LIAPOUNOFF'S THEOREM FOR NONATOMIC, FINITELY-ADDITIVE, BOUNDED, FINITE-DIMENSIONAL, VECTOR-VALUED MEASURES

BY

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ABSTRACT. Liapounoff's theorem states that if (X, Σ) is a measurable space and $\mu: \Sigma \rightarrow \mathbf{R}^d$ is nonatomic, bounded, and countably additive, then $\mathcal{R}(\mu) = \{\mu(A): A \in \Sigma\}$ is compact and convex. When Σ is replaced by a σ -complete Boolean algebra or an F -algebra (to be defined) and μ is allowed to be only finitely additive, $\mathcal{R}(\mu)$ is still convex. If Σ is any Boolean algebra supporting nontrivial, nonatomic, finitely-additive measures and Z is a zonoid, there exists a nonatomic measure on Σ with range dense in Z . A wide variety of pathology is examined which indicates that ranges of finitely-additive, nonatomic, finite-dimensional, vector-valued measures are fairly arbitrary.

1. Ranges need not be compact. Liapounoff's theorem, as proved by Halmos [10] or Lindenstrauss [13], state that if Σ is a σ -algebra of subsets of a set X and $\mu: \Sigma \rightarrow \mathbf{R}^d$ is a nonatomic countably additive function then the range, $\mathcal{R}(\mu) = \{\mu(A): A \in \Sigma\}$, of μ is a convex compact set. We are interested in the finitely additive analogue. Here Σ is replaced by some Boolean algebra $\mathcal{B}$ with supremum X and $\mu: \mathcal{B} \rightarrow \mathbf{R}^d$ denoted by $(\mu_1, \dots, \mu_d)$ where μ_j , the j th component of μ , is in $BA(\mathcal{B})$, the finitely-additive measures on $\mathcal{B}$ of bounded variation, and is nonatomic for all j . Recall [1], [14], [20], [21] that a *nonatomic* measure μ on $\mathcal{B}$ is one such that for any $\varepsilon > 0$ there is a partition $\{A_1, \dots, A_n\}$ of $\mathcal{B}$ such that the total variation, $|\mu|(A_j)$, of A_j under μ is at most ε . Such measures are also called continuous. A measure μ on $\mathcal{B}$ is nonatomic iff the corresponding Radon measure $\tilde{\mu}$ on the Stone space $X_{\mathcal{B}}$ is a nonatomic Radon measure. All measures absolutely continuous with respect to nonatomic measures are again nonatomic, both in $\mathcal{M}(X_{\mathcal{B}})$, the Radon measures on $X_{\mathcal{B}}$, and in $BA(\mathcal{B})$. If μ is a nonatomic vector-valued measure on $\mathcal{B}$ with values in $\mathbf{R}^d$ and l is any linear functional on $\mathbf{R}^d$, then $l \circ \mu$, defined by composition, is a nonatomic element of $BA(\mathcal{B})$. More generally, l may be a linear transformation of $\mathbf{R}^d$ into another vector space E , in which case $l \circ \mu$ is a nonatomic vector-valued measure with finite-dimensional range in E .

Liapounoff's theorem fails in the finitely-additive context only in the fact that $\mathcal{R}(\mu)$ need not be compact. Due to lack of compactness Lindenstrauss' elegant proof cannot be adapted to show convexity of $\mathcal{R}(\mu)$. An adaptation of Halmos' proof is available to establish this result (Theorem 2-2).

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LEMMA 1-1. Let $\mu: \mathfrak{B} \rightarrow \mathbb{R}^d$ be a bounded nonatomic measure on $\mathfrak{B}$. $\overline{\mathcal{R}(\mu)}$ is convex and compact. If $\mu = (\mu_1, \dots, \mu_d)$ and $\tilde{\mu} = (\tilde{\mu}_1, \dots, \tilde{\mu}_d)$ is the corresponding Baire measure on the Stone space $X_{\mathfrak{B}}$, then $\overline{\mathcal{R}(\mu)} = \mathcal{R}(\tilde{\mu}) = \{\tilde{\mu}(A): A \text{ Baire } \subset X_{\mathfrak{B}}\}$.

PROOF. There is a bijection $A \rightarrow [A]$ from $\mathfrak{B}$ to clopen sets of $X_{\mathfrak{B}}$. For any $A \in \mathfrak{B}$, $\mu(A) = \tilde{\mu}([A])$. Since clopen sets in $X_{\mathfrak{B}}$ are Baire, $\mathcal{R}(\mu) \subset \mathcal{R}(\tilde{\mu})$. Liapounoff's theorem guarantees that $\mathcal{R}(\tilde{\mu})$ is convex and compact, hence contains $\overline{\mathcal{R}(\mu)}$. Since $X_{\mathfrak{B}}$ is totally disconnected the Baire algebra is the monotone sequential closure of the clopen algebra. Let $\mathcal{E}$ denote those Baire sets $E \subset X_{\mathfrak{B}}$ with $\tilde{\mu}(E) \in \mathcal{R}(\mu)$. $\mathcal{E}$ contains the clopen algebra of $X_{\mathfrak{B}}$. If $\{E_n: n \in \mathbb{N}\}$ is a monotone sequence in $\mathcal{E}$ with limit E , then $\tilde{\mu}(E) = \lim_{n \rightarrow \infty} \tilde{\mu}(E_n) \in \overline{\mathcal{R}(\mu)}$. Thus, $\mathcal{E}$ is monotone sequentially closed, hence is the Baire algebra. This establishes the lemma. $\square$

REMARKS. (1) This is the strongest version of Liapounoff's theorem valid for general Boolean algebras. This is easily seen by considering the case where $\text{card}(\mathfrak{B}) = \aleph_0$, in which case $\mathcal{R}(\mu)$ is also countable.

(2) For a Boolean algebra $\mathfrak{B}$ to carry a nontrivial nonatomic measure it is necessary and sufficient that $\mathfrak{B}$ contain a countable nonatomic subalgebra $\mathfrak{B}_0$ (i.e., the clopen algebra of the Cantor set) [16, Theorem 1]. It is only in this context that the considerations in this article are nonvacuous.

COROLLARY 1-1-1. Let $\mu = (\mu_1, \mu_2, \dots, \mu_d)$ be a nonatomic measure on the Boolean algebra $\mathfrak{B}$. In order that $\{\mu_1, \dots, \mu_d\}$ be a mutually singular set of measures on $\mathfrak{B}$ it is necessary and sufficient that $\overline{\mathcal{R}(\mu)} = \prod_{j=1}^d [-\|\mu_j^-\|, \|\mu_j^+\|]$.

PROOF. We always have $-\|\mu_j^-\| \leq \mu_j(A) \leq \|\mu_j^+\|$ when $A \in \mathfrak{B}$. Hence, $\mathcal{R}(\mu)$ and, consequently, $\overline{\mathcal{R}(\mu)}$ are in $\prod_{j=1}^d [-\|\mu_j^-\|, \|\mu_j^+\|]$. The measures $\{\mu_1, \dots, \mu_d\}$ are mutually singular iff there exists a partition $\{A_1^+, A_1^-, \dots, A_d^+, A_d^-\}$ such that $\|\mu_j^+\| - \varepsilon \leq \mu_j(A_j^+) \leq \|\mu_j^+\|$ and $\|\mu_j^-\| - \varepsilon \leq -\mu_j(A_j^-) \leq \|\mu_j^-\|$ for all $j = 1, \dots, d$. By letting $A = \bigcup_{j=1}^d A_j^{\varepsilon(j)}$, where $\varepsilon: \{1, \dots, d\} \rightarrow \{+, -\}$ is arbitrarily chosen, we find that $\mu(A)$ is arbitrarily close to one of the corners of the rectangular parallelepiped $\prod_{j=1}^d [-\|\mu_j^-\|, \|\mu_j^+\|]$, hence each corner is in $\overline{\mathcal{R}(\mu)}$. Since $\overline{\mathcal{R}(\mu)}$ is convex the entire parallelepiped is in $\overline{\mathcal{R}(\mu)}$ if $\{\mu_1, \dots, \mu_d\}$ are mutually singular. Now suppose that μ is such that $\overline{\mathcal{R}(\mu)}$ is the entire parallelepiped. There is for any $i \neq j$ an $A \in \mathfrak{B}$ such that $\mu_i(A) \geq \|\mu_i^+\| - \varepsilon$ and $\mu_j(A) \geq \|\mu_j^+\| - \varepsilon$. We have then that $\mu_i^+(A) \geq \|\mu_i^+\| - \varepsilon$ and $\mu_j^-(A) \leq \varepsilon$. Since ε is arbitrary, $\mu_i^+ \perp \mu_j^-$. Similarly $\mu_i^+ \perp \mu_j^+$, $\mu_i^- \perp \mu_j^-$, and $\mu_i^- \perp \mu_j^+$. Thus $\mu_i \perp \mu_j$. Since $i \neq j$ are arbitrary the corollary is established. $\square$

