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ON SPECTRAL THEORY AND CONVEXITY

BY

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ABSTRACT. A compact convex set K in a locally convex algebra is said to be a spectral carrier if, for all $x, y \in K$, we have $xy = yx \in K$ and $x + y - xy \in K$. We show that if a compact convex set K is a spectral carrier, then the idempotents in K are exactly the extreme points of K and form a complete lattice. Conversely, if a compact set K is a closed convex hull of a lattice of commuting idempotents, then K is a spectral carrier. Furthermore, a metrizable spectral carrier is a Choquet simplex if and only if its extreme points form a chain of idempotents.

1. Introduction. The purpose of the present paper is to describe a general spectral theory for certain elements in a locally convex algebra over **C** or **R** from Choquet theory's point of view.

By a locally convex algebra we mean an algebra having an identity and a locally convex topology for which the multiplication is separately continuous. In §2, we will consider elements in a locally convex algebra A which are contained in some compact convex set K whose extreme points form a lattice of commuting idempotents. In §3, we show that such K is a simplex if and only if its extreme points form a chain of idempotents. If h is a hermitian operator on a Hilbert space H satisfying $0 \leqslant h \leqslant 1$ with $h = \int_0^1 \lambda de_\lambda$ as its spectral decomposition, we can show, by means of integration by parts, that h can be expressed as $\int_c e d\mu(e)$, where C is the weak closure of $\{e_\lambda\colon 0 \leqslant \lambda \leqslant 1\}$ and μ is a probability measure on C; thus h is contained in the simplex $\overline{\mathrm{co}}(C)$, the weak closure of the convex hull of C, whose extreme points form a chain of projections.

In case that the algebra A is finite-dimensional, the situation is much simpler, as shown in the following proposition.

PROPOSITION 1.1. *Let A be a finite-dimensional algebra* (*over* **C** *or* **R**, *with identity* 1) *and let x be in A. Then the following conditions are equivalent.*

(a) *x can be expressed as $\sum_{j=1}^m \mu_j f_j$, where μ_j are real numbers satisfying $0 \leqslant \mu_j \leqslant 1$ and f_j are idempotents in A such that $f_i f_j = 0$ if $i \neq j$.*

(b) *x can be expressed as $\sum_{k=1}^n \lambda_k e_k$ where $\lambda_k \geqslant 0$, $\sum_{k=1}^n \lambda_k = 1$ and e_k are idempotents satisfying $e_k e_j = e_j e_k = e_k$ if $k \geqslant j$.*

(c) *x can be expressed as a convex combination of commuting idempotents in A.*

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(d) *There is a compact convex set K containing x with the following property: if y, $z \in K$, then $yz = zy \in K$ and $y + z - yz \in K$.*

PROOF. (a) $\Rightarrow$ (b). Without the loss of generality, we may assume that

$$x = \sum_{j=1}^{m} \mu_j f_j \quad \text{with } 0 < \mu_1 < \mu_2 < \cdots < \mu_m < 1.$$

Let $\lambda_1 = \mu_1$ and $\lambda_j = \mu_j - \mu_{j-1}$ for $j = 2, 3, \ldots, m$. Then $\lambda_k > 0$ for all k and

$$\sum_{k=1}^{m} \lambda_k = \mu_m < 1.$$

Now we have

$$x = \sum_{j=1}^{m} \left(\sum_{k=1}^{j} \lambda_k \right) f_j = \sum_{k=1}^{m} \lambda_k e_k,$$

where $e_k = \sum_{j=k}^{m} f_j$. Then $e_k e_{k+1} = e_{k+1} e_k = e_{k+1}$ for $k = 1, 2, \ldots, m-1$. In case $\mu_m = 1$, we are done. Otherwise we let $e_{m+1} = 0$ and $\lambda_{m+1} = 1 - \mu_m$. Then $x = \sum_{k=1}^{m+1} \lambda_k e_k$, where $\lambda_k \geqslant 0$ and $\sum_{k=1}^{m+1} \lambda_k = 1$.

(b) $\Rightarrow$ (c). Obvious.

(a) $\Rightarrow$ (d). Let K be the set of those elements in A which can be expressed as $\sum_{j=1}^{m} \nu_j f_j$ with $0 \leqslant \nu_j \leqslant 1$. Then $x \in K$ and K has the required property.

(d) $\Rightarrow$ (c). Since x can be expressed as a convex combination of extreme points of K, it suffices to show that each extreme point e is an idempotent. By assumption, both e^2 and $2e - e^2$ are in K. Since e is an extreme point, from the identity

$$e = \tfrac{1}{2} e^2 + \tfrac{1}{2}(2e - e^2)$$

we obtain $e^2 = e$.

(c) $\Rightarrow$ (a). Suppose that $x = \sum_{k=1}^{m} \lambda_k e_k$ with $\lambda_k > 0$, $\sum_{k=1}^{m} \lambda_k = 1$ and $e_1, \ldots, e_m$ are commuting idempotents. We show (a) by induction on m. Assume that (a) holds if $m = s$. Now we consider the case $m = s + 1$. If $\lambda_{s+1} = 0$, then (a) follows from the induction hypothesis. Hence we assume $\lambda_{s+1} \neq 0$. Let

$$x_1 = \sum_{k=1}^{s} (1 - \lambda_{s+1})^{-1} \lambda_k e_k.$$

Since x_1 is a convex combination of $e_1, \ldots, e_s$, by the induction hypothesis, $x_1 = \sum_{k=1}^{r} \mu_k f_k$ for some reals μ_k with $0 \leqslant \mu_k \leqslant 1$ and some idempotents f_k with $f_j f_k = 0$ if $j \neq k$. We may assume $\sum_{k=1}^{r} f_k = 1$. Now

$$x = (1 - \lambda_{s+1}) x_1 + \lambda_{s+1} e_{s+1} = \sum_{k=1}^{r} (1 - \lambda_{s+1}) \mu_k f_k + \lambda_{s+1} e_{s+1}$$

$$= \sum_{k=1}^{r} ((1 - \lambda_{s+1}) \mu_k + \lambda_{s+1}) f_k e_{s+1} + \sum_{k=1}^{r} ((1 - \lambda_{s+1}) \mu_k) f_k (1 - e_{s+1})$$

with $0 \leqslant (1 - \lambda_{s+1}) \mu_k \leqslant ((1 - \lambda_{s+1}) \mu_k + \lambda_{s+1}) \leqslant 1$. Hence the statement (a) is true for $m = s + 1$. $\square$

Next we consider the uniqueness of the expression of x in (b) of the above proposition. Suppose that

$$x = \sum_{k=1}^{n} \lambda_k e_k = \sum_{k=1}^{n} \tilde{\lambda}_k e_k$$

where $e_k e_j = e_j e_k = e_k$ and $e_k \neq e_j$ if $k > j$. Then we have

$$(1 - e_2)x = \lambda_1 e_1(1 - e_2) = \tilde{\lambda}_1 e_1(1 - e_2).$$

Since $e_1 \neq e_2$, we have $e_1(1 - e_2) \neq 0$ and hence $\lambda_1 = \tilde{\lambda}_1$. By induction, we have $\lambda_k = \tilde{\lambda}_k$ for all k. This proves the following:

PROPOSITION 1.2. *The convex hull of idempotents $e_1, e_2, \ldots, e_n$ with the property $e_k e_{k+1} = e_{k+1} e_k = e_{k+1}$ for $k = 1, \ldots, n - 1$ is a simplex.*

2. Spectral carriers. The following definition is motivated by condition (d) in Proposition 1.1.

DEFINITION 2.1. A compact convex set K in a locally convex algebra A is called a *spectral carrier* if it satisfies the followng three conditions:

(a) $x, y \in K$ implies $xy = yx$,

(b) $x, y \in K$ implies $xy \in K$,

(c) $x, y \in K$ implies $x + y - xy \in K$.

