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THE COHOMOLOGY ALGEBRAS OF FINITE DIMENSIONAL HOPF ALGEBRAS

BY

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ABSTRACT. The cohomology algebra of a finite dimensional graded connected cocommutative biassociative Hopf algebra over a field K is shown to be a finitely generated K-algebra. Counterexamples to the analogue of a result of Quillen (that nonnilpotent cohomology classes should have nonzero restriction to some abelian sub-Hopf algebra) are constructed, but an elementary proof of the validity of this "detection principle" for the special case of finite sub-Hopf algebras of the mod 2 Steenrod algebra is given. As an application, an explicit formula for the Krull dimension of the cohomology algebras of the finite skeletons of the mod 2 Steenrod algebra is given.

If A is an augmented algebra over the field K, the cohomology algebra $H^*(A)$ is defined as $\mathrm{Ext}_A(K, K)$. If A is finite dimensional as a K-vector space, $H^*(A)$ may still fail to be a finitely generated K-algebra, e.g., Löfwall **[12]**. However, if $A = K[G]$, the group algebra of a finite group, then $H^*(A)$ is finitely generated, Evens **[6]** and Venkov **[21]**. Cocommutative Hopf algebras are one generalization of group algebras, and connected graded cocommutative Hopf algebras are closely analogous to finite p-groups. It is the intent of this work to push this analogy as far as possible. The first positive result is that finite generation holds in this context also (all Hopf algebras mentioned in this work are biassociative and either commutative or cocommutative).

THEOREM A. *If A_* is a finite dimensional graded connected cocommutative K-Hopf algebra, then $H^{**}(A_*) = \mathrm{Ext}^{**}_{A_*}(K, K)$ is a finitely generated K-algebra.*

The strategy of the proof is essentially that developed by Adams **[1]** and Liulevicius **[10]** for the computation of the cohomology of small sub-Hopf algebras of the Steenrod algebras. One resolves A_* as a sequence of iterated central extensions of Hopf algebras. Each such extension has an associated spectral sequence, and some hold on the differentials is provided by the transgression theorem relating Steenrod operations on the "fiber" and "base". The only philosophical difference between the present plan and that of **[1]**, **[10]** is that precise computational results are sacrificed for the sake of generality.

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Another analogue of a result of Evens **[6]** is a relative version of Theorem A that includes as Corollary C the analogue of a result of Swan **[20]**:

THEOREM B. *If A_* is a finite dimensional graded connected cocommutative K-Hopf algebra and B_* is a sub-Hopf algebra of A_*, then $H^{**}(B_*)$ is a finitely generated $H^{**}(A_*)$ module under the restriction map.*

COROLLARY C. (i) *If i: $B_* \to A_*$ is the inclusion of a sub-Hopf algebra with A_* as in Theorem* B, *then the restriction map $H^{s,t}(i)$: $H^{s,t}(A_*) \to H^{s,t}(B_*)/$nilpotents is nonzero in infinitely many positive bidegrees.*

(ii) *If Φ: $C_* \to A_*$ is onto, then Φ is an isomorphism if and only if $H^{s,t}(\Phi)$: $H^{s,t}(A_*) \to H^{s,t}(C_*)/$nilpotents is onto for all (s, t) with $s + t$ sufficiently large.*

COROLLARY D. *The Krull dimension of $H^{**}(A_*)$ is finite and greater than or equal to the Krull dimension of $H^{**}(B_*)$, for B_* any sub-Hopf algebra of A_*.*

Finally, pursuing the analogy with p-groups even further, we force it beyond its capacity. Quillen **[16]** proved that for finite groups, an element of $H^*(G, \mathbf{F}_p)$ is nilpotent if and only if its restriction to each p-elementary subgroup of G is zero. H. Miller speculated to the author that some approximation of this result might be valid in the context of Hopf algebras. A connected cocommutative commutative Hopf algebra E_* over a perfect field of characteristic p is said to be elementary if $(E_* \setminus K)^p \equiv 0$, and the Hopf algebra A_* is said to have the detection property if each nonnilpotent cohomology class has a nonzero restriction to at least one elementary sub-Hopf algebra of A_*. All the steps of the Quillen-Venkov proof **[17]** of the detection property for finite groups are valid for the Hopf algebra setting, except that the analogue of a key result of Serre **[18]** characterizing p-elementary groups cohomologically fails. Hence one obtains only a sufficient condition for the detection property to hold.

COUNTEREXAMPLE E. For each prime p, there exists a finite dimensional graded connected cocommutative Hopf algebra B_* over $\mathbf{F}_p$ and a nonnilpotent cohomology class u_B which restricts to zero on every abelian sub-Hopf algebra of B_*. If p is odd, B_* may be taken to be a sub-Hopf algebra of the cyclic reduced powers in the mod p Steenrod algebra.

In spite of these counterexamples to the universal validity of the detection property, the sufficient condition derived from the Quillen-Venkov proof can be directly verified in favorable cases:

THEOREM F. *Any finite sub-Hopf algebra of the* mod 2 *Steenrod algebra* $\mathcal{A}(2)$ *has the detection property.*

Theorem F was originally proved by W. H. Lin **[9]** from a somewhat different point of view.

COROLLARY G. (i) *If A_* has the detection property, then the Krull dimension of $H^{**}(A_*)$ is the maximal rank of the elementary sub-Hopf algebras of A_*, where the rank of E_* is* $\dim_K(H^{1,*}(E_*))$.

(ii) *The Krull dimension of the cohomology algebra of any finite sub-Hopf algebra of* $\mathcal{Q}(2)$ *can be calculated explicitly. For example, if* $\mathcal{Q}_n$ *denotes the sub-Hopf algebra of* $\mathcal{Q}(2)$ *generated by* $\mathrm{Sq}^1, \ldots, \mathrm{Sq}^{2^n}$, *the Krull dimension of its cohomology algebra is* $\varepsilon + r(2n + 5 - 3r)/2$, *where* r *is the greatest integer in* $(2n + 5)/6$ *and* $\varepsilon = 0$ *unless* 3 *divides* n, *in which case it is* 1.

Extensions of Quillen's results to $H^*(G, M)$ for M a G-module have been recently obtained by J. Alperin and L. Evens. These new results appear to have analogues for modules over Hopf algebras with the detection property, but a discussion of this material is postponed to a sequel to the present work.