THEOREM 1-2. If $\mathfrak{B}$ is a Boolean algebra admitting a nontrivial nonatomic measure there exists a $\mu: \mathfrak{B} \rightarrow \mathbb{R}^2$ which is nonatomic, nonnegative and finitely additive with $\mathcal{R}(\mu) \neq \overline{\mathcal{R}(\mu)}$.

PROOF. Assume to the contrary that $\mathcal{R}(\mu)$ is compact for all finitely-additive, nonatomic, nonnegative $\mu: \mathfrak{B} \rightarrow \mathbb{R}^2$. Let μ be a nonatomic element of $BA(\mathfrak{B})$ and let $\mu' = (\mu^+, \mu^-)$. There exists, by Corollary 1-1-1, an $A \in \mathfrak{B}$ such that $\mu^+(A) = \|\mu^+\|$ and $\mu^-(A) = 0$. As a consequence, $\mu(A) = \sup\{\mu(A'): A' \in \mathfrak{B}\} = \|\mu^+\|$.

Sobczyk and Hammer [20, Corollary 2.1] implies that μ restricted to A is μ^+ and μ restricted to A^c is $-\mu$. That is, $\mu(A') = \mu^+(A' \cap A) - \mu^-(A' \setminus A)$ for all A' . Thus, μ has a Hahn decomposition for all nonatomic $\mu \in BA^+(\mathfrak{B})$. As a consequence, if μ_1 and μ_2 are singular elements of $BA^+(\mathfrak{B})$ there is an $A \in \mathfrak{B}$ such that $\mu_1(A) = \|\mu_1\|$ and $\mu_2(A) = 0$, for if $\mu = \mu_1 - \mu_2$, then $\mu^+ = \mu_1$ and $\mu^- = \mu_2$. If $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are mutually singular elements of $\mathfrak{N}^+(X_{\mathfrak{B}})$, then there exists a clopen subset A of $X_{\mathfrak{B}}$ such that $\text{supp}(\tilde{\mu}_1) \subset A$ and $\text{supp}(\tilde{\mu}_2) \subset A^c$. Pick a nontrivial $\mu \in BA^+(\mathfrak{B})$ which is nonatomic. Let $\tilde{\mu}$ be the corresponding measure on $X_{\mathfrak{B}}$ and let $X_{\mu} = \text{supp}(\tilde{\mu})$. Since $\tilde{\mu} \neq 0$ is a nonatomic element of $\mathfrak{N}^+(X_{\mu})$; X_{μ} is not a scattered compact Hausdorff space (Semadeni [19, 19.7.6]). Since X_{μ} is nonscattered there is a closed separable perfect subset $Y \neq \emptyset$ such that $\tilde{\mu}(Y) = 0$ (Semadeni [19, 19.7.8(c)]). Since Y is nonscattered there exists a $\nu \in \mathfrak{N}^+(Y)$ which is nonatomic and nontrivial. Consider $\tilde{\nu}$ as an element of $\mathfrak{N}^+(X_{\mathfrak{B}})$ and let ν be the corresponding nonzero nonatomic element of $BA^+(\mathfrak{B})$. Since $\tilde{\mu}(Y) = 0$ and $\tilde{\nu}(Y) = \|\nu\|$, $\tilde{\mu}$ and $\tilde{\nu}$ are singular as are μ and ν . By assumption there is an $A \in \mathfrak{B}$ with $\mu(A) = 0$ and $\nu(A) = \|\nu\|$. If $[A]$ is the corresponding clopen subset of $X_{\mathfrak{B}}$ we have $Y \subset [A]$ and $X_{\mu} \cap [A] \neq \emptyset$, which is impossible, since $Y \subset X_{\mu}$. This contradicts our assumption that $\mathfrak{R}(\mu)$ is compact for all $\mu: \mathfrak{B} \rightarrow \mathbf{R}^2$ which are nonnegative and nonatomic. $\square$

COROLLARY 1-2-1. *If $\mathfrak{B}$ supports a nontrivial, nonatomic measure there is a nonatomic, nonnegative $\mu: \mathfrak{B} \rightarrow \mathbf{R}^2$ with singular components $\mu_1 \neq 0$ and $\mu_2 \neq 0$ so that $(\|\mu_1\|, 0)$ and $(0, \|\mu_2\|)$ are in $\overline{\mathfrak{R}(\mu)} \setminus \mathfrak{R}(\mu)$.*

COROLLARY 1-2-2. *With the same assumptions on $\mathfrak{B}$ there exists $\mu \in BA(\mathfrak{B})$ which is nonatomic with nontrivial positive and negative parts such that $\mathfrak{R}(\mu) = (-\|\mu^-\|, \|\mu^+\|)$.*

COROLLARY 1-2-3. *With the same assumptions on $\mathfrak{B}$ there is an uncountable family $\{\mu_{\alpha}\} \subset BA_1^+(\mathfrak{B})$ of nonatomic probabilities such that $\mu_{\alpha} \perp \mu_{\beta}$ if $\alpha \neq \beta$.*

PROOF. The proof of Theorem 1-2 demonstrated the existence, for any μ nonatomic on $\mathfrak{B}$, of a nonatomic ν with $\nu \perp \mu$. Let $\{\mu_{\alpha}\}$ be a maximal mutually singular collection in $BA_1^+(\mathfrak{B})$ consisting of nonatomic measures. If this collection is countable let μ be a countable convex combination so that $\mu_{\alpha} \ll \mu$ for all α . There is a ν nontrivial and nonatomic with $\nu \perp \mu$, hence with $\nu \perp \mu_{\alpha}$ for all α . Since ν may be chosen in $BA_1^+(\mathfrak{B})$, the maximality of $\{\mu_{\alpha}\}$ is contradicted. Thus, $\{\mu_{\alpha}\}$ is uncountable. $\square$

Recall that a nonnegative measure space (X, Σ, μ) is *semifinite* iff for any $A_0 \in \Sigma$ with $\mu(A_0) > 0$, one has $\mu(A_0) = \sup\{\mu(A): A \subset A_0, \mu(A) < \infty\}$. Also recall that $L^{\infty*+}(X, \Sigma, \mu)$ is the positive cone of the dual of $L^{\infty}(X, \Sigma, \mu)$.

