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HOMOTOPY GROUPS OF THE SPACE OF SELF-HOMOTOPY-EQUIVALENCES

BY

DARRYL McCULLOUGH

ABSTRACT. Let M be a connected sum of r closed aspherical manifolds of dimension $n > 3$, and let EM denote the space of self-homotopy-equivalences of M, with basepoint the identity map of M. Using obstruction theory, we calculate $\pi_q(EM)$ for $1 < q < n - 3$ and show that $\pi_{n-2}(EM)$ is not finitely-generated. As an application, for the case $n = 3$ and $r > 3$ we show that infinitely many generators of $\pi_1(EM^3, \mathrm{id}_M)$ can be realized by isotopies, to conclude that $\pi_1(\mathrm{Homeo}(M^3), \mathrm{id}_M)$ is not finitely-generated.

0. Introduction. Let EX be the H-space of homotopy equivalences from X to X, with the identity map of X as basepoint. It contains the basepoint-preserving self-homotopy-equivalences E_0X, the group of homeomorphisms Homeo(X), and, when X is a smooth manifold, the group of diffeomorphisms Diff(X). The inclusions of these subspaces are H-space homomorphisms. From knowledge of EX, one hopes to obtain information about these subspaces.

The groups $\pi_0(E_0X)$ and $\pi_0(EX)$ have been studied for various classes of spaces. It was shown by Sullivan [S] and, independently, Wilkerson [W] that when X is a simply-connected finite complex, $\pi_0(EX)$ is finitely-presented. In contrast, Frank and Kahn [F-K] showed that for $p \geqslant 2$, $\pi_0(E_0(S^1 \vee S^p \vee S^{2p-1}))$ is not finitely-generated. There are examples of finite aspherical 4-complexes K^4 with $\pi_0(E_0(K^4))$ not finitely-generated [M3].

Little is known about the homotopy groups $\pi_i(EX)$ for $i \geqslant 1$ except for two important cases. For X an aspherical complex, Gottlieb [G] proved that $\pi_1(EX) \cong \mathrm{center}(\pi_1(X))$ while $\pi_i(EX) = 0$ for $i \geqslant 2$. It follows that $\pi_j(E_0X) = 0$ for $j \geqslant 1$. The other case is that of the n-sphere S^n, for which $\pi_q(ES^n) = [S^q; \mathrm{Maps}(S^n, S^n)] \cong [S^q \wedge S^n; S^n] \cong \pi_{n+q}(S^n)$.

In this paper, I adapt the obstruction theory of Federer [F] to obtain some calculations of the homotopy groups of EM, where M is any connected sum of $r \geqslant 2$ (closed) aspherical (combinatorial) manifolds of dimension $n \geqslant 3$. Specifically:

(1) For $1 \leqslant q \leqslant n - 4$, $\pi_q EM \cong \bigoplus_{i=1}^{r-1} \pi_{n+q}(S^{n-1})$, hence is finite.

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(2) For $n \geqslant 4$, $\pi_{n-3}EM$ is a quotient of $\bigoplus_{i=1}^{r-1} \pi_{2n-3}(S^{n-1})$, and is finite.

(3) $\pi_{n-2}EM$ is infinitely-generated as an abelian group.

For the case $n = 3$ and $r \geqslant 3$, I show that infinitely many of the generators of $\pi_1(EM)$ can be realized as isotopies (which can be taken to be diffeotopies) of M. Therefore:

(4) For $n = 3$ and $r \geqslant 3$, $\pi_1(\mathrm{Homeo}(M), \mathrm{id}_M)$ and $\pi_1(\mathrm{Diff}(M), \mathrm{id}_M)$ are infinitely-generated.

The construction of these isotopies is very explicit. The results (1), (2), and (3) appeared in my dissertation, submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at the University of Michigan. I wish to thank my advisor Professor Frank Raymond for his patient encouragement and helpful suggestions. I also wish to thank the referee for suggesting several significant improvements to the manuscript of this paper.

Here is a description of the program I will use to make these calculations. Let Y be a CW complex and let Y^k denote its k-skeleton. If A and B are subcomplexes with $B \subset A \subset Y$, let $(Y; A, B)$ be the space of continuous maps from A to Y which restrict to the inclusion map on B. The inclusion map of A is the basepoint of $(Y; A, B)$. Let $Y^{[A,B]} \subset (Y; A, B)$ be the subspace of maps which extend to all of Y. Because A has the Homotopy Extension Property in Y, $Y^{[A,B]}$ consists of path components of $(Y; A, B)$. There are three fibrations in which the projection maps are restriction:

$$Y^{[Y,Y^2]} \to Y^{[Y,\varnothing]} \to Y^{[Y^2,\varnothing]}, \quad Y^{[Y,Y^2]} \to Y^{[Y,Y^1]} \to Y^{[Y^2,Y^1]},$$
$$Y^{[Y^2,Y^1]} \to Y^{[Y^2,\varnothing]} \to Y^{[Y^1,\varnothing]}.$$

These fit into the following diagram in which the row and columns are fibrations:

$$\begin{array}{ccccc}
\Omega(Y^{[Y^2,\varnothing]}) & \to & Y^{[Y,Y^2]} & \to & Y^{[Y,\varnothing]} \\
\downarrow & & \downarrow & & \\
\Omega(Y^{[Y^1,\varnothing]}) & \to & Y^{[Y,Y^1]} & & \\
\downarrow & & \downarrow & & \\
Y^{[Y^2,Y^1]} & = & Y^{[Y^2,Y^1]} & &
\end{array}$$

It is easy to produce from this diagram a long exact sequence:

$$\cdots \to J_q \xrightarrow{D_q} K_q \to \pi_q(Y^{[Y,\varnothing]}) \to J_{q-1} \xrightarrow{D_{q-1}} K_{q-1} \to \cdots$$

where

$$J_q = \operatorname{coker}\left(\pi_{q+1}(Y^{[Y^2,Y^1]}) \to \pi_{q+1}(Y^{[Y^2,\varnothing]})\right),$$
$$K_q = \operatorname{coker}\left(\partial\colon \pi_{q+1}(Y^{[Y^2,Y^1]}) \to \pi_q(Y^{[Y,Y^2]})\right),$$

and D_q is induced by $\partial\colon \pi_{q+1}(Y^{[Y^2,\varnothing]}) \to \pi_q(Y^{[Y,Y^2]})$.

In §1, under certain assumptions on Y, we will identify J_q and K_q as cohomology modules of Y and discuss the boundary homomorphism D_q. In §2, we list the properties of connected sums of aspherical manifolds which allow explicit calculations of the modules to be made. The results called (1), (2) and (3) above are obtained in §3, and the isotopies of 3-manifolds are constructed in the final section.

1. Obstruction theory preliminaries. We will denote by I^q the q-dimensional cube $[0, 1]^q$, by J^q the closure of the complement of $I^q \times \{0\}$ in ∂I^{q+1}.