Even in cases where A_* is not related to the mod p Steenrod algebras, there is a topological motivation for the study of $H^{**}(A_*)$. If X is a simply connected CW complex such that the loop space ΩX has finite $\mathbf{Z}/p\mathbf{Z}$ cohomology, then the Eilenberg-Moore or Rothenberg-Steenrod spectral sequence converging to $H^*(X, Z/pZ)$ has its E_2-term isomorphic to $\mathrm{Ext}^{**}_{H_*(\Omega X, Z/pZ)}(Z/pZ, Z/pZ)$. By Theorem A, this E_2-term is a finitely generated algebra. However, it is not yet known that the spectral sequence degenerates at any finite stage, so the question of finite generation of $H^*(X, Z/pZ)$ is still an important open question.

Finally, $H^{**}(A^*)$ is a function of only the algebra structure of A_*. J. Moore has asked if it is possible to abstract the properties of A_* forced by the Hopf algebra structure, e.g., the existence of a central series, in a nontautological way, so that Theorem A would be valid for this wider class of algebras. The present answer is no; the entire line of proof rests on the existence of Steenrod operations and on the transgression theorem. Corollary C(i) is essentially equivalent to Theorem A, and no large class of examples for which Corollary C(i) is valid springs to mind.

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1. The case char $K = 0$; **The central series.** The reader might well believe that the restrictions placed on the algebra structure of A_* by the concommitant coalgebra structure are so severe that Theorem A is trivially forced. This skepticism is perhaps justified if char $K = 0$, but in general the hypothesis manifests itself in a more subtle fashion.

PROPOSITION 1.1. *If* A_* *is as in Theorem* A *and* char $K = 0$, *then* A_* *is isomorphic to a tensor product of exterior algebras. Thus Theorem* A *is valid in this case.*

PROOF. Since char $K = 0$, the cocommutativity of A_* implies that A_* is primitively generated, Milnor-Moore [**15**]. Since A_* is finite dimensional, any nonzero primitive is odd dimensional, and hence the commutator of any two primitives is zero. Therefore A_* is commutative, and the Borel Structure Theorem implies the structure of A_*. $H^{**}(A_*)$ is the polynomial algebra on n generators, for n the K-dimension of the indecomposables of A_*. This is essentially the original argument of H. Hopf.

In general, the only immediate consequence of the Hopf algebra hypothesis is

PROPOSITION 1.2. *If $A_* \neq K$ is as in Theorem* A, *there exists a nontrivial central monogenic sub-Hopf algebra C_*.*

PROOF. The general case has the same proof as the special case given in Liulevicius [**10** p. 28]. Denote by A^* the graded K-linear dual of A_*. Let n be the largest dimension in which the indecomposable quotient QA^* is nonzero. Let I^* be the Hopf ideal of A^* generated by the elements of degree strictly less than n. Define B_* as the linear dual of A^*/I^*. A diagram chase shows that B_* is central in A_*. Take C_* to be the monogenic Hopf algebra generated by a nonzero element of B_* of lowest positive degree.

We will later need the result that in the category of connected cocommutative Hopf algebras morphisms have unique kernels:

PROPOSITION 1.3. *If Φ: $A_* \to B_*$ is a surjective morphism of graded connected cocommutative K-Hopf algebras, there exists a unique sub-Hopf algebra N_* of A_* such that* $\ker \Phi = A_* \bar{N}_*$, *the left ideal generated by the elements of positive dimension in N_*. If C_* is any sub-Hopf algebra of A_* such that $C_* \cap N_* = K$, then Φ restricted to C_* is one-to-one.*

PROOF. This is a restatement of Theorem 4.9 of Milnor-Moore [**15**]. N_* is the linear dual to $A^*/A^*\bar{B}^*$, and A_* is isomorphic to $N_* \otimes_K B_*$ as a left N_*-module. If $C_* \cap \ker \Phi \neq 0$, any lowest dimensional nonzero element in the intersection is primitive, and together with N_* would generate a strictly larger sub-Hopf algebra contained in $\ker \Phi$. This contradicts $A_* \approx N_* \otimes_K B_*$.

2. Steenrod operations and spectral sequences. Since A_* is graded, the cohomology algebra is bigraded by $H^{s,t}(A_*) = \operatorname{Ext}^{s,t}_{A_*}(K, K)$, where s is the homological degree and t is the internal grading. $H^{**}(A_*)$ can in theory be computed from the cobar construction on A^*, Adams [**1**]: $B^*(A_*)$ is the free tensor algebra on the K-vector space of elements of positive degree of A^*. The generators of $B^*(A_*)$ are denoted by $[z]$, for z in A^*, and the differential on generators is specified as $d[z] = \Sigma[z_i'][z_i'']$ where the reduced coproduct of z is $\Sigma z_i' \otimes z_i''$. d is extended to a graded differential on $B^*(A_*)$, where the grading is by total degree.

Steenrod operations appear in the guise of cup-i products on the cohomology of cocommutative Hopf algebras in Adams [**1**] and are defined explicitly in Liulevicius [**10**] by applying Steenrod's construction to the cobar construction. The spectral sequence associated to a central extension of cocommutative Hopf algebras appears in Adams [**1**] and Liulevicius [**10**]. We need basically the existence and formal properties of the Steenrod operations, the spectral sequence of a central extension, and the transgression formula. All this material is in May [**14**] and we follow that choice of notation and indexing. The only liberty taken is to extend the operations to the case K of characteristic p, but not equal to the prime field; then the operations are $\mathbf{Z}/p\mathbf{Z}$-linear, but not K-linear.

PROPOSITION 2.1. *Let A_* be a cocommutative Hopf algebra over K*, char $K > 2$. *There exist operations $\{\mathcal{P}^i, \beta\mathcal{P}^i$ for $i \geq 0\}$ with the following properties*:

(i) *$\mathcal{P}^i$ acting on $H^{s,t}(A_*)$ has bidegree $((2i - t)(p - 1), t(p - 1))$ and $\beta\mathcal{P}^i$ has bidegree $((2i - t)(p - 1) + 1, t(p - 1))$.*

(ii) *$\mathcal{P}^i \equiv 0$ if $2i < t$ or $2i > s + t$; $\beta\mathcal{P}^i \equiv 0$ if $2i < t$ or $2i \geq s + t$; $\mathcal{P}^i x_{st} = x^p$ if $s + t = 2i$.*

(iii) *The usual Cartan formulae and Adem relations hold if $\mathcal{P}^0$ is considered as an independent homomorphism not necessarily the identity.*

(iv) *The operations are natural for maps of Hopf algebras.*

For $p = 2$, the usual reindexing is required for the proper statement of Proposition 2.1. There is also a reindexing that is convenient for the operations on $H^{*,2*}$, $\tilde{\mathcal{P}}^i = \tilde{\mathcal{P}}^{i+t}$ on $H^{s,2t}$, and similarly for $\beta\tilde{\mathcal{P}}^i$. The Cartan formulae and Adem relations are satisfied for the new operations also. One operation of particular interest is the new $\tilde{\mathcal{P}}^0$. On the cobar construction it is represented by the map $[z] \to [z^p]$ on the generators.