COROLLARY 1-2-4. *Let (X, Σ, μ) be a nonnegative semifinite measure space. There exists in $L^{\infty*+}(X, \Sigma, \mu)$ an uncountable family of elements singular with respect to $L^1(X, \Sigma, \mu)$ provided that $L^1(X, \Sigma, \mu)$ is infinite dimensional.*

PROOF. By restriction to an element A of Σ with $0 < \mu(A) < \infty$, one may assume that μ is a finite nonatomic measure on (X, Σ) . $L^{\infty*+}(X, \Sigma, \mu)$ may be considered

as $BA^+(\Sigma_\mu)$ where Σ_μ is the quotient of Σ modulo μ -negligible sets. Apply Corollary 1-2-3 to $\mathfrak{B} = \Sigma_\mu$. If this assumption is not valid then there is a σ -finite $A \in \Sigma$ which is a union of $\aleph_0$ μ -atoms. By restriction of μ to A one can assume that $L^\infty(X, \Sigma, \mu)$ is Banach lattice isomorphic to $l^\infty \cong C(\beta N)$. Since βN supports nonatomic Radon probabilities, Corollary 1-2-3 is once again applicable since there is a nonatomic probability measure on Σ_μ . $\square$

REMARKS. (1) In Corollaries 1-2-3 and 1-2-4 one can obtain mutually singular families of cardinality c and 2^c , respectively, where $c = 2^{\aleph_0}$. See Corollaries 3-2-2 and 3-2-3.

(2) Do there exist uncountable mutually singular families of countably-additive nonatomic probabilities on a measurable space (X, Σ) ? When Σ is only required to be a σ -complete Boolean algebra, the answer is no. Take (X, Σ, μ) to be a nonatomic probability space. All countably additive measures on Σ_μ , the quotient of Σ with the μ -negligible sets, are absolutely continuous with respect to μ and the countable chain conditions on Σ_μ rules out having uncountably many, disjoint, countably-additive probabilities on Σ_μ . If Σ is the Baire algebra of a nonscattered compact Hausdorff space there do exist such families. One instance of a measurable space (X, Σ) not admitting such a family is when Σ contains all sets of outer measure 0 for some probability measure μ . In this case only countably many members of a mutually singular family of countably-additive probabilities on Σ are not singular with respect to μ . If the family is uncountable there is a μ -negligible set A and a nonatomic countably-additive probability ν with $\nu(A) = 1$. In this case ν is a nonatomic countably-additive probability defined on 2^A , hence $\text{card}(A)$ is a real-valued measurable cardinal [3]. The converse is also true.

2. Ranges are convex. Let $\mathfrak{B}$ be an F -algebra, which is defined to be an algebra whose Stone space is an F -space in the sense of Gillman and Jerison [8], [18]. Seever shows that $\mathfrak{B}$ is an F -algebra iff whenever $\{A_n\}$ is an increasing sequence and $\{B_n\}$ is a decreasing sequence in $\mathfrak{B}$ with $A_n \subset B_n$ for all n , there is a $C \in \mathfrak{B}$ with $A_n \subset C \subset B_n$ for all n . This may serve as our definition of an F -algebra. Any σ -complete algebra $\mathfrak{B}$ is an F -algebra. Seever shows that any closed subspace of an F -space is an F -space, hence it follows that any quotient algebra $\mathfrak{B}/\eta$ via an ideal η is an F -algebra if $\mathfrak{B}$ is an F -algebra. The chief means of constructing F -algebras is as quotients of σ -complete algebras. Under the assumption of Martin's Axiom and $c = \aleph_2$, van Douwen and van Mill [22] have constructed an F -algebra not arising in this fashion.

We shall need the following lemma, which is the one-dimensional version of Liapounoff's theorem on F -algebras.

LEMMA 2-1 (MAHARAM [14]). *Let $\mathfrak{B}$ be an F -algebra and let $\mu \in BA^+(\mathfrak{B})$. $\mathcal{R}(\mu) = [0, \|\mu\|]$ if μ is nonatomic.*

PROOF. Suppose we have found sequences $A_1 \subset A_2 \subset \cdots \subset A_n \subset B_n \subset \cdots \subset B_1$ in $\mathfrak{B}$ so that $\lambda - 2^{-n} \leq \mu(A_n) \leq \mu(B_n) \leq \lambda + 2^{-n}$ where $\lambda \in (0, \|\mu\|)$ is fixed beforehand. Find a partition $\{C_1, \dots, C_n\}$ of $B_n \setminus A_n$ into sets of

μ measure at most 2^{-n-1} . Let m_0 be the last integer, possibly 0, so that $\mu(A_n) + \sum_{i=1}^{m_0} \mu(C_i)$ is less than or equal to λ . Set $A_{n+1} = A_n \cup \bigcup_{i=1}^{m_0} C_i$ and set $B_{n+1} = A_n \cup \bigcup_{i=1}^{m_0+1} C_i$. We have $A_n \subset A_{n+1} \subset B_{n+1} \subset B_n$ and $\lambda - 2^{-n-1} < \mu(A_{n+1}) < \mu(B_{n+1}) \leq \lambda + 2^{-n-1}$. Find a $C \in \mathfrak{B}$ satisfying $A_n \subset C \subset B_n$ for all n . We have $\mu(C) = \lambda$. Since $\lambda \in (0, \|\mu\|)$ is arbitrary, $\mathfrak{R}(\mu) = [0, \|\mu\|]$. $\square$

REMARK. (1) Obvious modifications of the proof of Lemma 2-1 demonstrates that if μ is a finite-dimensional, nonatomic, nonnegative, finitely-additive measure on the F -algebra $\mathfrak{B}$, then, when $\{x^1, x^2\} \subset \mathfrak{R}(\mu)$, there exists a continuous path in $\mathfrak{R}(\mu)$ starting at x^1 and ending at x^2 . This hints that $\mathfrak{R}(\mu)$ is convex.

(2) Lemma 2-1 has been established by Granirer [9] for left invariant means on certain infinite semigroups. When suitably interpreted, L. J. Savage [17, Theorem 3.3.2], establishes Lemma 2-1 for nonatomic measures on the power set of a discrete set.

THEOREM 2-2. Let μ be a finite-dimensional, nonatomic, bounded, finitely-additive measure on the F -algebra $\mathfrak{B}$. $\mathfrak{R}(\mu)$ is convex.

PROOF. We first assume that the theorem is true for all n -dimensional nonnegative measures μ and demonstrate its validity for $(n+1)$ -dimensional measures. By induction and Lemma 2-1 the theorem is true for all finite dimensional nonnegative μ . To obtain the theorem for a general $(\mu_1, \dots, \mu_n) = \mu$, one appeals to its validity for $(\mu_1^+, \mu_1^-, \dots, \mu_n^+, \mu_n^-) = \nu$ and the fact that $\mathfrak{R}(\mu)$ and $\mathfrak{R}(\bar{\mu})$ are the linear images of $\mathfrak{R}(\nu)$ and $\mathfrak{R}(\bar{\nu})$ under the obvious transformations.

Let $(\mu_1, \dots, \mu_n, \mu_{n+1}) = \mu$ be a nonnegative, nonatomic measure on $\mathfrak{B}$ such that μ_{n+1} is absolutely continuous with respect to $(\mu_1, \dots, \mu_n) = \mu'$, so that if $\{A_n\}$ is a sequence in $\mathfrak{B}$ with $\mu'(A_n) \rightarrow 0$ then $\mu_{n+1}(A_n) \rightarrow 0$. By inductive assumption $\mathfrak{R}(\mu')$ is convex. For any $E \in \mathfrak{B}$ there is an $E_{1/2} \subset E = E_1$ with $\mu'(E_{1/2}) = \frac{1}{2}\mu'(E)$. There are, as a result, $E_{1/4}$ and $E_{3/4}$ with $E_{1/4} \subset E_{1/2}$, with $E_{1/2} \subset E_{3/4} \subset E_1$, with $\mu'(E_{1/4}) = \frac{1}{4}\mu'(E)$ and $\mu(E_{3/4}) = \frac{3}{4}\mu'(E)$. By induction, for all dyadic rationals k in $[0, 1]$, there is an E_k satisfying $E_k \subset E_{k'}$ for $k < k'$ with k' dyadic and satisfying $\mu'(E_k) = k\mu'(E)$. Here $E_0 = \emptyset$. Since $\mathfrak{B}$ is an F -algebra there exist $\{E_\lambda: \lambda \in [0, 1]\}$, with $E_\lambda \subset E_{\lambda'}$ if $\lambda \leq \lambda'$, and $\mu'(E_\lambda) = \lambda\mu'(E)$. If E and F are disjoint let $\{E_\lambda: \lambda \in [0, 1]\}$ and $\{F_\lambda: \lambda \in [0, 1]\}$ be the corresponding families obtained by the previous construction. If $C_\lambda = E_{1-\lambda} \cup F_\lambda$ then $\mu'(C_\lambda) = (1-\lambda)\mu'(E) + \lambda\mu'(F)$ for all $\lambda \in [0, 1]$, $C_0 = E$, and $C_1 = F$. We have