Condition (b) of the above definition says that K is closed under multiplication. If 1 is the identity of A, then condition (c) says that $1 - K = \{1 - x\colon x \in K\}$ is closed under multiplication.

REMARK. The term "spectral carrier" suggests that elements in K have certain spectral properties. To illustrate this point, next we show an analogue of the fact that if p is a positive operator on a Hilbert space H, $\xi \in H$ and $p^2\xi = 0$, then $p\xi = 0$.

PROPOSITION 2.1. *If K is a spectral carrier in a locally convex algebra A, $x \in K$, $y \in A$ and $x^2y = 0$, then $xy = 0$.* (*Note that we do not assume x and y commute.*)

PROOF. From $x \in K$ and condition (c) in Definition 2.1, we have $x_1 = 2x - x^2 \in K$. Since $x^2y = 0$, we have $x_1 y = 2xy$. Now $x_1^2 y = x_1(x_1 y) = x_1(2xy) = 2x(x_1 y) = 2x(2xy) = 4x^2y = 0$. On the other hand, since $x_1 \in K$, we have $2x_1 - x_1^2 \in K$. Hence $2x_1 y = 2x_1 y - x_1^2 y \in Ky$, or $4xy \in Ky$. By induction, we can show that $2^n xy \in Ky$ for all $n \geqslant 1$. Since Ky is compact, we must have $xy = 0$. $\square$

The argument used for proving (d) $\Rightarrow$ (c) in Proposition 1.1 gives the following result:

PROPOSITION 2.2. *Extreme points of a spectral carrier are idempotents.*

The main result of the present section is that, conversely, idempotents in a spectral carrier are extreme points and they form a complete lattice. The idea of the proof is to establish a one-one correspondence between extreme points of K and certain faces of K. In what follows, K always stands for a spectral carrier in a locally convex algebra and $\partial_e K$ stands for the set of all extreme points of K. Recall that a subset F of K is called a *face* of K if F is convex and extremal, that is, for x, $y \in K$, $(x + y)/2 \in F$ if and only if $x, y \in F$.

PROPOSITION 2.3. *If F is a face of K, then x, $y \in F$ implies $xy \in F$ and $x + y - xy \in F$. Hence closed faces of K are spectral carriers.*

PROOF. Let x, y be in F. Since F is convex, we have $(x + y)/2 \in F$. From x, $y \in K$ we have $xy \in K$ and $x + y - xy \in K$. From the trivial identity

$$\tfrac{1}{2}(xy + (x + y - xy)) = \tfrac{1}{2}(x + y) \in F$$

and the extremal property of F it follows that both xy and $x + y - xy$ are in F. $\square$

Since $e \in \partial_e K$ if and only if the singleton $\{e\}$ is a face of K, Proposition 2.2 is also a consequence of Proposition 2.3.

DEFINITION 2.2. A nonempty subset F of K is called a *facial ideal* of K if F is a closed face with the property $FK \subseteq F$, that is, if $x \in F$ and $y \in K$, then $xy \in F$.

PROPOSITION 2.4. *If $e \in \partial_e K$, then $F = eK$ is a facial ideal.*

PROOF. It is easy to see that F is compact, convex and $FK \subseteq F$. It remains to show that F is extremal in K. Suppose that $x, y \in K$ and $z = (x + y)/2 \in F$. By Proposition 2.2, we have $e^2 = e$. From $z \in eK$ and $e^2 = e$ it is easy to see that $ez = z$, that is

$$(ex + ey)/2 = (x + y)/2$$

or

$$e = ((e + x - ex) + (e + y - ey))/2.$$

Since $e + x - ex$ and $e + y - ey$ are in K and $e \in \partial_e K$, we have

$$e = e + x - ex = e + y - ey.$$

Hence $x = ex$ and $y = ey$. In other words, $x, y \in F$. $\square$

Next we show that K has a smallest idempotent.

PROPOSITION 2.5. *There exists an idempotent e_0 in $\partial_e K$ such that $e_0 x = e_0$ for all $x \in K$.*

PROOF. Consider the family

$$\mathfrak{F} = \{eK : e \in \partial_e K\}.$$

By Proposition 2.4, $\mathfrak{F}$ is a family of facial ideals. Hence the intersection $K_0 = \bigcap \mathfrak{F}$ is also a facial ideal, provided that it is nonempty.

Note that if $e_1, \ldots, e_n$ are in $\partial_e K$, then

$$(e_1 \cdots e_n)K \subseteq e_1 K \cap \cdots \cap e_n K.$$

Hence $\mathfrak{F}$ has the finite intersection property. The compactness of K guarantees that $K_0 = \bigcap \mathfrak{F}$ is nonempty.

By Proposition 2.3, K_0 is a spectral carrier. Let $e_0 \in \partial_e K_0$. Since K_0 is a face of K, we have $\partial_e K_0 \subseteq \partial_e K$. Therefore $e_0 \in \partial_e K$.

Finally we show that $e_0x = e_0$. If $e \in \partial_e K$, then $e_0 \in eK$ and hence $e_0 e = e_0$. If x is a convex combination of extreme points, say, $x = \sum_{k=1}^n \lambda_k e_k$ with $e_k \in \partial_e K$, $\lambda_k \geqslant 0$ and $\sum_{k=1}^n \lambda_k = 1$, then

$$e_0 x = \sum_{k=1}^{n} \lambda_k e_0 e_k = \sum_{k=1}^{n} \lambda_k e_0 = e_0.$$

In general, if $x \in K$, then, by Kreĭn-Mil'man's Theorem, there exists a net $\{x_\alpha\}$ of convex combinations of $\partial_e K$ converging to x and hence we obtain

$$e_0 x = e_0 \lim x_\alpha = \lim e_0 x_\alpha = e_0. \quad \square$$

COROLLARY 2.6. *If* $0 \in K$, *then* $0 \in \partial_e K$.

COROLLARY 2.7. *There exists an idempotent* e_1 *in* $\partial_e K$ *such that* $e_1 x = x$ *for all* $x \in K$.

PROOF. Note that $1 - K = \{1 - x \colon x \in K\}$ is also a spectral carrier. Apply Proposition 2.5 to $1 - K$, we can find an extreme point e of $1 - K$ such that $e(1 - x) = e$ for all $x \in K$. Let $e_1 = 1 - e$. Then $e_1 \in \partial_e K$ and $e_1 x = x$ for all x in K. $\square$

It follows from the proof of Proposition 2.5 that

COROLLARY 2.8. *The intersection of an arbitrary family of facial ideals is a facial ideal.*

The next result is the converse of Proposition 2.4.

PROPOSITION 2.9. *If* F *is a facial ideal of* K, *then there is an extreme point* e *of* K *such that* $F = eK$.