PROPOSITION 2.2. *If M_* is the monogenic Hopf algebra*

(i) *$\Lambda[x_{2n-1}]$ ($\Lambda[x_{n-1}]$ if $p = 2$) or*

(ii) *$\mathbf{F}_p[x_{2n}]/(x^p)$ for $p > 2$, then $H^{**}(M_*)$ is*

(a) *$\mathbf{F}_p[z]$, where bidegree z is $(1, 2n - 1)$, ($(1, n)$ for $p = 2$),*

(b) *$\Lambda[z] \otimes \mathbf{F}_p[u]$, where bidegree $z = (1, 2n)$ and bidegree $u = (2, 2np)$*

and the nonzero Steenrod operations which are not compositions are

(a) *$\mathcal{P}^n z = z^p$, ($\mathrm{Sq}^n z = z^2$),*

(b) *$\mathcal{P}^{pn+1} u = u^p$, $\beta\mathcal{P}^n z = ru$, for $r \neq 0$.*

PROOF. The structure of the cohomology algebras is standard, e.g., Liulevicius **[10]**, while the operations are forced from the axioms, except for $\beta\mathcal{P}^n$. By analogy to the topological case in $H^*(B\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z})$, one might think that there should be a Bockstein connecting z to u, and $\beta\mathcal{P}^n$ is the only possibility on dimensional grounds. Less wistfully, the iterated p-fold Massey product $\langle z, z, \ldots, z\rangle$ is shown in May (**[14]**, following Kraines **[8]**) to contain $-\beta\mathcal{P}^n z$ for any z in $H^{1,2n}$. A short calculation with the cobar construction shows that in this particular case, the obvious representative of $\langle z, z, \ldots, z\rangle$ is the definition of u given in Liulevicius **[10]**, up to sign. The indeterminancy is zero, so the proposition is established.

PROPOSITION 2.3 **[1]**, **[10]**, **[13]**. *Let $K \to C_* \to A_* \to B_* \to K$ be a central extension of graded connected cocommutative Hopf algebras. There is a first quadrant cohomology spectral sequence of algebras with E_2-term $H^{**}(B_*) \otimes H^{**}(C_*)$ and abutting to $H^{**}(A_*)$. The Steenrod operations defined via the Cartan formula on E_2 commute with the differentials in the sense that $d_r\theta x = \theta d_j x$, for θ a Steenrod operation of homological degree $r - j$, and x in E_j. That is, θx and $\theta d_j x$ are d_{r-i}-cycles for $i > 0$, and the equation holds in E_r. In particular, if x transgresses to y, then θx transgresses to θy.*

Actually, the spectral sequence is trigraded, but since the differentials preserve the internal degree, we suppress this degree whenever possible. We have also a spectral sequence in the case that C_* is not central in A_*, $\mathrm{Ext}_{B_*}^{**}(K, \mathrm{Ext}_{C_*}^{**}(K, K))$ abutting to $H^{**}(A_*)$, as in Cartan-Eilenberg **[5]**. This is used in later sections, but is not required for the proof of finite generation.

EXAMPLE 2.4. The simplest nontrivial example of the use of Propositions 2.1, 2.2 and 2.3 is provided by the sub-Hopf algebra of the mod 2 Steenrod algebra generated by Sq^1 and Sq^2. This algebra, denoted as $\mathcal{Q}_1$, has relations $(\mathrm{Sq}^1)^2 = 0$, $(\mathrm{Sq}^2)^2 = \mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1$, and the commutator of Sq^1 and Sq^2, Sq^{01}, is nonzero. Sq^{01} is central in $\mathcal{Q}_1$. The E_2-term of the spectral sequence is $\mathbf{F}_2[h_0, h_1, h_{01}]$, and $d_2h_{01} = h_0h_1$, where the tridegrees are (1, 0, 1), (1, 0, 2), and (0, 1, 3) respectively. $\widetilde{\mathrm{Sq}}^0h_0 = h_1$, $\widetilde{\mathrm{Sq}}^0h_1 = 0$. The behavior of the rest of the spectral sequence is now determined by the transgression theorem: $E_3 = \mathbf{F}_2[h_0, h_1, (h_{01}^2)]/(h_0h_1)$. d_3 is determined on h_{01}^2 as $\widetilde{\mathrm{Sq}}^1h_0h_1 = h_1^3$, and E_4 is the module over $\mathbf{F}_2[h_0, h_1, (h_{01}^4)]/(h_0h_1, h_1^3)$ generated by 1 and $h_0h_{01}^2$. By the transgression theorem, h_{01}^4 survives to E_5, where $d_5h_{01}^4 = \widetilde{\mathrm{Sq}}^2\widetilde{\mathrm{Sq}}^1h_0h_1 = 0$ in E_5. The extra indecomposable $h_0h_{01}^2$ is a d_r-cycle for all r, so the spectral sequence collapses after stage four. In general, the spectral sequences considered need not collapse at the same point that all differentials on the "fiber" vanish. The possibility of the creation at each stage of the spectral sequence of new indecomposables on which higher differentials are nontrivial lends content to Theorem A.

This example is well known and can be computed directly via resolutions, e.g., Liulevicius **[11]**. The above use of the transgression theorem for the computation was shown to me by H. Miller.

3. The proof of Theorem A. For connected graded algebras over a field K, finite generation as a K-algebra is equivalent to the Noetherian condition. The following lemma, used basically in all Noetherian arguments involving spectral sequences, was pointed out to me by L. Evens. It is the key to dealing with the creation of new indecomposables in the spectral sequence in an implicit way.

LEMMA 3.1. *If the first quadrant spectral sequence $\{E_r, d_r\}$ is a R_* module for some Noetherian ring R_*, and E_2 is a finitely generated R_* module, then E_∞ is a finitely generated R_* module.*

PROOF. We have the sequence of R_* submodules of E_2, $0 \subset B_2 \subset \cdots \subset B_n \subset \cdots \subset B_\infty \subset Z_\infty \subset \cdots \subset Z_2$, where Z_r is the set of elements that survive to stage r, and B_r consists of those elements that bound by stage r. Since E_2 is a Noetherian R_* module, Z_∞ is also a Noetherian R_* module, and therefore $E_\infty = Z_\infty/B_\infty$ is Noetherian and hence finitely generated.

