^m$ in which $H^n(a, Z)$ is represented by a diagonal matrix and use these bases together with the conditions on D to construct for every pair $(d, d') \in D$ maps h: $K(Z^m, n) \to K(Z^m, n)$ and k': $K(Z^l, n) \to K(Z^l, n)$ which realize d and d', respectively and satisfy $ah \sim ka$.

In the same way we obtain that $G_{K(Z^l,n)}(a) = G_{K(Z^m,n)}(a) = 0$.

Case 3. $F \to K(Z^l, n) \to K(Z_{p^k}, n)$ $(k, l > 1)$.

We prove that $G(a) = 0$ by constructing to each vector $(x_1, \ldots, x_l)$ $(x_i \in Z_{p^k})$ and to each number $d \in Z_{p^k}$ an $l \times l$ matrix A (over Z) so that $\det A = d$ and $A(x_1, \ldots, x_l) \equiv (dx_1, \ldots, dx_l) \pmod{p^k}$.

Consider the vector $(x_1, \ldots, x_l)$. Each x_i is of the form $x_i = a_i p^{k_i}$ where $(a, p) = 1$. Without loss of generality assume that $k_1 < k_i$ or every i. Let b be an integer satisfying $a_1 b \equiv 1 \pmod{p^k}$ and let $A = (a_{ij})$ be the following matrix:

$$A = \begin{bmatrix} d & & & \\ \vdots & & 0 & \\ b(d-1)a_i p^{k_i - k_1} & 1 & & \\ & & \ddots & \\ \vdots & 0 & & 1 \end{bmatrix}$$

namely

$$a_{ij} = \begin{cases} d, & i = 1, j = 1, \\ b(d-1)a_i p^{k_i - k_1}, & i \neq 1, j = 1, \\ 1, & i \neq 1, j = i, \\ 0, & \text{otherwise.} \end{cases}$$

Obviously $\det A = d$ and $A(x_1, \ldots, x_l) \equiv (dx_1, \ldots, dx_l) \pmod{p^k}$. ($a_{i1}x_1 \equiv (d-1)x_i \pmod{p^k}$ for $i > 1$.)

Case 4. $F \to K(G, m) \to K(H, m)$, G and H are finite p-groups.

It is obvious that $G(a) \cong G^{K(G,m)}(a) \cong G_{K(H,m)}(a) = 0$.

Case 5. $X_{n+1} \xrightarrow{h_n} X_n \xrightarrow{a} K(G, n+2)$, the Postnikov approximation of X.

It follows from Lemma 6.1 and from Example 3 in §5 that $G(a) \cong G(h_n) \cong G(X_{n+1})$.

Case 6. $X \xrightarrow{f} Y \xrightarrow{a} K(G, n)$, $\pi_n a$ is an epimorphism.

Suppose $X \xrightarrow{f'} Y \xrightarrow{a} K(G, n)$ is in $G_Y(a)$. Let $k, k'\colon X \to X$ be $\hat{t}$-equivalences satisfying $fk \sim f'$ and $f'k' \sim f$ (by Theorem 2.2 such $\hat{t}$-equivalences exist). We shall prove that $G_Y(a) \cong G(X)$ by showing that k is a homotopy equivalence.

It is obvious that $k_\#\colon \pi_m X \to \pi_m X$ is an isomorphism for $m \neq n, n-1$ and that $\pi_n k$ and $\pi_n f$ are monomorphisms. As $\pi_n a$ is onto $\pi_{n-1} k$ is also an isomorphism. Consequently in order to prove that k is a homotopy equivalence it is enough to prove that $\pi_n k$ is an epimorphism. But $\pi_n f \circ \pi_n k = \pi_n f'$ and $\pi_n f' \circ \pi_n k' = \pi_n f$; hence $\pi_n f(\pi_n k \circ \pi_n k') = \pi_n f$, $\pi_n k \circ \pi_n k' = 1$ and $\pi_n k$ is an epimorphism.

Case 7. $X \to Y \xrightarrow{a} K(G, n)$, G is a finitely generated free group, $H^n(Y, Z)$ is torsion free and $H^n(a, Z)$ is a surjection.

Assume that $d = (d_1, \ldots, d_{l(Y)}) \in Z^{l(Y)}$ can be realized by a map $h\colon Y \to Y$ and that $(d, d') \in D \subseteq Z^{l(Y)+1}$. As $\ker H^n(a, Z) = 0$ implies $d_n = d'$ and $\ker H^n(a, Z) \neq 0$ implies d_n / d', there exists a map $k\colon K(G, n) \to K(G, n)$ which realizes d' and satisfies $ka \sim ah$. Hence by Proposition 3.11, $\xi'(d, d') = a$ and consequently $G(a) \cong G(Y)$.