1.A. *The boundary homomorphism for fibrations of spaces of mappings*. Suppose that $C \subset B \subset A$ are subcomplexes of the CW complex Y. For the fibration $Y^{[A,B]} \to Y^{[A,C]} \to Y^{[B,C]}$ the boundary homomorphism ∂: $\pi_{q+1}(Y^{[B,C]}) \to \pi_q(Y^{[A,B]})$ can be described as follows [**F**, pp. 346–347]. Let $\langle w \rangle \in \pi_{q+1}(Y^{[B,C]})$; then w is defined on the subset $B \times I^q \times I \subset Y \times I^{q+1}$. Extend w to $A \times J^q \cup B \times I^{q+1}$ using the projection map to A. By the Homotopy Extension Property applied to the pair $(A \times I^q \times \{1\}, A \times \partial I^q \times \{1\} \cup B \times I^q \times \{1\})$, we obtain an extension to all of $A \times I^{q+1}$. If u denotes the restriction of this extension to $A \times I^q \times \{0\}$, then $\langle u \rangle = \partial \langle w \rangle$.

1.B. *Calculation of* $J_q = \operatorname{coker}(\pi_{q+1}(Y^{[Y^2, Y^1]}) \to \pi_{q+1}(Y^{[Y^2, \varnothing]}))$. All cochains and cohomology will be with local coefficients. We will denote by proj_y the projection map from $Y \times I^{q+1}$ to Y, or its restriction to any subspace of $Y \times I^{q+1}$. Let $* \in Y^0$ be the basepoint of Y.

LEMMA 1.B.1. *If* $\pi_{q+1}(Y^{[Y^2, *]}) \to \pi_{q+1}(Y^{[Y^2, \varnothing]})$ *is surjective, then*
$$J_q \cong H^1(Y; \pi_{q+2} Y).$$

PROOF. Let $\langle f \rangle \in \pi_{q+1}(Y^{[Y^2, \varnothing]})$; then $f|_{Y^2 \times \partial I^{q+1}} = \mathrm{proj}_y$. By assumption we may choose f so that $f|_{Y^0 \times I^{q+1}} = \mathrm{proj}_y$. Consider the difference cochain $d_{q+2}(\mathrm{proj}_y, f) \in C^{q+2}(Y^2 \times I^{q+1}, Y^2 \times \partial I^{q+1}; \pi_{q+2} Y) \cong C^1(Y; \pi_{q+2} Y)$. We have $\delta d_{q+2}(\mathrm{proj}_y, f) = c_{q+3}(\mathrm{proj}_y) - c_{q+3}(f) = 0$ since both proj_y and f admit extensions to $Y^2 \times I^{q+2}$. Thus we may define $d_1 \langle f \rangle = \{d_{q+2}(\mathrm{proj}_y, f)\} \in H^1(Y; \pi_{q+2} Y)$. Changing f by a homotopy on $Y^0 \times I^{q+1}$ alters $d_{q+2}(\mathrm{proj}_y, f)$ by a coboundary so d_1 is well defined, and it is easy to see that d_1 is a homomorphism which vanishes on $\mathrm{image}(\pi_{q+1}(Y^{[Y^2, Y^1]}))$. If $d_1 \langle f \rangle = 0$, then $f|_{Y^1 \times I^{q+1}}$ is homotopic to proj_y, so $\langle f \rangle \in \mathrm{image}(\pi_{q+1}(Y^{[Y^2, Y^1]}))$; thus d_1: $J_q \to H^1(Y; \pi_{q+2} Y)$ is injective. Given $\{c\} \in H^1(Y; \pi_{q+2} Y)$, define f: $Y^1 \times I^{q+1} \to Y$ so that $f|_{Y^0 \times I^{q+1}} = \mathrm{proj}_y$ and $\langle f|_{\sigma \times I^{q+1}} \rangle = c(\sigma)$ for each $\sigma \in Y^1 - Y^0$. Since $\delta c = 0$, f extends to $Y^2 \times I^{q+1}$, and $d_1 \langle f \rangle = \{c\}$. Therefore d_1 is surjective. $\square$

1.C. *Calculation of* $K_q = \operatorname{coker}(\partial$: $\pi_{q+1}(Y^{[Y^2, Y^1]}) \to \pi_q(Y^{[Y, Y^2]}))$.

LEMMA 1.C.1. *If* $H^p(Y; \pi_{p+q} Y) = H^{p-1}(Y; \pi_{p+q} Y) = 0$ *for* $3 \leqslant p \leqslant n-1$, *then* $K_q \cong H^n(Y; \pi_{n+q} Y)$.

PROOF. Define d_n: $K_q \to H^n(Y; \pi_{n+q} Y)$ as follows. Let $\langle f \rangle$ represent an element of K_q. If $n = 3$, let $d_n \langle f \rangle = \{d_{n+q}(\mathrm{proj}_y, f)\}$. Suppose $n > 3$. Then $\delta d_{q+3}(\mathrm{proj}_y, f) = c_{q+4}(\mathrm{proj}_y) - c_{q+4}(f) = 0$ and $\{d_{q+3}(\mathrm{proj}_y, f)\} \in H^3(Y; \pi_{q+3} Y) = 0$. Hence there is a homotopy F: $f \simeq f_1$ (rel $Y^1 \times I^q$) with $f_1|_{Y^3 \times I^q} = \mathrm{proj}_y$. Let g: $Y^2 \times I^{q+1} \to Y$ be $F|_{Y^2 \times (I^q \times I)}$. Then g represents an element of $\pi_{q+1}(Y^{[Y^2, Y^1]})$ such that $d_{q+3}(\mathrm{proj}_y, \partial \langle g \rangle) = d_{q+3}(\mathrm{proj}_y, f)$. Moreover $\langle f_1 \rangle = \langle f \rangle - \partial \langle g \rangle$ so $\langle f_1 \rangle$ represents the same element of K_q as $\langle f \rangle$ did. Inductively, for $4 \leqslant k \leqslant n-1$, assume $f_1|_{Y^{k-1} \times I^q} = \mathrm{proj}_y$. We have $\{d_{k+q}(\mathrm{proj}_y, f_1)\} \in H^k(Y, \pi_{k+q} Y) = 0$. Therefore f_1 is homotopic to a map, again called f_1, such that $f_1|_{Y^k \times I^q} = \mathrm{proj}_y$. This completes the induction. Let $d_n \langle f \rangle = \{d_{n+q}(\mathrm{proj}_y, f_1)\} \in H^n(Y; \pi_{n+q} Y)$.

We must show this assignment is well defined. Suppose $\langle f_1' \rangle$ is another homotopy class with $f_1'|_{Y^{n-1}\times I^q} = \mathrm{proj}_y$ and $f_1' \simeq f$ (rel $Y^1 \times I^q$). Then $f_1 \simeq f \simeq f_1'$ (rel $Y^1 \times I^q$) and we must show $f_1 \simeq f_1'$ (rel $Y^{n-2} \times I^q$). For $n = 3$, this is automatic, so assume $n > 3$. Let G: $Y \times I^q \times I \to Y$ be a homotopy from f_1 to f_1'. Inductively, for $2 \leqslant k \leqslant n-2$, suppose $G|_{Y^{k-1}\times I^{q+1}} = \mathrm{proj}_y$. Then $\{d_{(q+1)+k}(\mathrm{proj}_y, G)\} \in H^k(Y; \pi_{(q+1)+k}Y) = 0$ so G is homotopic (rel $Y \times \partial I^{q+1}$) to a new homotopy, also called G, with $G|_{Y^k \times I^{q+1}} = \mathrm{proj}_y$. This completes the induction; thus $f_1 \simeq f_1'$ (rel $Y^{n-2} \times I^q$) so $d_n\langle f_1 \rangle = d_n \langle f_1' \rangle$.