PROPOSITION 3.2. *If M_* is a monogenic Hopf algebra of height at most p, where p is* char K, *then for any central extension $K \to M_* \to A_* \to B_* \to K$ of cocommutative connected Hopf algebras with $H^{**}(B_*)$ finitely generated as a K-algebra, $H^{**}(A_*)$ is finitely generated as a K-algebra.*

PROOF. M_* is either $\Lambda[x_{2n-1}]$ or $K[x_{2n}]/(x^p)$ for p odd, or $\Lambda[x_{n-1}]$ for $p = 2$. Thus the structure of $H^{**}(M_*)$ is described by Proposition 2.2. The Steenrod operations given there imply that the p^sth powers of the polynomial generator of

$H^{**}(M_*)$ transgress to $E_r^{*,0}$, the base. But $H^{**}(B_*) = E_2^{*,0}$ is Noetherian by assumption. If one defines the increasing sequence of ideals $\{I_n\}$ so that I_n is the kernel of $E_2^{*,0} \to E_n^{*,0}$, then $I_N = I_{N+1} = \ldots$ for all $N \gg 0$. The p^sth powers of the polynomial generator of $H^{**}(M_*)$ therefore transgress to zero, for $s > S \gg 0$. Define R_* to be the subring of E_2 generated by $E_2^{*,0}$ and the p^Sth power of the polynomial generator of $H^{**}(M_*)$. Then R_* consists of infinite cycles, and E_2 is a finitely generated R_* module. By Lemma 3.1, E_∞ is finitely generated as a R_* module and hence as a K-algebra. But $H^{**}(A_*)$ is complete with respect to the exhaustive filtration giving rise to the spectral sequence, so the fact that its associated graded algebra is Noetherian implies that $H^{**}(A_*)$ is Noetherian, and hence a finitely generated K-algebra, Bourbaki [**4**, 3.2.9, Corollary 2 to Proposition 12].

Proof of Theorem A. We induce on $\dim_K(A_*)$. Theorem A is trivially true for $\dim_K(A_*)$ less than or equal to p. If $\dim_K(A_*)$ is greater than p, A_* must contain a nontrivial central monogenic sub-Hopf algebra of height at most p, so that $\dim_K(A_*/M_*)$ is less than $\dim_K(A_*)$. Applying the inductive hypothesis, Proposition 3.2 applies, so $H^{**}(A_*)$ is finitely generated. The proof of Theorem A given above is analogous to the p-group case given by Golod [**7**], but was derived from the proof in Evens [**6**]. The Steenrod operations and the transgression theorem are used as a poor man's substitute for a transfer argument of Evens.

Proof of Theorem B. We induce over $\dim_K(A_*)$. If $\dim_K(A_*) < \operatorname{char} K$, the result is clear. Now assume that the result is valid for all pairs (A'_*, B'_*) for which $\dim_K(A'_*) < \dim_K(A_*)$. By Proposition 1.1, there is a nontrivial central monogenic sub-Hopf algebra C_* of A_* with height at most p. Hence we have a diagram of central extensions.

$$\begin{tikzcd}
K \arrow[r] & B_* \cap C_* \arrow[r] \arrow[d] & B_* \arrow[r] \arrow[d, "i"] \arrow[drr, dashed, "\Phi"] & B_*//(B_* \cap C_*) \arrow[r] \arrow[d] & K & (1) \\
K \arrow[r] & C_* \arrow[r] & A_* \arrow[r] & A_*//C_* \arrow[r] & K & (2)
\end{tikzcd}$$

If $B_* \cap C_* = K$, then by Proposition 1.3, Φ is one-to-one, and the result follows from the inductive hypothesis, since image $\Phi^* \subset$ image i^*. If $B_* \cap C_* = C_*$, we apply the main argument of the proof of Theorem A again. The E_2-term of the spectral sequence for extension (2) is a finitely generated R_* module, for R_* the subalgebra of infinite cycles generated by $H^{**}(A_*//C_*)$ and a sufficiently large p^Sth power of the polynomial generator of $H^{**}(C_*)$. Applying the inductive hypothesis, the E_2-term of the spectral sequence of extension (1) is also a finitely generated R_*-module. By Lemma 3.1, the E_2-term of spectral sequence (1) is a finitely generated R_* module, and hence finitely generated over the associated graded algebra to $H^{**}(A_*)$. By Bourbaki [**4**, 3.2.9, Proposition 12], $H^{**}(B_*)$ is a finitely generated $H^{**}(A_*)$ module.

Before the proofs of Corollaries C and D we need to recall the definition of Krull dimension and some basic facts about Krull dimension, e.g., Chapter 5 of Matsumura [**13**]. If R_* is a graded commutative ring, consider chains of homogeneous prime ideals $\mathfrak{J}_0 \subset \mathfrak{J}_1 \subset \cdots \subset \mathfrak{J}_n$, where the inclusions are proper. If R_* is finitely generated over a field k, there exists a finite maximal length n, called the Krull dimension or simply the dimension of R_*. If $R_* = k[X_1, \ldots, X_n]$, a graded polynomial algebra, then dimension $R_* = n$. If $\sqrt{0}$ denotes the ideal of nilpotent elements, then $R_*/\sqrt{0}$ has the same dimension as R_*, for R_* f.g. over k. If $R_* \to R'_*$ is monic so that R'_* is a f.g. R_* module, dim $R'_* =$ dim R_* [**13**, Theorem 20, p. 81]. If φ: $R_* \to R'_*$ is surjective, then dim $R'_* \leqslant$ dim R_*.

Thus the Krull dimension is a rough measure of the size of R_*. In fact, if dim $R_* = n$, the E. Noether Normalization Theorem implies that there exists a polynomial subalgebra on n generators such that R_* is finitely generated as a module over the subpolynomial algebra [**13**, p. 91, Corollary 13]. If $R_* = k[X_1, \ldots, X_n]/\mathfrak{A}$, then the radical of $\mathfrak{A}$, $\sqrt{\mathfrak{A}} = \mathfrak{J}_1 \cap \cdots \cap \mathfrak{J}_r$, where $\mathfrak{J}_j$ are prime homogeneous ideals. Then $k[X_1, \ldots, X_n]/\mathfrak{J}_j$ is an integral domain, and dim $R_* = \max_j(\mathrm{Tr} \cdot \deg_k k[X_1, \ldots, X_n]/\mathfrak{J}_j)$. That is, geometrically the variety corresponding to R_* is a union of irreducible varieties of dimension $\leqslant$ dim R_*.

Finally one can consider the Poincaré series of R_*, $P_T(R_*) = \sum_{n=0}^{\infty} \dim_k(R_n)T^n$. If R_* is a f.g. algebra, dim R_* = order of pole of $P_T(R_*)$ at $T = 1$, Quillen [**16**]. This is not a particularly useful definition for the purposes of this paper but it does have the advantage of brevity.