$$C_\lambda \triangle C_\delta = (F_\lambda \setminus F_\delta) \cup (E_{1-\delta} \setminus E_{1-\lambda}) \quad \text{if } \lambda \geq \delta$$

so $\mu'(C_\lambda \triangle C_\delta) \rightarrow 0$ if $\lambda - \delta \rightarrow 0$. If E and F are not disjoint, replace them by $E \setminus F$ and $F \setminus E$ in the above construction. To each C_λ obtained adjoin the disjoint set $E \cap F$ to get a new C_λ . Again $\mu'(C_\lambda) = (1-\lambda)\mu'(E) + \lambda\mu'(F)$ and if $\lambda - \delta \rightarrow 0$ then $\mu'(C_\lambda \triangle C_\delta) \rightarrow 0$. Consequently, $\mu_{n+1}(C_\lambda \triangle C_\delta) \rightarrow 0$ if $\lambda - \delta \rightarrow 0$.

Let $A \in \mathfrak{B}$. We will show that there is an $A_{1/2} \subset A$ with $\mu(A_{1/2}) = \frac{1}{2}\mu(A)$. If this is the case for all $A \in \mathfrak{B}$, imitation of the previous paragraph shows that if $\{E, F\} \subset \mathfrak{B}$, there is a family $\{C_\lambda: \lambda \in [0, 1]\}$ such that $C_0 = E$, $C_1 = F$, $\mu(C_\lambda) = (1-\lambda)\mu(E) + \lambda\mu(F)$ for $0 \leq \lambda \leq 1$, and $\mu(C_\lambda \triangle C_\delta) \rightarrow 0$ as $\lambda - \delta \rightarrow 0$. Thus, in this case, $\mathfrak{R}(\mu)$ is convex. There is an $E \subset A$ with $\mu'(E) = \frac{1}{2}\mu'(A)$. Let $F = A \setminus E$

so $\mu'(F) = \frac{1}{2}\mu'(A)$. Let $\{C_\lambda\}$ be the family constructed in the previous paragraph with $\mu'(C_\lambda) = (1 - \lambda)\mu'(E) + \lambda\mu'(F) = \frac{1}{2}\mu'(A)$. Since μ_{n+1} is absolutely continuous with respect to μ' , the function $\lambda \rightarrow \mu_{n+1}(C_\lambda)$ is continuous in λ . If $\mu_{n+1}(E) > \frac{1}{2}\mu_{n+1}(A)$, then $\mu_{n+1}(C_0) > \frac{1}{2}\mu_{n+1}(A) > \mu_{n+1}(C_1)$, otherwise $\mu_{n+1}(C_0) < \frac{1}{2}\mu_{n+1}(A) < \mu_{n+1}(C_1)$. In any case there is a λ such that $\mu_{n+1}(C_\lambda) = \frac{1}{2}\mu_{n+1}(A)$. Let $A_{1/2} = C$ for this λ . Thus, when μ_{n+1} is absolutely continuous with respect to μ' , $\mathcal{R}(\mu)$ is convex.

Now suppose that $(\mu_1, \dots, \mu_{n+1})$ is any nonatomic, nonnegative, finitely-additive measure on $\mathcal{B}$. Set $\nu = \sum_{i=1}^{n+1} \mu_i$. The measure $\nu = (\nu_1, \dots, \nu_{n+1})$ is nonatomic and has ν_{n+1} absolutely continuous with respect to $(\nu_1, \dots, \nu_n)$. As in Halmos [10, Lemma 6], $\mathcal{R}(\mu)$ is a 1-1 linear image of $\mathcal{R}(\nu)$. Since $\mathcal{R}(\nu)$ is convex, $\mathcal{R}(\mu)$ is also convex. This establishes that the range of any finite-dimensional vector measure is convex. $\square$

COROLLARY 2-2-1. (a) Let $\mathcal{B}$ be an F -algebra and μ a nonnegative and nonatomic finitely-additive map from $\mathcal{B}$ to R^d with $d < \infty$. For any $E \in \mathcal{B}$ there is an element f_E of the continuous functions $\mathcal{C}(X_{\mathcal{B}})$ on the Stone space $X_{\mathcal{B}}$ of $\mathcal{B}$ such that $0 \leq f_E \leq \chi_{[E]}$ (where $[E]$ is the clopen set in $X_{\mathcal{B}}$ corresponding to E), such that $\tilde{\mu}\{f_E < \lambda\} = \lambda\mu(E)$ for $0 \leq \lambda \leq 1$. Here $\tilde{\mu}$ corresponds to μ under the Stone correspondence.

(b) If $\mathcal{B}$ is an F -algebra of subsets of a set X there is a $\mathcal{B}$ -measurable function f_E on X with $0 \leq f_E \leq \chi_E$, for an $E \in \mathcal{B}$, such that $\mu\{f < \lambda\} = \lambda\mu(E)$ for $0 \leq \lambda \leq 1$.

PROOF. $\mathcal{B}$ is an F -algebra iff $\{[E]: E \in \mathcal{B}\} = [\mathcal{B}]$ is an F -algebra of subsets of $X_{\mathcal{B}}$. The continuous functions on $X_{\mathcal{B}}$ as uniform limits of step functions based on clopen subsets of $X_{\mathcal{B}}$ are precisely the bounded $[\mathcal{B}]$ -measurable functions on $X_{\mathcal{B}}$. Thus, (a) follows from (b). To establish (b) one takes the family E_λ constructed in the proof of Theorem 2-2 and sets f_E equal to the uniform limit of $\{f_n: n \in N\}$ where f_n is the step function equal to $k \cdot 2^{-n}$ on $E_{(k+1)2^{-n}} \setminus E_{k2^{-n}}$ for $k < 2^n$.

REMARKS. (1) The proof of Theorem 2-2 is a simplification of Halmos' proof [10] of the convexity of the range of a countably-additive, finite-dimensional, nonatomic, vector-valued measure. Weiss [23] establishes Theorem 2-2 for the case $\mathcal{B} = 2^X$ for a set X . His proof closely follows Halmos' proof.

(2) Lemma 1-1 is a consequence of Theorem 2-2.

(3) Margolies [15] has shown that all left invariant means on infinite groups are nonatomic. Theorem 2-2 shows that the joint range, $(\mu_1(A), \dots, \mu_d(A))$ as A ranges over subsets of the group, of finitely many left invariant means $(\mu_1, \dots, \mu_d)$ is convex. Results of Granirer [9] allow this result to be extended to left invariant means on right cancellative semigroups.

In the proof of Theorem 2-2 it is not necessary to use the full strength of the interpolation property defining F -algebras. It was noted by Margolies [15] that what is necessary is that if $\mu = (\mu_1, \dots, \mu_d)$, $\{A_n: n \in N\}$ and $\{B_n: n \in N\}$ are sequences in the algebra $\mathcal{B}$ with $A_n \subset A_{n+1} \subset B_{n+1} \subset B_n$ for all n , and if j is such that $\mu_j^+(B_n \setminus A_n) \rightarrow 0$, or $\mu_j^-(B_n \setminus A_n) \rightarrow 0$, there exists a C in $\mathcal{B}$ satisfying $A_n \subset C \subset B_n$ for all n . This property is called μ -completeness of $\mathcal{B}$. Then $\mathcal{B}$ is μ -complete iff it is μ_j^+ and μ_j^- complete for all j .

