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UNIQUENESS OF PRODUCT AND COPRODUCT DECOMPOSITIONS IN RATIONAL HOMOTOPY THEORY

BY

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ABSTRACT. Let X be a nilpotent rational homotopy type such that (1) $S(X)$, the image of the Hurewicz map has finite total rank, and (2) the basepoint map of M, a minimal algebra for X, is an element of the Zariski closure of $\mathrm{Aut}(M)$ in $\mathrm{End}(M)$ (i.e. X has "positive weight"). Then (A) any retract of X satisfies the two properties above, (B) any two irreducible product decompositions of X are equivalent, and (C) any two irreducible coproduct decompositions of X are equivalent.

Introduction. This paper is primarily concerned with the uniqueness of irreducible factors in decompositions of rational homotopy types as products and as coproducts (one-point unions). Uniqueness is demonstrated with respect to both products and coproducts, for simply-connected, positive weight, rational homotopy types for which the spherical cohomology is finite dimensional.

There are no known examples of rational homotopy types which fail to satisfy either unique factorization property. Indeed, we conjecture that no such examples exist. This contrasts sharply with the situation in certain more structured geometric contexts; specifically, the noncancellation examples due to Hilton and Roitberg (see **[10]**) illustrate the failure of unique factorization with respect to products among differentiable manifolds, topological spaces and integral homotopy types. A similar situation exists for coproducts. Hilton and others have constructed examples of integral homotopy types which exhibit noncancellation with respect to the formation of pointed coproducts.

For product decompositions of simply-connected rational homotopy types having positive weights, previous results **[3]** confirmed unique factorization in the "finitary" case ("finitary" means that either the homotopy or the cohomology are required to be finite dimensional). The "finitary" restriction is removed in this paper, and replaced by the much weaker hypothesis that the Hurewicz homomorphisms are eventually trivial (i.e., have a finite image for all sufficiently high degrees). Moreover, the results reported here represent a significant generalization of the topological results obtained in **[1]**, where (product) unique factorization is proved for formal, simply-connected, rational homotopy types having rational cohomology finitely generated as an algebra.

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Recall that a simply-connected, rational homotopy type may be viewed as a "minimal model" M; here, M is a differential graded algebra, whose underlying algebra is a simply-connected, free, graded-commutative Q-algebra, and whose differential is a degree 1, graded derivation with decomposable image. For details regarding minimal models **[8]**, **[9]** or **[17]** should be consulted.

$\mathrm{End}(M)$, the set of differential graded algebra endomorphisms of M, is an algebraic variety (defined over Q) and it is equipped with a Zariski topology. $\mathrm{Aut}(M)$, the set of invertible endomorphisms, is an open subset of $\mathrm{End}(M)$. The minimal model M is said to have *positive weight*, if the trivial (basepoint) endomorphism is in the Zariski closure of $\mathrm{Aut}(M)$ in $\mathrm{End}(M)$.

Many familiar spaces satisfy the positive weight condition. For example, all H-spaces and co-H-spaces, many homogeneous spaces, all formal spaces, and all nonsingular complex varieties **[13]** have positive weight.

The unique factorization problem may be posed more generally by considering nilpotent rational homotopy types. In this case, for products, the same methods yield an affirmative solution. However, any nilpotent rational homotopy type which is not simply-connected must be irreducible with respect to coproduct. Thus, the unique factorization of nilpotent rational homotopy types with respect to coproducts trivially reduces to the simply connected case.

1. Products, coproducts and splitting idempotents. In this preliminary section an easily verifiable categorical criterion is discussed. Definition (1.1)–(1.4) below is motivated by the question: Under what conditions are product and coproduct splittings determined by idempotents?

Let Δ be a category with C and P objects of Δ.

Recall that a *product structure on* P is a collection of morphisms $\{\pi_\alpha\colon P \to P_\alpha | \alpha \in J\}$ such that

$$\mathrm{Hom}_\Delta(X, P) \underset{(\pi_{\alpha *})}{\rightarrow} \prod_{\alpha \in J} \mathrm{Hom}_\Delta(X, P_\alpha)$$

is a bijection for all objects X of Δ.

Dually, a *coproduct structure on* C is a collection of morphisms $\{i_\alpha\colon C_\alpha \to C | \alpha \in I\}$ such that

$$\mathrm{Hom}_\Delta(C, X) \underset{(i_\alpha^*)}{\rightarrow} \prod_{\alpha \in I} \mathrm{Hom}_\Delta(C_\alpha, X)$$

is a bijection for all objects X of Δ.

DEFINITION. A category Δ is called *I-split*, if it satisfies

(1.1) Δ has a zero object.

(1.2) Δ has finite products and finite coproducts.

(1.3) If $f \in \mathrm{Hom}_\Delta(A, A)$ satisfies $f \circ f = f$ then there is a factorization

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "p"'] & & A \\ & B \arrow[ur, "i"'] & \end{tikzcd}$$

such that $p \circ i = 1_B$.

(1.4) For each object A of Δ there is a finite cardinal $n(A)$ such that if $A \cong \prod_{i \in I} A_i$ (or $A \cong \coprod_{i \in I} A_i$), then $\operatorname{card}(I) < n(A)$.

For the remainder of this section Δ will always denote an I-split category.

Note. The factorization of (1.3) above is unique up to canonical equivalence in Δ because

$$B \xrightarrow[i]{} A \overset{f}{\underset{1}{\rightrightarrows}} A$$

is an equalizer and

$$A \overset{f}{\underset{1}{\rightrightarrows}} A \xrightarrow[p]{} B$$

is a coequalizer.

For any object A of Δ let $E_A = \operatorname{Hom}_\Delta(A, A)$ and $G_A = \operatorname{Aut}_\Delta(A)$.

Because of (1.3) above a product structure (resp., coproduct structure) on P, $P \to \prod P_\alpha$ (resp., $\coprod P_\alpha \to P$) is equivalent to a collection $\{e_\alpha\} \subseteq E_P$ such that

(1.5) $e_\alpha \circ e_\alpha = e_\alpha$ for each α.

(1.6) $e_\alpha \circ e_\beta = 0$ if $\alpha \neq \beta$.

(1.7) $P \xrightarrow[(p_\alpha)]{} \prod P_\alpha$ (resp., $\coprod P_\alpha \xrightarrow[(i_\alpha)]{} P$) is an isomorphism where $e_\alpha = i_\alpha \circ p_\alpha$ is the factorization of (1.3).

Product structures (resp., coproduct structures) will be called $\prod$-*splittings* (resp., $\coprod$-*splittings*). A $\prod$-splitting (or $\coprod$-splitting) will be called *irreducible* if each factor P_α is nontrivial and has no nontrivial $\prod$-splitting (resp., $\coprod$-splitting).

