NEIGHBORHOOD FIXED PENDANT VERTICES

BY

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ABSTRACT. If x is pendant in G , then x^* denotes the unique vertex of G adjacent to x . Such an x is said to be *neighborhood-fixed* whenever x^* is fixed by $A(G - x)$. It is shown that if G is not a tree and has a pendant vertex, but no $*$ -fixed pendant vertex, then there is a subgraph $G^\#$ of G such that for some $y \in V(G^\#)$, $O(A(G^\#)_y) > t!$ where t is the maximum number of edges in a tree rooted in $G^\#$.

Let G denote a finite connected graph without loops or multiple edges. Let $V(G)$, $E(G)$ and $A(G)$ denote respectively the vertex set, the edge set, and the automorphism group of G . Let $x \in V(G)$. The valency of x in G is denoted by $\text{val}(G, x)$, and x is defined to be pendant in G if $\text{val}(G, x) = 1$. The subgraph of G obtained by deleting x and all edges incident with x is denoted by $G - x$. If x is pendant in G , then x^* denotes the unique vertex of G adjacent to x . Such an x is said to be *neighborhood-fixed* whenever x^* is fixed by $A(G - x)$. Neighborhood-fixed will be denoted by $*$ -fixed for the remainder of the paper.

1. Tree growth number. A *pruning* of G is a decomposition:

$$G = G^\# \cup T_1 \cup T_2 \cup \cdots \cup T_k,$$

where $G^\#$ is the maximal subgraph of G , each vertex of which has valency ≥ 2 , and where $\{T_i: 1 \leq i \leq k\}$ is a nonempty set of disjoint nontrivial rooted trees each having only its root vertex x_i in $G^\#$. Note that G has a pruning if and only if it is not a tree and has a pendant vertex. Moreover, $G^\#$ is connected because G is connected, and the decomposition is unique up to the order of the T_i .

In [1], Robertson and Zimmer conjectured: *If G is a finite graph having a pruning such that*

$$\max\{|E(T_i)|: 1 \leq i \leq k\} > \max\{\text{val}(G^\#, x): x \in V(G^\#)\}$$

then G has a $$ -fixed pendant vertex.*

There are counterexamples to the conjecture. One class of counterexamples is found among G with $G^\#$ a rooted tree with triangles affixed to the pendant vertices.

FIGURE 1

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Another counterexample can be constructed from Tutte's 8-cage, mentioned in [2]. Let $G^\#$ be the 8-cage. There exists a subset $\{x_0, x_1, x_2, x_3, x_4\}$ of $V(G^\#)$ on which $A(G^\#)$ acts as the symmetric group S_5 . Then a pruning $G = G^\# \cup T_1 \cup T_2 \cup T_3 \cup T_4$, where T_i is an i -star with center the root x_i for $1 \leq i \leq 4$, defines a graph G with no $*$ -fixed pendant vertex.

Let A be a finite group acting on a finite set X . Let $Y \subseteq X$ and $x \in X$. Denote the order of A by $o(A)$, the subgroup of A fixing x by A_x , and the subgroup of A fixing Y setwise by $A_{(Y)}$. The restriction of $A_{(Y)}$ to Y is denoted by A^Y . Denote the group generated by elements $\varphi_1, \varphi_2, \dots, \varphi_k$ by $\langle \varphi_1, \dots, \varphi_k \rangle$ and the group generated by subgroups $H_1, \dots, H_k$ by $\langle H_1, \dots, H_k \rangle$.

An ordered partition $X = \bigcup_{i=0}^s X_i$ is *reducible* if $s > 1$, $X_s \neq \emptyset$, and for every $x \in X_i$, $i \geq 1$, there exists a *reducing* map $\alpha \in A$ such that $(x)\alpha \in X_{i-1}$ and α stabilizes the blocks of the partition $X_0 \cup \dots \cup (X_{i-1} \cup x) \cup (X_i - x) \cup \dots \cup X_s$. Let $t(A, X)$ denote the largest s such that a reducible partition exists.

Note that for $x \in X_i$, $i \geq 1$, there may be many maps that reduce x . Let α_x be one such reducing map. If $Z \subset X_i$, $i \geq 1$, let $I(Z) = \{(z)\alpha_z : z \in Z\}$. If H is a subgroup of A that contains the group generated by the reducing maps $\{\alpha_y : y \in Y\}$ then H *reduces* Y . An orbit O of a subgroup H of A is a *reducing orbit* if there exists a $z \in O$ such that $\alpha_z \in H$.

Let G be a graph with a pruning. Suppose that G has no $*$ -fixed pendant vertex. Let T_i , $1 \leq i \leq k$, be the trees of the pruning of G . Replace each tree T_i by a star S_i with $|E(S_i)| = |E(T_i)|$ rooted at its center x_i to form a new graph G' with a pruning. Then G' also has no $*$ -fixed pendant vertex. To see this, suppose on the contrary that $x \in V(S_i)$ is a $*$ -fixed pendant vertex. Then x_i is fixed by $A(G' - x_i)$. Let the weight, $w(A)$, of $A \in E(T_i)$ be the number of edges in the component of $(T_i)_A$ (formed by deleting A from T_i), which does not contain x_i . Suppose $(x_i, A_1, a_1, A_2, a_2, \dots, a_{k-1}, A_k, x)$ is a path in T_i from x_i to a pendant vertex x following successive lightest possible edges. Then x is a $*$ -fixed pendant vertex of G , contrary to assumptions.

G will be assumed to have a pruning such that $G^\#$ is vertex-transitive and such that T_i , $1 \leq i \leq k$, is a star rooted at its center x_i . If G has a pruning and no $*$ -fixed pendant vertex, the star roots induce a reducible partition of $V(G^\#)$ under $A(G^\#)$ by the rule: $X_j = \{x : x \text{ is the root of a } j\text{-star}\}$. Conversely, any reducible partition of $V(G^\#)$ leads to a graph G with a pruning which induces the given reducible partition. Because of this equivalence $t(A(G^\#), V(G^\#))$ is termed the *tree growth number* of $G^\#$.

2. Lemmas on permutation groups.

LEMMA 1. Let A be a group acting on a set X with a reducible partition $X = \bigcup_{i=0}^s X_i$. Let $Z \subseteq X_i$, for some $i \geq 1$, and let H be a subgroup such that either (1) $H \subset A_{(X_i)}$ where $Z \subseteq X'_i \subseteq X_i$ and $(z)\alpha_z \in X'_i$ for all distinct $z, z' \in Z$, or (2) $H \subset A_{(X'_{i-1})}$ where $I(Z) \subseteq X'_{i-1} \subseteq X_{i-1}$. If $K = \langle H, \{\alpha_z : z \in Z\} \rangle$ then $o(K) > (|Z| + 1)o(H)$.

PROOF. Note that if $o(H) < 1$ or $|Z| < 1$ the result is clear. Let $\psi, \varphi \in H$ and $z, z' \in Z$, where $\psi \neq \varphi$ and $z \neq z'$. Then $\alpha_z \psi \neq \alpha_z \varphi$, $\alpha_z \psi \neq \varphi$, and $\alpha_z \psi \neq \alpha_{z'} \psi$. Suppose $\alpha_z \psi = \alpha_{z'} \varphi$. Suppose $H \subset A_{(X_j)}$. Then $(z)\alpha_z \psi \notin X'_i$, but $(z)\alpha_{z'} \varphi \in X'_i$. Suppose $H \subset A_{(X'_{i-1})}$. Then $(z)\alpha_z \psi \in X'_{i-1}$, but $(z)\alpha_{z'} \varphi \notin X'_{i-1}$. Hence $o(K) > (|Z| + 1)o(H)$.

LEMMA 2. Let $\{n_i\}_{i=1}^s$ be a sequence of positive integers with $s \geq 2$. If $\sum_{i=1}^s n_i = N$ then $\prod_{i=1}^s (n_i + 1) \geq 2N$.

PROOF. By induction on s .

LEMMA 3. Let A be a group acting on a set X with a reducible partition $X = \bigcup_{j=0}^s X_j$. Suppose $Z \subseteq X_j$ for $1 \leq j \leq t$ and let $H = \langle \alpha_z: z \in Z \rangle$ and for every reducing orbit O of H $O \cap X_j \subseteq Z$. Then either $O(H) = 2^a = |Z| + 1$ for some positive integer a or $O(H) \geq 2|Z|$.

PROOF. Let H have $k \geq 2$ orbits $O_1, \dots, O_k$ such that $Z_i = O_i \cap Z \neq \emptyset$. Let $H_1 = \langle \alpha_z: z \in Z_1 \rangle$. Clearly, $o(H_1) \geq |Z_1| + 1$. Since $H_1 \subset H_{(Z_2)}$ by Lemma 1 (L1), $o(H_2) \geq (|Z_2| + 1)o(H_1)$ where $H_2 = \langle \alpha_z: z \in Z_1 \cup Z_2 \rangle$. By the repeated application of (L1), $o(H) \geq \prod_{i=1}^k (|Z_i| + 1)$. By (L2), $o(H) \geq 2|Z|$.