PROOF. Apply Corollary 2.7 to the spectral carrier F, we obtain an idempotent e in $\partial_e F$ such that $F = eF$. From the fact that F is a face of K, we have $\partial_e F \subseteq \partial_e K$ and hence $e \in \partial_e K$. Since $e \in F$, we have $eK \subseteq Fk \subseteq F$. On the other hand, since $F \subseteq K$, we have $F = eF \subseteq eK$. Therefore $F = eK$. $\square$

THEOREM 2.10. *An element in a spectral carrier* K *is an idempotent if and only if it is an extreme point of* K.

PROOF. The "if" part, which is the easier part, is just Proposition 2.2. To show the "only if" part, suppose that $e \in K$ and $e^2 = e$. We claim that $F = eK$ is a facial ideal. It is clear that F is compact, convex and $FK \subseteq F$. Suppose that $x, y \in K$ and $z = (x + y)/2$ is in F. We have to show that x, y are also in F. Since $e \in K$, we have $0 = e(1 - e) \in (1 - e)K$. It is easy to check that $(1 - e)K$ is a spectral carrier. Hence, by Corollary 2.6, 0 is an extreme point of $(1 - e)K$. On the other hand, from $z = (x + y)/2 \in eK$ we have $ez = z$ or

$$((1 - e)x + (1 - e)y)/2 = 0.$$

It follows that $(1 - e)x = (1 - e)y = 0$ or $x = ex$ and $y = ey$. Thus we have shown F is a face and hence a facial ideal of K. By Proposition 2.7, there exists an extreme point e_1 of K such that $F = e_1 K$. From the fact that both e_1 and e are idempotents and $eK = e_1 K$ it is easy to see that $e = e_1$. Therefore $e \in \partial_e K$. $\square$

REMARK. It follows from the above theorem that if K_1 and K_2 are spectral carriers, then $K_1 \subseteq K_2$ if and only if $\partial_e K_1 \subseteq \partial_e K_2$. Also, if $\{K_\alpha\}$ is a family of spectral carriers and its intersection $K = \bigcap_\alpha K_\alpha$ is nonempty, then K is also a spectral carrier and

$$\partial_e K = \bigcap_\alpha \partial_e K_\alpha.$$

We define an ordering among idempotents in an algebra as follows. For idempotents e_1, e_2 in A, we put $e_1 \leqslant e_2$ if $e_1 e_2 = e_2 e_1 = e_1$. With this ordering, it is easy to check that if e_1, e_2 are commuting idempotents in A, then $e_1 \wedge e_2$ (the infimum of $\{e_1, e_2\}$) and $e_1 \vee e_2$ exist, in fact,

$$e_1 \wedge e_2 = e_1 e_2, \qquad e_1 \vee e_2 = e_1 + e_2 - e_1 e_2.$$

Thus the idempotents in K form a lattice. Let $\mathcal{F}$ be the family of all facial ideals of K. Then the mapping Φ: $\partial_e K \to \mathcal{F}$ given by $\Phi(e) = eK$ is a one-one correspondence between the idempotents and the facial ideals of K. Obviously $e_1 \leqslant e_2$ if and only if $\Phi(e_1) \subseteq \Phi(e_2)$. Also

$$\Phi(e_1 \wedge e_2) = \Phi(e_1) \cap \Phi(e_2).$$

If $\{e_\alpha\}$ is a family of idempotents in K, then, by Corollary 2.8, the intersection $\bigcap_\alpha \Phi(e_\alpha)$ is a facial ideal and hence equals $\Phi(e)$ for some $e \in \partial_e K$. It is easy to see that e is the infimum of $\{e_\alpha\}$ and thus

$$\Phi\Big(\bigwedge_\alpha e_\alpha\Big) = \bigcap_\alpha \Phi(e_\alpha).$$

This proves part (a) and (b) of the following theorem.

THEOREM 2.11. *Let K be a spectral carrier. Then the following statements hold.*

(a) *The extreme boundary $\partial_e K$ forms a complete lattice.*

(b) *If $\{e_\alpha\}$ is a subset of $\partial_e K$ with e as its infimum, then $eK = \bigcap_\alpha e_\alpha K$.*

(c) *If $\{e_\alpha\}$ is a decreasing (or increasing) net of idempotents in K, then* $\lim e_\alpha = e$ *exists and $e \in \partial_e K$.*

PROOF OF (c). Let e be the infimum of the decreasing net $\{e_\alpha\}$. Suppose for this moment that $\lim e_\alpha = x$ does exist. For $\beta > \alpha$, we have $e_\alpha e_\beta = e_\beta$. Hence, when α is fixed, we have

$$x = \lim_\beta e_\beta = \lim_\beta e_\alpha e_\beta = e_\alpha \lim_\beta e_\beta = e_\alpha x.$$

Therefore $x \in e_\alpha K$ for all α. By (b), we have $x \in eK$ from which it follows that $xe = x$. On the other hand, since $e \in e_\alpha K$ for all α, we have $ee_\alpha = e$. Hence

$$e = \lim_\alpha ee_\alpha = e \lim_\alpha e_\alpha = ex.$$

Therefore $e = x$. The same argument shows that if a subnet of $\{e_\alpha\}$ is convergent, then it must converge to e. By the compactness of K, it follows that $\lim e_\alpha = e$. □

REMARK. In general, $\partial_e K$ is not a closed set in K. For example, let $A = L^\infty[0, 1]$ with the weak*-topology. Then $K = \{x \in L^\infty[0, 1]\colon 0 \leqslant x \leqslant 1\}$ is a spectral carrier and $\partial_e K$ is the set of all indicator functions, which is not closed under the weak*-topology.

COROLLARY 2.11′. *If K is a spectral carrier satisfying $K = 1 - K$ then $\partial_e K$ is a complete Boolean algebra of idempotents.*

Next we show that, under a suitable condition, the closed convex hull of a lattice of idempotents is a spectral carrier.

PROPOSITION 2.12. *If E is a set of commuting idempotents in A such that, for all e_1, $e_2 \in E$, both $e_1 \wedge e_2 \equiv e_1 e_2$ and $e_1 \vee e_2 \equiv e_1 + e_2 - e_1 e_2$ are in E and the closed convex hull $J = \overline{\mathrm{co}}(E)$ is compact, then J is a spectral carrier.*

PROOF. We write $\mathrm{co}(E)$ for the convex hull of E. Let $x, y \in \mathrm{co}(E)$. Then x, y can be expressed as $\sum \lambda_k e_k$ and $\sum \mu_j f_j$ respectively, where $e_k, f_j \in E$, $\lambda_k \geqslant 0$, $\mu_j \geqslant 0$ and $\sum \lambda_k = \sum \mu_j = 1$. Hence

$$xy = \sum_{j,k} (\lambda_k \mu_j)(e_k f_j)$$

with $e_k f_j \in E$, $\lambda_k \mu_j \geqslant 0$ and

$$\sum_{j,k} \lambda_k \mu_j = \Big(\sum_k \lambda_k\Big)\Big(\sum_j \mu_j\Big) = 1.$$

Therefore $xy \in \mathrm{co}(E)$. Now suppose that $x \in \overline{\mathrm{co}}(E)$ and $y \in \mathrm{co}(E)$. Then there exists a net $\{x_\alpha\}$ in $\mathrm{co}(E)$ such that $\lim x_\alpha = x$. Since $x_\alpha y \in \mathrm{co}(E)$ for all α, we have $xy = \lim x_\alpha y \in \overline{\mathrm{co}}(E)$. Finally, suppose that both x, y are in $\overline{\mathrm{co}}(E)$. Then there is a net $\{y_\alpha\}$ in $\mathrm{co}(E)$ such that $\lim y_\alpha = y$. Since $xy_\alpha \in \overline{\mathrm{co}}(E)$ for all α, we have $xy = \lim xy_\alpha \in \overline{\mathrm{co}}(E)$. Thus we have shown that J is closed under multiplication. Replace J by $1 - J$ and E by $1 - E$, it follows that $1 - J$ is also closed under multiplication. Therefore J is a spectral carrier. $\square$

From the above proposition, we see that, if E is a lattice of commuting idempotents contained in a compact convex set in A, then E is contained in a complete lattice of commuting idempotents in A. From the same proposition, we see that, if E is a sublattice of $\partial_e K$, where K is a spectral carrier, then $\overline{\mathrm{co}}(E)$ is a spectral carrier contained in K. It is not hard to see that, conversely, every spectral carrier contained in K is of the form $\overline{\mathrm{co}}(E)$, where E is a sublattice of $\partial_e K$.