Clearly, d_n is a surjective homomorphism. It remains to show d_n is injective. Suppose $d_n\langle f \rangle = 0$. By the preceding argument, we can find $\langle f_1 \rangle$ with $f_1|_{Y^{n-1}\times I^q} = \mathrm{proj}_y$, $d_n\langle f_1 \rangle = 0$, and $f \simeq f_1$ (rel $Y^1 \times I^q$). Let H: $f \simeq f_1 \simeq \mathrm{proj}_y$ (rel $Y^1 \times I^q$). Letting $g = H|_{Y^2 \times I^{q+1}}$, we have $\langle f \rangle = \partial \langle g \rangle$, so $\langle f \rangle$ represents the zero element of K_q. $\square$

1.D. *An important example*. The following example illustrates the techniques we use for computing homotopy groups of mapping spaces, and it is pertinent to the manifolds we will be considering. Let $X = S^{n-1} \times I$. We regard it as a cell complex with six cells: two 0-cells $* \times \{0\}$ and $* \times \{1\}$, one 1-cell $\sigma = * \times I$ connecting the 0-cells, two $(n-1)$-cells $S^{n-1} \times \{0\}$ and $S^{n-1} \times \{1\}$, and one n-cell τ. Letting $C = \partial X$, $B = \sigma \cup \partial X$, and $A = X$, the fibration of §1.A becomes

$$X^{[X, \sigma \cup \partial X]} \to X^{[X, \partial X]} \to X^{[\sigma \cup \partial X, \partial X]}.$$

It is not difficult to observe that $X^{[\sigma \cup \partial X, \partial X]} = X^{[\sigma, \partial \sigma]} \simeq \Omega X$ and, because the attaching map of τ is null-homotopic, that $X^{[X, \sigma \cup \partial X]} \simeq \Omega^n X$. Therefore the homotopy exact sequence for the fibration becomes

$$\cdots \to \pi_{q+2}(X) \xrightarrow{D_q} \pi_{n+q}(X) \to \pi_q(X^{[X,\partial X]}) \to \pi_{q+1}(X) \xrightarrow{D_{q-1}} \pi_{n+q-1}(X) \to \cdots.$$

It is a lengthy exercise (written out in **[M2]**) to check that, up to sign, $D_q(u)$ equals the Whitehead product $[z, u]$ where z is a generator of $\pi_{n-1}(X) \cong \mathbf{Z}$.

For the calculations of §§3 and 4, we should describe the isomorphisms d_1: $\pi_{q+1}(X^{[\sigma \cup \partial X, \partial X]}) \cong \pi_{q+2}(X)$ and d_n: $\pi_q(X^{[X, \sigma \cup \partial X]}) \cong \pi_{n+q}(X)$ more explicitly. Let f: $(\sigma \cup \partial X) \times I^{q+1} \to X$ represent an element of $\pi_{q+1}(X^{[\sigma \cup \partial X, \partial X]})$; then the restriction of f to $\partial X \times I^{q+1} \cup (\sigma \cup \partial X) \times \partial I^{q+1}$ equals the projection map to X. We define $d_1\langle f \rangle$ to be the value of the difference cochain $d_{q+2}(f, \mathrm{proj}_X)$ on the $(q+2)$-cell $\sigma \times I^{q+1}$. The definition of d_n is similar.

2. Connected sums of aspherical manifolds. The letter M will always denote a connected sum $M_1 \# M_2 \# \cdots \# M_r$ of $r \geqslant 2$ closed aspherical manifolds of dimension $n \geqslant 3$. We note that $\pi_1 M = \pi_1 M_1 * \pi_1 M_2 * \cdots * \pi_1 M_r$ is torsion-free, since each $\pi_1 M_i$ is (being the fundamental group of a finite-dimensional aspherical complex).

2.A. *The homotopy groups of M*. We will state some results and notation to be used later. Except where otherwise noted, detailed proofs may be found in **[M1]**.

The following theorem extends Bloomberg's **[B]** description of the universal cover of a connected sum.

THEOREM 2.A.1 *The universal cover $\tilde{M}$ of M is homotopy-equivalent to a 1-point union $\bigvee_{i=1}^{r-1} \bigvee_{g\in\pi_1 M} S_g^i$ of $(n-1)$-spheres. Furthermore, the action of $\pi_1 M$ on $\tilde{M}$ corresponds to the left permutation action of $\pi_1 M$ on the indices. That is, $g_1 \in \pi_1 M$ sends S_g^i homeomorphically to $S_{g_1 g}^i$.*

We will use e to denote the identity element of a group π.

DEFINITION. A $\mathbf{Z}$-module $A = A_e$ is called a π-basis for a $\mathbf{Z}\pi$-module N if

1. $N \cong \bigoplus_{g\in\pi} A_g$
2. $g\colon A_e \xrightarrow{\cong} A_g$ is the action of π on $A_e \subset N$.

It follows that $g\colon A_h \xrightarrow{\cong} A_{gh}$ for all $g, h \in \pi$, and that any element of N can be written uniquely (up to order of summands) as $\sum_{i=1}^{s} g_i a_i$, where $g_i \in \pi$, $a_i \in A_e$. Let X be a connected simplicial complex with universal cover $\tilde{X}$. Let $\pi = \pi_1 X$ and denote by $H_f^i(X; N)$ the ith cohomology of X with local coefficients in N (and finite cochains). The following lemma is standard for the case $N = \mathbf{Z}\pi$, $A_e = \mathbf{Z}\cdot e$.

LEMMA 2.A.2. (a) $H_f^i(X; N) \cong H_f^i(\tilde{X}; A)$,
(b) $H_i(X; N) \cong H_i(\tilde{X}; A)$.

The proof parallels the proof of the standard case (for details, see the appendix of **[M2]**). Using Theorem 2.A.1 and Lemma 2.A.2(a) together with Poincaré Duality in $\tilde{M}$, one obtains

LEMMA 2.A.3. *Let q be a dimension in which $\pi_q M$ has a $\pi_1 M$-basis A_q. Then*
(a) $H^1(M; \pi_q M) \cong \bigoplus_{i=1}^{r-1} \pi_q M$.
(b) $H^j(M; \pi_q M) = 0$ *for* $2 \leqslant j \leqslant n-1$.
(c) $H^n(M; \pi_q M) \cong A_q$.

We will first describe A_q for $2 \leqslant q \leqslant 2n-4$. Order the elements of π arbitrarily as $g_1, g_2, \ldots$. For $k \geqslant 1$ let $T_k = \bigvee_{j=1}^{k} \bigvee_{i=1}^{r-1} S_{g_j}^i \subset \bigvee_{g\in\pi} \bigvee_{i=1}^{r-1} S_g^i = T$. Then for all $m \geqslant 2$, $\pi_m(M, *) \cong \pi_m(T) \cong \operatorname{ind\,lim}_k \pi_m(T_k)$. According to Hilton **[H2]**, $\pi_q(T_k) \cong \bigoplus_{j=1}^{k} \bigoplus_{i=1}^{r-1} \pi_q(S_{g_j}^i)$ and thus $\pi_q(T) \cong \bigoplus_{g\in\pi} \bigoplus_{i=1}^{r-1} \pi_q(S_g^i)$. Since $g_1 S_g^i = S_{g_1 g}^i$, it is clear that $A_q = \bigoplus_{i=1}^{r-1} \pi_q(S_e^i)$ is a π-basis for $\pi_q T \cong \pi_q(M)$.