Let O be the only orbit of H such that $Z \cap O \neq \emptyset$. Let $|$ denote divides. Now $|O| | o(H)$. If $|O| < o(H)$ then $o(H) \geq 2|O| > 2|Z|$. If $|O| = o(H)$ then H is regular on O . Suppose $|O| = |Z| + 1$. Then $(z)\alpha_z^2 = z$ and by regularity α_z has order 2, thus by Cayley's Theorem $o(H) = 2^a$. Suppose $|O| > |Z| + 1$. Let $X'_{j-1} = O \cap X_{j-1}$. A reducing map sends only one element of X'_{j-1} into Z . The remaining images of X'_{j-1} lie in X'_{j-1} . Since H is regular on O , $|\{(p, q) | p \in X'_{j-1} \text{ and } q = (p)\psi \in X'_{j-1}, \text{ where } \psi \in H\}| = |X'_{j-1}|^2$. Each α_z induces $|X'_{j-1}| - 1$ of these pairs. The identity map induces $|X'_{j-1}|$ pairs. Hence $(|X'_{j-1}| - 1)|Z| \leq |X'_{j-1}|^2 - |X'_{j-1}|$ so that $|Z| \leq |X'_{j-1}|$. Hence $o(H) = |O| \geq |X'_{j-1}| + |Z| \geq 2|Z|$.

COROLLARY 3.1. Let A be a group acting on a set X . Let $s \geq 2$ and let $\bigcup_{i=0}^s X_i$ be a reducible partition of X . Let $H \subseteq A$ and suppose H has an orbit O such that $O \cap X_j \neq \emptyset$ and $O \cap X_{j+1} \neq \emptyset$, for some $1 \leq j < s$. Suppose $\langle \alpha_z: z \in O \cap (X_j \cup X_{j+1}) \rangle \subset H$. Then H is not regular on O .

PROOF. Let $W = O - (X_j \cup X_{j+1})$. Suppose H is regular on O . Since H is regular, each point $x \in O$ is the image of $y \in O$ by a single map of H . Reducing $X_j \cap O$ and $X_{j+1} \cap O$ requires $|(X_j \cup X_{j+1}) \cap O|$ distinct nonidentity maps of H by definition. Let $v \in X_{j+1}$. No reducing map α_x for $x \in O \cap (X_j \cup X_{j+1})$ is such that $(w)\alpha_x = v$ where $w \in W$. Hence there are $|W|$ such maps in H . However, the total number of nonidentity maps in H is only $|O| - 1$, a contradiction. Hence H is not regular on O .

COROLLARY 3.2. Let A be a group acting on a set X . Let $X = \bigcup_{i=0}^s X_i$ be a reducible partition. Let $H = \langle \alpha_z: z \in X_k \rangle$ for some $k > 0$.

(1) Suppose H has a single reducing orbit O . Then

(a) $o(H) = |X_k| + 1$ when $|X_k| = 2^a - 1$ for $a \geq 1$.

(b) $o(H) = 2|X_k|$ when $|O \cap X_{k-1}| = |X_k|$, and

- (c) $o(H) \geq 2(|X_k| + 1)$ otherwise.
- (2) Suppose H has exactly two reducing orbits O_1 and O_2 . Then
- (a) $o(H) > 2|X_k|$, or
- (b) $o(H) = 2|X_k|$, $|O_1| = 2$, and $|O_2| = |X_k| = 2^b$ for $b \geq 1$.
- (3) Suppose H acts on $l \geq 3$ reducing orbits. Then
- (a) $o(H) \geq 3|X_k|$ if $|X_k| > p$, and
- (b) $o(H) \geq 2^p$ if $|X_k| = p$.

PROOF. Suppose H has a single reducing orbit O . If H does not act regularly on O then $o(H_x) \geq 2$ for $x \in O$. Since O contains a point of X_{k-1} , $o(H) > 2(|X_k| + 1)$. Suppose H acts regularly on O . By (L3), $o(H) = |X_k| + 1$ only if $|X_k| = 2^a - 1$, for $a \geq 1$. By (L3), $|X_{k-1} \cap O| \geq |X_k|$ if $|X_{k-1} \cap O| \geq 2$. Hence if $o(H) = 2|X_k|$ then $|X_{k-1} \cap O| = |X_k|$. Suppose H acts on two reducing orbits O_1 and O_2 . Let $|O_1 \cap X_k| = b$ and $|O_2 \cap X_k| = a$ and suppose $a \geq b$. By (L1), $o(H) \geq (a+1)(b+1) \geq ab + a + b + 1$. If $b \geq 2$ then $o(H) > 2|X_k|$. If $b = 1$ then $o(H) \geq 2(a+b)$. Hence, if $o(H) = 2|X_k|$ then $|O_1 \cap X_k| = 1$. The unique point of $O_1 \cap X_k$ is fixed by $H' = \langle \alpha_z : z \in O_2 \cap X_k \rangle$. Since $o(H) \leq o(H')|O_1|$ it follows that $|O_1| = 2$. Since $o(H') = |O_2 \cap X_k| + 1$ it follows by (L3) that $|X_k| = 2^c$ for $c \geq 1$.

Suppose H acts on $p \geq 3$ orbits $O_1, \dots, O_p$. By (L1), $o(H) \geq \prod_{j=1}^p (a_j + 1)$ where $a_j = |X_k \cap O_j|$ for $1 \leq j \leq p$. Without loss of generality, $\{a_j\}_{j=1}^p$ is ordered so that $a_j \geq a_{j-1}$ for $2 \leq j \leq p$. Suppose $a_2 \geq 2$. Then

$$\begin{aligned} & (a_p + 1)(a_{p-1} + 1) \cdots (a_1 + 1) \\ & \geq a_p + \cdots + a_1 + a_p(a_{p-1} + \cdots + a_1) + a_p(a_{p-1}a_{p-2} + \cdots + a_2a_1) \\ & \geq 3|X_k| \quad \text{if } p \geq 4. \end{aligned}$$

If $p = 3$ and $a_2 \geq 2$ then

$$\begin{aligned} & (a_3 + 1)(a_2 + 1)(a_1 + 1) \geq a_2a_1 + a_1 + a_2 + 1 + a_3a_1 + a_3a_2 + a_3 \\ & \geq (a_1 + a_2 + a_3) + (a_2a_1 + a_3a_2 + a_3a_1a_2) \geq 3|X_k|. \end{aligned}$$

If $a_2 = 1$ then $(a_3 + 1)(a_2 + 1)(a_1 + 1) = (a_3 + 1)4 \geq 3a_3 + a_3 + 4 \geq 3|X_k|$.

If $a_p = 1$ then $|X_k| = p$ and $o(H) \geq 2^p$ by (L1).

LEMMA 4. Let A be a group acting faithfully and transitively on a set X . Suppose $X = X_0 \cup X_1 \cup X_2$ is a reducible partition. Then neither $A_{(X_2)}$ nor $A_{(X_0)}$ contains a nontrivial normal subgroup of A .

PROOF. Let N be a nontrivial normal subgroup of A . The orbits of N are blocks of A . Since A is faithful these orbits are not singletons. Let $x_2 \in X_2$ and let $(x_2)\alpha_{x_2} = x_1$. Suppose $N \subset A_{(X_2)}$ so there exists $\psi \in N$ with a cycle $(x_2x'_2 \cdots)$, $x'_2 \in X_2$. Then $(x'_2)\alpha_{x_2} \in X_2$ by definition, and $\alpha_{x_2}^{-1}\psi\alpha_{x_2}$ has a cycle $(x_1(x'_2)\alpha_{x_2} \cdots)$. However, $\alpha_{x_2}^{-1}\psi\alpha_{x_2} \in N$ while $\alpha_{x_2}^{-1}\psi\alpha_{x_2} \notin A_{(X_2)}$, a contradiction. Suppose $N \subset A_{(X_0)}$. Let $x_1 \in X_1$ and $(x_1)\alpha_{x_1} = x_0$. There exists a $\psi \in N$ with a cycle $(x_1x'_1 \cdots)$ with $x'_1 \in X_1 \cup X_2$. Then $(x'_1)\alpha_{x_1} \in X_1 \cup X_2$ and $\alpha_{x_1}^{-1}\psi\alpha_{x_1}$ has a cycle $(x_0(x'_1)\alpha_{x_1} \cdots)$. However, $\alpha_{x_1}^{-1}\psi\alpha_{x_1} \in N$ and $\alpha_{x_1}^{-1}\psi\alpha_{x_1} \notin A_{(X_0)}$, a contradiction.

REMARK. Actually, the hypothesis that $X_0 \cup X_1 \cup X_2$ is reducible can be weakened to the existence of some reducing maps α_{x_2} and α_{x_1} .

COROLLARY 4.1. *Let $H = \langle \alpha_z : z \in X_2 \rangle$ and let $K = \langle \alpha_z : z \in X_1 \rangle$. Then $o(A)[A : K]!$ and $o(A)[A : H]!$.*

PROOF. By (L4), K and H cannot contain normal subgroups of A . However, by the core theorem of algebra (Herstein, *Topics in algebra*, Blaisdell, Waltham, Mass., 1964, p. 62), $o(A)[A : K]!$ and $o(A)[A : H]!$.