Two splittings are *equivalent* if they have isomorphic factors.

Observe that $G_A \times E_A \to E_A$, $(g, f) \mapsto gfg^{-1}$ is a group action for each object A of Δ. If the $\prod$-splitting $P \xrightarrow[(p_\alpha)]{} \prod P_\alpha$ (resp., $\coprod$-splitting $\coprod P_\alpha \xrightarrow[(i_\alpha)]{} P$) is given by $\{e_\alpha\}$ as in (1.5)–(1.7), then the $\prod$-splitting (resp., $\coprod$-splitting) obtained from $\{ge_\alpha g^{-1}\}$ is equivalent to the one obtained from $\{e_\alpha\}$ for any $g \in G_A$.

1.8. Lemma. *Let* $\{e_\alpha\} \subseteq E_A$ *be a* $\prod$-*splitting* (*resp.*, $\coprod$-*splitting*) *of* A *and let* $f \in E_A$ *satisfy* $f = f \circ f$ *and* $f \circ e_\alpha = e_\alpha \circ f$ *for all* α. *Then* $\{g_\alpha\} \subseteq E_B$ *is a* $\prod$-*splitting* (*resp.*, $\coprod$-*splitting*) *of* B, *where*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "q"'] & & A \\ & B \arrow[ur, "j"'] & \end{tikzcd}$$

(*from* (1.3)) *and* $g_\alpha = q \circ e_\alpha \circ j$.

Proof. $\{g_\alpha\} \subseteq E_B$ is a collection of commuting idempotents. Let $g_\alpha = \nu_\alpha \circ \mu_\alpha$ where

$$\begin{tikzcd} B \arrow[rr, "g_\alpha"] \arrow[dr, "\mu_\alpha"'] & & B \\ & B_\alpha \arrow[ur, "\nu_\alpha"'] & \end{tikzcd} \quad \text{and} \quad \mu_\alpha \circ \nu_\alpha = 1_{B_\alpha}.$$

Also let $s_\alpha = p_\alpha \circ j \circ \nu_\alpha$: $B_\alpha \rightarrowtail A_\alpha$, $r_\alpha = \mu_\alpha \circ q \circ i_\alpha$: $A_\alpha \twoheadrightarrow B_\alpha$. Observe that $r_\alpha \circ s_\alpha = 1_{B_\alpha}$. Thus we have

$$\begin{array}{ccccc}
\amalg B_\alpha & \overset{\amalg(s_\alpha)}{\rightarrow} & \amalg A_\alpha & \overset{\amalg(r_\alpha)}{\rightarrow} & \amalg B_\alpha \\
(\nu_\alpha)\downarrow & & \downarrow(i_\alpha) & & \downarrow(\nu_\alpha) \\
B & \underset{j}{\rightarrow} & A & \underset{q}{\rightarrow} & B \\
(\mu_\alpha)\downarrow & & \downarrow(p_\alpha) & & \downarrow(\mu_\alpha) \\
\prod B_\alpha & \underset{\prod(s_\alpha)}{\rightarrow} & \prod A_\alpha & \underset{\prod(r_\alpha)}{\rightarrow} & \prod B_\alpha
\end{array}$$

where $(\amalg(r_\alpha)) \circ (\amalg(s_\alpha)) = 1_{\amalg B_\alpha}$, $q \circ j = 1_B$, and $(\prod(r_\alpha)) \circ (\prod(s_\alpha)) = 1_{\prod B_\alpha}$.

Thus the result follows in either case because a retract of an isomorphism is an isomorphism. Q.E.D.

1.9. PROPOSITION. *Let A be an object of Δ. Suppose $\{e_i, \ldots, e_n\}$ and $\{f_1, \ldots, f_m\}$ are irreducible $\prod$-splittings (resp., $\amalg$-splittings) of A. If $e_\alpha \circ f_\beta = f_\beta \circ e_\alpha$ for all α and β, then $m = n$ and the splittings of A are equivalent.*

The proof of Proposition 1.9 is a straightforward application of Lemma 1.8 and is left to the reader. (A model for the proof may be found in the proof of Theorem 2 in **[2]**.)

We say that two splittings (either $\prod$ or $\amalg$) $\{e_\alpha\}$ and $\{f_\beta\}$ of A are *compatible* if there exists $g \in G_A$ such that $e_\alpha \circ g \circ f_\beta \circ g^{-1} = g \circ f_\beta \circ g^{-1} \circ e_\alpha$ for each α and β.

1.10. COROLLARY. *Let A be an object of Δ. If all irreducible $\prod$-splittings (resp., $\amalg$-splittings) of A are compatible, then they are all equivalent. In fact, any two such splittings are conjugate.*

Corollary 1.10 follows directly from Proposition 1.9.

REMARK. Assume that Δ is skeletally small. Let Δ' be the set of isomorphism classes of objects of Δ. Then the conclusion of Corollary 1.10 is equivalent to the assertion that Δ' is a free commutative semigroup under $\prod$ (resp., $\amalg$). (Here, "skeletally small" means that the class of isomorphism classes of objects in the category form a set.)

2. Splittings of minimal algebras. In this section and the next we prove our main results. Most of the discussion is focussed on proving the following two assertions.

(A) The category $\mathcal{P}$ of minimal algebras with positive weights is I-split (Definition (1.1)–(1.4)).

(B) Every object M of $\mathcal{P}$ satisfies the assumptions of Corollary 1.10.

Before proceeding to the details we outline the structure of the proof of (B).

Observe that $\mathrm{Aut}(M) \subseteq \mathrm{End}(M)$ is an open imbedding of an affine Q-group in an affine variety (cf. Remark 3.1, below).

If M is an object of $\mathcal{P}$ and $M \to \prod M_\alpha$ is an isomorphism, then (up to homotopy) $\{e_\alpha\} \subseteq \overline{T} \subseteq \overline{\mathrm{Aut}(M)}$ (Zariski closure) where

$$\begin{array}{ccc}
M & \overset{e_\alpha}{\rightarrow} & M \\
& p_\alpha \searrow \quad \nearrow i_\alpha & \\
& M_\alpha &
\end{array}$$

as in (1.7) and T is a Q-split torus. (This is part of the reason for our categorical strategy of §1.) Thus any splitting adheres to a maximal Q-split torus.

Maximal Q-split tori are conjugate. Thus any two irreducible splittings are compatible; in fact, they are conjugate.

2.1. *Notation and definitions.* A *differential graded algebra* (d.g.a.) is a pair (A, d_A) where $A = \bigoplus_{n \geqslant 0} A^n$ is a nonnegatively graded, associative algebra over Q with identity, such that $a \cdot b = (-1)^{pq} b \cdot a$ for $a \in A^p$ and $b \in A^q$, and d_A is a degree 1 derivation such that $d_A^2 = 0$.