In dimensions $2n-3 \leqslant q \leqslant 3n-6$ the first Whitehead products appear. As above, we have $\pi_{n-1} M \cong \bigoplus_{i=1}^{r-1} \bigoplus_{g\in\pi} \pi_{n-1}(S_g^i)$, and we may choose generators z_g^i of $\pi_{n-1}(S_g^i)$ so that $g_1 z_g^i = z_{g_1 g}^i$. For each $(\alpha, \beta) \in \pi \times \pi$ and $1 \leqslant i, j \leqslant r-1$, let $z_{\alpha,\beta}^{i,j}$ be a generator of $\pi_{2n-3}(S_{\alpha,\beta}^{i,j})$, where $S_{\alpha,\beta}^{i,j}$ is a copy of the $(2n-3)$-sphere mapped to T in such a way that the induced homomorphism sends $z_{\alpha,\beta}^{i,j}$ to the Whitehead product $[z_\alpha^i, z_\beta^j]$. We will always exclude the case of both $i = j$ and $\alpha = \beta$. In all the remaining cases, according to Hilton **[H2]**, the image of $\pi_m(S_{\alpha,\beta}^{i,j})$ is a direct summand of $\pi_m T$ for all m, and it will be regarded as a subgroup. Moreover, using direct limits again, there is a direct sum decomposition when $2n-3 \leqslant q \leqslant 3n-6$:

$$\pi_q(T) \cong \bigoplus_{g\in\pi} \bigoplus_{i=1}^{r-1} \pi_q\left(S_g^i\right) \oplus \bigoplus_{1\leqslant i<j} \bigoplus_{l=1}^{r-1} \pi_q\left(S_{g_i,g_j}^{l,l}\right)$$
$$\oplus \bigoplus_{(\alpha,\beta)\in\pi\times\pi} \bigoplus_{1\leqslant k<l\leqslant r-1} \pi_q\left(S_{\alpha,\beta}^{k,l}\right).$$

Since $[z^i_\alpha, z^j_\beta] = (-1)^{n-1}[z^j_\beta, z^i_\alpha]$ there are commutative diagrams for all m:

$$\begin{array}{ccc} \pi_m\left(S^{i,j}_{\alpha,\beta}\right) & \rightarrow & \pi_m(\tilde{M}) \\ (-1)^{n-1}\downarrow & \nearrow & \\ \pi_m\left(S^{j,i}_{\beta,\alpha}\right) & & \end{array}$$

The action of π on $\pi_{2n-3}(\tilde{M})$ satisfies $g \cdot z^{l,m}_{\alpha,\beta} = z^{l,m}_{g\alpha,g\beta}$. Now let Γ be a subset of π having the following properties:

1. $e \notin \Gamma$.

2. For every $g \in \pi$ with $g \neq e$, exactly one of g and g^{-1} is contained in Γ.

Since π is torsion-free, the second condition makes sense. In **[M1]** the following was proved.

LEMMA 2.A.4. *For $2 \leqslant q \leqslant 3n-6$, $\pi_q M$ has a π-basis A_q given in the following table*:

range of q	$r = 2$	$r > 2$
$2 \leqslant q \leqslant n-2$	0	0
$n-1 \leqslant q \leqslant 2n-4$	$\pi_q(S_e)$	$\bigoplus_{i=1}^{r-1} \pi_q(S^i_e)$
$2n-3 \leqslant q \leqslant 3n-6$	$\pi_q(S_e) \oplus \bigoplus_{g\in\Gamma} \pi_q(S_{e,g})$	$\bigoplus_{i=1}^{r-1} \pi_q(S^i_e) \oplus \bigoplus_{g\in\Gamma} \bigoplus_{l=1}^{r-1} \pi_q(S^{l,l}_{e,g}) \oplus \bigoplus_{g\in\pi_1 M} \bigoplus_{1\leqslant i<j\leqslant r-1} \pi_q(S^{i,j}_{e,g})$

We will also need the following observation, immediate from Theorem 2.A.1 and the fact that $\pi_1 M$ is infinite.

LEMMA 2.A.5. *Let $q \geqslant 2$. For every nonzero x in $\pi_q M$, there is a g in $\pi_1 M$ such that $gx \neq x$.*

2.B. *The relation between E_0M and EM.* M^k will denote the k-skeleton of M. The evaluation map ev: $f \to f(*)$ gives a surjection from EM to M which is a fibration with fiber E_0M.

THEOREM 2.B.1. *The exact homotopy sequence for the fibration $E_0M \to EM \to M$ decomposes into short exact sequences for every $q > 1$*:

$$0 \to \pi_{q+1}M \to \pi_q E_0 M \to \pi_q EM \to 0.$$

REMARK. This holds for $q = 0$ also, since $\pi_1 M$ is centerless.

PROOF OF THE THEOREM. This will be a consequence of

LEMMA 2.B.2. *Suppose g: $M^1 \times I^q \to M$ and $g|_{M^1\times\partial I^q} = \mathrm{proj}_M$. Then g is homotopic* (rel $M^1 \times \partial I^q$) *to a map g_1 with $g_1|_{M^0\times I^q} = \mathrm{proj}_M$.*

Deferring the proof of the lemma for a moment, we consider an element $\langle f \rangle \in \pi_q EM$. Then f: $M \times I^q \to M$ with $f|_{M\times\partial I^q} = \mathrm{proj}_M$. Applying the lemma to $f|_{M^1 \times I^q}$, we can homotop $f|_{M^1 \times I^q}$ and hence f (rel $M \times \partial I^q$) so that $f|_{* \times I^q} = \mathrm{proj}_M$. Thus $\mathrm{ev}_\#\langle f \rangle = \langle f(* \times I^q) \rangle = 0 \in \pi_q M$. $\square$

Proof of Lemma 2.B.2. For $q = 1$, $\langle f(* \times I) \rangle$ is central in $\pi_1(M, *)$, which has trivial center, and the result follows easily. Assume $q \geq 2$. Consider $d_q = d_q(\mathrm{proj}_M, g) \in C^q(M \times I^q; \pi_q(M))$. We have $\delta d_q = c_{q+1}(\mathrm{proj}_M) - c_{q+1}(g) = 0$ since both extend to the $(q+1)$-skeleton. We will show that $\delta d_q = 0$ only if $d_q([* \times I^q]) = 0$.

We may assume that the paths used to define the local coefficient system are the unique paths in some maximal tree in the 1-skeleton of M. Let σ be a 1-simplex in the tree with $\partial\sigma = \tau - *$. Then $0 = \delta d_q[\sigma \times I^q] = d_q[\tau \times I^q] - d_q[* \times I^q]$; hence $d_q[\tau \times I^q] = d_q[* \times I^q]$. By induction on the distance of τ from $*$ in the maximal tree, we have $d_q[\tau \times I^q] = d_q[* \times I^q]$ for every 0-simplex τ of M.