COROLLARY 4.2. *Let A act on a set X . Let $X = \bigcup_{i=0}^t X_i$ be a reducible partition. Suppose $2 \leq j \leq t$. Let $H_1 = \langle \alpha_z : z \in X_j \rangle$ and $H_2 = \langle \alpha_z : z \in X_j \cup X_{j-1} \rangle$. Suppose $O_1, \dots, O_k$ are the reducing orbits of H_2 and N_2 is the pointwise stabilizer of $\bigcup_{v=1}^k O_v$. Let $\bar{H}_2 = H_2/N_2$. Let $H'_1 = \langle H_1, N_2 \rangle$ and $\bar{H}'_1 = H'_1/N_2$. Then $\bar{H}'_1$ contains no normal nontrivial subgroup of $\bar{H}_2$.*

PROOF. Suppose N is a normal nontrivial subgroup of $\bar{H}_2$ contained in $\bar{H}'_1$. Then N acts nontrivially on an orbit O_s for $1 \leq s \leq k$. Also $O_s \cap X_{j-1} \neq \emptyset$, since O_s is a reducing orbit. Let $x_{j-1} \in O_s \cap X_{j-1}$. Since the orbits of N are blocks of $\bar{H}_2$ there exists a $g \in N$ containing a nontrivial cycle $(x_{j-1}x'_{j-1} \dots)$ where $x'_{j-1} \in X_{j-1} \cup X_j$. Then $(x_{j-1})\alpha_{x_{j-1}} \in X_{j-2}$ and $(x'_{j-1})\alpha_{x_{j-1}} \in X_{j-1} \cup X_j$ by definition. Since $\alpha_{x_{j-1}}^{-1} \cdot g\alpha_{x_{j-1}}$ contains the cycle $((x_{j-1})\alpha_{x_{j-1}}(x'_{j-1})\alpha_{x_{j-1}} \dots)$ a contradiction follows from $N \subset \bar{H}'_1 \subset \bar{H}_{2(X_{j-2})}$. Hence, $\bar{H}'_1$ contains no normal subgroup of $\bar{H}_2$. As in (L4 C1) this means $o(\bar{H}_2)[\bar{H}_2 : \bar{H}'_1]!$.

LEMMA 5. *Let A be a group acting on a set X . Let $X = \bigcup_{i=0}^s X_i$ be a reducible partition. Suppose $1 \leq j \leq s$ is such that $|X_j| \geq 2$. Suppose H is a subgroup of A that stabilizes X_j . Let $K = \langle H, \{\alpha_z : z \in X_j\} \rangle$. Then there exists $x \in X_j$ such that $o(K_x) \geq 2o(H)/|X_j|$.*

PROOF. Suppose X_j is not an orbit of H . There exists an orbit O of H such that $O \subseteq X_j$ and $|O| \leq |X_j|/2$. Hence $o(K_x) \geq o(H_x) \geq 2o(H)/|X_j|$ for any $x \in O$. Suppose X_j is an orbit of H . Let x, x' be distinct elements of X_j . Since X_j is an orbit of H there are $o(H)/|X_j|$ maps of H sending x' to $(x')\alpha_x$. Let ψ be such a map. Then $(x')\alpha_x\psi^{-1} = x'$ and also $\alpha_x\psi^{-1} \notin H_{x'}$. Hence $o(K_x) \geq 2o(H_x) \geq 2o(H)/|X_j|$.

Let A act on a set X . Let $X = \bigcup_{i=0}^t X_i$ be a reducible partition. Suppose $1 \leq j \leq t$ and $t \geq 2$. Let

$$H_l = \langle \alpha_z : z \in X_j \cup \dots \cup X_{j-(l-1)} \rangle \quad \text{if } 1 \leq l \leq j$$

and

$$H_l = \langle \alpha_z : z \in X_1 \cup \dots \cup X_l \rangle \quad \text{if } j < l \leq t. \quad (2.1)$$

Let O_l be a reducing orbit of H_l that intersects X_l . Let O_{l-1} be a reducing orbit of H_{l-1} contained within O_l which intersects X_{l-1} if $j < l$ and intersects X_l if $j = l$. Let $O_{l-i} \subseteq O_{l-(i-1)}$ be a reducing orbit of H_{l-i} for $2 \leq i \leq t-1$, such that O_{l-i} intersects X_{l-i} if $j \leq t-i$ and intersects X_j if $t-i < j$. For $2 \leq l \leq t$ define N_l to

be the subgroup of H_l that stabilizes O_l pointwise. Let

$$\begin{aligned}\bar{H}_l &= H_l/N_l \quad \text{if } 1 \leq l \leq t-1, \\ H'_l &= \langle H_l, N_{l+1} \rangle \quad \text{if } 2 \leq l \leq t, \text{ and} \\ \bar{H}'_l &= H'_l/N_{l+1} \quad \text{if } 2 \leq l \leq t.\end{aligned}\tag{2.2}$$

LEMMA 6. *Let A act on a set X . Let $X = \bigcup_{i=0}^t X_i$ be a reducible partition. Suppose $1 \leq j \leq t$ and $t \geq 2$. For $2 \leq l \leq t$, u_l is defined to be the least positive integer such that $u_l! \geq u_l o(\bar{H}'_{l-1})$ where $\bar{H}'_{l-1}$ is defined in (2.2). Then for $l \geq i \geq 2$, $o(H_l) \geq u_l \cdots u_i o(H_{i-1})$ where $u_l > \cdots > u_i \geq i+1$ and where H_l and H_{i-1} are defined in (2.1). Moreover, $o(H_2) \geq o(\bar{H}_2) \geq 3!|X_j|$.*

PROOF. First it is shown that $o(H_{l+1}) \geq u_{l+1} o(H_l)$ for $1 \leq l \leq t-1$. By (L4 C1), $o(\bar{H}_{l+1})|[\bar{H}_{l+1} : \bar{H}'_l]|$. Hence $[\bar{H}_{l+1} : \bar{H}'_l]! \geq [\bar{H}_{l+1} : \bar{H}_l] o(\bar{H}'_l)$. Hence $[\bar{H}_{l+1} : \bar{H}'_l] \geq u_{l+1}$. By the correspondence theorem $[\bar{H}_{l+1} : \bar{H}_l] = [H_{l+1} : H'_l] \geq u_{l+1}$. Since $H'_l > H_l$, $[H_{l+1} : H_l] \geq [H_{l+1} : H'_l]$. Hence $o(H_{l+1}) \geq u_{l+1} o(H_l)$.

Next it is shown that $u_{l+1} > u_l$ for $2 \leq l \leq t-1$. Suppose $u_{l+1} \leq u_l$. By definition $u_{l+1}! \geq u_{l+1} o(\bar{H}'_l)$. Hence $u_l! \geq u_l o(\bar{H}'_l)$ and $(u_l - 1)! \geq o(\bar{H}'_l)$. Since $[\bar{H}_l : \bar{H}'_{l-1}] \geq u_l$, $o(\bar{H}_l) \geq u_l o(\bar{H}'_{l-1})$.

$$o(\bar{H}'_l) = \frac{o(H_l) o(N_{l+1})}{o(H_l \cap N_{l+1}) o(N_{l+1})} = \frac{o(H_l)}{o(H_l \cap N_{l+1})}$$

and

$$o(\bar{H}_l) = \frac{o(H_l)}{o(H_l \cap N_l)}.\tag{2.3}$$

Now $H_l \cap N_l \supset H_l \cap N_{l+1}$ since if $g \in H_l \cap N_{l+1}$ it fixes all points O_l and since it is in H_l it is in N_l . Hence $o(\bar{H}'_l) \geq o(\bar{H}_l)$. But then, $(u_{l-1})! \geq o(\bar{H}_l)$, whence $(u_{l-1})! \geq u_l o(\bar{H}'_{l-1})$, a contradiction of the choice of u_l .

Finally, it is shown that $o(H_2) \geq 6|X_j|$. Note since $o(\bar{H}'_1) \geq 2$, $u_2 \geq 3$. If $o(H_1) \geq 2|X_j|$ then, since $u_2 \geq 3$, $o(H_2) \geq 3!|X_j|$. Otherwise, by (L3), $o(H_1) = 2^a = |X_j| + 1$ for some $a \geq 1$.

Suppose $a = 1$. In this case $|X_j| = 1$. Since $o(H_1) = 2$ and $u_2 \geq 3$, $o(H_2) \geq 6|X_j|$.

Suppose $a \geq 4$. In this case, $|X_j| \geq 15$. Since H_1 is regular $H_1 \cap N_2 = \langle e \rangle$ so $o(\bar{H}'_1) = o(H_1)/o(H_1 \cap N_2) = o(H_1)$. Hence $16|o(\bar{H}'_1)|$ and hence $[\bar{H}_2 : \bar{H}'_1] \geq 7$ by (L4 C1). Thus $o(H_2) \geq 6|X_j|$.