Proof of Corollary C. (i) Suppose that $H^{s,t}(i)$ is zero for all (s, t) with $s + t$ sufficiently large. Then the Krull dimension of image i^* is zero, since image i^*/nilpotents is K. But if B_* is not K, then B_* contains a central monogenic sub-Hopf algebra C_*. Clearly $H^{**}(C_*)$ is not finitely generated as a module over image $i^* = K$, contradicting Theorem B, so it must not be the case that $H^{s,t}(i)$ is zero mod nilpotents for all (s, t) with $s + t$ sufficiently large. Part (ii) is proved in the next section as Proposition 4.2.

Proof of Corollary D. Since $H^{**}(B_*)$ is a finitely generated $H^{**}(A_*)$ module, $H^{**}(B_*)$ is integral over $H^{**}(A_*)$, and dim $H^{**}(B_*)$ = dim image $i^* \leqslant$ dim $H^{**}(A_*)$.

4. Homological characterization of isomorphisms and elementary Hopf algebras. $A_*, B_* \ldots$ continue to denote finite dimensional graded connected cocommutative Hopf algebras over K. We first have an easy analogue of a theorem of Stallings [**19**].

Proposition 4.1. *If Φ: $A_* \to B_*$ such that $H^{1,*}(\Phi)$ is an isomorphism and $H^{2,*}$ is a monomorphism, then Φ is an isomorphism.*

Proof. The proposition is valid for graded connected algebras, but the proof here is valid only for the Hopf algebra case. Denote by N_* the Hopf algebra kernel of Φ. Since $H^{1,*}(\Phi)$ is monic, Φ is onto, and we have an extension: $K \to N_* \to A_* \to B_* \to K$. Grading the spectral sequence [**5**] $\{H^{**}(B_*; H^{**}(N_*)) \Rightarrow H^{**}(A_*)\}$ by homological degree, we have d_2: $H^{0,*}(B_*, H^{1,*}(N_*)) \to H^{2,*}(B_*)$ must be identically zero, since $H^{2,*}(\Phi)$ is monic. Thus $H^{0,*}(B_*; H^{1,*}(N_*))$ survives to E_∞. But $H^{1,*}(A_*) = H^{1,*}(B) \oplus H^{0,*}(B_*;\ H^{1,*}(N_*))$, so $H^{1,*}(\Phi)$ onto implies that $H^{0,*}(B_*; H^{1,*}(N_*)) = 0$. Since B_* is nilpotent, $H^{1,*}(N_*) = 0$, and $N_* = K$.

PROPOSITION 4.2. *If $\Phi\colon A_* \to B_*$ is onto, then Φ is an isomorphism if and only if $H^{s,t}(\Phi)$ is onto $H^{s,t}(A_*)/$nilpotents for all (s, t) with $s + t$ sufficiently large.*

PROOF. $\Rightarrow$ Trivial.

$\Leftarrow$ Let N_* be the Hopf algebra kernel of Φ. Then $i^*\colon H^{s,t}(A_*) \to H^{s,t}(N_*)/$nilpotents is zero for all (s, t) with $s + t \gg 0$. By Corollary C(i), we must have $N_* = K$, since otherwise i^* would be nontrivial. Hence Φ is an isomorphism.

DEFINITION 4.3. The Hopf algebra E_* is elementary $\Leftrightarrow$ E_* is commutative and $(\bar{E}_*)^p = 0$. The coalgebra structure is unspecified but cocommutative.

At this point, we restrict to considering K a perfect field of characteristic p, since we will use the Borel Structure Theorem.

PROPOSITION 4.4. *The coalgebra structure of an elementary Hopf algebra E_* over a perfect field of characteristic p is uniquely determined by $H^{1,*}(E_*)$ as a $\tilde{\mathcal{P}}^0$ module. Any $\tilde{\mathcal{P}}^0$ module structure in which $\tilde{\mathcal{P}}^0$ acts nilpotently, and with the proper degree, is realizable.*

PROOF. Since E_* is finitely generated, $\tilde{\mathcal{P}}^0$ is a nilpotent transformation on $H^{1,\text{even}}(E_*)$. $H^{1,\text{even}}(E_*)$ is thus the direct sum of cyclic $\tilde{\mathcal{P}}^0$ modules of the form $\{x_{2n_i}, \tilde{\mathcal{P}}^0 x_{2n_i}, \ldots, (\tilde{\mathcal{P}}^0)^{r_i - 1} x_{2n_i}$, where $(\tilde{\mathcal{P}}^0)^{r_i} x_{2n_i} = 0\}$, for some $\{x_{2n_i}\} \in H^{1,2n_i}(E_*)$. Each such cyclic module corresponds to a tensor product factor in $(E_*)^*$ of the form $K[x_{2n_i}]/(x_{2n_i}^{p^{r_i}})$, so the coalgebra structure can be read off from the $\tilde{\mathcal{P}}^0$ module structure, and conversely.

PROPOSITION 4.5. *For any A_*, there exists a $\Phi\colon A_* \to E_*$ elementary, such that $H^{1,*}(\Phi)$ is an isomorphism.*

PROOF. If there is such a map, it factors through the abelianization of A_*, and it must be $A_*(\text{ab})/p$th powers. An explicit construction is also possible: Choose a graded vector space basis $\{f_1, \ldots, f_n\}$ of $\text{Alg}(A_*, M_*)$, where $M_* = \Lambda_K[x_{2n-1}]$ or $K[x_{2n}]/(x^p)$.

Define

$$\Phi\colon A_* \xrightarrow{\Delta^{N-1}} A_* \otimes \cdots \otimes A_* \xrightarrow{\otimes f_i} \otimes M_*(f_i).$$

Then $H^{1,*}(\Phi)$ is an isomorphism by construction. Put the coalgebra structure on $\otimes M_*(f_i)$ induced by the $\tilde{\mathcal{P}}^0$-module structure on $H^{1,\text{even}}(A_*)$.

PROPOSITION 4.6. *The following conditions are equivalent*:

(a) *A_* is elementary.*

(b) *$H^{**}(A_*)/$nilpotents is generated by $H^{1,\text{odd}}$ and $\beta\tilde{\mathcal{P}}^0 H^{1,\text{even}}$ for all sufficiently large dimensions.*

(c) *The degree 2 monomials in $H^{1,\text{odd}}$, $H^{1,\text{even}}$ and $\beta\tilde{\mathcal{P}}^0 H^{1,\text{even}}$ are linearly independent.*

PROOF. a $\Rightarrow$ b and c. c $\Rightarrow$ a by Proposition 4.1 applied to $\Phi\colon A_* \to E_*$. b $\Rightarrow$ a by Proposition 4.2 applied to $\Phi\colon A_* \to E_*$.