Now suppose σ is any 1-simplex not in the maximal tree, representing an element $g_\sigma \in \pi_1 M$. Then $0 = \delta d_q[\sigma \times I^q] = g_\sigma d_q[\sigma(1) \times I^q] - d_q[\sigma(0) \times I^q]$ so $d_q[* \times I^q] = g_\sigma d_q[* \times I^q]$. Therefore $g d_q[* \times I^q] = d_q[* \times I^q]$ for every $g \in \pi_1 M$. By Lemma 2.A.5, this implies $d_q[* \times I^q] = d_q[\tau \times I^q] = 0$ for every $\tau \in M^0$. Therefore the image of $\tau \times I^q$ is a null-homotopic q-sphere based at τ, so we can homotop f (rel $M \times \partial I^q$) so that $f|_{M^0 \times I^q} = \mathrm{proj}_M$, which was to be proved. $\square$

2.C. *A cell structure for M.* We describe a cell structure for M that will facilitate our calculations. For $1 \leq i \leq r$ let $M_i' = M_i$-open ball, and let $S_i = \partial M_i'$. For $1 \leq i \leq r-1$ let $X_i = S^{n-1} \times I$ be a collar neighborhood of S_i in M_i', so that $S_i = S^{n-1} \times \{0\}$. Give each X_i a cell structure as in §1.D. Let σ_i, $1 \leq i \leq r-1$, be the 1-cell in X_i, and assume that $\sigma_i \cap S_i$ is the basepoint of M_i'. Give the rest of M_i' any triangulation, for $1 \leq i \leq r-1$, and give M_r' any triangulation. Form the 1-point union of the M_i' for $1 \leq i \leq r-1$, and glue S_r to its boundary $\bigvee_{i=1}^{r-1} S_i$ to form M.

The convenience of this construction stems from the following observation. From the proof of Theorem 2.A.1, the inclusion $\bigvee_{i=1}^{r-1} S_i \to M$ sends $\pi_{n-1}(\bigvee_{i=1}^{r-1} S_i)$ isomorphically to $A_{n-1} = \bigoplus_{i=1}^{r-1} \pi_{n-1}(S_e^i)$, a π-basis for $\pi_{n-1}M$. If $\pi_q M$ has a π-basis, then an element of $H_{n-1}(M; \pi_q M) \cong H_{n-1}(\tilde{M}; A_q)$ (by Lemma 2.A.2) can be represented as $\sum_{i=1}^{r-1} \sum_{j=1}^{N} a_{ij}(g_{ij}[S_e^i]) = \sum_{i=1}^{r-1}(\sum_{j=1}^{N} a_{ij} g_{ij})[S_e^i] = \sum_{i=1}^{r-1} x_i[S_e^i]$, where $x_i \in \pi_q M$.

3. Calculations of $\pi_q(EM)$.

All cohomology will be with local coefficients.

3.A. *An exact sequence for $\pi_q(EM)$.*

Theorem 3.A.1. *For $1 \leq q \leq 2n-5$ there is an exact sequence*

$$\ldots \to H^1(M; \pi_{q+2}M) \xrightarrow{D_q} H^n(M; \pi_{n+q}M) \to \pi_q(EM)$$
$$\to H^1(M; \pi_{q+1}M) \xrightarrow{D_{q-1}} H^n(M; \pi_{n+q-1}M) \to \ldots .$$

Proof. Using the diagram of fibrations discussed in the introduction with $Y = M$, and noting that for $q \geq 1$, $\pi_q(M^{[M, \varnothing]}) = \pi_q(EM)$, we obtain an exact sequence for each $q \geq 1$:

$$J_q \xrightarrow{D_q} K_q \to \pi_q(EM) \to J_{q-1} \xrightarrow{D_{q-1}} K_{q-1}.$$

In this sequence,

$$J_q = \operatorname{coker}\left(\pi_{q+1}(M^{[M^2,M^1]}) \to \pi_{q+1}(M^{[M^2,\varnothing]})\right),$$
$$K_q = \operatorname{coker}\left(\partial\colon \pi_{q+1}(M^{[M^2,M^1]}) \to \pi_q(M^{[M,M^2]})\right),$$

and D_q is induced by $\partial\colon \pi_{q+1}(M^{[M^2,\varnothing]}) \to \pi_q(M^{[M,M^2]})$. The theorem is immediate from the following two lemmas.

LEMMA 3.A.2. *For $q \geqslant 0$, $J_q \cong H^1(M; \pi_{q+2}M)$.*

PROOF. By Lemma 2.B.2, $\pi_{q+1}(M^{[M^2,*]}) \to \pi_{q+1}(M^{[M^2,\phi]})$ is surjective. Therefore Lemma 1.B.1 applies. □

LEMMA 3.A.3. *For $0 \leqslant q \leqslant 2n-5$, $K_q \cong H^n(M; \pi_{n+q}M)$.*

PROOF. By Lemma 2.A.4, $\pi_{p+q}M$ has a π-basis for $2 \leqslant p+q \leqslant 3n-6$. Therefore when $3 \leqslant p \leqslant n-1$, the condition $0 \leqslant q \leqslant 2n-5$ guarantees that $\pi_{p+q}(M)$ has a π-basis. By Lemma 2.A.3, $H^p(M; \pi_{p+q}M) = H^{p-1}(M; \pi_{p+q}M) = 0$, so Lemma 1.C.1 applies. □

COROLLARY 3.A.4. *For $1 \leqslant q \leqslant n-4$, $\pi_q EM \cong A_{n+q}$, hence is finite.*

PROOF. For these dimensions, $A_{q+2} = 0 = A_{q+1}$ by Lemma 2.A.4. Therefore $H^1(M; \pi_{q+2}M) = 0 = H^1(M; \pi_{q+1}M)$, so $\pi_q(EM) \cong H^n(M; \pi_{n+q}M) \cong A_{n+q}$, using Lemma 2.A.3.

3.B. *Calculation of D_q.* To compute D_q, we first define a homomorphism $k\colon H^1(M; \pi_{q+2}M) \to \pi_{q+1}(M^{[M^2,\varnothing]})$ such that $d_1 \circ k =$ identity, where d_1 is the homomorphism of Lemma 1.B.1. Recall the cell structure for M described in §2.C. Given a generator $x_i[S^i] \in H_{n-1}(M; \pi_{q+2}M) \cong H^1(M; \pi_{q+2}M)$, let $f\colon ((M - \operatorname{int}(X_i)) \cup \sigma_i) \times I^{q+1} \to M$ be a map such that

$$f|_{(M-\operatorname{int}(X_i))\times I^{q+1}} = \operatorname{proj}_M, \text{ and } d_{q+2}(\operatorname{proj}_M, f)\left[\sigma_i \times I^{q+1}\right] = x_i \in \pi_{q+2}(M, *).$$

Define $k(x_i[S^i]) = \langle f|_{M^2\times I^{q+1}}\rangle$.