Suppose $a = 2$. In this case, $|X_j| = 3$ and $o(H_1) = 4$. Since H_1 is regular $o(H_1) = o(\bar{H}'_1) = 4$. By (L4 C1), $o(\bar{H}_2)|[\bar{H}_2 : \bar{H}'_1]|$. Hence $[\bar{H}_2 : \bar{H}'_1] \geq 5$ since $4^2 \nmid 4!$. Thus $o(H_2) \geq 5 \cdot 4 \geq 6|X_j|$.

Suppose $a = 3$. In this case, $|X_j| = 7$ and $o(H_1) = 8$. Since H_1 is regular $o(\bar{H}'_1) = 8$. If $[\bar{H}_2 : \bar{H}'_1] \geq 6$ the result follows. Hence $[\bar{H}_2 : \bar{H}'_1] = 5$. Hence $o(\bar{H}_2) = 40$.

Suppose $j = 1$. Then $|X_2| \leq 4$ for otherwise using (L1), $o(H_2) > (|X_2| + 1)o(H_1) \geq 6 \cdot 8 \geq 3!|X_1|$. Suppose $|X_2| = 3$ or 4 . Since $|O_2|$ must divide 40 and $|O_2| \geq 11$, $|O_2| = 20$ or 40 . Since X_2 is stabilized setwise by $\bar{H}'_1$, by (L5), $o((\bar{H}_2)_{x_2}) \geq 4$, a contradiction of $|O_2| = 20$ or 40 by (P7.4). If $|X_2| = 2$ or 1 then by (L5), $o((\bar{H}_2)_{x_2}) \geq 8$, a contradiction to $|O_2| \geq 10$.

Suppose $j > 1$. Let x_{j-1} be the point of X_{j-1} in the reducing orbit of $\bar{H}'_1$. Suppose $|X_{j-1}| \neq 1$. Note $X_{j-1} - \{x_{j-1}\}$ is stabilized by $\bar{H}'_1$ and $\alpha_{x_{j-1}}$. If $\bar{H}$ is generated by $\bar{H}'_1$ and $\alpha_{x_{j-1}}$, then $o(\bar{H}) \geq 40$ by (L4 C1), a contradiction to $o(\bar{H}) < o(\bar{H}_2) = 40$. Let $|X_{j-1}| = 1$. Then $|O_2| \mid 40$. Since $|X_j \cup X_{j-1}| = 8$, $|X_{j-2} \cap O_2| = 2, 12$, or 32 . Suppose $|O_2 \cap X_{j-2}| = 2$. Since $3 \nmid o(\bar{H}_2)$, $\alpha_{x_{j-1}}$ has cycles $(x_{j-2})(x'_{j-2}x_{j-1})$. Since $X_{j-2} \cap O_2$ is stabilized by $\bar{H}'_1$, $o((\bar{H}'_1)_{x_{j-2}}) \geq 4$. Hence $o((\bar{H}_2)_{x_{j-2}}) \geq 8$, a contradiction to $o((\bar{H}_2)_{x_{j-2}}) = o(\bar{H}_2)/|O_2| = 4$. Let $|X_{j-2} \cap O_2| = 12$. Since C_5 is a normal subgroup of H_2 the orbits of C_5 are a complete block system of $\bar{H}_2$. No point of X_j in a block with x_{j-1} can be reduced since the reducing maps of X_j do not have order 5. Hence the block containing x_{j-1} contains 4 points of X_{j-2} . However, if $(x_j)\alpha_{x_j} = x_{j-1}$, x_j is in a block containing 4 distinct points of X_{j-2} . Since this can be true for only two x_j , a contradiction follows. Let $|X_{j-2} \cap O_2| = 32$. Then $\bar{H}_2$ is regular on O_2 and hence by (L3 C1) cannot reduce $O_2 \cap (X_j \cup X_{j-1})$. Hence (L6) is proved.

COROLLARY 6.1. *Let A act on a set X . Let $X = \bigcup_{i=0}^t X_i$ be a reducible partition. Then $o(A) \geq (t+1)!$ and $o(A) = (t+1)!$ only if there is a reducing orbit O of A such that $|O| = t+1$ and $A^0 \cong S_{t+1}$.*

PROOF. Let O be a reducing orbit of A such that $|O \cap X_i| \geq 1$. Let $H_t = \langle \alpha_z : z \in O \cap \bigcup_{i \geq 1} X_i \rangle$. If $t = 1$ then $o(A) \geq 2$ and $o(A) = 2$ implies $A^0 \cong C_2$ and the result is clear. If $t > 1$ then by (L6) for every $j \geq 1$, $o(H_j) \geq (t+1)!|X_j|$. Hence $o(A) \geq (t+1)!$ and each X_j is a singleton for $j \geq 1$ if $o(A) = (t+1)!$. Suppose $o(A) = (t+1)!$. Let $H_{t-1} = \langle \alpha_z : z \in O \cap \bigcup_{i=1}^{t-1} X_i \rangle$. Then $o(H_{t-1}) \geq t!$ by (L6). But H_{t-1} fixes X_t . Thus $o(A^0) \leq |O|t!$ and hence $|O| = t+1$. Hence $A^0 \cong S_{t+1}$.

COROLLARY 6.2. *Let A act on a set X . Let $X = \bigcup_{i=0}^t X_i$ be a reducible partition. Let $H_1 = \langle \alpha_z : z \in X_j \rangle$. Suppose either*

$$H_2 = \langle \alpha_z : z \in X_j \cup X_{j+1} \rangle \quad \text{and} \quad H_3 = \langle \alpha_z : z \in X_j \cup X_{j+1} \cup X_{j+2} \rangle \quad (2.4)$$

and there is a reducing orbit O_2 of H_2 meeting X_j in at least 2 points, or

$$H_2 = \langle \alpha_z : z \in X_{j-1} \cup X_j \rangle \quad \text{and} \quad H_3 = \langle \alpha_z : z \in X_{j-2} \cup X_{j-1} \cup X_j \rangle \quad (2.5)$$

and $|X_j| > 2$.

Then $o(H_3) \geq o(\bar{H}_3) \geq 120$ where $\bar{H}_3$ is defined as in (2.2).

PROOF. Let H_2 and H_3 be as in (2.4). Let $\bar{H}_2$ and $\bar{H}'_1$ be as in (2.2) acting on the orbit O_2 . Since $|X_j \cap O_2| \geq 2$, $o(\bar{H}'_1) \geq 4$ by (L3). Suppose $o(\bar{H}_1) = 4$. By (L4 C1), $[\bar{H}_2 : \bar{H}'_1] \geq 5$. Also $o(\bar{H}'_1) \neq 5$ since C_5 has only a single reducing map. Suppose $o(\bar{H}_1) = 6$; then $[\bar{H}_2 : \bar{H}'_1] \geq 4$ by (L4 C1). Suppose $o(\bar{H}_1) \geq 7$; then $[\bar{H}_2 : \bar{H}'_1] \geq 5$. Using the calculation of (2.3), $o(H'_2) \geq o(\bar{H}_2)$. Suppose $o(\bar{H}_2) \geq 24$; then by (L4 C1), $[\bar{H}_3 : \bar{H}'_2] \geq 5$ and $o(\bar{H}_3) \geq 120$.

Let H_3 and H_2 be as in (2.5). If there is a reducing orbit of H_2 meeting X_j in at least 2 points the argument follows as above. Suppose not. Let $O_{21}, \dots, O_{2k}$ be the reducing orbits of H_2 . Let N_2 be the pointwise stabilizer of $\bigcup_{i=1}^k O_{2i}$. Let $\bar{H}_2 = H_2/N_2$. Let $H'_1 = \langle H, N_2 \rangle$ and let $\bar{H}'_1 = H'_1/N_2$. By (L3), $o(\bar{H}'_1) \geq 4$. This case now follows to the point of bounding $o(\bar{H}_2)$ as above using (L4 C2) in place of (L4 C1). Let $O_{31}, \dots, O_{3k'}$ be the reducing orbits of H_3 . Let N_3 be the pointwise

stabilizer of $\bigcup_{i=1}^{k'} O_{3i}$. Let $\bar{H}_3 = H_3/N_3$. Let $H'_2 = \langle H_2, N_3 \rangle$ and let $\bar{H}'_2 = H'_2/N_3$. The case concludes as above using (L4 C2) in place of (L4 C1).