The same definition may be used to define μ -complete subsets $\mathcal{E}$ of $\mathcal{B}$. One may ask: "For which algebras $\mathcal{B}$ does one have μ -completeness for all nonatomic μ ?"

If $A \subset B$ are in $\mathcal{B}$ the restriction of a nonatomic μ to $\{C: C \in \mathcal{B}, C \subset B \setminus A\}$ is a nonatomic measure on this Boolean algebra. Hence $\{\mu(C): C \in \mathcal{B}, A \subset C \subset B\}$ is convex when the order interval $\langle A, B \rangle = \{C \in \mathcal{B}: A \subset C \subset B\}$ is μ -complete. If $\mathcal{F}$ is a filter on $\mathcal{B}$ and $\mathcal{G}$ is an ideal then $\mathcal{F} \cap \mathcal{G}$ is an increasing union of order intervals $\langle A, B \rangle \subset \mathcal{F} \cap \mathcal{G}$. Thus we have the following corollary.

COROLLARY 2-2-2. *Let $\mathcal{F}$ be a filter and $\mathcal{G}$ be an ideal in the Boolean algebra $\mathcal{B}$. If $\mathcal{F} \cap \mathcal{G}$ is μ -complete for the nonatomic, finite-dimensional, finitely-additive measure μ then $\mu(\mathcal{F} \cap \mathcal{G}) = \{\mu(C): C \in \mathcal{F} \cap \mathcal{G}\}$ is convex.*

REMARK. Margolies has extended Theorem 2-2 to show that $\mu(\mathcal{E})$ is convex if $\mathcal{E}$ is a μ -complete linear system in a Boolean algebra $\mathcal{B}$ on which μ is nonatomic. A linear system is defined as a subset of $\mathcal{B}$ so that the set of characteristic functions $\{\chi_{[A]}: A \in \mathcal{E}\}$ of the clopen sets $[A]$ in the Stone space of $\mathcal{B}$ corresponding to elements A of $\mathcal{E}$ are the $\{0, 1\}$ -valued elements in a linear variety of real functions on the Stone space. Linear systems $\mathcal{E}$ are closed under complementation and disjoint union but not necessarily under intersection. Lack of closure under intersection causes our proof of Theorem 2-2 to break down. Given Margolies' result, an extension of Corollary 2-2-2 should be possible for affine systems $\mathcal{E}$ where $\{\chi_{[A]}: A \in \mathcal{E}\}$ are the $\{0, 1\}$ -valued elements in an affine variety of functions on the Stone space of $\mathcal{B}$.

If $\mathcal{B}$ is an infinite F -algebra and $\mu \in BA_1^+(\mathcal{B})$, the quotient $\mathcal{B}_\mu$ of $\mathcal{B}$ modulo μ -negligible sets is an F -algebra which satisfies the countable chain condition. From this it may be shown that $\mathcal{B}_\mu$ is complete. This can be deduced from a result of Seever that the support of any measure on an F -space is Stonian [18, Theorem 2.2].

PROPOSITION 2-3. *If $\mathcal{B}$ is an infinite F -algebra there exists a nonatomic $\mu \in BA_1^+(\mathcal{B})$. There are no infinite scattered F -spaces.*

PROOF. If not, the Stone space $X_{\mathcal{B}}$ is scattered and infinite, hence it contains a sequence $\{x_n: n \in N\}$ of distinct isolated points. Let Y be the closure of this sequence. Y is extremely disconnected and $\{x_n\}$ are isolated in Y . For any $A \subset N$, $Y_A = \overline{\{x_n: n \in A\}}$ is a clopen subset of Y . The mapping $A \rightarrow Y_A$ is an isomorphism of the clopen algebra of 2^N into the clopen algebra of Y . Since there is a nonatomic probability on 2^N there is one on the clopen algebra of Y . Hence Y has a nonatomic Radon probability, hence $X_{\mathcal{B}}$ has a nonatomic probability Radon measure, contradicting the scatteredness of $X_{\mathcal{B}}$. Thus, $X_{\mathcal{B}}$ is nonscattered. Thus, $\mathcal{B}$ admits a nonatomic probability measure.

If X is an F -space which is infinite and scattered it is totally disconnected and is the Stone space of its F -algebra $\mathcal{B}$ of clopen sets. Since $\mathcal{B}$ admits a nonatomic probability, X could not have been scattered. $\square$

REMARK. This is established in [16, Corollary 1] for σ -complete algebras.

If $\mathcal{B}$ is a σ -algebra and μ is countably additive then $\mathcal{R}(\mu)$ is compact. It is not known whether this is true if $\mathcal{B}$ is an F -algebra. The following proposition states

that this is true iff all countably additive elements of $BA(\mathfrak{B})$ have Hahn decompositions.

PROPOSITION 2-4. *Let $\mathfrak{B}$ be an F -algebra. Let F be a vector space of nonatomic measures on $\mathfrak{B}$. These two statements are equivalent.*

- (1) $\mathcal{R}(\mu)$ is compact if all components of μ are in F .
- (2) If $\mu \in F$ then μ has a Hahn decomposition.

PROOF. The implication (1) $\rightarrow$ (2) is immediate. To see that (2) $\rightarrow$ (1) note that we need only show that $\mathcal{R}(\mu)$ contains all extreme points of $\overline{\mathcal{R}(\mu)}$, for $\overline{\mathcal{R}(\mu)}$ is the convex hull of its extreme points and $\mathcal{R}(\mu)$ is convex. If x is an extreme point let l be a linear functional attaining its maximum on $\mathcal{R}(\mu)$ at x . The function $l \circ \mu: A \rightarrow l(\mu(A))$ is an element of F since all components of μ are in F . Since $l \circ \mu$ has a Hahn decomposition there is an $A \in \mathfrak{B}$ such that $l(\mu(A)) = l(x)$. We have $x \in \mathcal{R}(\mu)$. $\square$

If $\mathfrak{B}$ is a Boolean algebra, and μ is a bounded, finitely-additive, finite-dimensional, nonatomic vector measure on $\mathfrak{B}$, let $\tilde{\mu}$ be the corresponding measure on $X_{\mathfrak{B}}$. If we set $\mathcal{R}'(\mu) = \{ \int f d\tilde{\mu} : f \in \mathcal{C}(X_{\mathfrak{B}}) : 0 \leq f \leq 1 \}$, then $\mathcal{R}'(\mu)$ is convex and $\mathcal{R}(\mu) \subset \mathcal{R}'(\mu) \subset \overline{\mathcal{R}(\mu)}$. These notations are understood in Propositions 2-5 and 2-6.

PROPOSITION 2-5. $\mathcal{R}'(\mu) = \text{conv}(\mathcal{R}(\mu))$.