Proposition 4.5 is precisely the point at which the analogy with finite p-groups begins to falter. Serre **[18]** gives a third equivalent condition, deduced from (c) via

Steenrod operations. "d": The p-group G is elementary if and only if $u_G = \Pi_{v \neq 0}\, \beta v$ in $H^1(G)$ is not nilpotent. Condition "d" is not valid for the cohomology of Hopf algebras.

5. Detection of cohomology classes. The aim of this section is to trace through the proof of Quillen's Theorem given in Quillen-Venkov **[17]**, translating into the Hopf algebra setting. The entire translation is successful, except for the reference to Serre **[18]** mentioned at the end of §4. The outcome then is a sufficient condition for a Hopf algebra to have the detection property with respect to its elementary sub-Hopf algebras. In the important special case of sub-Hopf algebras of $\mathcal{Q}(2)$, this sufficient condition is easily verified, (see §6), and in general the condition points out what properties a counterexample must have. K is to be a *finite* field of char p.

PROPOSITION 5.1. (1) *If $v \in \ker \tilde{\mathcal{P}}^0 \cap H^{1,2n}(A_*)$ there exists a Hopf algebra morphism $\Phi_v\colon A_* \to K[x_{2n}]/(x_{2n}^p)$ such that $\Phi_v^*(z) = v$, for z the generator of $H^{1,2n}(K[x]/(x^p))$.*

(2) *If $v \in H^{1,2n-1}(A_*)$, there exists $\Phi_v\colon A_* \to \Lambda_K[x_{2n-1}]$ such that $\Phi_v^*(z) = v$, for z the generator of $H^{1,2n-1}(\Lambda_K[x_{2n-1}])$.*

In either case, define $B_*(v)$ as the Hopf algebra kernel of Φ_v, and i_v as the inclusion $B_*(v) \to A_*$.

PROOF. (1) $H^{1,*}(A_*)$ is naturally isomorphic to the primitives of A^*, so regard v as a primitive of A^*. v generates a sub-Hopf algebra V_* of A^*, and since $\tilde{\mathcal{P}}^0 v = v^p = 0$, V_* is a truncated at height p polynomial algebra. Hence its dual is also a truncated at height p polynomial algebra. Φ_v is then dual to $V_* \to A^*$. Case (2) is similar.

PROPOSITION 5.2. *If $v \in \ker \tilde{\mathcal{P}}^0 \cap H^{1,2n}(A_*)$ or $(H^{1,2n-1}(A_*)$ respectively), and $u \in H^{**}(A_*)$ such that $i_v^* u = 0$, then $u^N \in H^{**}(A_*)(\beta\tilde{P}^0 v)$ or $H^{**}(A_*)(v)$ respectively, for some $N > 0$.*

PROOF. This is the exact analogue of the first lemma of Quillen-Venkov **[17]**. Consider the spectral sequence associated to

$$K \to B_*(v) \xrightarrow{i_v} A_* \xrightarrow{\Phi_v} K[x_{2n}]/(x^p) \to K.$$

Each E_r is a module over $H^{**}(K[x_{2n}]/(x^p)) = \Lambda_K[z] \otimes K[w]$, where $w = \beta\tilde{\mathcal{P}}^0 z$. Multiplication by w on E_2, $\operatorname{Ext}_{M_*}^{s_1,t_1}(K, \operatorname{Ext}_{B_*}^{s_2,t_2}(K, K))$ is a surjection for $s_1 > 0$ and an injection for $s_1 \geqslant 1$, by the periodicity of the cohomology of $M_* = K[x]/(x^p)$. By induction one proves that

$$w\colon E_r^{s_1,s_2,t} \to E_r^{s_1+2,s_2,t+2np}$$

is surjective for $s_1 \geqslant 0$ and injective for $s_1 > r - 1$. Hence $E_\infty^{s_1,s_2,t} \to E_\infty^{s_1+2,s_2,t+2np}$ is surjective for $s_1 > 0$. Now

$$E_\infty^{s_1,s_2,t} = F_{s_1}H^{s_1+s_2,t}(A_*)/F_{s_1+1}H^{s_1+s_2,t}(A_*),$$

where $F_*H^{**}(A_*)$ is the filtration induced on $H^{**}(A_*)$ from the filtration on $B^*(A_*)$. By decreasing induction over s_1,

$$w \cdot F_{s_1}H^{s_1+s_2,t}(A_*) = F_{s_1+2}H^{s_1+s_2+2,\,t+2np}(A_*).$$

If $i_v^*u = 0$, $u \in F_1H^{s,t}(G)$, so $u^N \in wH^{**}(A_*)$ where $N \geqslant 2$ and $tN \geqslant 2np$. The case for p equals 2 or the quotient an exterior algebra is virtually the same. Alternately, the long exact sequence of Theorem 3.2 of [3] applies.

DEFINITION 5.3. (i) A_* has the *detection property* if for each $u \in H^{**}(A_*)$ such that u restricts to zero on every elementary sub-Hopf algebra u is nilpotent.

(ii) The fundamental class of A_* is $\Pi v \Pi \beta \tilde{\mathcal{P}}^0 u$, where the product is taken over *all* nonzero v in $H^{1,\text{odd}}(A_*)$ and *all* nonzero u in $\ker \tilde{\mathcal{P}}^0 \cap H^{1,\text{even}}(A_*)$. Denote this class by U_A.

LEMMA 5.4. *If i: $B_* \to A_*$ is the inclusion of a proper sub-Hopf algebra, then $i^*u_A = 0$.*

PROOF. Since $B_* \neq A_*$, $\ker H^{1,*}(i) \neq 0$. Let $x \in \ker i^*$. If x has odd internal degree, it appears in the product u_A, and i^*u_A is clearly zero. If x has even internal degree, there exists $r > 0$ such that $(\tilde{\mathcal{P}}^0)^r x \neq 0$ is in $\ker i^* \cap \ker \tilde{\mathcal{P}}^0$. Then $\beta \tilde{\mathcal{P}}^0(\tilde{\mathcal{P}}^0)^r x$ is in $\ker i^*$, and is a term in the product, so $i^*u_A = 0$.

PROPOSITION 5.5. *If every sub-Hopf algebra of A_* of the form $B_*(v)$, for $v \neq 0$ in* $\ker \tilde{\mathcal{P}}^0 \cap H^{1,*}(A_*)$, *has the detection property, then A_* has the detection property if and only if either A_* is elementary or u_A is nilpotent.*

PROOF. By Lemma 5.4, u_A restricts to zero on any proper sub-Hopf algebra, so if A_* has the detection property and is not elementary, u_A is nilpotent. Conversely, if u restricts to zero on each elementary sub-Hopf algebra of A_*, this is also true for its restrictions i_v^*u, $v \in H^{1,*}(A_*) \cap \ker \tilde{\mathcal{P}}^0$. Therefore, since there are a finite number of such v, and i_v^* is nilpotent for each v, we can assume $i_v^*u^N = 0$ for all such v, for $N \gg 0$. By Proposition 5.2, $u^M \in H^{**}(A_*)u_A$ for $M \gg 0$. But since u_A is nilpotent, u is nilpotent.