For the rest of this section, we consider some examples and applications to operators defined on a Hilbert space H.

For real numbers α, β with $\alpha < \beta$, we write $\mathcal{IP}[\alpha, \beta]$ for the set of polynomials with real coefficients such that $p(\alpha) = 0$, $p(\beta) = 1$ and p is increasing on $[\alpha, \beta]$. It follows from the spectral theory for hermitian operators that, if h is a hermitian operator on H with its spectrum $\sigma(H)$ contained in $[\alpha, \beta]$, then, for each p in $\mathcal{IP}[\alpha, \beta]$, we have $\|p(h)\| \leqslant 1$. The converse also holds and thus we have a characterization of hermitian operators as follows:

PROPOSITION 2.13. *An operator h on a Hilbert space H is a hermitian operator with $\sigma(h) \subseteq [\alpha, \beta]$ if and only if for all $p \in \mathcal{IP}[\alpha, \beta]$, $\|p(h)\| \leqslant 1$.*

PROOF. To show the "if" part, let K be the closure of $\{p(h) \colon p \in \mathcal{IP}[\alpha, \beta]\}$ in the weak operator topology. Since both $\mathcal{IP}[\alpha, \beta]$ and $1 - \mathcal{IP}[\alpha, \beta]$ are convex and closed under multiplication and K is contained in the unit ball of $B(H)$ which is compact in the weak operator topology, K is a spectral carrier. Suppose that $e \in \partial_e K$. Then, by Proposition 2.2, $e^2 = e$. Since we also have $\|e\| \leq 1$, e must be a projection. In particular, e is hermitian. By Kreĭn-Mil'man's Theorem, all elements in K are hermitian. Let p_0 be the polynomial defined by

$$p_0(x) = (\beta - \alpha)^{-1}(x - \alpha).$$

Then $p_0 \in \mathcal{I}\mathcal{P}[\alpha, \beta]$. Thus $p_0(h) = (\beta - \alpha)^{-1}(h - \alpha) \in K$ and hence h is hermitian. Since K is the closed convex hull of a set of projections, every element k in K satisfies $0 \leq k \leq 1$. In particular, $0 \leq (\beta - \alpha)^{-1}(h - \alpha) \leq 1$ from which it follows $\alpha \leq h \leq \beta$, or $\sigma(h) \subseteq [\alpha, \beta]$. □

REMARK. If the condition $\|p(h)\| \leq 1$ for all p in $\mathcal{I}\mathcal{P}[\alpha, \beta]$ is replaced by the weaker condition that there exists a positive number M such that $\|p(h)\| \leq M$ for all p in $\mathcal{I}\mathcal{P}[\alpha, \beta]$, then K, the closure of $\{p(h): p \in \mathcal{I}\mathcal{P}[\alpha, \beta]\}$, is still a spectral carrier. If h is a well-bounded operator on H according to Smart [9], that is, there exist constants α, β, M with $\alpha < \beta$ and $M > 0$ such that

$$\|p(h)\| \leq M(|p(\alpha)| + \text{total variation of } p \text{ over } [\alpha, \beta])$$

for every polynomial p, then

$$\|p(h)\| \leq M \quad \text{if } p \in \mathcal{I}\mathcal{P}[\alpha, \beta]$$

and hence $(\beta - \alpha)^{-1}(h - \alpha)$ is contained in a spectral carrier.

Now we give an alternative proof of Theorem XVII.2.5 in Dunford and Schwartz [4] in case that the underlying space is a Hilbert space.

PROPOSITION 2.14. *Let A be an algebra in $B(H)$ which is the image under a continuous homomorphism ϕ of the algebra $C(\Lambda)$ of all complex continuous functions on a compact space Λ. Then there exists an invertible element s in $B(H)$ such that $s^{-1}As = \{s^{-1}as: a \in A\}$ is a commutative C^*-algebra of normal operators.*

PROOF. By assumption, there exists a positive number M such that $\|\phi(f)\| \leq M\|f\|_\infty$ for all $f \in C(\Lambda)$. Let K be the closure of $\{\phi(f): f \in C(\Lambda), 0 \leq f \leq 1\}$ in the weak operator topology. Then it is easy to check that K is a spectral carrier with $K = 1 - K$. Let E be the set of all idempotents in K. Then it is easy to see that E is a bounded Boolean algebra of idempotents. By [4, Lemma XV.6.2], there is an invertible operator s such that $s^{-1}Es$ consists of projections. By Proposition 2.2, $\partial_e K \subseteq E$ and hence, by Kreĭn-Mil'man's Theorem, $s^{-1}Ks$ consists of hermitian operators. Since every element in $s^{-1}As$ is a linear combination of elements in $s^{-1}Ks$, $s^{-1}As$ consists of commuting normal operators. Now it is clear that the mapping $\psi: C(\Lambda) \to B(H)$ given by $\psi(f) = s^{-1}\phi(f)s$ is a homomorphism from $C(\Lambda)$ into a commutative C^*-algebra. Hence ψ must be *-preserving and $s^{-1}As$, the image of ψ, must be a C^*-algebra. □

3. Simplex and chain. If K is a metrizable carrier in a locally convex algebra A, then, by Choquet's theory, $\partial_e K$ is a G_δ-set and, for each $x \in K$, there exists a probability measure μ on K such that $\mu(\partial_e K) = 1$ and

$$x = \int_{\partial_e K} e\,d\mu(e).$$

The last identity means that, for every continuous linear functional ϕ,

$$\phi(x) = \int_{\partial_e K} \phi(e) d\mu(e).$$

The compact convex set K is said to be a simplex if, for each x, the measure μ described as above is uniquely determined by x. For the case when K is not necessarily metrizable, the above statements have appropriate generalizations. For details, see **[1]**, **[2]**.

By Proposition 1.1, the spectral carrier K is a simplex if the lattice $\partial_e K$ is finite and totally ordered. The main result of the present section is: a metrizable spectral carrier is a simplex if and only if the lattice $\partial_e K$ is totally ordered. The "only if" part is straightforward to prove

PROPOSITION 3.1. *If a spectral carrier K is a simplex, then the lattice $\partial_e K$ of idempotents is totally ordered.*

PROOF. Let $e_1, e_2 \in K$. Then $e_1 e_2$ and $e_1 + e_2 - e_1 e_2$ are in $\partial_e K$. Since

$$\tfrac{1}{2}(e_1 + e_2) = \tfrac{1}{2}(e_1 e_2 + (e_1 + e_2 - e_1 e_2)),$$

by the assumption that K is a simplex, we have either $e_1 = e_1 e_2$ or $e_2 = e_1 e_2$. □

For convenience, we introduce the following definition.