A *morphism* of d.g.a.'s is a graded algebra map which commutes with the differentials.

A d.g.a. A is called *connected* if $A^0 \cong Q$ and *simply-connected* if in addition $A^1 = 0$.

Note that if A is connected then A has a unique augmentation $\varepsilon_A: A \to Q$.

For the purposes of this paper we make the following definition.

A *minimal algebra* is a d.g.a. M such that M is simply-connected, M^n is finite-dimensional for all n, M is free as a graded commutative algebra, and $d_M(M) \subseteq M^+ \cdot M^+$ (the image of d_M is decomposable).

A minimal algebra M has *positive weights* if there is a direct sum decomposition $M = \bigoplus_{\alpha, n \geqslant 0} {}_\alpha M^n$ such that $M^n = \bigoplus_{\alpha \geqslant 0} {}_\alpha M^n$, $d_M({}_\alpha M^n) \subseteq {}_\alpha M^{n+1}$, ${}_\alpha M \cdot {}_\beta M \subseteq {}_{\alpha+\beta} M$ and ${}_0 M = M^0$. Such a direct sum decomposition is called a *positive weight splitting*.

In general a given minimal algebra may possess many distinct positive weight splittings, or none at all.

2.2. Remark. A very satisfying topological interpretation of this condition is given in [4]. One striking feature is the following: Let X be a finitary simply-connected C.W. complex and let M_X be its minimal algebra. Then M_X has positive weights if and only if there is a prime p and a map $\lambda_p: X \to X$ such that $\tilde{\lambda}_p: M_X \to M_X$ is an isomorphism and

$$\lambda_{p*} \otimes 1: \pi_*(X) \otimes \mathbf{Z}/p\mathbf{Z} \to \pi_*(X) \otimes \mathbf{Z}/p\mathbf{Z}$$

is the zero morphism. Furthermore, this is independent of the prime p.

2.3. *The category* $\mathcal{P}$. Let M be a minimal algebra and let $Z(M) = \{x \in M \mid dx = 0\}$, $M^+ = \bigoplus_{k>0} M^k$ and $Q(M) = M^+/(M^+ \cdot M^+)$.

There is a linear map

$$Z^+(M) \xrightarrow[i]{} M^+ \xrightarrow[\pi]{} Q(M)$$

which is inclusion followed by projection (the "Hurewicz map"). Let $S(M) = \operatorname{Image}(\pi \circ i)$.

The category $\mathcal{P}$ has as its objects the minimal algebras M such that M has positive weights (2.1), and $\dim_Q S(M) < \infty$. The morphisms of $\mathcal{P}$ are the d.g.a. homotopy classes of d.g.a. maps.

Note that d.g.a. homotopies can be defined and computed with either a path or a cylinder object. This is, if $f, g: M \to N$ are d.g.a. maps then f is homotopic to g

$(f \simeq g)$ if either

$$\begin{array}{ccc} & & N \otimes I \\ & \overset{F}{\nearrow} & \downarrow (e_0, e_1) \\ M & \underset{(f,g)}{\longrightarrow} & N \times N \end{array}$$

commutes for some d.g.a. map F, or

$$\begin{array}{ccc} M^I & & \\ (\lambda_0, \lambda_1) \uparrow & \searrow G & \\ M \otimes M & \underset{(f,g)}{\longrightarrow} & N \end{array}$$

commutes for some d.g.a. map G.

An explicit development of both these notions and their interrelationship can be found in Chapters 5 and 11 of [**9**].

In order to prove that the category $\mathcal{P}$ is I-split (so that we are able to apply Corollary 1.10) we shall have to exploit the good behaviour of homotopies under the passage to inverse limits (a property peculiar to the rational homotopy categories).

2.4. LEMMA. *Let $f, g\colon M \to N$ be d.g.a. maps, where M is minimal and both M and N are finite type. Then $f \simeq g$ if and only if $f_n \simeq g_n$ for all n, where $f_n = f|M_n$ and M_n is the subalgebra generated by elements of degree $\leqslant n$.*

PROOF. In [**6**] the functors $A\colon \mathbb{S} \to \mathrm{DGA}$ and $F\colon \mathrm{DGA} \times \mathrm{DGA} \to \mathbb{S}$ are defined where $\mathbb{S}$ is the category of simplicial sets.

A is the simplicial de Rham functor and F is the simplicial function space construction for DGA. Further [**6**, p. 25] the adjunction

$$\Phi\colon \mathrm{Hom}_{\mathrm{DGA}}(M, A(W) \otimes N) \underset{\cong}{\to} \mathrm{Hom}_{\mathbb{S}}(W, F(M, N))$$

is proved under mild assumptions (W finite or M finite type). The following facts are then deduced:

(i) If $i\colon M \to M'$ is a cofibration, then $F(M', N) \to F(M, N)$ is a Kan fibration [**6**, p. 26].

(ii) If M is minimal then $[M, N] \cong \pi_0 F(M, N)$ where $[\ ,\ ]$ denotes d.g.a. homotopy classes [**6**, p. 32].

An induction argument (similar to Lemma 2.7 of [**16**]) shows that for M' finitely generated minimal, $\pi_r(F(M', N), f)$ is a Q-nilpotent group for $r \geqslant 1$ (N finite type).

We can write for M finite type minimal $Q \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq \mathrm{ind\,lim}_n M_n = M$.

By (i) above we obtain a tower of Kan fibrations $F(M, N) = \mathrm{proj\,lim}_n F(M_n, N) \to \cdots \to F(M_n, N) \to \cdots \to *$.

If further we let $f_n \in F(M_n, N)$ be the basepoint we have the following exact sequence of sets (see [**7**, p. 254])

$$* \to \mathrm{proj\,lim}^1_n \pi_1(F(M_n, N)) \to \pi_0(F(M, N)) \to \mathrm{proj\,lim}_n \pi_0(F(M_n, N)) \to * .$$

But $\mathrm{proj\,lim}^1_n \pi_1(F(M_n, N)) \cong *$ because the inverse tower of nilpotent groups $\{\pi_1(F(M_n, N), f_n)\}$ satisfies a Mittag-Leffler condition. Thus,

$$[M, N] \cong \pi_0(F(M, N)) \cong \mathrm{proj\,lim}_n \pi_0(F(M_n, N)) \cong \mathrm{proj\,lim}_n [M_n, N].$$

REMARK. The proof above relies on simplicial techniques. Alternatively, a topological argument, based on Lemma 2.7 of **[16]**, can be constructed using elementary obstruction theory. It is curious that no elementary algebraic proof is available.

2.5. PROPOSITION. *The category $\mathcal{P}$ is I-split.*

PROOF. Q, the d.g.a. concentrated in degree zero, is the zero object of $\mathcal{P}$.