7. Noncancellation. In the following proposition we use the notations of Theorem 3.2.

7.1. PROPOSITION. *Let X and Y be H_0-spaces so that $QH^{n_i}(X, Q) \neq 0$ for $1 < i < l(X)$ and $QH^{m_i}(Y, Q) \neq 0$ for $1 < i < l(Y)$, and let $\psi\colon X \to K(X)$ and $\varphi\colon Y \to K(Y)$ be rational equivalences. Denote by t the least common multiple of $t(X, \psi)$ and $t(Y, \varphi)$. Assume that*

$$d = (d_{n_1}, \ldots, d_{n_{l(X)}}) \in [(Z_t^*)/\pm 1]^{l(X)} \quad \textit{and}$$
$$d' = (d'_{m_1}, \ldots, d'_{m_{l(Y)}}) \in [(Z_t^*)/\pm 1]^{l(Y)}$$

satisfy the following conditions:

$$\text{(a)}\quad d'_{m_i} = \begin{cases} d_{n_j}, & m_i = n_j, \\ 1, & m_i \neq n_j \textit{ for every } j. \end{cases}$$

$$\text{(b)}\quad d_{n_i} = \begin{cases} d'_{m_j}, & n_i = m_j, \\ 1, & n_i \neq m_j \textit{ for every } j. \end{cases}$$

If $X' = \xi(d) \in G(X)$ *and* $Y' = \xi(d') \in G(Y)$ *then* $X' \times Y \approx X \times Y'$.

PROOF. It follows from the definition of ξ that if $Y' = \xi(d') \in G(Y)$ then $Y = \xi((d')^{-1}) \in G(Y')$. Consequently $X' \times Y = \xi(1, \dots, 1) \in G(X \times Y')$ and $X' \times Y \approx X \times Y'$.

An immediate consequence of this proposition is

7.2. COROLLARY. *Let* X *be an* H*-space and let* Y *be an* H_0*-space. If* $l(X) = l(Y) + 1$ *then for every* $X' \in G(X)$ *there exists a* $Y' \in G(Y)$ *so that* $X' \times Y \approx X \times Y'$.

Using Corollary 7.2 together with Theorem 2.2 one obtains

7.3. LEMMA. *Let* $F \xrightarrow{j} X \xrightarrow{f} Y$ *be a fibration satisfying the conditions of Theorem* I.

(a) *For every fibration* $F' \xrightarrow{j'} X' \xrightarrow{f'} Y$ *in* $G_Y(f)$, $X \times F' \approx X' \times F$.

(b) *If* f *is a rational equivalence then for every fibration* $F' \xrightarrow{j'} X' \xrightarrow{f'} Y'$ *in* $G(f)$, $X \times Y' \approx X' \times Y$.

PROOF. (a) Choose bases for $\pi_* X/\text{torsion}$ and $\pi_* Y/\text{torsion}$ in which $\pi_* f/\text{torsion}$ is represented by a diagonal matrix A. Let t be an integer divisible by $|{}^t\pi_n X| \cdot |{}^t\pi_n Y|$ for $n < \max\{N(X), N(Y)\}$ and by the nonzero elements of A. By Theorem 2.2 there exist t-equivalences $h\colon X' \to X$ and $k\colon F' \to F$ so that $fh \sim f'$ and $jk \sim hj'$.

The choice of t together with the commutativity of the diagram

$$\begin{tikzcd}
& \pi_{n+1} Y \arrow[dl] \arrow[dr] & \\
\pi_n F' \arrow[rr, "k_\#"] \arrow[d, "j'_\#"] & & \pi_n F \arrow[d, "j_\#"] \\
\pi_n X' \arrow[rr, "h_\#"] \arrow[dr, "f'_\#"] & & \pi_n X \arrow[dl, "f_\#"] \\
& \pi_n Y &
\end{tikzcd}$$

imply that $\det(k_\# | \ker j'_\#) = 1$, $\det(h_\# | [(\pi_* X'/\text{torsion})/\ker f'_\#]) = 1$ and

$$\det(h_\# | \ker f'_\#) = \det\Big(k_\# \Big| \Big[\big(\pi_* F'/\text{torsion}\big)/\ker j_\#\Big]\Big).$$

Consequently one obtains from Proposition 7.1 that $X \times F' \approx X' \times F$.

(b) Let t be as in 3.1 and let $h\colon X' \to X$, $k\colon Y' \to Y$ be t-equivalences satisfying $fh \sim kf'$ (by Theorem 2.2 such t-equivalences exist). Since $\det(QH^*(h, Z)/\text{torsion}) = \det(QH^*(k, Z)/\text{torsion})$, $X' \times Y \approx X \times Y'$.

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