DEFINITION 3.1. A set C of commuting idempotents in an algebra is said to be a *chain*, if, for all e_1, e_2 in C, either $e_1 \leqslant e_2$ or $e_2 \leqslant e_1$.

PROPOSITION 3.2. *If K is a spectral carrier and $\partial_e K$ is totally ordered, then $\partial_e K$ is closed and the multiplication in $\partial_e K$ is jointly continuous.*

PROOF. Let $\{e_\alpha : \alpha \in D\}$ be a convergent net of idempotents in K and $x = \lim e_\alpha$. We claim that $\{e_\alpha\}$ has a monotone subnet. In fact, if there exists some $\alpha_0 \in D$ such that $\{e_\alpha : \alpha \geqslant \alpha_0\}$ is decreasing, then we are done. Otherwise, for each $\alpha_1 \in D$, there exists some $\alpha_2 \in D$ such that $e_{\alpha_2} \geqslant e_{\alpha_1}$ and, by means of Zorn's lemma, we can choose an increasing subnet from $\{e_\alpha\}$. By Theorem 2.11(c), every monotone net in $\partial_e K$ converges to an idempotent. Hence $x \in \partial_e K$. This shows that $\partial_e K$ is closed. By using a similar argument, we can show the ordering $\leqslant$ in $\partial_e K$ is closed, that is, $\{(e, f) : e, f \in \partial_e K, e \leqslant f\}$ is a closed subset of $\partial_e K \times \partial_e K$. Now the second part of the proposition follows from the following lemma.

LEMMA 3.3. *Let K be a spectral carrier. Suppose that $\partial_e K$ is closed and the ordering $\leqslant$ in $\partial_e K$ is closed, then the multiplication in $\partial_e K$ is jointly continuous.*

PROOF. Since $\partial_e K$ is compact, it suffices to show that if $\{e_\alpha\}$ and $\{f_\alpha\}$ are nets with the same directed set, $\lim e_\alpha = e$, $\lim f_\alpha = f$ and $\lim e_\alpha f_\alpha = g$, then $ef = g$. From the fact that $e_\alpha f_\alpha \leqslant e_\alpha$ and the assumption that $\leqslant$ is closed, we have

$$g = \lim e_\alpha f_\alpha \leqslant \lim e_\alpha = e.$$

Similarly, we have $g \leqslant f$. Hence $g \leqslant ef$. On the other hand, since $\partial_e K$ is closed and

$$e_\alpha \vee f_\alpha = e_\alpha + f_\alpha - e_\alpha f_\alpha \in \partial_e K$$

for all α, $e + f - g = \lim e_\alpha \vee f_\alpha \in \partial_e K$. Hence the element $e + f - g$ is an idempotent. Therefore

$$\begin{aligned} e + f - g = (e + f - g)^2 &= e + f + g + 2ef - 2eg - 2fg \\ &= e + f + g + 2ef - 4g \end{aligned}$$

from which we obtain $g = ef$. $\square$

COROLLARY 3.4. *If C is a chain of idempotents in a locally convex algebra and if $K = \overline{\mathrm{co}}(C)$ is compact, then K is a spectral carrier and $\partial_e K$ is a chain of idempotents containing C as a dense subchain.*

PROOF. By Proposition 2.12, K is a spectral carrier. By the proof of Proposition 3.2, we can show that $\bar{C}$, the closure of C, is a chain of idempotents. By a well-known result (e.g. [3, V.8.5]), we have $\partial_e K \subseteq \bar{C}$. On the other hand, by Theorem 2.10, $\bar{C} \subseteq \partial_e K$. Hence we have $\partial_e K = \bar{C}$. $\square$

The proof of the main result of the present section, which is the converse of Proposition 3.2 (under the extra assumption that K is metrizable), is divided into two stages. First we prove a special case: if S is a spectral carrier in $B(H)$ with $\partial_e S$ forming a chain of projections, then S is a simplex. Secondly, we treat the general case by establishing a "covering simplex" S in $B(H)$ and showing that the "covering map" is an "isomorphism" between S and K.

Now, let h be a hermitian operator defined on a Hilbert space H with the property that $0 \leqslant h \leqslant 1$. Let $h = \int_0^1 \lambda de_\lambda$ be its spectral decomposition, where $\{e_\lambda\}$ is a resolution of unity which is continuous from the right. Let C be the closure of the chain $\{1 - e_\lambda : 0 \leqslant \lambda \leqslant 1\}$. Then it follows from Corollary 3.4 that all elements in C are idempotents. On the other hand, $\|e\| \leqslant 1$ for all e in C. Therefore C is a chain of projections. Let μ be the measure defined on C by assigning $\mu(A)$ to be the Lebesgue measure of the set $\{\lambda: 0 \leqslant \lambda \leqslant 1, 1 - e_\lambda \in A\}$ for every Borel set A in C. Then, for $\xi \in H$, we have

$$\begin{aligned} \langle h\xi, \xi\rangle &= \int_0^1 \lambda d\langle e_\lambda \xi, \xi\rangle = \lambda\langle e_\lambda\xi, \xi\rangle|_0^1 - \int_0^1 \langle e_\lambda \xi, \xi\rangle d\lambda \\ &= \int_0^1 \langle (1 - e_\lambda)\xi, \xi\rangle d\lambda = \int_C \langle e\xi, \xi\rangle d\mu(e). \end{aligned}$$

Since linear functionals of the form $x \to \langle x\xi, \xi\rangle$ with $\xi \in H$ separate points of $B(H)$, we have

$$h = \int_C e d\mu(e).$$

Thus we have shown that h is the barycenter of a probability measure supported by a closed chain of projections, an expression obtained from the spectral decomposition of h by means of integration by parts. Conversely, assuming that $h = \int_C e d\mu(e)$, where C is a closed chain of projections and μ is a probability measure supported by C, it is considerably more difficult to recover the resolution of identity $\{e_\lambda\}$ from C and μ directly such that $h = \int_0^1 \lambda de_\lambda$; otherwise, we would prove the uniqueness of μ by means of the uniqueness of the spectral decomposition of h, and thus would show that the closed convex hull of C is a simplex. The last statement, laid out as a theorem as follows, is proved in an indirect way.

THEOREM 3.5. *If C is a closed chain of projections in $B(H)$, where H is a separable Hilbert space, then the closed convex hull of C (in the weak operator topology) is a simplex.*

PROOF. *Step* I. We assume here that C contains 0, 1 and has no gap. (By a gap in C we mean a pair (e_1, e_2) of elements in C with $e_1 \leqslant e_2$ and $e_1 \neq e_2$ such that, for all $e \in C$, either $e \leqslant e_1$ or $e \geqslant e_2$. See [5, Chapter I].)