$\mathcal{P}$ has finite coproducts. For if M and N are objects of $\mathcal{P}$ then so too is $M \otimes N$. If $M = \bigoplus_{\alpha>0}\, {}_\alpha M$ and $N = \bigoplus_{\beta>0}\, {}_\beta N$ are positive weight splittings of M and N respectively, then $M \otimes N = \bigoplus_{\gamma>0} (\bigoplus_{\alpha+\beta=\gamma} ({}_\alpha M \otimes {}_\beta N))$ is a positive weight splitting of $M \otimes N$ (where ${}_\gamma(M \otimes N) = \bigoplus_{\alpha+\beta=\gamma}\, {}_\alpha M \otimes {}_\beta N$).

$\mathcal{P}$ has finite products. For if M and N are objects of $\mathcal{P}$ let $M \times_Q N$ be the product of simply-connected d.g.a.'s. That is,

$$\begin{array}{ccc} M \times_Q N & \xrightarrow{P'_N} & N \\ {\scriptstyle P'_M}\downarrow & & \downarrow{\scriptstyle \varepsilon_N} \\ M & \xrightarrow[\varepsilon_M]{} & Q \end{array}$$

is a pull-back diagram. $M \times_Q N$ is not minimal, but by Theorem 6.1 of **[9]** there is a minimal algebra $M \amalg N$ and a map ρ: $M \amalg N \to M \times_Q N$ such that ρ^*: $H^*(M \amalg N) \to H^*(M \times_Q N)$ is an isomorphism.

Thus we have

$$\begin{array}{ccccccc} M \times_Q N & \xleftarrow{\rho} & & M \amalg N & & \xrightarrow{\rho} & M \times_Q N \\ {\scriptstyle P'_M}\downarrow & & \swarrow {\scriptstyle P_M} & & {\scriptstyle P_N} \searrow & & \downarrow {\scriptstyle P'_N} \\ M & & & & & & N \end{array}$$

where P_M and P_N are defined by the commutativity of the diagram.

If L is a minimal algebra and f: $L \to M$ and g: $L \to N$ are d.g.a. maps, then there is a unique d.g.a. map ϕ: $L \to M \times_Q N$ such that

$$P'_M \circ \phi = f \text{ and } P'_N \circ \phi = g.$$

By Proposition 6.4 of **[6]** the lifting problem

$$\begin{array}{ccc} & & M \amalg N \\ & \nearrow & \downarrow \\ L & \xrightarrow[\phi]{} & M \times_Q N \end{array}$$

has a solution (f, g): $L \to M \amalg N$ unique up to d.g.a. homotopy.

If $f \simeq f'$ and $g \simeq g'$ then $(f, g) \simeq (f', g')$. For let

$$F\colon L^I \to M, \qquad F\colon f \simeq f',$$

$$G\colon L^I \to N, \qquad G\colon g \simeq g'.$$

Then $L^I \xrightarrow[\Delta]{} L^I \times L^I \xrightarrow[F \times_Q G]{} M \times_Q N$ is a homotopy: $(f, g) \simeq (f', g')$.

Thus

$$\begin{array}{ccccc} & & M \amalg N & & \\ & {\scriptstyle P_M} \swarrow & & \searrow {\scriptstyle P_N} & \\ M & & & & N \end{array}$$

is a product diagram in the homotopy category of minimal algebras. $M \amalg N$ is an object of $\mathcal{P}$ since $S(M \amalg N) \cong S(M) \oplus S(N)$ and by Lemma 3.13 below we can "lift positive weight splittings over weak equivalences" in this situation. (If $M = \bigoplus_{\alpha \geqslant 0} {}_{\alpha}M$ and $N = \bigoplus_{\beta \geqslant 0} {}_{\beta}N$ are positive weight splittings, then $M \times_Q N = Q \oplus (\bigoplus_{\gamma > 0}({}_{\gamma}M \oplus_{\gamma} N))$ is a positive weight splitting of $M \times_Q N$. Now apply Lemma 3.13 to the map ρ: $M \amalg N \to M \times_Q N$.)

If M is an object of $\mathcal{P}$ and $f \in \operatorname{Hom}_{\mathcal{P}}(M, N)$ satisfies $f \circ f = f$, then there is a factorization (in $\mathcal{P}$)

$$\begin{array}{ccccc} M & & \underset{f}{\rightarrow} & & M \\ & p \searrow & & \nearrow i & \\ & & N & & \end{array}$$

such that $p \circ i = 1_N$. This follows from 3.12 below.

If M is an object of $\mathcal{P}$ then by definition (see 2.3) $\dim_Q S(M) < \infty$. So if $M \cong M_1 \amalg M_2$ (or $M_1 \amalg M_2$) then $S(M) \cong S(M_1) \oplus S(M_2)$. Also if $S(M) \cong (0)$ then $M \cong Q$. Thus put $n(M) = \dim_Q S(M)$ and M will satisfy the conditions of (1.4).

Summing up, we have shown that $\mathcal{P}$ satisfies conditions (1.1)–(1.4). Q.E.D.

Because of Proposition 2.5 our questions about unique factorization in $\mathcal{P}$ have been reduced to the study of conjugacy properties of idempotents (by Corollary 1.10). This necessitates the study of the topological properties of $\operatorname{Aut}(M) \subseteq \operatorname{End}(M)$ for each object M of $\mathcal{P}$.

3. Algebraic groups of automorphisms. If A is a finitely generated d.g.a. then $\operatorname{End}(A)$ is an affine Q-variety and $\operatorname{Aut}(A)$ is an affine Q-group. Further, $\operatorname{Aut}(A) \subseteq \operatorname{End}(A)$ is an open imbedding.

This follows from the following elementary observations:

$\operatorname{End}(A)$ is faithfully represented on a finite dimensional subspace of A.

Preservation of the multiplicative structure is a quadratic relation (hence algebraic) and commutation with d_A is a linear relation.

$\operatorname{Aut}(A) = \det^{-1}(Q \setminus \{0\})$ is open because det: $\operatorname{End}(A) \to Q$ is continuous.

3.1. REMARK. Properly speaking $\operatorname{Aut}(A)$ and $\operatorname{End}(A)$ are not algebraic varieties, but are the Q-rational points of the Q-varieties $\operatorname{Aut}(A \otimes_Q K)$ and $\operatorname{End}(A \otimes_Q K)$, respectively, where K is an algebraic closure of Q. This is the point of view adopted in [5] which is our basic reference for the theory of algebraic groups.

Our slight heresy shall not cause difficulties. (To translate from our terminology to that of [5], one need only insert the words, "the Q-rational points of", where appropriate.)