Since D_q is induced by $\partial\colon \pi_{q+1}(M^{[M^2,\varnothing]}) \to \pi_q(M^{[M,M^2]})$, we have $D_q = d_n \circ \partial \circ k$, where $d_n\colon \pi_q(M^{[M,M^2]}) \to H^n(M; \pi_{n+q}M)$ is defined in Lemma 1.C.1. The calculation of $\partial\langle f|_{M^2\times I^{q+1}}\rangle$ is exactly analogous to the calculation of D_q in the example of §1.D. The generator z there of $\pi_{n-1}(X)$ corresponds to the element z_e^i of $\pi_{n-1}(M, *)$ (defined in §2.A). The group $\pi_{q+2}X$ is replaced by $\pi_{q+2}(M, *)$ and $\pi_q(X^{[X,\sigma\cup\partial X]})$ is replaced by $\pi_q(M^{[M,M^2]})$. Therefore $\partial k(x_i[S^i])$ is representable by a map $\bar{f}$ which equals proj_M on $(M - (\text{small ball in } X_i)) \times I^q$ and such that $d_{n+q}(\operatorname{proj}_M, \bar{f})([M \times I^q, M \times \partial I^q]) = [z_e^i, x_i]$. Hence $D_q(x_i[S^i]) = d_n\langle \bar{f}\rangle = \{[z_e^i, x_i]\}$ where the curly brackets indicate an equivalence class in $H^n(M; \pi_{n+q}M) \cong \pi_{n+q}(M)/\pi_1(M) \cong A_{n+q}$. We have shown

PROPOSITION 3.B.1. $D_q(\Sigma_{i=1}^{r-1} x_i[S^i]) = \Sigma_{i=1}^{r-1} \{[z_e^i, x_i]\} \in A_{n+q}$.

We will now determine $\ker D_{n-3}$ and $\operatorname{coker} D_{n-3}$ using the results and notation of §2.A. For $q = n-3$, we have $[z_e^i, z_\gamma^j] = z_{e,\gamma}^{ij}$, so

$$\begin{aligned}D_{n-3}\colon H_{n-1}(M;\pi_{n-1}M) &\cong \bigoplus_{i=1}^{r-1}\pi_{n-1}M \to A_{2n-3}\\ &= \left(\bigoplus_{i=1}^{r-1}\pi_{2n-3}(S_e^i)\right)\oplus\left(\bigoplus_{i=1}^{r-1}\bigoplus_{\gamma\in\Gamma}\pi_{2n-3}(S_{e,\gamma}^{i,i})\right)\\ &\quad\oplus\left(\bigoplus_{1\leqslant i<j\leqslant r-1}\bigoplus_{\gamma\in\pi_1 M}\pi_{2n-3}(S_{e,\gamma}^{i,j})\right)\end{aligned}$$

given on generators by $D_{n-3}(0,\ldots,(z_\gamma^j)_i,\ldots,0)=\{z_{e,\gamma}^{i,j}\}$ (where $(\ )_i$ indicates that z_γ^j appears in the ith slot) in all cases except both $\gamma=e$ and $i=j$. We will describe the inverse image of each of these summands in order to determine the kernel and cokernel of D_{n-3}. Let $B_{n-3}^i=\ker(D_{n-3}|_{\pi_{n-1}(S_e^i)}=[z_e^i,-]\colon \pi_{n-1}(S_e^i)\to\pi_{2n-3}(S_e^i))$, which is 0 if n is odd and has index $\leqslant 2$ if n is even. Let $C_{n-3}^i=\operatorname{coker}(D_{n-3}|_{\pi_{n-1}(S_e^i)})$, which is well known to be finite. Observe that $\bigoplus_{l=1}^{r-1}\bigoplus_{\gamma\in\Gamma}\pi_{2n-3}(S_{e,\gamma}^{l,l})$ is in the image of D_{n-3} since $D_{n-3}(0,\ldots,(z_\gamma^i)_i,\ldots,0)=\{z_{e,\gamma}^{i,i}\}$, and the inverse image of $\{z_{e,\gamma}^{i,i}\}$ consists of $(0,\ldots,(z_\gamma^i)_i,\ldots,0)$ and $(0,\ldots,(-1)^{n-1}(z_{\gamma-1}^i)_i,\ldots,0)$. Therefore the kernel contains $\bigoplus_{\gamma\in\Gamma}(\mathbf{Z})_\gamma$, where $(\mathbf{Z})_\gamma\cong\mathbf{Z}$ is generated by $(0,\ldots,(z_\gamma^i-(-1)^{n-1}z_{\gamma-1}^i)_i,\ldots,0)$. Explicitly, we have

$$\begin{aligned}D_{n-3}\big(0,\ldots,\big(z_\gamma^i-(-1)^{n-1}z_{\gamma-1}^i\big)_i,\ldots,0\big)&=\big\{\big[z_e^i,z_\gamma^i\big]\big\}-\big\{(-1)^{n-1}\big[z_e^i,z_{\gamma-1}^i\big]\big\}\\ &=\big\{\big[z_e^i,z_\gamma^i\big]\big\}-\big\{\big[z_{\gamma-1}^i,z_e^i\big]\big\}=\big\{\big[z_e^i,z_\gamma^i\big]\big\}-\big\{\gamma^{-1}\big[z_e^i,z_\gamma^i\big]\big\}=0.\end{aligned}$$

Finally, $\bigoplus_{\gamma\in\pi}\bigoplus_{1\leqslant i<j\leqslant r-1}\pi_{2n-3}(S_{e,\gamma}^{i,j})$ is in the image of D_{n-3} and the kernel of the inverse image of $\{z_{e,\gamma}^{i,j}\}$ is generated by

$$\big(0,\ldots,\big(z_\gamma^j\big)_i,\ldots,\big(-(-1)^{n-1}z_{\gamma-1}^i\big)_j,\ldots,0\big).$$

Explicitly, we have

$$\begin{aligned}&D_{n-3}\big(0,\ldots,z_\gamma^j,\ldots,-(-1)^{n-1}z_{\gamma-1}^i,\ldots,0\big)\\ &\quad=\big\{\big[z_e^i,z_\gamma^j\big]\big\}-\big\{(-1)^{n-1}\big[z_e^j,z_{\gamma-1}^i\big]\big\}\\ &\quad=\big\{\big[z_e^i,z_\gamma^j\big]\big\}-\big\{\big[z_{\gamma-1}^i,z_e^j\big]\big\}=\big\{\big[z_e^i,z_\gamma^j\big]\big\}-\big\{\gamma^{-1}\big[z_e^i,z_\gamma^j\big]\big\}=0.\end{aligned}$$

Collecting this information, we state

LEMMA 3.B.2. *$\operatorname{coker}(D_{n-3})\cong\bigoplus_{i=1}^{r-1}C_{n-3}^i$ and*

$$\ker(D_{n-3})\cong\left(\bigoplus_{i=1}^{r-1}B_{n-3}^i\right)\oplus\left(\bigoplus_{\gamma\in\Gamma}\bigoplus_{i=1}^{r-1}\mathbf{Z}\right)\oplus\left(\bigoplus_{\gamma\in\pi}\bigoplus_{1\leqslant i<j\leqslant r-1}\mathbf{Z}\right).$$

COROLLARY 3.B.3. *For $n\geqslant 4$, $\pi_{n-3}EM$ is finite.*

PROOF. In this case, the exact sequence of Theorem 3.A.1 yields $\pi_{n-3}EM\cong\operatorname{coker}(D_{n-3})\cong\bigoplus_{i=1}^{r-1}C_{n-3}^i$. □

COROLLARY 3.B.4. *$\pi_{n-2}EM$ is infinitely-generated as an abelian group.*

PROOF. In Theorem 3.A.1 we have a surjection $\pi_{n-2}(EM)\to\ker(D_{n-3})$ and $\ker(D_{n-3})$ is infinitely-generated by Lemma 3.B.2. □

COROLLARY 3.B.5. *Let M^3 be a connected sum of aspherical 3-manifolds. Then $\pi_1(EM^3)$ is infinitely-generated.*

PROOF. Take $n = 3$ in Corollary 3.B.4. □

4. Homeomorphisms of nonirreducible 3-manifolds. Let HX denote the path component of id_X in the group $\mathrm{Homeo}(X)$; then $HX \subset EX$. Throughout this section, we will assume $M = M_1 \# \cdots \# M_r$ is a connected sum of $r \geqslant 3$ aspherical 3-manifolds.