PROOF. That $\text{conv}(\mathcal{R}(\mu)) \subset \mathcal{R}'(\mu)$ is immediate. If $x \in \mathcal{R}'(\mu) \setminus \text{conv}(\mathcal{R}(\mu))$ there is a linear functional l such that $l(x)$ is strictly larger than $l(y)$ if $y \in \text{conv}(\mathcal{R}(\mu))$. If $\mu_l = l \circ \mu$ then $\mu_l \in BA(\mathfrak{B})$ with $\mathcal{R}(\mu_l) = (-\|\mu_l^-\|, \|\mu_l^+\|)$ with $\|\mu_l^+\| = l(x)$. Since $x \in \mathcal{R}'(\mu)$ we have $l(x) \in \mathcal{R}'(\mu_l)$, so there is an $f \in \mathcal{C}(X_{\mathfrak{B}})$ with $\int f d\tilde{\mu}_l^+ = \|\mu_l^+\|$ and $0 \leq f \leq 1$. Since $\int f d\tilde{\mu}_l \leq \int f d\tilde{\mu}_l^+ \leq \|\mu_l^+\|$ we have $\int f d\tilde{\mu}_l^+ = \|\mu_l^+\|$ and $\int f d\tilde{\mu}_l^- = 0$. Thus, $\tilde{\mu}_l^+ \{f = 1\} = \|\mu_l^+\|$. Let $A \in \mathfrak{B}$ be such that $[A]$ is a clopen set containing $\{f = 1\}$ and disjoint from $\{f = 0\}$. We have $\mu_l^+(A) = \tilde{\mu}_l^+([A]) = \|\mu_l^+\|$ and $\mu_l^-(A) = \tilde{\mu}_l^-([A]) = 0$, so $\mu_l(A) = l(x)$, which is impossible. Hence, $\mathcal{R}'(\mu) \subset \text{conv}(\mathcal{R}(\mu))$, which establishes the proposition. $\square$

PROPOSITION 2-6 (LINDENSTRAUSS). *If $x \in \mathcal{R}'(\mu)$ is such that $\{f \in \mathcal{C}(X_{\mathfrak{B}}) : \int f d\tilde{\mu} = x\}$ has an extreme point f_0 , then $f_0 = \chi_{[A]}$ for some $A \in \mathfrak{B}$. Thus, $x \in \mathcal{R}(\mu)$.*

PROOF. Lindenstrauss [13]. $\square$

3. Ranges of two-dimensional, nonatomic, nonnegative measures with singular components. If $\mu = (\mu_1, \mu_2)$ is a nonatomic, nonnegative measure with singular components on an infinite F -algebra $\mathfrak{B}$, then $\mathcal{R}(\mu)$ is a convex subset of the closed rectangle $[0, \|\mu_1\|] \times [0, \|\mu_2\|]$ which contains the open rectangle $(0, \|\mu_1\|) \times (0, \|\mu_2\|)$ and the corners $(0, 0)$ and $(\|\mu_1\|, \|\mu_2\|)$. We will examine which convex sets in the closed rectangle are actually possible for such a μ . We will assume without loss of generality that $\|\mu_1\| = \|\mu_2\| = 1$ so we are dealing with convex subsets of the unit square $[0, 1] \times [0, 1]$. We remark that the involution $A \rightarrow A^c$ on $\mathfrak{B}$ requires that $\mathcal{R}(\mu)$ be symmetric with respect to the center $(\frac{1}{2}, \frac{1}{2})$ of the square.

One possibility for $\mathfrak{R}(\mu)$ is that it be the entire closed square, which is true iff it contains either $(1, 0)$ or $(0, 1)$, hence the other. This is the case iff μ_1 and μ_2 assign full measure to disjoint elements of $\mathfrak{B}$. Such μ_1 and μ_2 exist on any infinite F -algebra by Proposition 2-3.

Is it possible for $\mathfrak{R}(\mu)$ to omit only $(1, 0)$ and $(0, 1)$? The answer is negative.

PROPOSITION 3-1. *If $\mu = (\mu_1, \mu_2)$ with μ_1 and μ_2 mutually singular nonatomic probabilities on the infinite F -algebra, it is impossible that $\mathfrak{R}(\mu)$ omits only $(0, 1)$ and $(1, 0)$ from the closed unit square.*

PROOF. Suppose that for every $\varepsilon > 0$ the points $(1 - \varepsilon, 0)$ and $(0, 1 - \varepsilon)$ belong to $\mathfrak{R}(\mu)$. We assert that there are disjoint $A(\varepsilon)$ and $B(\varepsilon)$ in $\mathfrak{B}$ such that $\mu(A(\varepsilon)) = (1 - \varepsilon, 0)$ and $\mu(B(\varepsilon)) = (0, 1 - \varepsilon)$. Let $\mu(\tilde{A}(\varepsilon)) = (1 - \varepsilon, 0)$ and $\mu(\tilde{B}(\varepsilon)) = (0, 1 - \varepsilon)$. Set $A(\varepsilon) = \tilde{A}(\varepsilon) \setminus \tilde{B}(\varepsilon)$ and $B(\varepsilon) = \tilde{B}(\varepsilon) \setminus \tilde{A}(\varepsilon)$. Since $A(\varepsilon) \subset \tilde{A}(\varepsilon)$, $\mu_2(A(\varepsilon)) = 0$ and

$$\mu_1(A(\varepsilon)) = \mu_1(\tilde{A}(\varepsilon)) + \mu_1(\tilde{A}(\varepsilon) \cap \tilde{B}(\varepsilon)) = \mu_1(\tilde{A}(\varepsilon)) = 1 - \varepsilon.$$

Thus, $\mu(A(\varepsilon)) = (1 - \varepsilon, 0)$ and, similarly, $\mu(B(\varepsilon)) = (0, 1 - \varepsilon)$ with $A(\varepsilon)$ and $B(\varepsilon)$ disjoint. Suppose increasing sequences $\{A_k: k = 1, \dots, n\}$ and $\{B_k: k = 1, \dots, n\}$ have been constructed with $\mu(A_k) = (1 - \varepsilon^k, 0)$, $\mu(B_k) = (0, 1 - \varepsilon^k)$ and $A_k \cap B_k = \emptyset$ for all k , starting with $A_1 = A(\varepsilon)$, and $B_1 = B(\varepsilon)$. Let $A'_n = A_n \cup [A(\varepsilon^{n+1}) \setminus B_n]$ and $B'_n = B_n \cup [B(\varepsilon^{n+1}) \setminus A_n]$. We have $A'_n \cap B'_n = \emptyset$, $A_n \subset A'_n$, $B_n \subset B'_n$, $\mu_1(B'_n) = 0 = \mu_2(A'_n)$ and $\mu_1(A'_n) \geq 1 - \varepsilon^{n+1} \leq \mu_2(B'_n)$. There exist $A_n \subset A_{n+1} \subset A'_n$ and $B_n \subset B_{n+1} \subset B'_n$ with $\mu(A_{n+1}) = (1 - \varepsilon^{n+1}, 0)$ and $\mu(B_{n+1}) = (0, 1 - \varepsilon^{n+1})$. Thus, by induction, disjoint increasing sequences $\{A_n: n \in N\}$ and $\{B_n: n \in N\}$ are constructed with $\mu(A_n) = (1 - \varepsilon^n, 0)$ and $\mu(B_n) = (0, 1 - \varepsilon^n)$ for all n . Let $C_n = B_n^c$ so $\mu(C_n) = (1, \varepsilon^n)$ and $A_n \subset C_n$ for all n . Since $\mathfrak{B}$ is an F -algebra there is a $D \in \mathfrak{B}$ with $A_n \subset D \subset C_n$ for all n . For this D we have $\mu(D) = (1, 0)$. Thus, if $\mathfrak{R}(\mu)$ contains $(1 - \varepsilon, 0)$ and $(0, 1 - \varepsilon)$ for all $\varepsilon > 0$ it contains $(1, 0)$ and $(0, 1)$. This completes the proof of this proposition. $\square$