(3.2) A positive weight splitting on a finitely generated minimal algebra $M = \bigoplus_{\alpha \geqslant 0} {}_{\alpha}M$ induces a morphism of Q-algebraic groups (Q-group) λ: $Q^* \to \operatorname{Aut}(M)$ where $\lambda(t)(x) = t^{\alpha} \cdot x$ for $x \in {}_{\alpha}M$, and $t \in Q^*$. Further, λ extends to a morphism of varieties $\bar{\lambda}$: $Q \to \operatorname{End}(M)$ such that $\bar{\lambda}(0) = O_M$.

Thus, if M is an object of $\mathcal{P}$, then $O_M \in \overline{\operatorname{Aut}(M)}$ (Zariski closure) if M is finitely generated.

Conversely, if $O_M \in \overline{\operatorname{Aut}(M)}$ then M is in $\mathcal{P}$ (for M finitely generated).

The proof of this assertion will only be outlined. A general discussion of toroidal symmetry of minimal algebras is advanced in **[14]**.

Step 1. If $O_M \in \overline{\mathrm{Aut}(M)}$ then $O_M \in \overline{T}$ where $T \subseteq \mathrm{Aut}(M)$ is any maximal Q-split torus. This follows from the following more general result about certain representation of algebraic groups.

Let $G \subseteq \mathrm{Gl}(V) \subseteq \mathrm{Hom}_Q(V, V)$ be an algebraic subgroup such that $O_V \in \overline{G}$. Then $O_V \in \overline{T}$ where $T \subseteq G$ is any maximal Q-split torus. A detailed proof is contained in **[14]**.

Step 2. If $O_V \in \overline{T}$ (as above) then there is a one-parameter subgroup (1-p.s.g.) λ: $Q^* \to T$ such that λ extends to $\bar{\lambda}$: $Q \to \overline{T}$ with $\bar{\lambda}(0) = O_V$.

This follows from the following result proved in **[11]**.

If $T \to \overline{T}$ is a toroidal imbedding ($T \to \overline{T}$ is an open dominant imbedding and, $T \times T \to T$ extends to an action $T \times \overline{T} \to \overline{T}$), then for any orbit X in $\overline{T} \setminus T$ there is a 1-p.s.g. λ: $Q^* \to T$ extending to $\bar{\lambda}$: $Q \to \overline{T}$ with $\bar{\lambda}(0) \in X$.

Thus, the assertion above follows because $O_V = T \cdot O_V$ is the unique point in its orbit.

Step 3. If λ: $Q^* \to T \subseteq \mathrm{Aut}(M)$ is a 1-p.s.g. that extends to $\bar{\lambda}$: $Q \to \mathrm{End}(M)$ with $\lambda(0) = O_M$, then this induces a positive weight splitting on M.

This requires an elementary proof from linear algebra. For each $t \in Q^*$, $\lambda(t)$ is Q-diagonalizable with eigenvalues t^α for various integers α. Thus $M = \bigoplus {}_\alpha M$, where ${}_\alpha M$ is the eigenspace of $\lambda(t)$ with eigenvalue t^α. All weights are nonnegative because λ extends to $\bar{\lambda}$. ${}_0M = M^0$ because $\bar{\lambda}(0) = O_M$.

Thus, we have established the following characterization of finitely generated minimal algebras M with positive weights:

M has positive weights if and only if $O_M \in \overline{\mathrm{Aut}(M)}$ (Zariski closure) within $\mathrm{End}(M)$.

3.3. Proposition. *If M is a finitely generated object of $\mathcal{P}$ and N is a retract of M, then N is in $\mathcal{P}$.*

Proof. From the remarks above, we need only prove that $O_N \in \overline{\mathrm{Aut}(N)}$ (because retracts of minimal algebras are minimal).

Let $N \xrightarrow[i]{} M \xrightarrow[r]{} N$ satisfy $r \circ i = 1_N$. This induces

$$\mathrm{End}(M) \xrightarrow[\phi]{} \mathrm{End}(N), \qquad \phi(f) = r \circ f \circ i,$$

a morphism of varieties. Now $O_M \in \overline{\mathrm{Aut}(M)}$. Thus $O_M \in \overline{(\mathrm{Aut}(M))^0}$ (the irreducible component of 1). $\phi((\mathrm{Aut}(M))^0) \subseteq \mathrm{End}(N)$ is irreducible, as is its closure, $\overline{\phi((\mathrm{Aut}(M))^0)}$. Moreover,

$$1_N \in (\mathrm{Aut}(N)) \cap \overline{\phi((\mathrm{Aut}(M))^0)} = S.$$

Further, $S \subseteq \overline{\phi((\mathrm{Aut}(M))^0)}$ is open. Thus,

$$O_N \in \phi\left(\overline{(\mathrm{Aut}(M))^0}\right) \subseteq \overline{\phi((\mathrm{Aut}(M))^0)} = \overline{S}.$$

Therefore, $O_N \in \overline{\mathrm{Aut}(N)}$. Q.E.D.

Thus, to complete the proof of Proposition 2.5 we still have to prove the following two assertions.

(3.4) (Homotopy) idempotents split in $\mathcal{P}$.

(3.5) Proposition 3.3 is true without assuming M is finitely generated. (Thus $\mathcal{P}$ is closed under retracts.)

Assertion (3.4) follows easily with the aid of Lemma 2.4. Let e: $M \to M$ represent an idempotent homotopy class (thus, $e \simeq e \circ e$). Now let

$$N = \bigcap_{n \geq 1} \text{Image } e^n \quad \text{and} \quad J = \bigcup_{n \geq 1} \text{kernel } e^n.$$

Then $N \subseteq M$ is a d.g. subalgebra and $J \subseteq M$ is a d.g. ideal. Futher $M = N \oplus J$. Thus

$$N \underset{i}{\rightarrowtail} M \underset{p}{\twoheadrightarrow} N \text{ is a retract.}$$

A routine verification shows that

$$i \circ p|_{M_n} \simeq e|_{M_n} \quad \text{for all } n \geq 0.$$

By Lemma 2.4 $i \circ p \simeq e$.

The proof of assertion (3.5) requires the following discussion of pro-algebraic groups.

(3.6) If M is not finitely generated, then $\text{Aut}(M)$ and $\text{End}(M)$ are no longer algebraic varieties. However, the situation can be remedied as follows. Every object M of $\mathcal{P}$ has a canonical series

$$Q = M_1 \subseteq M_2 \subseteq \cdots \subseteq \text{ind lim}_n M_n = M$$

(where M_n is the subalgebra generated by elements of degree $< n$) which induces

$$\begin{array}{ccccc} \text{Aut}(M_2) & \leftarrow \cdots \leftarrow & \text{proj lim}_n \text{Aut}(M_n) & = & \text{Aut}(M) \\ \cap & & \cap & & \cap \\ \text{End}(M_2) & \leftarrow \cdots \leftarrow & \text{proj lim}_n \text{End}(M_n) & = & \text{End}(M). \end{array}$$

Thus $\text{Aut}(M)$ and $\text{End}(M)$ are pro-algebraic varieties (inverse limits of algebraic varieties). In order to make use of this observation we shall need the following facts from algebraic group theory and general topology.