Let $\{\xi_n\}$ be an orthonormal basis of H. Then the mapping $\phi: C \to [0, 1]$ defined by

$$\phi(e) = \sum_{n=1}^{\infty} 2^{-n}(e\xi_n, \xi_n)$$

is one-one, continuous and order-preserving. Since C has no gap, C is connected and hence ϕ must be surjective. Now suppose that μ_1, μ_2 are probability measures supported by C and

$$h = \int_C e d\mu_1(e) = \int_C e d\mu_2(e).$$

Let $\nu_j = \mu_j \circ \phi^{-1}$, $e_\lambda = \phi^{-1}(\lambda)$ and $f_j(\lambda) = \nu_j[0, \lambda]$ for $j = 1, 2$ and $0 \leqslant \lambda \leqslant 1$. Then, for $\xi \in H$,

$$\begin{aligned}\langle h\xi, \xi\rangle &= \int_C \langle e\xi, \xi\rangle d\mu_j(e) = \int_0^1 \langle e_\lambda \xi, \xi\rangle d\nu_j(\lambda)\\ &= f_j(\lambda)\langle e_\lambda\xi, \xi\rangle|_0^1 - \int_0^1 f_j(\lambda) d\langle e_\lambda \xi, \xi\rangle\\ &= \|\xi\|^2 - \left\langle\left(\int_0^1 f_j(\lambda) de_\lambda\right)\xi, \xi\right\rangle.\end{aligned}$$

Hence we have $\int_0^1 f_1(\lambda) de_\lambda = \int_0^1 f_2(\lambda) de_\lambda$. Since both f_1 and f_2 are nondecreasing, continuous from the right and the map $\lambda \to e_\lambda$ is continuous, strictly increasing, it is easy to check that $f_1 = f_2$. Therefore $\nu_1 = \nu_2$ which in turn implies $\mu_1 = \mu_2$.

Step II. Now we consider the general case. Let $\phi: B(H) \to B(H \otimes H)$ be the mapping defined by $\phi(x) = x \otimes 1$. Then $\phi(C)$ is closed chain of projections in $B(H \otimes H)$. It is easy to check that, for $e_1, e_2 \in C$, the pair (e_1, e_2) is a gap in C if and only if $(\phi(e_1), \phi(e_2))$ is a gap in $\phi(C)$ and, in such case, the rank of the projection $\phi(e_2) - \phi(e_1) = +\infty$. Hence there is a closed chain $\tilde{C}$ of projections in $B(H \otimes H)$ such that $\phi(C) \subseteq \tilde{C}$ and $\tilde{C}$ has no gap. (For details, see [5, pp. 17–18].) Now suppose that

$$h = \int_C e d\mu_j(e) \qquad (j = 1, 2).$$

Let $\nu_j = \mu_j \circ \phi^{-1}$. Then ν_j is a measure on $\phi(C)$ and hence can be regarded as a measure on $\tilde{C}$. We have

$$h \otimes 1 = \int_C (e \otimes 1) d\mu_j(e) = \int_{\tilde{C}} f d\nu_j(f).$$

By Step I, we have $\nu_1 = \nu_2$. Since ϕ is a homeomorphism between C and $\phi(C)$, we must have $\mu_1 = \mu_2$. □

PROPOSITION 3.6. *Let C be a chain of projections in $B(H)$ and $K = \overline{\mathrm{co}}(C)$, where the Hilbert space H is not necessarily separable. Then the strong operator topology in K coincides with the weak operator topology.*

PROOF. From a topological consideration, we see that it suffices to show that K is compact under the strong operator topology. Since the unit ball of $B(H)$ is complete in the strong operator topology, by a well-known fact concerning compact sets in a topological vector space (e.g., see [**8**, p. 50, Corollary II 4.3]) it suffices to show that C is compact in the strong operator topology. Now, if $\{e_\nu\}$ is a net in C, then, by an argument used in the proof of Proposition 3.4, $\{e_\nu\}$ has a monotone subnet. It is well known that every monotone net of projections on H is strongly convergent. Therefore C is compact. □

Now we return to the general theory. For the rest of this section, we always assume that K is a metrizable spectral carrier with $\partial_e K$ totally ordered. For technical reasons, we also assume that K contains 0 and 1. Our goal is to show that K is a simplex.

LEMMA 3.7. *There is an order-preserving homeomorphism ψ from $\partial_e K$ onto some compact set M in $[0, 1]$ such that $\psi(0) = 0$ and $\psi(1) = 1$.*

PROOF. By modifying the proof of Urysohn's lemma, for given $e_1, e_2 \in \partial_e K$ with $e_1 \leqslant e_2$, $e_1 \neq e_2$, one can construct a continuous increasing function ϕ: $\partial_e K \to [0, 1]$ such that $\phi(e_1) = 0$ and $\phi(e_2) = 1$. (For details, see [**6**, Theorem 1.2.1].) It suffices to show that there exists a sequence $\{\psi_n\}$ of continuous increasing functions from $\partial_e K$ into $[0, 1]$ which separates points of $\partial_e K$ with $\psi_n(0) = 0$ and $\psi_n(1) = 1$; for then we can set

$$\psi = \sum_{n=1}^{\infty} 2^{-n}\psi_n$$

which is a function having the required properties. To this end, first we show that $\partial_e K$ has at most countably many gaps. Let ρ be a metric on K. For each positive integer k, let G_k be the collection of all gaps (e, f) with $\rho(e, f) > k^{-1}$. Then G_k is a finite collection. Otherwise, by an argument used in Proposition 3.2, we can show that there exists a sequence (e_n, f_n) in G_k such that $\{e_n\}$ is strictly monotone. For definiteness, we assume that $\{e_n\}$ is strictly increasing. We have

$$e_1 \leqslant f_1 \leqslant e_2 \leqslant f_2 \leqslant e_3 \leqslant f_3 \leqslant \cdots.$$

By Theorem 2.11(c), both $\{e_n\}$ and $\{f_n\}$ are convergent to the same limit. But, on the other hand,

$$\rho(\lim e_n, \lim f_n) = \lim \rho(e_n, f_n) \geqslant k^{-1}.$$

Thus we arrive at a contradiction. Now it is clear that the collection of all gaps, namely, $\bigcup_k G_k$, is at most countable. Let D be a countable dense subset of $\partial_e K$. Let P be the collection of all those pairs (e, f) with $e, f \in C$, $e \leqslant f$, $e \neq f$, such that either (e, f) is a gap of C or both e, f are in D. Then D is countable and hence can be arranged into a sequence, say, $D = \{(e_n, f_n): n = 1, 2, \ldots\}$. For each n, let ψ_n be a continuous increasing function on $\partial_e K$ into $[0, 1]$ such that $\psi_n(e_n) = 0$ and $\psi_n(f_n) = 1$. It is not hard to see that $\{\psi_n\}$ separates points of $\partial_e K$. □

LEMMA 3.8. *There exists a closed chain C of projections on a separable Hilbert space H and a homeomorphism ϕ: $C \to \partial_e K$ such that $\phi(0) = 0$, $\phi(1) = 1$ and $\phi(p_1 p_2) = \phi(p_1)\phi(p_2)$ for all $p_1, p_2 \in C$.*

PROOF. Let ψ, M be the same as those in the previous lemma. Let $H = L^2[0, 1]$. For $\lambda \in M$, let p_λ be the projection sending $\xi \in L^2[0, 1]$ to $\chi_{[0,\lambda]}\xi$. (Here $\chi_{[0,\lambda]}$ stands for the characteristic function of the closed interval $[0, \lambda]$.) Then $\lambda \to p_\lambda$ is a one-one, continuous and increasing mapping from M onto a chain C of projections. It is straightforward to check that the inverse mapping of $e \to p_{\psi(e)}$, where $e \in \partial_e K$, is the required mapping ϕ. (Note that the condition $\phi(p_1 p_2) = \phi(p_1)\phi(p_2)$ for all p_1, p_2 means the same as that ϕ is increasing.) □