The proof of Theorem 1-2 demonstrated that if μ_1 is a nonatomic probability measure on a Boolean algebra $\mathfrak{B}$ with corresponding $\tilde{\mu}_1$ on the Stone space $X_{\mathfrak{B}}$, there is a nonatomic probability μ_2 on $\mathfrak{B}$ with corresponding measure $\tilde{\mu}_2$ on $X_{\mathfrak{B}}$ such that $\text{supp}(\tilde{\mu}_2)$ is a nowhere dense subset of $\text{supp}(\tilde{\mu}_1)$ with $\tilde{\mu}_1(\text{supp}(\tilde{\mu}_2)) = 0$. Consequently, when $\mu_1(A) = 0$ then $\mu_2(A) = 0$ and there is an increasing sequence $\{A_n: n \in N\} \subset \mathfrak{B}$ such that $\mu_1(A) = 0$ converges to 1, yet $\mu_2(A_n) = 0$ for all n . If $\mathfrak{B}$ is an F -algebra then $\mathfrak{R}(\mu)$ contains $(1 - \varepsilon, 0)$ for all $\varepsilon > 0$, yet contains no $(0, \lambda)$ with $\lambda > 0$. By averaging two such measures living on disjoint elements of $\mathfrak{B}$ where one has had its coordinates reversed, one may obtain a μ such that $\mathfrak{R}(\mu) \cap ([0, 1] \times \{0\}) = [0, \lambda] \times \{0\}$ and $\mathfrak{R}(\mu) \cap (\{0\} \times [0, 1]) = \{0\} \times [0, 1 - \lambda]$ for $0 \leq \lambda \leq 1$. By averaging with yet another measure living on a disjoint element of $\mathfrak{B}$ which has the closed unit square as its range, one may obtain a μ such that $\mathfrak{R}(\mu) \cap ([0, 1] \times \{0\}) = [0, \alpha] \times \{0\}$ and $\mathfrak{R}(\mu) \cap (\{0\} \times [0, 1]) = \{0\} \times (0, \beta)$ where $\{\alpha, \beta\} \subset [0, 1]$ and $\alpha + \beta \geq 1$. One may also find μ with $\mathfrak{R}(\mu) \cap ([0, 1] \times \{0\}) = [0, 1] \times \{0\}$ and $\mathfrak{R}(\mu) \cap (\{0\} \times [0, 1]) = [0, \beta]$ or with $\mathfrak{R}(\mu) \cap (\{0\} \times [0, 1]) = [0, \beta)$ with $\beta < 1$.

Does there exist a μ on $\mathfrak{B}$ so that $\mathcal{R}(\mu)$ is as small as possible so it consists of the open unit square $(0, 1) \times (0, 1)$ together with the corners $(0, 0)$ and $(1, 1)$? Equivalently, do there exist mutually singular nonatomic μ_1 and μ_2 such that if $A \in \mathfrak{B}$ then $\mu_1(A) > 0$ iff $\mu_2(A) > 0$? This latter question becomes, on the Stone space $X_{\mathfrak{B}}$, “Do there exist mutually singular nonatomic $\{\tilde{\mu}_1, \tilde{\mu}_2\} \subset \mathfrak{M}_1^+(X_{\mathfrak{B}})$ with $\text{supp}(\tilde{\mu}_1) = \text{supp}(\tilde{\mu}_2)$?” We may replace $X_{\mathfrak{B}}$ by any compact Hausdorff space X and ask whether such $\{\tilde{\mu}_1, \tilde{\mu}_2\} \subset \mathfrak{M}_1^+(X)$ exist. If $X = [0, 1]$ the existence of $\{\tilde{\mu}_1, \tilde{\mu}_2\}$ which are singular nonatomic and have $\text{supp}(\tilde{\mu}_1) = \text{supp}(\tilde{\mu}_2) = [0, 1]$ is well known.

PROPOSITION 3-2. *Let X be a nonscattered compact Hausdorff space. There exist $\{\tilde{\mu}_1, \tilde{\mu}_2\} \subset \mathfrak{M}_1^+(X)$ which are nonatomic, mutually singular and have the same support.*

PROOF. We may assume that X is perfect. Semadeni [19, 8.5.4 Theorem] shows that there is a continuous surjection $f: X \rightarrow [0, 1]$. Let Y be a minimal closed subset of X with $f(Y) = [0, 1]$. Thus, f is irreducible on Y . As a result, if $\tilde{\mu}$ is any measure on Y whose image, $\tilde{\mu} \circ f^{-1}$, under f has $[0, 1]$ as support then $\text{supp}(\tilde{\mu}) = Y$. If $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are measures on Y such that $\tilde{\mu}_1 \circ f^{-1} \perp \tilde{\mu}_2 \circ f^{-1}$ then $\tilde{\mu}_1 \perp \tilde{\mu}_2$. If $\tilde{\mu}$ is a measure on Y which has $\tilde{\mu} \circ f^{-1}$ nonatomic then $\tilde{\mu}$ is nonatomic. Since $[0, 1]$ has mutually singular nonatomic $\{v_1, v_2\} \subset \mathfrak{M}_1^+([0, 1])$ with $\text{supp}(v_i) = [0, 1]$ for $i = 1, 2$, we may find $\tilde{\mu}_i \in \mathfrak{M}_1^+(Y)$, via Hahn-Banach, with $\tilde{\mu}_i \circ f^{-1} = v_i$ for $i = 1, 2$. The measures $\{\tilde{\mu}_1, \tilde{\mu}_2\}$ are mutually singular nonatomic elements of $\mathfrak{M}_1^+(Y)$ with $\text{supp}(\tilde{\mu}_1) = Y = \text{supp}(\tilde{\mu}_2)$. This establishes the proposition. $\square$

COROLLARY 3-2-1. *There exists a family $\{\tilde{\mu}_t: t \in [0, 1]\} \subset \mathfrak{M}_1^+(X)$ such that $\tilde{\mu}_t \perp \tilde{\mu}_s$ if $t \neq s$, $\tilde{\mu}_t$ is nonatomic for all t and $\text{supp}(\tilde{\mu}_t) = \text{supp}(\tilde{\mu}_s)$ for all t, s .*

COROLLARY 3-2-2. *There exists on any Boolean algebra admitting a nontrivial nonatomic measure a family $\{\mu_t: t \in [0, 1]\} \subset BA_1^+(\mathfrak{B})$ such that $\mu_t \perp \mu_s$ if $t \neq s$, each μ_t is nonatomic and $\mu_t(A) = 0$ iff $\mu_s(A) = 0$ for all $A \in \mathfrak{B}$ and $\{t, s\} \subset [0, 1]$.*

For many algebras $\mathfrak{B}$ much larger mutually singular families as in Corollary 3-2-3 are possible.

COROLLARY 3-2-3. *If $\mathfrak{B}$ is an infinite F -algebra there exists a family $\{\mu_\alpha\} \subset BA_1^+(\mathfrak{B})$ of mutually singular nonatomic measures of cardinality 2^c with the property that if $A \in \mathfrak{B}$ then $\mu_\alpha(A) = 0$ iff $\mu_\beta(A) = 0$ for all α and β .*

PROOF. There exists a family $\{\nu_\alpha\}$ of mutually nonatomic Radon probability measures on $\{0, 1\}^c$ of cardinality 2^c each with support equal to $\{0, 1\}^c$. This may be constructed by using, for $\{\nu_\alpha\}$, coin flip measures with c flips and with varying probabilities of heads. All that is needed to establish the corollary is to find a closed subset of $X_{\mathfrak{B}}$ admitting $\{0, 1\}^c$ as a continuous image. To do this pick a nonatomic element of $\mathfrak{M}^+(X_{\mathfrak{B}})$ and let Z be its support. Seever [18] shows that Z is Stonian, hence has a complete (infinite) clopen algebra $\mathfrak{B}_Z$. One may find a disjoint sequence in $\mathfrak{B}_Z \setminus \{\emptyset\}$. The σ -complete algebra $\mathfrak{B}_0$ it generates is isomorphic to 2^N . There is, as a result, a continuous map from Z onto βN . Since $\{0, 1\}^c$ is

separable (Comfort and Negrepontis [6, Theorem 3.20]) there is a continuous map of βN onto $\{0, 1\}^c$. Thus, Z has $\{0, 1\}^c$ as a continuous image. $\square$

REMARK. It is possible to choose the measures in Corollary 3-2-4 so that if $\{\mu_{n_1}: n \in N\}$ and $\{\mu_{n_2}: n \in N\}$ are disjoint subsets there is an $A \subset N$ such that $\mu_{n_1}(A) \geq 1 - \varepsilon$ and $\mu_{n_2}(A) \leq \varepsilon$ for all n for a given $\varepsilon > 0$. This requires the use of some results from the theory of independent sets.