Let A act on a set X . Let $X = \bigcup_{i=0}^t X_i$ be a reducible partition. For $1 \leq l \leq t$ let

$$H_l = \langle \alpha_z : z \in X_{t-(l-1)} \cup \cdots \cup X_t \rangle. \quad (2.6)$$

For $2 \leq l \leq t$ let $O_{l1}, \dots, O_{lk_l}$ be the reducing orbits of H_l . Let N_l be the pointwise stabilizer $\bigcup_{v=1}^{k_l} O_{lv}$. For $2 \leq l \leq t$, let

$$\begin{aligned} \bar{H}_l &= H_l/N_l \quad \text{and for } 1 \leq l \leq t-1 \text{ let} \\ H'_l &= \langle H_l, N_{l+1} \rangle \quad \text{and let} \\ \bar{H}'_l &= H'_l/N_{l+1}. \end{aligned} \quad (2.7)$$

COROLLARY 6.3. *Let A act on a set X . Let $X = \bigcup_{i=0}^t X_i$ be a reducible partition. Suppose $2 \leq l \leq t$ and H_1 has at least two reducing orbits. For $2 \leq l \leq t$, u_l is defined to be the least positive integer such that $u_l! \geq u_l o(\bar{H}'_{l-1})$ where $\bar{H}'_{l-1}$ is defined in (2.7). Then for $2 \leq i \leq l$, $o(H_i) \geq u_i \cdots u_i o(H_{i-1})$ where $u_i > \cdots > u_i \geq i+1$ and where H_i and H_{i-1} are defined in (2.6). Moreover $u_2! \geq u_2 o(\bar{H}'_1)$.*

PROOF. By (L4 C2), $[\bar{H}_l : \bar{H}'_{l-1}] \geq u_l$ for $2 \leq l \leq t$ where $\bar{H}_l$ and $\bar{H}'_{l-1}$ are defined. By the correspondence theorem $[\bar{H}_l : H'_{l-1}] \geq u_l$. Since $o(H'_{l-1}) \geq o(H_{l-1})$, $o(H_l) \geq u_l o(H_{l-1})$. Using the calculation in (2.3), it can be shown as in (L6) that for $3 \leq l \leq t$, $u_l \geq u_{l-1}$ or the defining property of u_{l-1} is contradicted. Hence for $2 \leq l \leq t$ and $l \geq i \geq 2$, $o(H_i) \geq u_i \cdots u_i o(H_{i-1})$ and $u_i > \cdots > u_2 \geq 3$. Since $\bar{H}'_1$ acts on several reducing orbits, $o(\bar{H}'_1) \geq 2|X_1|$ by (L3). Hence $u_2! \geq u_2 \cdot 2 \cdot |X_1|$.

COROLLARY 6.4. *Let A act on a set X . Let $X = \bigcup_{i=0}^t X_i$ be a reducible partition. Suppose $t \geq 3$ and $|X_t| \geq 2$. Let H_l and $\bar{H}_l$ be defined as in (2.6) and (2.7) for $1 \leq l \leq t$. Then for $3 \leq l \leq t$, $o(H_l) \geq o(\bar{H}_l) \geq (l+2)!$.*

PROOF. By (L6 C2), $o(\bar{H}_3) \geq 5!$ if $|X_t| \geq 3$. Suppose it has been shown for $3 \leq l \leq t-1$ that $o(\bar{H}_l) \geq (l+2)!$. By the calculation of (2.3), $o(\bar{H}'_l) \geq o(\bar{H}_l)$. By the definition of u_{l+1} , $u_{l+1} \geq l+3$ since $o(\bar{H}'_l) \geq (l+2)!$. Hence $o(H_{l+1}) \geq o(\bar{H}_{l+1}) \geq (l+3)o(H_l) \geq (l+3)!$.

LEMMA 7. *Let A be a group acting on a set X . Let $\bigcup_{i=0}^t X_i$ be a reducible partition of X . Let $H \subseteq A$. Suppose Y is an orbit of H where $Y \subset X_j$, $j \geq 1$. Suppose $Z \subseteq X_{j+1}$ such that $I(Z) \subset Y$. Let $K = \langle H_1, \langle \alpha_z : z \in Z \rangle \rangle$. Suppose that if O is the orbit of K containing Y , then $O \cap X_j = Y$. Let $L = \langle K, \langle \alpha_y : y \in Y \rangle \rangle$. Suppose that if O' is the orbit of L containing Y then $O' \cap X_j = Y$. Let*

$$m = \min\{|Y| - 1, |Z|\}.$$

It is the case that $o(K_y) \geq (m+1)o(H_y)$, for $y \in Y$. Moreover, $o(L_y) \geq |Y|o(K_y)$.

PROOF. The following propositions are used in the proof.

PROPOSITION 7.1. *Let A be a group acting on a set X . Let $X = \bigcup_{i=0}^t X_i$ be a reducible partition. Suppose that $H \subseteq A$ and that O is an orbit of H . Let $x \in O$ and let $(x)\psi_x \notin O$. Then if $h \in H$, $h\psi_x^{-1} \notin H$.*

PROOF. Since $x \in O$ and $(x)\psi_x \notin O$, $\psi_x \notin H$. Hence $h\psi_x^{-1} \notin H$.

PROPOSITION 7.2. Let A be a group acting on a set X . Let $X = \bigcup_{i=0}^s X_i$ be a reducible partition. Suppose that $H \subseteq A$ and that O is an orbit of H . Suppose $\{z_r: 1 \leq r \leq k\} \subset O$ is such that $(z_r)\alpha_{z_r} \notin O$ and $(z_s)\alpha_{z_s} \in O$ if $r \neq s$ for $1 \leq r \leq k$ and $1 \leq s \leq k$. If $h \in H$, α_{z_r} and α_{z_s} are such that $r \neq s$, then $h\alpha_{z_r}^{-1} \notin H$.

PROOF. Since $(z_s)\alpha_{z_s} \in O$ and O is an orbit of H , $(z_s)\alpha_{z_s}h^{-1} \notin O$. Now $((z)\alpha_{z_r}h^{-1})h\alpha_{z_r}\alpha_{z_s} = (z_s)\alpha_{z_s} \in O$ by assumption. Since O is an orbit of H , $h\alpha_{z_r}^{-1}\alpha_{z_s} \notin H$.

PROPOSITION 7.3. Let A be a group acting on a set X . Let $X = \bigcup_{i=0}^s X_i$ be a reducible partition. Let $H \subseteq A$. Let Y be part of an orbit of H . Suppose $\alpha \notin H$ and $(y)\alpha \in Y$. Then there exists $h \in H$ such that $h\alpha^{-1} \in K_y$.

PROOF. Since Y is part of an orbit of H there exists an h such that $(y)h = (y)\alpha_x$. Hence $h\alpha_x^{-1} \in K_y$.

PROPOSITION 7.4. Let A be a group acting on a set X . Suppose $H \subseteq A$ and O is an orbit of H . Then for every $x \in O$, $o(H_x) = o(H)/|O|$.

PROOF. This is a basic fact of permutation group theory.

The main argument of the proof begins.

Since $|Z| \geq m$ there exists a set of distinct points $\{z_1, \dots, z_m\}$ with $\{\alpha_{z_1}, \dots, \alpha_{z_m}\}$ the set of corresponding reducing maps. Since $O \cap X_j = Y$, $(Y - (z_k)\alpha_{z_k}^{-1})\alpha_{z_k} \subset Y$, for any $1 \leq k \leq m$. Hence at most m distinct points of Y are mapped outside Y by one or more of the α_{z_i} , $1 \leq i \leq m$. Hence since $|Y| - 1 \geq m$, there exists a $y \in Y$ such that $(y)\alpha_{z_i} \in Y$, for $1 \leq i \leq m$. By (P7.3) there exists an $h_i \in H$ for $1 \leq i \leq k$ such that $h_i\alpha_{z_i}^{-1} \in K_y$. By (P7.1), $h_i\alpha_{z_i}^{-1} \notin H_y$, for $1 \leq i \leq k$. Let h and h' be two elements of H_y . If $hh_i\alpha_{z_i}^{-1} = h'h_k\alpha_{z_k}^{-1}$, $i \neq k$, then $hh_i = h'h_k\alpha_{z_k}^{-1}\alpha_{z_i}$. By (P7.2), $h'h_k\alpha_{z_k}^{-1} \notin H$, a contradiction. Thus for every map of H_y there are $m+1$ maps of K_y , namely $h, h\alpha_{z_1}^{-1}, \dots, h\alpha_{z_m}^{-1}$. The maps arising from different elements of H_y are clearly distinct. Hence $o(K_y) \geq (m+1)o(H_y)$.

If $|Y| = 1$, then $o(L_y) \geq o(K_y)$ since $L \supset K$. Suppose $|Y| > 1$. Fix $y^- \in Y$. Let $y \in Y - y^-$. By (P7.3) there exists $k_y \in K$ such that $k_y\alpha_{y^-}^{-1} \in L_{y^-}$. By (P7.1) $k_y\alpha_{y^-}^{-1} \notin K_{y^-}$. There are $|Y| - 1$ such maps. Suppose k and k' are elements of K_y . Suppose $kk_y\alpha_{y^-}^{-1} = k'k_y\alpha_{y^-}^{-1}$ where y and $y' \in Y - \{y^-\}$. Then $kk_y = k'k_y\alpha_{y^-}^{-1}\alpha_{y'}$, a contradiction of (P7.2). Hence for every map of K_y there are $|Y|$ maps of L_y . Hence $o(L_y) \geq |Y|o(K_y)$.