Let C and ϕ be those described in Lemma 3.8. Let S be the closure of $\mathrm{co}(C)$ in the weak operator topology. By Theorem 3.5, S is a simplex. Hence, for each $x \in S$, there exists a unique probability measure μ_x on C such that $x = \int_C p\, d\mu_x(p)$. Define $\tilde{\phi}$: $S \to K$ by putting

$$\tilde{\phi}(x) = \int_{\partial_e K} e\, d(\mu_x \circ \phi^{-1})(e).$$

It is not hard to check that $\tilde{\phi}$ is continuous and affine. (For details, see [**1**, Theorem II 4. 1].) Note that if $x \in \mathrm{co}(C)$, say, $x = \sum \lambda_k p_k$ with $\lambda_k > 0$ and $\sum \lambda_k = 1$, then $\tilde{\phi}(x) = \sum \lambda_k \phi(p_k)$. From the fact that $\phi(pq) = \phi(p)\phi(q)$ for all $p, q \in C$, it is easy to check that, if $x, y \in \mathrm{co}(C)$, then $\tilde{\phi}(x)\tilde{\phi}(y) = \tilde{\phi}(xy)$. By the continuity of $\tilde{\phi}$ and the denseness of $\mathrm{co}(C)$ in S, the last identity holds for all $x, y \in S$. Thus we obtain

PROPOSITION 3.9. *There exists a continuous affine mapping ϕ from $S = \overline{\mathrm{co}}(C)$ onto K such that for all $x, y \in S$, $\phi(xy) = \phi(x)\phi(y)$ and the restriction of ϕ to C is one-one and onto $\partial_e K$.*

COROLLARY 3.10. *The multiplication in K is jointly continuous.*

PROOF. By the compactness of K, it suffices to show that if a_n, $b_n \in K$, $\lim b_n = b$ and $\lim a_n b_n = c$, then $c = ab$. Let $x_n, y_n \in S$ be such that $\phi(x_n) = a_n$ and $\phi(y_n) = b_n$. By taking a subsequence if necessary, we may assume that both $\{x_n\}$ and $\{y_n\}$ are convergent in the weak operator topology, say $x = \lim x_n$ and $y = \lim y_n$. By Proposition 3.6, the weak operator topology and the strong operator topology in S coincide. Therefore we have $xy = \lim x_n y_n$. Now

$$\begin{aligned} ab &= \phi(x)\phi(y) = \phi(xy) = \lim \phi(x_n y_n) \\ &= \lim \phi(x_n)\phi(y_n) = \lim a_n b_n = c. \quad \square \end{aligned}$$

Next we develop a functional calculus for elements in K. We denote by $\mathcal{G}[0, 1]$ the set of all real-valued functions f defined on $[0, 1]$ such that $f(0) = 0$, $f(1) = 1$

and f is monotonely increasing on [0, 1]. Recall that $\mathcal{I}\mathcal{P}[0, 1]$ is the set of those functions in $\mathcal{I}[0, 1]$ which are polynomials. We write $\mathcal{I}\mathcal{C}[0, 1]$ for all those functions in $\mathcal{I}[0, 1]$ which are continuous.

LEMMA 3.11. *If $a \in K$ and $p \in \mathcal{I}\mathcal{P}[0, 1]$, then $p(a) \in K$.*

PROOF. Using summation by parts (see the proof of (a) $\Rightarrow$ (b) in Proposition 1.1), we can show that an element b in K is a convex combination of $\partial_e K$ if and only if it can be expressed as $b = \sum_{j=1}^{n} \mu_j f_j$, where $0 \leqslant \mu_1 < \mu_2 < \cdots < \mu_n \leqslant 1$, f_j are idempotents satisfying $f_j f_k = 0$ if $j \neq k$ and $e_k = \sum_{j=k}^{n} f_j \in \partial_e K$ for $k = 1, \ldots, n$. For such b, we have

$$p(b) = \sum_{j=1}^{n} p(\mu_j) f_j$$

with $0 \leqslant p(\mu_1) < p(\mu_2) < \cdots < p(\mu_n) \leqslant 1$. Thus the lemma holds for convex combination of $\partial_e K$. In general, we take a sequence $\{b_n\}$ of convex combinations of $\partial_e K$ such that $\lim b_n = a$. By Corollary 3.10, we have $\lim p(b_n) = p(a)$. Since $p(b_n) \in K$ for each n, we have $p(a) \in K$. $\square$

LEMMA 3.12. (a) *If $g \in \mathcal{I}\mathcal{C}[0, 1]$, then there is a sequence $\{q_n\}$ in $\mathcal{I}\mathcal{P}[0, 1]$ such that*

$$\|q_n - g\|_\infty = \sup_{0 \leqslant \lambda \leqslant 1} |q_n(\lambda) - g(\lambda)| \to 0 \quad \text{as } n \to \infty.$$

(b) *If $f \in \mathcal{I}[0, 1]$, then there exists a sequence $\{r_n\}$ in $\mathcal{P}\mathcal{I}[0, 1]$ such that $\lim r_n(\lambda) = f(\lambda)$ for all $\lambda \in [0, 1]$.*

PROOF. (a) By means of a smoothing process, it is not difficult to show that g can be uniformly approximated by C^∞-functions in $\mathcal{I}[0, 1]$. Since g is increasing, $g'(\lambda) \geqslant 0$ for all $\lambda \in [0, 1]$. Consider the Bernstein polynomials

$$b_n(\lambda) = \sum_{k=0}^{n} g'\left(\frac{k}{n}\right)\binom{n}{k}\lambda^k(1 - \lambda)^{n-k} \qquad (0 \leqslant \lambda \leqslant 1)$$

and let $p_n(\lambda) = \int_0^\lambda b_n(\xi)\,d\xi$. Then $b_n(\lambda) \geqslant 0$ for all λ and hence p_n is a polynomial increasing on [0, 1]. Since $\|b_n - g'\|_\infty \to 0$, we have $\|p_n - g\|_\infty \to 0$. Let $q_n = p_n(1)^{-1}p_n$. Then q_n is the required sequence.

(b) Note that f can be expressed as

$$f = \varepsilon_0 f_0 + \sum_{n=1}^{\infty} \varepsilon_n \chi_n$$

where $\varepsilon_n \geqslant 0$ for all $n \geqslant 0$, $\sum_{n=0}^{\infty} \varepsilon_n = 1$, $f_0 \in \mathcal{I}\mathcal{C}[0, 1]$ and, for each $n \geqslant 1$, χ_n is a function in $\mathcal{I}[0, 1]$ such that, for all $\lambda \in [0, 1]$, possibly with one exceptional point, the value of $\chi_n(\lambda)$ is either 0 or 1. It is not difficult to show that each χ_n is the pointwise limit of a sequence in $\mathcal{I}\mathcal{C}[0, 1]$. Now (b) follows from (a). $\square$

PROPOSITION 3.13. *For each $a \in K$, there is a mapping from $\mathcal{G}[0, 1]$ into K, denoted by $f \to f(a)$, such that*:

(a) *If $p \in \mathcal{G}\mathcal{P}[0, 1]$, then $p(a)$ has the usual meaning.*