From Theorem 3.A.1 we have a homomorphism

$$D_0\colon H^1(M; \pi_2 M) \to H^3(M; \pi_3 M)$$

and a surjective homomorphism $j\colon \pi_1(EM) \to \mathrm{kernel}(D_0)$. Let $i\colon \pi_1(HM, \mathrm{id}_M) \to \pi_1(EM, \mathrm{id}_M)$ denote the homomorphism induced by inclusion. The remainder of this section will be devoted to the proof of

THEOREM 4.1. *The image of $j \circ i$ contains an infinitely-generated direct summand of kernel (D_0). Hence $\pi_1(HM)$ is not finitely-generated.*

4.A. *Isotopies of $S^2 \times I$.* Let $X = S^2 \times I$. We regard S^2 as the unit sphere in R^3. Since $SO(3)$ preserves S^2 we have $SO(3) \subset HS^2$ (it is actually a deformation retract [K]). Let $SO(2) \subset SO(3)$ be the subgroup of rotations that leave the points $(0, 0, 1)$ and $(0, 0, -1)$ fixed. Let $\tau\colon (I, 0, 1) \to (SO(2), \mathrm{id}, \mathrm{id})$ be the path such that $\tau(t)$ is rotation through an angle of $2\pi t$; then τ represents a generator of $\pi_1(SO(2)) \cong \mathbf{Z}$ (and hence represents a generator of $\pi_1(SO(3)) \cong \mathbf{Z}/2\mathbf{Z}$). We will now use the results and notation of §1.D. We define a level-preserving homeomorphism $f\colon X \to X$ by $f(x, s) = (\tau(s)(x), s)$. Assuming that the 1-cell σ equals $(0, 0, 1) \times I$, we see that f represents an element of $\pi_0(X^{[X, \partial X \cup \sigma]}) \cong \pi_3(X) \cong \mathbf{Z}$. It is known that the difference class $d_3(f, \mathrm{id}_X)$ is a generator of this group (see [H1, p. 85]). In the exact sequence of §1.D,

$$\pi_2 X \xrightarrow{[z,-]} \pi_3 X \to \pi_0(X^{[X,\partial X]}) \to 0$$

the homomorphism $[z, -]$ is well known to have image $2\mathbf{Z} \subset \mathbf{Z}$; hence $\pi_0(X^{[X,\partial X]})$ is isomorphic to $\mathbf{Z}/2\mathbf{Z}$ and f represents a generator of this quotient group. Now if $\theta\colon (I \times I, \partial I \times I) \to (SO(3), \mathrm{id})$ is a nullhomotopy with $\theta(s, 0) = \tau^2(s)$, $\theta(s, 1) = \mathrm{id}$, then $F\colon X \times I \to X$ defined by $F_t(x, s) = F((x, s), t) = (\theta(s, t)(x), s)$ is an isotopy from f^2 to id_X. Under the identification $\pi_1(X^{[\sigma \cup \partial X, \partial X]}) \cong \pi_2(X)$, the restriction $F|_{\sigma \cup \partial X}$ represents a generator since $\partial\langle F|_\sigma\rangle = \langle F_0\rangle = \langle f^2\rangle$. We choose the generator z of $\pi_2(X)$ such that $d_1(\langle F|_\sigma\rangle)([\sigma \times I]) = z$. Note that F^{-1} is an isotopy from $(f^{-1})^2$ to id_X, with $d_1(\langle F^{-1}|_\sigma\rangle)([\sigma \times I]) = -z$.

4.B. *The 3-manifold Z.* Let X_1 and X_2 be two copies of $S^2 \times I$, and let $B = D^2 \times I$. Let Z denote the 3-manifold-with-boundary obtained by identifying $D_1 = D^2 \times \{0\}$ with a disc in $S^2 \times \{1\} \subset \partial X_1$ and $D_2 = D^2 \times \{1\}$ with a disc in $S^2 \times \{1\} \subset \partial X_2$, by orientation-reversing homeomorphisms. We assume these discs do not contain the 0-cells $\sigma_1 \cap S^2 \times \{1\} \subset \partial X_1$ and $\sigma_2 \cap S^2 \times \{1\} \subset \partial X_2$. We will use S_1 to denote $S^2 \times \{0\} \subset \partial X_1$ and S_2 for $S^2 \times \{0\} \subset \partial X_2$. Let S_3 denote the remaining boundary component of Z, so that the oriented boundary of Z is $\partial Z = S_1 \cup S_2 \cup (-S_3)$. We choose a nice collar neighborhood X_3 of S_3 so that $\sigma_1 \cap X_3 = \sigma_1$ has the form $(\sigma_1 \cap S_3) \times I = (0, 0, 1) \times I$ and $\sigma_2 \cap X_3 = \sigma_2$

has the form $(\sigma_2 \cap S_3) \times I = (0, 0, -1) \times I$. We will define two isotopies of Z. The isotopy G is defined by

$$G_t(z) = G(z, t) = \begin{cases} F_t(x, s) & \text{if } z = (x, s) \in X_3, \\ z & \text{if } z \notin X_3, \end{cases}$$

and the isotopy H is defined by

$$H_t(z) = H(z, t) = \begin{cases} F_t(x, s) & \text{if } z = (x, s) \in X_1, \\ F_t^{-1}(x, s) & \text{if } z = (x, s) \in X_2, \\ z & \text{if } (z, t) \in B \times I. \end{cases}$$

We make the important observation that G_0 is isotopic to H_0 by an isotopy which is fixed on $\sigma_1 \cup \sigma_2 \cup \partial Z$. This observation appears in the thesis of Hendriks **[H1**, p. 103].

We will now construct a nontrivial element of $\pi_1(Z^{[Z,\partial Z]})$. Let $\sigma = \sigma_1 \cup \sigma_2 \subset Z$. From the fibration $Z^{[Z,\sigma\cup\partial Z]} \to Z^{[Z,\partial Z]} \to Z^{[\sigma\cup\partial Z,\partial Z]}$ we obtain a commutative diagram

$$\begin{array}{ccccccc} \pi_1(Z^{[Z,\partial Z]}) & \to & \pi_1(Z^{[\sigma\cup\partial Z,\partial Z]}) & \xrightarrow{\partial} & \pi_0 Z^{[Z,\sigma\cup\partial Z]} & \to & \pi_0(Z^{[Z,\partial Z]}) \\ & j\searrow & d_1\downarrow\cong & & d_3\downarrow\cong & \nearrow & \\ & & H^1(Z, \partial Z; \pi_2 Z) & \xrightarrow{D_0} & H^3(Z, \partial Z; \pi_3 Z) & & \end{array}$$