COROLLARY 7.1. Let A be a group acting on a set X . Let $X = \bigcup_{i=0}^s X_i$ be a reducible partition of X . Let $H \subseteq A$. Suppose $Y = O \cap X_j$ for some $j \geq 1$ and for some orbit O of H . Suppose Z is contained within an orbit O_Z of H distinct from O and disjoint from X_0 . Suppose for all distinct $z, z' \in Z$ that $(z)\alpha_z \in Z$. Suppose $I(Z) \cap O_Z = \emptyset$. Let $K = \langle H, \langle \alpha_z: z \in Z \rangle \rangle$. Let O' be the orbit of K containing O . Suppose $O' \cap X_1 = Y$. Suppose $\{(z)\alpha_z^{-1}: z \in Z\} \cap Y = \emptyset$. Then $o(K_y) \geq (|Z| + 1)o(H_y)$.

PROOF. Let $y \in Y$. Let $z \in Z$. Suppose $(y)\alpha_z \notin Y$. Then since $O' \cap X_j = Y$, $(y)\alpha_z = z$. However, this contradicts the hypothesis that $\{(z)\alpha_z^{-1} : z \in Z\} \cap Y = \emptyset$. Hence $(y)\alpha_z \in Y$. By (P7.3) there exists an $h \in H$ such that $h\alpha_z^{-1} \in K_y$. Then $(z)\alpha_z \notin O_Z$, since $I(Z) \cap O_Z = \emptyset$. Hence by (P7.1), $h\alpha_z^{-1} \notin H_y$. Let h and $h' \in H_y$. Suppose $hh_y\alpha_z^{-1} = h'h_y\alpha_z^{-1}$ for $z \neq z'$. Then $hh_y = h'h_y\alpha_z^{-1}\alpha_z$. This contradicts (P7.2). Thus for every element $h \in H_y$ there are $|Z| + 1$ elements of K_y , namely $\{h\} \cup \{h\alpha_z^{-1} : z \in Z\}$. Hence $o(K_y) > (|Z| + 1)o(H_y)$.

LEMMA 8. Let A be a group acting on a set X . Let $X = \bigcup_{i=0}^s X_i$ be a reducible partition. Suppose $Y = X_i$, $i \geq 1$, breaks into at least two orbits under H . Let O be an orbit of H on Y . Suppose $|O| \geq 2$. Then if $y \in O$ and $K = \langle H, \langle \alpha_y : y \in Y \rangle \rangle$, then $o(K_y) > o(H_y)$.

PROOF. Let $y \in O$. Suppose there exists a $y^* \in Y - y$ such that $(y)\alpha_{y^*} \in O$. By (P7.3) there exists an $h \in H$ such that $h\alpha_{y^*}^{-1} \in K_y$. Since $(y^*)\alpha_{y^*} \notin O$ by (P7.1), $h\alpha_{y^*}^{-1} \notin H_y$. Hence $o(K_y) > o(H_y)$. Suppose for all $y^* \in Y - y$, $(y)\alpha_{y^*} \in O$. Since $|Y| - 1 > |Y| - |O|$ by the pigeon-hole principle some y^- is the image of y under $\alpha_{y^-}^{-1}$ and α_{y^-} . Hence $\alpha_{y^-}^{-1}\alpha_{y^-} \in K_k \setminus H_y$. Hence $o(K_y) > o(H_y)$.

Let A be a group acting on a set X . Suppose that $X = \bigcup_{i=0}^s X_i$ is a reducible partition of X . Suppose $Y \subset X_j$ for some $j \geq 1$. Suppose $H \subseteq A$ and suppose $O_1, \dots, O_u$ are the orbits of H where $Y \subset \bigcup_{k=1}^u O_k$. Suppose $I(Y) \cap \bigcup_{k=1}^u O_k = \emptyset$. Suppose for every $y \in Y$ and $a \in \bigcup_{k=1}^u O_k$ that $(a)\alpha_y \notin U_y$ where U_y is the orbit of H containing $(y)\alpha_y$. Let

$$K = \langle H, \langle \alpha_z : z \in Y \rangle \rangle. \quad (2.8)$$

Let $Q_1, \dots, Q_{u'}$ be the orbits of K . Let M be the pointwise stabilizer of $\bigcup_{m=1}^{u'} Q_m$. Let

$$\begin{aligned} K^- &= K/M, \\ H' &= \langle H, M \rangle, \text{ and} \\ H^- &= H'/M. \end{aligned} \quad (2.9)$$

LEMMA 9. Let K and H be as in (2.8). Let K^- and H^- be as in (2.9). If f is the least integer such that $fo(H^-) < f!$ then $[K^- : H^-] \geq f$ and $[K : H] \geq f$.

PROOF. The proof follows the proof of (L4 C2). Suppose H^- contains a nontrivial normal subgroup N^- of K^- . Since N^- is nontrivial it acts nontrivially on one of the orbits Q_m , $1 \leq m \leq u'$. However, the orbits of N are blocks of K on Q_m . By assumption, there exists a k such that $Q_m \supset O_k$. Hence $Q_m \cap Y \neq \emptyset$. Since N has nontrivial orbits on Q_m there exists a nontrivial orbit O of N containing $y \in O_s \cap Y$. Hence there exists an element $n \in N$ containing a cycle $(ya \dots)$ where $a \in O_k$. The cycle $((y)\alpha_y(a)\alpha_y \dots)$ is contained by $\alpha_y^{-1}n\alpha_y$. By definition $(y)\alpha_y \in U$ and $(a)\alpha_y \notin U_y$. Since $\alpha_y^{-1}n\alpha_y \in N \subset H^-$ a contradiction follows. As in (L4 C1) H^- containing no nontrivial normal subgroup of K^- means that $o(K^-)[K^- : H^-]!$. Hence $[K^- : H^-] \geq f$. By the correspondence theorem $[K : H'] \geq f$.

Let A be a group acting on a set X . Let $X = \bigcup_{i=0}^s X_i$ be a reducible partition. Let $H = \langle \alpha_z : z \in \bigcup_{i=r}^s X_i \rangle$. Suppose H has reducing orbits $O_1, \dots, O_p$ such that $X_s \cap O_v \neq \emptyset$ for $1 \leq v \leq p$. For $1 \leq u \leq p$ and $1 \leq k \leq r - s + 1$ let

$$H_{uk} = \left\langle \alpha_z : z \in \left(\bigcup_{v=1}^{u-1} \bigcup_{m=r}^s O_v \cap X_m \right) \cup \left(\bigcup_{m=s-(k-1)}^s O_u \cap X_m \right) \right\rangle. \quad (2.10)$$

If $k > 1$ then let H_{uk-1} have reducing orbits $O_1^-, \dots, O_c^-$ with $\bigcup_{w=1}^c O_w^- \subseteq O_u$. Let H_{uk} act on reducing orbits $Q_1, \dots, Q_d$ with $\bigcup_{e=1}^d Q_e \subseteq O_u$. Let N_{uk} be the pointwise stabilizer of $\bigcup_{e=1}^d Q_e$. Let

$$\begin{aligned} H_{uk}^- &= H_{uk} / N_{uk}, \\ H'_{uk-1} &= \langle H_{uk-1}, N_{uk} \rangle, \quad \text{and} \\ H_{uk-1}^{-'} &= H'_{uk-1} / N_{uk}. \end{aligned} \quad (2.11)$$

COROLLARY 9.1. *Let A be a group acting on a set X . Let $X = \bigcup_{i=0}^s X_i$ be a reducible partition. For $1 \leq u \leq p$ and $1 \leq k \leq r - s + 1$ let H_{uk} be defined in (2.10). If $u = 1$ and $k = 1$ then $o(H_{11}) \geq (|X_s \cap O_1| + 1)$. If $u > 1$ and $k = 1$ then $o(H_{u1}) \geq (|X_s \cap O_s| + 1)o(H_{u-1s-r+1})$. Let H_{uk}^- and $H_{uk-1}^{-'}$ be defined in (2.11). Let f_{uk} be the least integer such that $f_{uk}! \geq f_{uk} o(H_{uk-1}^{-'})$, where for $3 \leq k \leq s - r + 1$ and $1 \leq u \leq p$, $f_{uk} > f_{uk-1}$ and $f_{u2} \geq 3$. Then (1) $[H_{uk}^- : H_{uk-1}^{-'}] \geq f_{uk}$, $[H_{uk} : H_{uk-1}] \geq f_{uk}$, and $o(H_{u2}^-) \geq 6|X_s \cap O_u|$. (2) $o(H_{uk-1}) \geq f_{um'} \cdots f_{un} o(H_{un-1})$ where $1 \leq u \leq p$ and $2 \leq n \leq m' \leq s - r + 1$. (3) $o(H) \geq \prod_{v=1}^p (\prod_{k=2}^{s-r+1} f_{vk} (|X_s \cap O_v| + 1))$.*

PROOF. Suppose $u = 1$ and $k = 1$. Then $o(H_{11}) \geq |X_s \cap O_1| + 1$ by (L3). Suppose $u > 1$ and $k = 1$. By (L1), $o(H_{u1}) \geq (|X_s \cap O_u| + 1)o(H_{u-1s-r+1})$. Suppose $k > 1$. Then H_{uk}^- and $H_{uk-1}^{-'}$ satisfy the hypothesis of (L9). Hence $[H_{uk}^- : H_{uk-1}^{-'}] \geq f_{uk}$ and by the correspondence theorem $[H_{uk} : H_{uk-1}] \geq f_{uk}$.