Let G be a Q-group. If $\mathcal{W}_G = \{F | F$ is a finite union of cosets of Q-closed subgroups of $G\} \cup \{\varnothing\}$ then $\mathcal{W}_G$ is the set of closed sets for a T_1 compact topology on G, the *$\mathcal{W}$-topology*. If ϕ: $G \to H$ is a morphism of Q-groups then ϕ is continuous for the $\mathcal{W}$-topology. If Kernel ϕ is a unipotent subgroup of G then $\phi(G)$ is a Q-closed subgroup of H [**5**, p. 363]. Consequently, for such ϕ: $G \to H$, ϕ is a closed map in the $\mathcal{W}$-topology.

3.7. Projective Limit Theorem. *If $\{X_i, \pi_{ij}\}$ is an inverse system of topological spaces and continuous maps such that*

(a) *for each i, $X_i \neq \varnothing$ is compact and T_1,*

(b) *π_{ij}: $X_i \to X_j$ is closed for each $i \geq j$,*

then proj $\lim_i X_i$ *is nonempty.*

Such a system will be called a *proper system* and proj lim X_i will be called a *proper limit*. For a proof, see [**14**, Theorem 1.2.1] or [**15**, Theorem 3].

(3.8) If M is a minimal algebra, then the natural map $S\colon \mathrm{Aut}(M) \to \mathrm{Aut}(S(M))$ has a locally unipotent kernel. That is, for all $\phi \in$ Kernel S and all $x \in M$ there exists n_x such that $(\phi - 1)^{n_x}(x) = 0$.

This is easily proved by induction along the canonical series of M. See for example [**17**]. Thus we have

$$\begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ \mathrm{Aut}(M_n) & \xrightarrow{S} & \mathrm{Aut}(S(M_n)) \\ \downarrow & & \downarrow \\ \mathrm{Aut}(M_{n-1}) & \xrightarrow{S} & \mathrm{Aut}(S(M_{n-1})) \\ \downarrow & & \downarrow \\ \vdots & & \vdots \end{array}$$

If $S(M) = S(M_N)$ for some N (for example, if M is an object of $\mathcal{P}$) we have

$$\begin{array}{ccc} \mathrm{Aut}(M_{N+k}) & & \\ {\scriptstyle r_{N+k,N}}\downarrow & \searrow & \\ \mathrm{Aut}(M_N) & \underset{S}{\to} & \mathrm{Aut}(S(M_N)) \end{array}$$

for all $k \geqslant 0$. Hence, Ker $r_{N+k,N}$ is unipotent.

3.9. Proposition. *Let M be a minimal algebra such that* $\dim_Q S(M) < \infty$ *and let* $G_n = \mathrm{Aut}(M_n)$. *Then for sufficiently large integers N,*

$$* \leftarrow G_N \leftarrow G_{N+1} \leftarrow \cdots$$

is a proper system of topological spaces for the $\mathfrak{W}$-topology.

Proof. (3.6), 3.7 and (3.8) above.

Remark. If $A_k \subseteq G_k$ are nonempty $\mathfrak{W}$-closed sets and $r_{k,l}(A_k) \subseteq A_l$, then $\{A_k, r_{k,l}|_{A_k}\}$ is also a proper system.

Much of the structure theory of Q-groups remains valid for proper limits of Q-groups [**14**]. In particular, the following theorem is central to our discussion of splitting idempotents of minimal algebras.

3.10. Theorem. *If $\{G_i, \pi_{ij}\}$ is a proper system of Q-groups, then there exists $T \subseteq G =$ proj lim G_i such that*

(a) *T is a closed subgroup of G (in the $\mathfrak{W}$-topology).*

(b) *$\pi_i(T) \subseteq G_i$ is a Q-split torus for all i.*

(c) *T is maximal with respect to conditions* (a) *and* (b).

If $S \subseteq G$ satisfies (a), (b) *and* (c) *then there exists $g \in G$ such that $gSg^{-1} = T$.*

Proof. In the algebraic case ($G_i = G_j$ for all i and j) this is a standard result (see [**5**, p. 263]). The general case follows from two applications of the Projective Limit Theorem 3.7.

Assume for the moment that $\pi_{ij}\colon G_i \to G_j$ is onto for $i > j$. Then each π_{ij} induces $\mathfrak{T}(\pi_{ij})\colon \mathfrak{T}(G_i) \to \mathfrak{T}(G_j)$ where $\mathfrak{T}(G_i) = \{T \subseteq G_i | T \text{ is a maximal } Q\text{-split torus}\}$. $\mathfrak{T}(G_i)$ can be given the stucture of a homogeneous space over G_i in such a way that $\{\mathfrak{T}(G_i), \mathfrak{T}(\pi_{ij})\}$ is a proper system. Thus

$$\text{proj lim } \mathfrak{T}(G_i) \text{ is nonempty.}$$

Given $\{T_i\} \in \text{proj lim } \mathfrak{T}(G_i)$ it is a routine verification to show that $T = \text{proj lim } T_i$ satisfies (a), (b) and (c).

If $S \subseteq \text{proj lim } G_i$ satisfies (a), (b) and (c) then $S_i = \pi_i(S) \subseteq G_i$ is a maximal Q-split torus for each i, and $S = \text{proj lim } \pi_i(S)$. But

$$\mathrm{Tran}_{G_i}(S_i, T_i) = \{ g \in G_i | \, g S_i g^{-1} = T_i \}$$

is closed and nonempty for all i. Thus $\mathrm{Tran}_G(S, T) = \text{proj lim } \mathrm{Tran}_{G_i}(S_i, T_i)$ is nonempty by the Projective Limit Theorem 3.7.

Hence, the proof is complete once we have justified the assumption that each $\pi_{ij}\colon G_i \to G_j$ is onto.

Observe that $G \cong \text{proj lim } \pi_i(G)$ and that $\pi_i(G) = \bigcap_{j > i} \pi_{ji}(G_j)$.

Thus G is a proper limit of the proper system $\{\pi_i(G), \pi_{ij}\}$. Q.E.D.

For the first application of Theorem 3.10 we complete the proof of the Proposition 2.5.

3.11. Proposition. *M is an object of $\mathcal{P}$ if and only if M_n is an object of $\mathcal{P}$ for all n and* $\dim_Q S(M) < \infty$.

Proof. If M is in $\mathcal{P}$, then any positive weight splitting on M restricts to a positive weight splitting on M_n.