COROLLARY 5.6. *If every sub-Hopf algebra B_* of A_* is either elementary or has u_B nilpotent, then A_* has the detection property.*

COROLLARY 5.7. *If A_* has the detection property, then Krull dimension $H^{**}(A_*)$ is the maximum of the ranks of the elementary sub-Hopf algebras of A_*.*

PROOF. Let $\{E_*(j)\}$ be the maximal elementary sub-Hopf algebras of A_*. If A_* has the detection property, then the diagonal map $H^{**}(A_*) \to \bigoplus_j H^{**}(E_*(j))$ is a monomorphism modulo nilpotents. Hence the Krull dimension of $H^{**}(A_*)$ is less than that of the direct sum, which is the maximum of that of $H^{**}(E_*(j))$. The Krull dimension of $H^{**}(E_*(j))$ is the rank of $E_*(j)$, since modulo nilpotents, the cohomology of E_* is a polynomial algebra on rank E_* generators. Corollary D gives $\max_j$ rank $(E_*(j))$ as a lower bound for Krull dimension $H^{**}(A_*)$. Hence this is an equality.

6. Counterexamples and finite sub-Hopf algebras of $\mathcal{Q}(2)$. We need the explicit determination of the sub-Hopf algebras of the mod p Steenrod algebra $\mathcal{Q}(p)$ given by Adams-Margolis [2] and Anderson-Davis for $p = 2$ [3]. The characterization is that the only finite sub-Hopf algebras are the obvious ones. More explicitly

PROPOSITION 6.1. *Each finite sub-Hopf algebra B_* of $\mathcal{Q}(p)$ determines functions e: $\{1, 2, \ldots\} \to \{0, 1, 2, \ldots\}$ and k: $\{0, 1, 2, \ldots\} \to \{1, 2\}$ such that*
(1) $e(r) \geqslant \min(e(r - i) - i, e(i))$ for $0 \leqslant i < r$,
(2) if $k(i + j) = 1$, then either $e(i) < j$ or $k(j) = 1$ for all $i \geqslant 1, j \geqslant 0$,
(3) $e(r) = 0$ and $k(r) = 1$ for almost all r.

B_* is isomorphic to the dual of the quotient of

$$\mathcal{Q}(p)^* = \Big\{\mathbf{F}_p[\xi_1, \ldots, \xi_n, \ldots] \otimes \Lambda[\tau_0, \tau_1, \ldots],$$
$$\psi\xi_n = \sum \xi_{n-1}^{p^i} \otimes \xi_i, \psi\tau_n = \sum \xi_{n-i}^{p^i} \otimes \tau_i + \tau_n \otimes 1\Big\}$$

by the ideal generated by $\{\xi_1^{p^{e(1)}}, \ldots, \xi_j^{p^{e(j)}}, \ldots, \tau_0^{k(0)}, \ldots, \tau_i^{k(i)}, \ldots\}$ (for $p = 2$, set $k(i) \equiv 1$). Conversely, any functions e and k satisfying (1), (2) and (3) determine a finite sub-Hopf algebra of $\mathcal{Q}(p)$, denoted by $B_*(e, k)$ in the following.

LEMMA 6.2. *Let $B_*(e, k)$ be as described in Proposition 6.1. If $\xi_1, \ldots, \xi_{r-1}$ are zero but $\xi_r \neq 0$, in $(B_*(e, k))^*$, and $\xi_r, \ldots, \xi_{n-1}$ are primitive, then the reduced coproduct $\bar{\psi}\xi_n = \xi_{n-r}^{p^r} \otimes \xi_r$. If all ξ_i are primitive, $e(i) \leqslant r$ for all i.*

PROOF. $\xi_r, \ldots, \xi_{2r-1}$ are primitive since $\xi_1, \ldots, \xi_{r-1}$ are zero. By induction, $e(r + i) \leqslant r$ if $i < n - 2r$. Hence in the reduced coproduct

$$\bar{\psi}\xi_n = \xi_{n-r}^{p^r} \otimes \xi_r + \xi_{n-r-1}^{p^{r+1}} \otimes \xi_{r+1} + \cdots + \xi_1^{p^{n-1}} \otimes \xi_{n-1}$$

and all the terms except the first are zero.

COUNTEREXAMPLE 6.3. $B_*(e, k)$ for p odd with the exponent sequences $e(1) = 1$, $e(2) = 2$, $e(3) = 1$, and $e(3 + i) = 0$; $k(i) \equiv 1$ does not have the detection property. Hence the sub-Hopf algebra of $\mathcal{Q}(p)$ generated by $\mathcal{P}^1$, $\mathcal{P}^p$, and $\mathcal{P}^{p^2}$ does not have the detection property.

PROOF. View $B_*(e, k)$ as the central extension dual to

$$\mathbf{F}_p \to \mathbf{F}_p[\xi_1, \xi_2]/\big(\xi_1^p, \xi_2^{p^2}\big) \to \mathbf{F}_p[\xi_1, \xi_2, \xi_3]/\big(\xi_1^p, \xi_2^{p^2}, \xi_3^p\big) \to \mathbf{F}_p[\xi_3)/(\xi_3^p) \to \mathbf{F}_p.$$

Then the E_2-term of the spectral sequence of the extension is

$$\Lambda\big[[\xi_1], [\xi_2], [\xi_2^p], [\xi_3]\big] \otimes \mathbf{F}_p[u_1, u_2, u_3, u_4]$$

where $\beta\tilde{\mathcal{P}}^0[\xi_1] = u_1$, $\beta\tilde{\mathcal{P}}^0[\xi_2] = u_2$, $\beta\tilde{\mathcal{P}}^0[\xi_2^p] = u_3$ and $\beta\tilde{\mathcal{P}}^0[\xi_3] = u_4$. d_2 is determined by $d_2[\xi_3] = [\xi_2^p][\xi_1]$. The d_2-cycles are the module over $\Lambda[[\xi_1], [\xi_2], [\xi_2^p]] \otimes \mathbf{F}_p[u_1 u_2, u_3, u_4]$ generated by 1, $[\xi_2^p][\xi_3]$, and $[\xi_1][\xi_3]$, so E_3 is this module with the relations generated by $[\xi_2^p][\xi_1]$. The module generators other than 1 have bidegree (1, 1) in the spectral sequence and hence are d_r-cycles for all r. $d_3 u_4 = d_2\beta\tilde{\mathcal{P}}^0[\xi_3] = \beta\tilde{\mathcal{P}}^0[\xi_2^p][\xi_1] = u_3 \cdot 0 + 0 \cdot u_1 = 0$ by the Cartan formula for $\beta\tilde{\mathcal{P}}^0$. That is, the spectral sequence collapses at E_3, since all higher differentials vanish for dimensional reasons. Thus the element $u_B = (u_1 u_3)^{p-1}$ is not nilpotent in $H^{**}(B_*(e, k))$, since it is not nilpotent in the associated graded algebra to $H^{**}(B_*(e, k))$, E_∞. This