Suppose for some $3 \leq k \leq s - r + 1$ and $1 \leq u \leq p$, $f_{uk} < f_{uk-1}$. By definition $o(H_{uk-1}^-) \geq f_{uk-1} o(H_{uk-2}^-)$. Hence $(f_{uk-1} - 1)! \geq o(H_{uk-1}^{-'})$ since $f_{uk} o(H_{uk-1}^-) < f_{uk}!$ and $f_{uk} < f_{uk-1}$. To obtain a contradiction the proof proceeds as in (L6) with the following elaboration of the calculation made in (2.3). It is necessary to show that $H_{uk-1} \cap N_{uk} \subseteq H_{uk-1} \cap N_{uk-1}$. If $\varphi \in H_{uk-1} \cap N_{uk}$ then φ fixes every point of $\bigcup_{e=1}^d Q_e$. However $\bigcup_{e=1}^d Q_e \supset \bigcup_{w=1}^c O_w^-$. Since $\varphi \in H_{uk-1}$ and φ fixes every point of $\bigcup_{w=1}^c O_w^-$, $\varphi \in H_{uk-1} \cap N_{uk-1}$ and the result holds.

Since $o(H_{u1}^{-'}) \geq 2$, $f_{u2} \geq 3$. Repeated application of $o(H_{uk}) \geq f_{uk} o(H_{uk-1})$ gives $o(H_{um'}) \geq f_{um'} \cdots f_{un} o(H_{un-1})$ where $1 \leq u \leq p$ and $2 \leq n \leq m' \leq s - r + 1$. By the argument of (L6), $o(H_{u2}^-) \geq 6|X_s \cap O_u|$ for $1 \leq u \leq p$. In summary $o(H) \geq \prod_{v=1}^p (\prod_{k=2}^{s-r+1} f_{vk} (|X_s \cap O_v| + 1))$.

Let A be a group acting on a set X . Let $X = \bigcup_{i=0}^s X_i$ be a reducible partition of X . Let $H \subseteq A$. Let O_1 and O_2 be two orbits of H such that $O_1 \subset \bigcup_{i=1}^{r+1} X_i$ where $0 \leq r \leq s$, and $O_2 \subset \bigcup_{i=r+1}^s X_i$ where $O_2 \cap X_{r+1} \neq \emptyset$. Suppose there exists a $z \in O_2 \cap X_{r+1}$ such that $(z)\alpha_z \in O_1$. Let $K_0 = H$ and let $Q_0 = O$. For $1 \leq j \leq v$ let $K_j = \langle K_{j-1}, \alpha_{z_j} \rangle$ where $z_j \in Q_{j-1} \cap X_{r+1}$ and $\alpha_{z_j} \notin K_j$ and let Q_j be the orbit of K_j containing Q_{j-1} . Since A is finite, v is bounded. Let $K = K_v$ and $Q = Q_v$. K is termed an O_1 extension of H .

COROLLARY 9.2. *Suppose A is a group acting on a set X . Suppose $X = \bigcup_{i=0}^s X_i$ is a reducible partition. Let $H \subseteq A$ with orbits $O_1 \subset \bigcup_{i=1}^{r+1} X_i$ where $0 < r < s$ and $O_2 \subset \bigcup_{i=r+1}^s X_i$ with $O_2 \cap X_{r+1} \neq \emptyset$. Let K be an O_1 extension of H . Let $N = K_Q$ and $H' = \langle H, N \rangle$. Let $K^- = K/N$ and $H'^- = H'/N$. Then if a is the least integer such that $a! \geq ao(H'^-)$ then $[K : H] \geq a$.*

PROOF. This is a direct consequence of (L4 C1).

3. A lower bound for $o(A_x)$.

THEOREM 1. *Let A be a group acting on a set X . Let $X = \bigcup_{i=0}^t X_i$ be a reducible partition of X . Then there exists an $x \in X$ such that $o(A_x) \geq t!$.*

PROOF. The case $t = 1$ is clear. Let $t = 2$. Let $H = \langle \alpha_z : z \in X_2 \cup X_1 \rangle$. Suppose H acts on a single orbit $O = X_0 \cup X_1 \cup X_2$. By (L3 C1), H is not regular. Hence for $x \in O$, $o(A_x) \geq 2$.

Let $t \geq 3$. Suppose $\{X_i\}_{i=1}^{t-1}$ are singletons. Let $u = |X_t|$. Let $H_j = \langle \alpha_z : z \in X_j \cup \dots \cup X_{t-(j-1)} \rangle$, $1 \leq j \leq t-1$. By (L3), $o(H_1) \geq u+1$. Since H_j fixes $X_{t-(j+1)}$, $o(H_{j+1}) \geq (j+1+u)o(H_j)$, $1 \leq i \leq t-2$. Hence

$$o(H_{t-1}) \geq (t-1+u)o(H_{t-2}).$$

Hence $o(H_{t-1}) \geq (t-1+u) \cdots (u+2)(u+1)$. Let $x_1 \in X_1$. Since $\bigcup_{i=1}^t X_i$ is an orbit of H_{t-1} and $|\bigcup_{i=1}^t X_i| = u+t-1$, by (P7.4),

$$o(A_{x_1}) \geq o(H_{t-1}) / (t+u-1) \geq (t+u-2) \cdots (u+1).$$

Since α_{x_1} fixes x_2 by (L7 C1), $o(A_{x_2}) \geq 2 \cdot (t+u-2) \cdots (u+1)2 \geq t!$ provided $u \geq 2$. If $u = 1$ let $H = \langle \alpha_z : z \in \bigcup_{i=1}^{t-1} X_i \rangle$. Since x_t is fixed by H , $o(A_{x_t}) \geq o(H)$. However, $o(H) \geq t!$ by (L6).

Suppose not all of the $\{X_i\}_{i=1}^{t-1}$ are singletons. Suppose $t \geq 4$. Let $|X_t| \leq t+1$. Suppose $|X_{t-1}| \geq 2$. Let $H_1 = \langle \alpha_z : z \in X_{t-1} \rangle$ and $H_2 = \langle \alpha_z : z \in X_{t-1} \cup X_{t-2} \rangle$. If H_3 reduces X_{t-2} , in addition $o(H_3) \geq 120$ by (L6 C4). Let $H_{t-1} = \langle \alpha_z : z \in \bigcup_{i=1}^{t-1} X_i \rangle$. By (L6 C4), $o(H_{t-1}) \geq (t+1)!$. Since $H_{t-2} \subseteq A_{(x_t)}$ and $|X_t| \leq t+1$, $o(A_{x_t}) \geq t!$, where $x_t \in X_t$.

Suppose $|X_{t-1}| = 1$. Suppose $t \geq 5$. The arguments just applied to the case $|X_{t-1}| \geq 2$ show that if $H_{t-2} = \langle \alpha_z : z \in \bigcup_{i=1}^{t-2} X_i \rangle$ is generated reducing from the top then $o(H_{t-2}) \geq t!$. As $H_{t-2} \subseteq A_{x_{t-1}}$ this implies $o(A_{x_{t-1}}) \geq t!$. Hence suppose $|X_{t-2}| = 1$ if $t \geq 5$.

Suppose $|X_1| \geq |X_t|$. If $|X_1| = 1$ then $|X_t| = 1$ and the result follows by (L6).

Suppose $|X_1| \geq 2$. Let $H_{t-1} = \langle \alpha_z : z \in \bigcup_{i=1}^{t-1} X_i \rangle$. By (L6), $o(H_{t-1}) \geq t!|X_1|$ and the result follows using $H_{t-1} \subseteq A_{(x_t)}$.

Suppose $|X_t| > |X_1|$. By the above argument $|X_{t-1}| = 1$. Let $|X_t| = 2$. Then $H_1 \cong S_3$. If $H_{t-2} = \langle \alpha_z : z \in \bigcup_{i=3}^t X_i \rangle$, then by (L6 C4), $o(H_{t-2}) \geq t!$. Since $|X_1| = 1$ the result follows.

Suppose $|X_1| = 1$ and $|X_t| \geq 3$. By (L6 C4), $o(H_3) \geq 5!$ and so if $t \geq 5$ then $o(H_{t-2}) \geq t!$ by (L6 C4). Hence $o(A_{x_t}) \geq t!$ since $H_{t-2} \subseteq A_{x_t}$. If $t = 4$ then $o(H_2) \geq 20$ by (L3) and (L4 C2). Since $o((H_3)_{x_3}) = o((H_3)_{x_1}) \geq o(H_2) \geq 20$ and x_3 is fixed by α_{x_1} , $o(A_{x_3}) \geq 4!$.