Conversely, if M_n is in $\mathcal{P}$ for all n and $\dim_Q S(M) < \infty$ we shall construct a 1-p.s.g. $\lambda\colon Q^* \to \mathrm{Aut}(M)$ such that λ extends to $\bar{\lambda}\colon Q \to \mathrm{Aut}\, M$ with $\bar{\lambda}(0) = O_M$.

By Theorem 3.10 and Proposition 3.9 there exist tori $T_i \subseteq \mathrm{Aut}\, M_i$ for each $i > 0$ such that $\text{proj lim } T_i = T$ is a maximal torus in $\mathrm{Aut}(M)$. Since tori are faithfully represented on $S(M)$ (which is finite dimensional) there exists N such that $\dim T_n = \dim T_N$ and $S(M_n) = S(M_N)$ for all $n > N$. Since $\ker r_{k,N}$ is unipotent for $k > N$ it follows that $r_{i,j}\colon T_i \to T_j$ is an isomorphism for $i > j > N$.

By assumption that exist $\lambda_i\colon Q^* \to T_i$ $(i > N)$ such that λ_i extends to $\bar{\lambda}_i\colon Q \to \overline{T_i}$ (Zariski closure) with $\bar{\lambda}_i(0) = O_{M_i}$.

For $i > N$ we have

$$\begin{tikzcd}
& \vdots \arrow[d, "\simeq"] & & \vdots \arrow[d] \\
& T_{i+2} \arrow[d, "\simeq"] & \subseteq & \mathrm{Aut}(M_{i+2}) \arrow[d] \\
& T_{i+1} \arrow[d, "\simeq"] & \subseteq & \mathrm{Aut}(M_{i+1}) \arrow[d] \\
Q^* \arrow[r] \arrow[ur, dashed] \arrow[uur, dashed] & T_i & \subseteq & \mathrm{Aut}(M_i)
\end{tikzcd}$$

Thus, there exist unique liftings which make the whole diagram commute. Since $S^n(M) = 0$ for all $n > N$, $\lambda = \text{proj lim } \lambda_i$ extends to $\bar{\lambda}$: $Q \to \bar{T} = \text{proj lim } \overline{T_i}$ with $\bar{\lambda}(0) = O_M$ (since all eigenvalues of λ are multiplicatively generated on $S(M)$). Q.E.D.

3.12. Conclusion of Proposition 2.5. *Suppose $N \underset{i}{\to} M \underset{r}{\to} N$ satisfies $r \circ i = 1_N$, where M is in $\mathcal{P}$. Then $N_n \underset{i_n}{\to} M_n \underset{r_n}{\to} N_n$, $r_n \circ i_n = 1_{N_n}$. By Proposition* 3.3, *N_n is in $\mathcal{P}$ for all n. Then by Proposition* 3.11, *N is in $\mathcal{P}$.*

Hence we have completed the proof that $\mathcal{P}$ is an I-split category.

In order to prove unique factorization in $\mathcal{P}$ we need to prove that any two irreducible Π-splittings (resp., $\amalg$-splittings) of a fixed object M of $\mathcal{P}$ are compatible. This too we deduce with the aid of Proposition 2.5.

Let A be a simply-connected d.g.a. of finite type such that ρ: $M_A \to A$ is onto, where M_A is minimal and ρ^*: $H^*(M_A) \to H^*(A)$ is an isomorphism. Let $T \subseteq \text{Aut}(A)$ be a Q-split torus and let

$$B = \{(\alpha, \beta) \in (\text{Aut}(M_A)) \times T | \rho \circ \alpha = \beta \circ \rho\}.$$

3.13. Lemma. *In the pull-back diagram*

$$\begin{array}{ccc} B & \xrightarrow{r} & T \\ q\downarrow & & \downarrow\rho^* \\ \text{Aut}(M_A) & \underset{\rho_*}{\to} & \text{Hom}(M_A, A) \end{array}$$

(a) *q is one-to-one.*
(b) *r is onto.*
(c) *$B \xrightarrow{r} T$ splits as a morphism of Q-groups.*

Proof. If $q(\alpha, \beta) = q(\alpha, \beta')$ then $\beta \circ \rho = \beta' \circ \rho$. Thus $\beta = \beta'$ because ρ is onto. This proves (a).

Let $\beta \in T$. Then in

$$\begin{array}{ccc} M_A & \dashrightarrow & M_A \\ \rho\downarrow & & \downarrow\rho \\ A & \underset{\beta}{\to} & A \end{array}$$

the dotted arrow exists because ρ is a fibration and a weak equivalence. (See [**6**].) This proves (b).

$$\text{Kernel}(r) = \{(\alpha, \beta) | \beta = 1\}.$$

That is,

$$\begin{array}{ccc} M_A & \xrightarrow{\alpha} & M_A \\ \rho\downarrow & & \downarrow\rho \\ A & \underset{1}{\to} & A \end{array} \quad \text{if } \alpha \in \text{Kernel}(r).$$

Thus $\alpha \simeq 1$ because the lifting problem has a solution unique up to homotopy. In particular, Kernel(r) is unipotent. Thus by the Proposition on p. 219 of [5], $B \underset{r}{\twoheadrightarrow} T$ splits because B is a Q-split Q-group. Q.E.D.

Let M be an object of $\mathcal{P}$ and suppose $M \underset{(p_\alpha)}{\to} \Pi M_\alpha$ is an isomorphism (in $\mathcal{P}$, where morphisms are homotopy classes, as in 2.3). Let $\{g_\alpha\} \subseteq \mathrm{End}_{\mathcal{P}}(M)$ be the corresponding splitting idempotents (since $\mathcal{P}$ is I-split). Thus

$$\begin{array}{ccc} M & \xrightarrow{g_\alpha} & M \\ & p_\alpha \searrow \quad \nearrow i_\alpha & \\ & M_\alpha & \end{array}$$

commutes in $\mathcal{P}$ and $g_\alpha = g_\alpha \circ g_\alpha$ in $\mathcal{P}$ for all α. So if we choose d.g.a. maps $\{e_\alpha\} \subseteq \mathrm{End}(M)$, e_α representing g_α for each α, we have $e_\alpha \simeq e_\alpha \circ e_\alpha$.