(b) *For $f, g \in \mathcal{G}[0, 1]$ and $0 \leqslant \lambda \leqslant 1$, we have*

$$(fg)(a) = f(a)g(a),$$
$$(f \circ g)(a) = f(g(a)),$$
$$(\lambda f + (1 - \lambda)g)(a) = \lambda f(a) + (1 - \lambda)g(a).$$

(c) *If $f_n \in \mathcal{G}[0, 1]$ and $f_n \to f$ pointwisely, then $f_n(a) \to f(a)$.*

PROOF. Take any $x \in S$ such that $\phi(x) = a$. (The symbols S and ϕ are the same as those in Proposition 3.9.) For $f \in \mathcal{G}[0, 1]$, let $f(a)$ be defined by putting $f(a) = \phi(f(x))$. We have to show that $f(a)$ does not depend on the choice of x. By Lemma 3.12(b), there exists a sequence $\{p_n\}$ in $\mathcal{G}\mathcal{P}[0, 1]$ such that $\lim p_n(\lambda) = f(\lambda)$ for all $\lambda \in [0, 1]$. By Lebesgue's dominated convergence theorem, one can show that $p_n(x)$ converges to $f(x)$ in the strong operator topology. Hence

$$\phi(f(x)) = \phi(\lim p_n(x)) = \lim \phi(p_n(x)) = \lim p_n(a).$$

The limit $\lim p_n(a)$ is certainly independent of the choice of x. Hence the expression $f(a)$ is well defined. The rest of the proof is routine and hence omitted. □

Now we can state and prove the main theorem of the present section.

THEOREM 3.14. *If K is a metrizable spectral carrier and if $\partial_e K$ is a chain of idempotents, then K is a simplex.*

PROOF. Let ϕ: $S \to K$ be the affine mapping constructed in Proposition 3.9. Since, by Theorem 3.5, S is a simplex, it suffices to show that ϕ is one-one. Let x_1, $x_2 \in S$ be such that $\phi(x_1) = \phi(x_2) = a$. For $\lambda \in (0, 1]$, define f_λ by

$$f_\lambda(\xi) = \begin{cases} 0 & \text{if } \xi < \lambda, \\ 1 & \text{if } \xi \geqslant \lambda. \end{cases}$$

Then, according to the proof of Proposition 3.13, $f_\lambda(a) = \phi(f_\lambda(x_j))$, $j = 1, 2$. Since $f_\lambda^2 = f_\lambda$, $f_\lambda(x_j)$ is a projection and hence is in C. Since ϕ is one-one on C, we have $f_\lambda(x_1) = f_\lambda(x_2)$ for all $\lambda \in (0, 1]$. Now, by the spectral theorem for hermitian operators, we have $x_1 = x_2$. Hence ϕ is one-one and thus K is a simplex. □

A spectral carrier which is also a simplex is naturally called a simplicial spectral carrier, or simply called a simplicial carrier. Theorem 3.14 and Proposition 3.1 say that a metrizable spectral carrier is simplicial if and only if its extreme points form a chain of idempotents. From Choquet Theory's point of view, elements in a simplicial carrier are nice. A natural question is, when is an element contained in a simplicial carrier? In particular, if an element is contained in a spectral carrier, is it necessarily contained in a simplicial carrier? Here we only give a rather modest partial answer to this question.

PROPOSITION 3.15. *If K is a metrizable spectral carrier such that* (a) *$\partial_e K$ is closed and* (b) *the ordering $\leqslant$ in $\partial_e K$ is closed, then each element in K is contained in a simplicial carrier.*

PROOF. Let ρ be a metric in K and let $x \in K$. By Kreĭn-Mil'man's Theorem, there is a sequence $\{x_n\}$ in $\mathrm{co}(\partial_e K)$ such that $\lim \rho(x, x_n) = 0$. For each n, x_n is a convex combination of commuting idempotents and hence, by Proposition 1.1, x_n is contained in a finite-dimensional simplex, say, $\mathrm{co}(C_n)$, where C_n is a finite chain of idempotents. Let $S_n = \mathrm{co}(C_n) \cap K$. It is straightforward to check that S_n is a spectral carrier. Since an element is an idempotent in S_n if and only if it is an idempotent in both $\mathrm{co}(C_n)$ and K, we have $\partial_e S_n = C_n \cap \partial_e K$. Therefore $\partial_e S_n$ is a chain and thus S_n is a simplex. Without the loss of generality, we may assume $\mathrm{co}(C_n) = S_n$ and thus $x_n \in S_n$ and $\partial_e S_n = C_n \subseteq \partial_e K$. Recall that $\lim\sup_n C_n$ (resp. $\lim\inf_n C_n$) is the set of all those elements y in K such that, for every neighborhood V_y of y, $V_y \cap C_n \neq \varnothing$ for infinitely many n (resp., for all except finitely many n). By a well-known result in general topology (see, e.g., [10, Theorem I.7.1]), $\{C_n\}$ has a subsequence $\{C_{n_k}\}$ such that

$$\limsup_k C_{n_k} = \liminf_k C_{n_k} \ (= C, \text{say}).$$

From $\lim\sup_k C_{n_k} = C$ and the compactness of C, we see that, for a given $\varepsilon > 0$, there exists some k_0 such that, for $k \geqslant k_0$, we have

$$C_{n_k} \subseteq \{y : \rho(y, C) \leqslant \varepsilon\}$$

from which we obtain

$$\mathrm{co}(C_{n_k}) \subseteq \{y : \rho(y, \overline{\mathrm{co}}(C)) \leqslant \varepsilon\}.$$

Since $x_{n_k} \in \mathrm{co}(C_{n_k})$ for all k and $\rho(x_{n_k}, x) \to 0$ as $k \to \infty$, we have $\rho(x, \overline{\mathrm{co}}(C)) \leqslant \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $x \in \overline{\mathrm{co}}(C)$. It remains to show that C is a chain of idempotents.

By assumption (a), we have $C \subseteq \partial_e K$. Let $e_1, e_2 \in C$ with $e_1 \neq e_2$. Since $\lim\inf_k C_{n_k} = C$, there exist sequences $\{p_k\}$, $\{q_k\}$ such that $p_k, q_k \in C_{n_k}$ and $\lim p_k = e_1$, $\lim q_k = e_2$. For each k, we have either $p_k \leqslant q_k$ or $p_k \geqslant q_k$. Hence we have either $p_k \leqslant q_k$ for infinitely many k or $p_k \geqslant q_k$ for infinitely many k. By assumption (b), we have either $e_1 \geqslant e_2$ or $e_1 \leqslant e_2$. Therefore C is a chain. $\square$

COROLLARY 3.16. *If K is a metrizable spectral carrier in which the multiplication is jointly continuous, then each element in K is contained in a simplicial carrier.*

REMARK. Let x be an element in a locally convex algebra A. Suppose that x is contained in a simplicial carrier. Then, by Lemma 3.11, the set

$$S_x = \overline{\mathrm{co}}\{p(x) : p \in \mathcal{G}\mathcal{P}[0, 1]\}$$

is the smallest simplicial carrier containing x. Since $\mathcal{G}[0, 1]$ is compact in the pointwise-convergence topology, by Proposition 3.13(c), we have

$$S_x = \{f(x) : f \in \mathcal{G}[0, 1]\}.$$

We may call S_x the support of x. It is easy to see that, if x is contained in a simplicial carrier, $y \in A$ and $yx = xy$, then $ys = sy$ for every s in the support of x. Thus we obtain a version of Fuglede's theorem for such x.

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