Suppose $|X_t| \geq 3$ and $|X_1| > 1$. Let $|X_t| = 3$. If $t > 5$ then by (L6 C4), $o(H_{t-2}) > t!$. If X_1 is an orbit of H_{t-2} then $o(A_{x_1}) \geq t!$ by (L7). If $|X_1|$ is not an orbit by (P7.4), $o(A_{x_1}) \geq 2t!/|X_1|$. Note that $|X_1| \leq 2$ since $|X_t| > |X_1|$. If $t = 4$ then if $H_2 = \langle \alpha_z: z \in X_1 \cup X_2 \rangle$, $o(H_2) \geq 12$ by (L6). Let $H_3 = \langle \alpha_z: z \in X_1 \cup X_2 \cup X_4 \rangle$. H_2 fixes x_3 so $o((H_3)_{x_4}) = o((H_3)_{x_3}) \geq 12$. By (L7 C1), $o((H_4)_{x_4}) \geq 24$. Let $|X_t| = 4, 5$, or 6 . By (L3), $o(H_1) > 2|X_t|$. By (L6 C1), $o(H_{t-2}) > (t+1) \cdots 5 \cdot 2|X_t|$. By (L8),

$$\begin{aligned} o(A_{x_1}) &\geq 2o(H_{t-2})/|X_1| \geq 2o(H_{t-2})/(|X_t| - 1) \\ &\geq 2(t+1) \cdots 5 \cdot 2|X_t|(|X_t| - 1) > t!. \end{aligned}$$

Let $|X_t| \geq 7$. Since $t \geq 6$ as $|X_t| \leq t+1$, $|X_{t-2}| = 1$. By (L3), $o(H_1) > |X_t| + 1 > 8$. Since x_{t-2} is fixed by H_1 , $o(H_2) \geq 9(|X_t| + 1) \geq 72$. By (L6), $o(H_{t-2}) > (t+1) \cdots 6 \cdot 9(|X_t| + 1)$. By (L8),

$$\begin{aligned} o(A_{x_1}) &\geq 2o(H_{t-2})/|X_1| \geq 2o(H_{t-2})/|X_t| \\ &\geq 2(t+1) \cdots 6 \cdot 9(|X_t| + 1)/|X_t| > t!. \end{aligned}$$

Let $|X_t| \geq t+2$ and let $|X_1| \geq \frac{1}{2}|X_t|$. Let $H_t = \langle \alpha_z: z \in \bigcup_{i=1}^{t-1} X_i \rangle$ be generated reducing from the bottom. By (L6), $o(H_{t-1}) \geq t!|X_1|$. Since X_t is not a singleton (L5) gives $o(A_{x_1}) \geq 2o(H_{t-1})/|X_t| \geq t!2|X_1|/|X_t| \geq t!$.

Suppose $|X_1| < \frac{1}{2}|X_t|$. Let $H_{t-2} = \langle \alpha_z: z \in \bigcup_{i=3}^{t-1} X_i \rangle$. Suppose $t > 4$. Hence $|X_t| \geq 7$ as $|X_t| \geq t+2$. By (L3), $o(H_1) \geq 8$. By (L6 C3), $o(H_{t-2}) \geq (t+1) \cdots 5|X_t| \geq (t+1) \cdots 5 \cdot 2|X_1|$. If X_1 is a singleton, then $o(A_{x_1}) \geq o(H_{t-2}) \geq (t+1) \cdots 5 \cdot 8 \geq t!$. If not, apply (L8) to obtain $o(A_{x_1}) \geq 2o(H_{t-2})/|X_1| \geq t!$. Let $t = 4$ and let $|X_4| = 6$. By (L3), $o(H_1) \geq 12$. By (L6 C3), $o(H_2) \geq 5o(H_1)$. If $|X_4| \geq 7$, $o(H_1) \geq 8$ by (L3). By (L6 C3), $o(H_2) \geq 5o(H_1) \geq 40$. If X_1 is a singleton, the result follows. Suppose X_1 is not a singleton. Let $|X_4| = 6$. By (L8), $o(A_{x_1}) \geq 2 \cdot 5o(H_1)/|X_1| \geq 20|X_4|/|X_1| \geq 40$. Let $|X_4| = 7$. By (L6 C3), $o(H_2) \geq 40$. By (L8), $o(A_{x_1}) \geq (2 \cdot 40)/3 \geq 4!$. Let $8 \leq |X_4| \leq 14$. By (L3) and (L6 C3), $o(H_2) \geq 5 \cdot 2|X_4|$. By (L8), $o(A_{x_1}) \geq 2 \cdot 5 \cdot 2|X_4|/|X_1| \geq 40$. Let $|X_4| = 15$. By (L3) and (L6 C3), $o(H_2) \geq 7 \cdot 16$. By (L8), $o(A_{x_1}) \geq 2 \cdot 7 \cdot 2|X_4|/|X_1| \geq 24$. Let $|X_4| \geq 16$. By (L3), $o(H_1) \geq 24$. By (L6 C3), $o(H_2) \geq 6|X_4|$. By (L8), $o(A_{x_1}) \geq 2 \cdot 6 \cdot 2|X_1|/|X_1| = 24!$.

Let $t = 3$. The proposition has been shown in the case X_1 and X_2 are singletons. Suppose $|X_3| \leq 4$. Let $H_1 = \langle \alpha_z: z \in X_2 \rangle$ and $H_2 = \langle \alpha_z: z \in X_2 \cup X_1 \rangle$. If X_3 is a singleton then $o((H_2)_{x_3}) \geq 3!$ by (L6). Suppose X_3 is not a singleton. Suppose $|X_2| \geq 2$. By (L6 C2), $o(H_2) \geq 20$. By (L5), $o(A_{x_3}) \geq 2o(H_2)/|X_3| \geq 20/2 \geq 10$. Suppose X_2 is a singleton and X_1 is not one. Let $H_1 = \langle \alpha_z: z \in X_1 \rangle$. By (L6), $o(H_2) \geq 6|X_1|$. By (L5),

$$o(A_{x_3}) \geq 2o(H_2)/|X_3| \geq 6 \cdot 2|X_1|/4 \geq 6.$$

Let $|X_3| \geq 5$. Suppose $2|X_1| \geq |X_3|$. Let $H_2 = \langle \alpha_z: z \in X_1 \cup X_2 \rangle$. By (L6), $o(H_2) \geq 6|X_1|$. By assumption, $|X_3| \geq 5$. By (L5),

$$o((H_3)_{x_3}) \geq \frac{2o(H_2)}{|X_3|} \geq 6.$$

Suppose $|X_3| \geq 2|X_1|$. Let $H_1 = \langle \alpha_z: z \in X_3 \rangle$. If X_1 is a singleton, then since $o(H_1) \geq 8$, by (L3) the result follows. If X_1 is a doubleton then by (L8), $o(A_{x_1}) \geq 2o(H_1)/2 \geq 8$. Suppose $|X_1| \geq 3$. If X_1 splits into 3 orbits under H_1 then there exists an orbit $O \subseteq X_1$, with $3|O| < |X_1|$. Hence for $x_1 \in O$, $o((H_1)_{x_1}) \geq o(H_1)/|O| \geq (|X_3| + 1)/|O| \geq 2|X_1|/|O| \geq 6$. Suppose X_1 splits into 2 orbits. If either orbit is a singleton, then $o(A_{x_1}) \geq o((H_1)_{x_1}) \geq 8$. Suppose neither is a singleton. Suppose only one of the orbits of H_1 on X_1 is part of a reducing orbit of H_2 or both are part of different reducing orbits. Let O be an orbit such that $2|O| < |X_1|$. Either O is part of the reducing orbit of H_2 or it is stabilized by H_2 . Since $|O| \geq 2$ and $\alpha_{x_2}^{-1}$ sends all but at most one element of O into O for $x_2 \in X_2$, there exists an $x_1 \in O$ such that $(x_1)\alpha_{x_2}^{-1} \in O$ for some $x_2 \in X_2$. By (P7.3) and (P7.1), there exists a $\psi \in H_1$ such that $\alpha_{x_2}^{-1}\psi^{-1} \notin (H_1)_{x_1}$. However

$$o((H_1)_{x_1}) \geq o(H_1)/|O| \geq (|X_3| + 1)/|O| \geq 2 \cdot 2 \cdot (|X_1| + 1)/|X_1| > 4.$$

Hence $o(A_{x_1}) \geq 8$. Suppose both orbits are part of a single reducing orbit of H . Since $o((H_1)_{x_1}) \geq 4$ for $x_1 \in O$, $o((H_2)_{x_3}) > 4$. Since X_3 is part of a single reducing orbit of H_2 and if $H_3 = \langle \alpha_z: z \in \bigcup_{i=0}^3 X_i \rangle$ then $o((H_3)_{x_3}) \geq 2o((H_2)_{x_3}) > 8$ by (L7 C1).

Suppose X_1 is a single orbit under H_1 . Since $o(H_1) \geq |X_3| + 1 \geq 2|X_1|$, $o((H_1)_{x_1}) \geq 2$. Since X_1 is not a singleton, $o((H_2)_{x_1}) \geq 4$ by (L7). Since X_3 is part of a single reducing orbit of H_2 , $o((H_2)_{x_3}) \geq 4$. By (L7 C1), $o((H_3)_{x_3}) \geq 8$.

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