Now $Z \simeq S_1 \vee S_2$, so using the results of Hilton **[H2]** we may write $\pi_2(Z) \cong \pi_2(S_1) \oplus \pi_2(S_2)$ and $\pi_3(Z) \cong \pi_3(S_1) \oplus \pi_3(S_2) \oplus \pi_3(S_{1,2})$ where $S_{1,2}$ is a 3-sphere. Let z_1, z_2 be generators of $\pi_2(S_1)$ and $\pi_2(S_2)$, respectively, such that the homotopy class represented by the oriented sphere S_3 equals $z_1 + z_2 \in \pi_2(Z)$. Then the Whitehead product $z_{1,2} = [z_1, z_2]$ corresponds to a generator of $\pi_3(S_{1,2}) \subset \pi_3(Z)$. Let $z_{i,i}$, $1 \leqslant i \leqslant 2$, be the generators of $\pi_3(S_i)$ so that $[z_i, z_i] = 2z_{i,i}$. We have $H^1(Z, \partial Z;\ \pi_2 Z) \cong \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ where the summands are generated by the cocycles $d_{i,j}$, $1 \leqslant i, j \leqslant 2$, such that $d_{i,j}[\sigma_i \times I] = z_j$ and $d_{i,j}[\sigma_k \times I] = 0$ if $k \neq i$. We also have $H^3(Z, \partial Z;\ \pi_3 Z) \cong \pi_3(Z) \cong \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ generated by the cocycles $c_{1,1}$, $c_{2,2}$, and $c_{1,2}$, where $c_{1,1}([Z, \partial Z]) = z_{1,1}$ and so on. Let $c_{2,1} = c_{1,2}$. As in §3.B, it follows that the homomorphism D_0 is given by $D_0(d_{i,j}) = c_{i,j}$ if $i \neq j$ while $D_0(d_{i,i}) = 2c_{i,i}$. Therefore the kernel of D_0 is generated by $d_{1,2} - d_{2,1}$. From the discussion of $F\colon X \times I \to X$, we have $d_1\langle H|_\sigma\rangle = d_{1,1} - d_{2,2}$. To find $d_1\langle G|_\sigma\rangle$, we see from the definition of G that $d_1\langle G|_\sigma\rangle([\sigma_1 \times I]) = z_1 + z_2$; hence $d_1\langle G|_\sigma\rangle = d_{1,1} + d_{1,2} + \alpha d_{2,1} + \beta d_{2,2}$ for some $\alpha, \beta \in Z$. Since G_0 is isotopic to H_0 with σ held fixed, we have $D_0(d_1(\langle G|_\sigma\rangle)) = D_0(d_1(\langle H|_\sigma\rangle)) = 2c_{1,1} - 2c_{2,2}$. From our formula for D_0 we also have

$$\begin{aligned} D_0(d_1(\langle G|_\sigma\rangle)) &= D_0(d_{1,1} + d_{1,2} + \alpha d_{2,1} + \beta d_{2,2}) \\ &= 2c_{1,1} + c_{1,2} + \alpha c_{2,1} + 2\beta c_{2,2}; \end{aligned}$$

hence $\alpha = \beta = -1$ and $d_1(\langle G|_\sigma\rangle)([\sigma_2 \times I]) = -z_1 - z_2$. The identity map of Z is isotopic to $G_0 \circ H_0^{-1}$ by an isotopy which is fixed on $\sigma \cup \partial Z$. Define an isotopy J: $Z \times I \to Z$ so that for $0 \leqslant t \leqslant \frac{1}{2}$, J is such an isotopy, while $J(z, t) = (G_{2t-1} \circ H_{2t-1}^{-1})(z, t)$ for $\frac{1}{2} \leqslant t \leqslant 1$. Then J is a loop in Homeo(Z) and regarding J

as a loop in $Z^{[Z,\partial Z]}$ we have $d_1\langle J|_\sigma\rangle = d_1\langle G|_\sigma\rangle - d_1\langle H|_\sigma\rangle = d_{1,2} - d_{2,1}$. That is, $j\langle J\rangle$ is a generator of kernel(D_0).[1]

4.C. *Isotopies of M*. To construct isotopies of M, we will use the cell-complex structure of M defined in §2.C. Since $\bigvee_{i=1}^{r-1} X_i \subset M$ is simply-connected and contains the basepoint $*$ of M, a path in M with endpoints in $\bigvee_{i=1}^{r-1} X_i$ represents a well-defined element of $\pi_1(M, *)$. We may represent any element $\langle\alpha\rangle \in \pi_1(M_3) * \cdots * \pi_1(M_r) \subset \pi_1(M)$ by a nicely-imbedded arc α in M that runs from X_1 to X_2 intersecting them only in its boundary. We can imbed Z in M so that

1. $X_1 \subset Z$ is mapped homeomorphically to $X_1 \subset M$, carrying the basepoint $\sigma_1 \cap S_3$ to $*$.

2. $D^2 \times I \subset Z$ is mapped to a tubular neighborhood of α that intersects $X_1 \cup X_2$ in $D^2 \times \partial I$.

3. $X_2 \subset Z$ is mapped homeomorphically into $X_2 - *$. (This will reverse the local orientation, when α is orientation-reversing.)

Such an imbedding induces an injection $\pi_2(Z, \sigma_1 \cap S_3) \to \pi_2(M, *)$ given on generators by $z_1 \to z_e^1$ and $z_2 \to z_\alpha^2$.

The isotopy J_α of M is of course defined to be J on $Z \subset M$ and the identity outside Z. Now $d_1(\langle J_\alpha|_{M^2}\rangle) = d_\alpha$ is the element of $H^1(M;\ \pi_2 M)$ such that $d_\alpha([\sigma_k \times I]) = 0$ if $3 \leqslant k \leqslant r - 1$, $d_\alpha([\sigma_1 \times I]) = z_\alpha^2$, and $d_\alpha([\sigma_2 \times I]) = -z^1_{\alpha^{-1}}$. The last formula differs by the action of α^{-1} from the corresponding calculation for $\langle J|_\sigma\rangle \in \pi_1(Z^{[\sigma,\partial\sigma]})$, since in M we use a path in X_2 to base the homotopy class that is the value of the difference cohomology class $d_1(J_\alpha \operatorname{proj}_M)$ on $[\sigma_2 \times I]$, rather than a path in Z that follows along α back to X_1. Under the isomorphism $H^1(M;\ \pi_2 M) \cong \bigoplus_{i=1}^{r-1} \pi_2(M)$, $d_1(\langle J_\alpha|_{M^2}\rangle)$ corresponds to $(z_\alpha^2, -z^1_{\alpha^{-1}}, 0, \ldots, 0)$. Regarding the J_α as loops in EM, we have $j\langle J_\alpha\rangle = (z_\alpha^2, -z^1_{\alpha^{-1}}, 0, \ldots, 0)$; hence the elements $j\langle J_\alpha\rangle$, $\alpha \in \pi_3(M) * \cdots * \pi_r(M)$, generate an infinitely-generated summand of kernel (D_0), using Lemma 3.B.2. This concludes the proof of Theorem 4.1. □

Question. Can the generators of kernel(D_0) of the form $(z_\gamma^1 - z^1_{\gamma^{-1}}, 0, \ldots, 0)$, or of the form $(z_\gamma^2, -z^1_{\gamma^{-1}}, 0, \ldots, 0)$ where γ involves elements of $\pi_1 M_1 * \pi_1 M_2$, be realized as the images $j\langle J_\gamma\rangle$ of loops J_γ in HM?

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OKLAHOMA, NORMAN, OKLAHOMA 73019