$B_*(e, k)$ is a sub-Hopf algebra of the Hopf algebra generated by 1, $\mathcal{P}^1$, $\mathcal{P}^p$, and $\mathcal{P}^{p^2}$ corresponding to the exponent sequence $(3, 2, 1, 0, \ldots), (1, 1, \ldots)$. Therefore the Krull dimension of $H^{**}(B_*(3, 2, 1, 0, \ldots; 1, 1, \ldots))$ is greater than or equal to the Krull dimension of $H^{**}(B_*(1, 2, 1, 0, \ldots; 1, 1, \ldots))$, which is 4. The two Hopf algebras have the same elementary sub-Hopf algebras,

$$B_*(1, 1, 1, 0, \ldots; 1, 1, \ldots) \quad \text{and} \quad B_*(0, 2, 1, 0, \ldots; 1, 1, \ldots),$$

each of which has rank 3. Therefore, by the contrapositive to Corollary 5.7, $B_*(3, 2, 1, 0, \ldots; 1, 1, \ldots)$ does not have the detection property.

THEOREM 6.4. *Any finite sub-Hopf algebra of $\mathcal{Q}(2)$ has the detection property.*

PROOF. By Lemma 6.2, the first nontrivial coproduct in $(B_*(e))^*$ is $\bar{\psi}\xi_n = \xi_{n-r}^{2^r} \otimes \xi_r$, where ξ_r and ξ_{n-r} are primitive. I claim that by the application of Steenrod operations, the relation $[\xi_{n-r}^{2^r}][\xi_r] = 0$ in $H^{2,*}(B_*)$ generates a relation of the form $(xy)^N = 0$ for $x, y \in H^{1,*}(B_*) \cap \ker \widetilde{\mathrm{Sq}}^0$. That is, $u_B^N = 0$.

Case 1. $e(r) = e(n - r) - r$. Apply $(\widetilde{\mathrm{Sq}}^0)^{e(r)-1}$ to obtain

$$\left[\xi_{n-r}^{2^{e(n-r)-1}}\right]\left[\xi_r^{2^{e(r)-1}}\right] = 0.$$

Case 2. $e(r) - k = e(n - r) - r$, for $k > 0$. Apply $\widetilde{\mathrm{Sq}}^{2^{k-1}} \ldots \widetilde{\mathrm{Sq}}^1(\widetilde{\mathrm{Sq}}^0)^{e(n-r)-r-1}$ to obtain $[\xi_{n-r}^{2^{e(n-r)-1}}]^{2^k}[\xi_r^{2^{e(r)-1}}] = 0$.

Case 3. $e(r) = e(n - r) - r - k$, for $k > 0$. This is similar to Case 2.

Thus, each finite sub-Hopf algebra B_* of $\mathcal{Q}(2)$ is either elementary or u_B is nilpotent. By Corollary 5.6, each finite sub-Hopf algebra of $\mathcal{Q}(2)$ has the detection property, by induction over its sub-Hopf algebras.

COUNTEREXAMPLE 6.5. For $p = 2$, the Hopf algebra with dual $\Lambda[x_1, x_2, x_3, x_4, x_5]$, degree $x_i = i$, $\bar{\psi}x_i = 0$, $i < 5$, $\bar{\psi}x_5 = x_1 \otimes x_4 + x_2 \otimes x_3$ does not have the detection property.

PROOF. In the spectral sequence of $(x_5)^* \to A_* \to A_*//(x_5)^*$, the E_3-term is E_∞, and $E_\infty = \mathbf{F}_2[z_1, z_2, z_3, z_4, z_5^2]/(z_1z_4 + z_2z_3)$ is an integral domain. Hence $u_A = z_1z_2z_3z_4$ is not nilpotent.

S. Priddy has observed that 6.5 is the universal enveloping algebra for its restricted Lie algebra of primitives, and hence it is a counterexample to a detection principle for connected graded Lie algebras, with respect to abelian sub-Lie algebras.

PROPOSITION 6.6. *The Krull dimension of $H^{**}(\mathcal{Q}_n)$, for $\mathcal{Q}_n$ the sub-Hopf algebra of $\mathcal{Q}(2)$ given by the exponent sequence $(n + 1, n, n - 1, \ldots, 1, 0, \ldots)$, is given by the formula $g_n(r) = \varepsilon + r(2n + 5 - 3r)/2$, where r is the greatest integer in $(2n + 5)/6$ and ε is 0 if n is not divisible by 3, and 1 if n is divisible by 3.*

PROOF. The work has been done; it remains only to count. The maximal elementary sub-Hopf algebras have exponent sequences $(0, 0, \ldots, 0, r, r, \ldots, r - 1, r - 2, \ldots, 1, 0, \ldots)$ where r first occurs in the rth position, and $r - 1$ occurs in the $(n - r + 3)$th position. We have only to add up the entries and maximize over r to compute the Krull dimension. The continuous maximum occurs at $(2n + 5)/6$, and the discrete maximum at either the greatest integer in $(2n + 5)/6$ or

$((2n+5)/6)+1$, since the function $g_n(r)$ is quadratic in r. Analysis of the cases gives the definition of ε.

PROPOSITION 6.7. *If $B_* \subset D_*$ are finite sub-Hopf algebras of the* mod 2 *Steenrod algebra with the same family of maximal elementary sub-Hopf algebras, then the restriction map is monic* mod *nilpotents.*

PROOF. This was proved by W. H. Lin [9] as the main argument in his proof of Theorem 6.4 for the special case that B_* is the intersection of D_* with the (nonfinite) sub-Hopf algebra of $\mathcal{Q}(2)$ with exponent sequence $(1, 2, 3, 4, 5, \ldots)$. In the logic of the present paper however, it is a corollary of 6.4.

In view of the counterexamples, the seeking of detection theorems for general Hopf algebras seems futile. However, there remains the hope that for finite sub-Hopf algebras of the mod p Steenrod algebra there exist suitable families of sub-Hopf algebras which detect nonnilpotent cohomology elements.

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