3.14. LEMMA. *Let M and $\{e_\alpha\}$ be as above. Then there exist a Q-split torus $T \subseteq \mathrm{Aut}(M)$ and a collection $\{e'_\alpha\} \subseteq \overline{T}$ such that*
(a) *$e'_\alpha \simeq e_\alpha$ for all α, and*
(b) *$e'_\alpha = e'_\alpha \circ e'_\alpha$ for all α.*

PROOF. We have a commutative diagram in $\mathcal{P}$

$$\begin{array}{ccc} & \Pi M_\alpha & \\ (p_\alpha) \nearrow & & \searrow \\ M & \underset{(p'_\alpha)}{\longrightarrow} & \times_Q M_\alpha \end{array}$$

and a homotopy-commutative diagram of d.g.a. morphisms

$$\begin{array}{ccc} M & \xrightarrow{e_\alpha} & M \\ \rho\downarrow & & \downarrow\rho \\ \times_Q M_\alpha & \underset{f_\alpha}{\longrightarrow} & \times_Q M_\alpha \end{array}$$

where f_α is projection followed by inclusion and ρ represents (p'_α). Define

$$\begin{array}{ccc} Q^* \times \cdots \times Q^* & \xrightarrow{\lambda} & \mathrm{Aut}(\times_Q M_\alpha) \\ \Pi\lambda_\alpha \searrow & & \nearrow j \\ & \Pi \mathrm{Aut}(M_\alpha) & \end{array}$$

as follows. Let $M_\alpha = \bigoplus_{k>0} {}_k(M_\alpha)$ be a positive weight splitting of M_α and let $\lambda_\alpha(t)(x) = t^k \cdot x$ for $x \in {}_k(M_\alpha)$. $\Pi\lambda_\alpha$ and j are the obvious maps.

By Lemma 3.13 there is a Q-split torus $T \subseteq \mathrm{Aut}(M)$ such that the lifting problem

$$\begin{array}{ccc} M & \dashrightarrow & M \\ \rho\downarrow & & \downarrow\rho \\ \times_Q M_\alpha & \underset{\beta}{\longrightarrow} & \times_Q M_\alpha \end{array}$$

has a unique solution in T for each $\beta \in \lambda(Q^* \times \cdots \times Q^*) = S$.

Since each weight splitting is nonnegative, $\{f_\alpha\} \subseteq \overline{S} \subseteq \mathrm{Aut}(\times_Q M_\alpha)$ and there are 1-p.s.g.'s ω_α: $Q^* \to S$ which extend to $\overline{\omega_\alpha}$: $Q \to \overline{S}$ with $\overline{\omega_\alpha}(0) = f_\alpha$. Thus each

1-p.s.g. $\omega_\alpha\colon Q^* \to S$ has a unique lifting

$$\begin{tikzcd} & T \arrow[d, "\simeq"] \\ Q^* \arrow[ur, "\omega'_\alpha"] \arrow[r, "\omega_\alpha"'] & S \end{tikzcd}$$

Each ω'_α extends to $\overline{\omega'_\alpha}\colon Q \to \overline{T}$ again because the weight splittings are nonnegative.

Let $e'_\alpha = \overline{\omega'_\alpha(0)}$ for each α. Then we have the following strictly commutative diagram of d.g.a. morphisms:

$$\begin{tikzcd} M \arrow[r, "e'_\alpha"] \arrow[d, "\rho"'] & M \arrow[d, "\rho"] \\ \times_Q M_\alpha \arrow[r, "f_\alpha"'] & \times_Q M_\alpha. \end{tikzcd}$$

Then $e_\alpha \simeq e'_\alpha$ because they both solve the same lifting problem and $e'_\alpha = e'_\alpha \circ e'_\alpha$ because $e'_\alpha \in \overline{T}$. Q.E.D.

Thus for any irreducible $\prod$-splitting $\{g_\alpha\}$ of M we can find representative $\{e_\alpha\} \subseteq \operatorname{End}(M)$ and a maximal Q-split torus $T \subseteq \operatorname{Aut}(M)$ such that $\{e_\alpha\} \subseteq \overline{T}$. Another irreducible $\prod$-splitting $\{h_\alpha\}$ yields a collection $\{f_\alpha\}$ of representatives and a maximal Q-split torus $S \subseteq \operatorname{Aut}(M)$ such that $\{f_\alpha\} \subseteq \overline{S}$. By Theorem 3.10 there exists $x \in \operatorname{Aut}(M)$ such that $x \circ \overline{T} \circ x^{-1} = \overline{S}$. Thus in the category $\mathcal{P}$ the two splittings are compatible. By Corollary 1.10 the two splittings are equivalent. Thus each object M of $\mathcal{P}$ satisfies unique factorization with respect to the formation of products. The dual statement (coproducts) is somewhat easier as no lifting lemmas are needed (the coproduct does not have to be constructed). The details will be left to the reader.

3.15. THEOREM. *Let M be an object of $\mathcal{P}$. Then any two irreducible $\prod$-splittings of M are equivalent.*

3.16. THEOREM. *Let M be an object of $\mathcal{P}$. Then any two irreducible $\coprod$-splittings of M are equivalent.*

REMARK. The question of unique factorization for p-local (or p-adic) positive weight spaces remains open. There is an added challenge to this problem arising from the fact that there exists no p-local analogue to the algebraic nature of rational homotopy types.

REFERENCES

1. R. Body and R. Douglas, *Tensor products of graded algebras and unique factorization*, Amer. J. Math. **101** (1979), 909–914.

2. ______, *Rational homotopy and unique factorization*, Pacific J. Math. **68** (1977).

3. ______, *Unique factorization of rational homotopy types*, Pacific J. Math. (to appear).

4. R. Body and D. Sullivan, *Zariski dynamics of a homotopy type* (unpublished notes).

5. A. Borel, *Linear algebraic groups*, Benjamin, New York, 1969.

6. A. Bousfield and V. Gugenheim, *On PL de Rham theory and rational homotopy type*, Mem. Amer. Math. Soc., No. 179 (1976).

7. A. Bousfield and D. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Math., vol. 304, Springer-Verlag, New York, 1972.

8. A. Deschner, *Sullivan's theory of minimal models*, Thesis, Univ. of British Columbia, 1976.

9. S. Halperin, *Lecture notes on minimal models*, Université des Sciences et Techniques de Lille, 1977.

10. P. Hilton, G. Mislin and J. Roitberg, *Localization of nilpotent spaces and groups*, North-Holland Math. Studies, no. 15, North-Holland, Amsterdam, 1975.

11. G. Kempf, F. Knudsen, D. Mumford and B. Saint-Donat, *Toroidal imbeddings*. I, Lecture Notes in Math., vol. 339, Springer-Verlag, New York, 1973.

12. M. Mimura and H. Toda, *On p-equivalences and p-universal spaces*, Comment. Math. Helv. **4** (1971), 87–97.

13. J. Morgan, *The algebraic topology of smooth algebraic varieties*, Inst. Hautes Études Sci. Publ. Math. **48** (1978), 137–204.

14. L. Renner, *Automorphism groups of minimal algebras*, Thesis, Univ. of British Columbia, 1978.

15. A. Stone, *Inverse limits of compact spaces*, General Topology Appl. **10** (1979), 204–211.

16. D. Sullivan, *Genetics of homotopy theory and the Adams conjecture*, Ann. of Math. (2) **100** (1974), 1–79.

17. ______, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. **47** (1977) 269–332.

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