One example of an F -algebra is gotten by taking the power set 2^κ of a cardinal κ and the ideal $[\kappa]^{<\kappa}$ of subsets of cardinality less than κ . The quotient algebra $\mathfrak{B} = 2^\kappa / [\kappa]^{<\kappa}$ is an F -algebra which is not σ -complete when κ is of countable cofinality. The measures on $\mathfrak{B}$ are precisely those arising from uniform measures on κ (i.e., measures which annihilate all subsets of κ of smaller cardinal).

COROLLARY 3-2-4. *There is, for any infinite cardinal, κ , a collection $\{\mu_t\}$ of cardinality 2^c of uniform measures on κ which are nonatomic, have $\mu_t(\kappa) = 1$ for all t , are mutually singular and have the property that if $A \subset \kappa$, then $\mu_t(A) = 0$ iff $\mu_s(A) = 0$ for all s and t .*

PROPOSITION 3-3. *Let $\mathfrak{B}$ be an infinite F -algebra. The only convex subset A of $[0, 1] \times [0, 1]$ symmetric about $(\frac{1}{2}, -\frac{1}{2})$ which contains $((0, 1, 1) \times (0, 1)) \cup \{(0, 0), (1, 1)\}$ which is not $\mathcal{R}(\mu)$ for some $\mu = (\mu_1, \mu_2)$ with μ_1 and μ_2 mutually singular nonatomic elements of $BA^+(\mathfrak{B})$ is $([0, 1] \times [0, 1]) \setminus \{(0, 1), (1, 0)\}$.*

PROOF. This is easily verified based on Proposition 3-1, Corollary 3-2-2 and remarks in this section. We only remark that to obtain $((0, 1) \times (0, 1)) \cup ([0, \lambda] \times \{0\}) \cup ([1 - \lambda, 1] \times \{1\})$ one takes a pair $(\mu_1, \mu_2) = \mu$ with

$$\mathcal{R}(\mu) = ((0, 1) \times (0, 1)) \cup \{(0, 0), (1, 1)\}$$

by Corollary 3-2-2 then one picks an $A_0 \in \mathfrak{B}$ with $\mu_1(A_0) = \lambda = \mu_2(A_0)$. Finally set

$$\nu(A) = (\nu_1(A_1), \nu_2(A)) = (\mu_1(A), \mu_2(A \setminus A_0)(1 - \lambda)^{-1})$$

for $A \in \mathfrak{B}$. $\square$

Consider the case of $\mu = (\mu_1, \dots, \mu_n)$ with $\{\mu_1, \dots, \mu_n\}$ mutually singular nonatomic probability measures. $\mathcal{R}(\mu)$ is dense in $[0, 1]^n$ and is convex if μ is defined on an F -algebra, hence contains $(0, 1)^n$, $(0, \dots, 0)$, and $(1, \dots, 1)$. Use of Corollary 3-2-3 shows that it is possible that $\mathcal{R}(\mu)$ contains only the points $(0, \dots, 0)$ and $(1, \dots, 1)$ on the boundary. It is also possible that $\mathcal{R}(\mu)$ be all of $[0, 1]^n$. In general, to examine $\mathcal{R}(\mu)$ requires the examination of $\mathcal{R}(\mu) \cap F$ for faces F of the cube $[0, 1]^n$. From the results of [2, Lemma 3] a face F of $[0, 1]^n$ is of the form $\square_{A,B}^+ = \{(x_1, \dots, x_n): x_i = 0 \text{ if } i \notin B \text{ and } x_i = 1 \text{ if } i \in A\}$. $E \in \mathfrak{B}$ has $\mu(E) \in F$ iff $\mu_i(E) = 0$ if $i \notin B$ and $\mu_i(E) = 1$ if $i \in A$. Thus, if $\mathcal{G}_B$ is the ideal $\{E \in \mathfrak{B}: \mu_i(E) = 0 \text{ if } i \notin B\}$ and $\mathcal{F}_A$ is the filter $\{E \in \mathfrak{B}: \mu_i(E) = 1 \text{ if } i \in A\}$, then $\mathcal{R}(\mu) \cap F = \mu(\mathcal{G}_B \cap \mathcal{F}_A)$.

PROPOSITION 3-4. *Let $\mu = (\mu_1, \dots, \mu_n)$ with $\mu_1, \dots, \mu_n$ mutually singular probability measures on the F -algebra $\mathfrak{B}$. For any face F of $[0, 1]^n$ there is a closed order interval (for the usual order on R^n) $\langle y_F, x_F \rangle \subset F$ so that $\mathcal{R}(\mu) \cap F$ is a dense (for the relative topology on F) subset of $\langle y_F, x_F \rangle$. Furthermore if $F \subset G$ are faces of $[0, 1]^n$ then $\langle y_F, x_F \rangle \subset \langle y_G, x_G \rangle$.*

PROOF. If $F \cap \mathcal{R}(\mu) = \mu(\mathcal{F} \cap \mathcal{G})$ for a filter $\mathcal{F}$ and an ideal $\mathcal{G}$, let $x_F = \sup\{\mu(A): A \in \mathcal{F} \cap \mathcal{G}\}$ and $y_F = \inf\{\mu(A): A \in \mathcal{F} \cap \mathcal{G}\}$ (for the order of R^n). Note that if $A_1 \subset A_2$ are in $\mathcal{F} \cap \mathcal{G}$ then $\mathcal{R}(\mu) \cap F$ contains the relative interior of the order interval $\langle \mu(A_1), \mu(A_2) \rangle$ since μ has singular components on $A_2 \setminus A_1$. As A_1 decreases and A_2 increases in $\mathcal{F} \cap \mathcal{G}$ these relative interiors increase to cover the relative interior of $\langle y_F, x_F \rangle$. This suffices to show that $\mathcal{R}(\mu) \cap F$ is a dense subset of $\langle y_F, x_F \rangle$. That, when $F \subset G$, $\langle y_F, x_F \rangle \subset \langle y_G, x_G \rangle$ is an immediate consequence of the convexity of $\mathcal{R}(\mu)$. $\square$

REMARKS. In essence Proposition 3-4 is an analysis of the facial structure of $\mathcal{R}(\mu)$. If one examines relative interiors of faces $\mathcal{R}(\mu)$ one obtains the Gleason parts of $\mathcal{R}(\mu)$. Corollary to Proposition 3-4 we may say that the Gleason parts of $\mathcal{R}(\mu)$ are all "open rectangles". Thus, in the n -dimensional case $\mathcal{R}(\mu)$ is far from being an arbitrary convex subset of $[0, 1]^n$ symmetric about $(\frac{1}{2}, \dots, \frac{1}{2})$ and containing $(0, \dots, 0)$ and $(1, \dots, 1)$.

4. Zonoids. In §2 we exhaustively considered the possible ranges of a two-dimensional, finitely-additive vector measure on an F -algebra with mutually singular probabilities as components. The conclusion was that "nearly all" dense convex subsets of the unit square containing $(0, 0)$ and $(1, 1)$ are possible. We conjecture that the same sort of results are true in general for nonatomic vector valued measures with nonsingular components.

We recall from [4] that a *zonoid* is the range of a finite-dimensional, nonatomic, countably-additive, vector-valued measure on a measurable space (X, Σ) . Lemma 1-1 assures us that the closure of the range of a finite-dimensional, finitely-additive, nonatomic measure on a Boolean algebra is always a zonoid. A converse also is true as we shall see in Proposition 4-2. We shall assume that all zonoids are in $\mathbf{R}^d$ for some d . When a zonoid lies in the nonnegative orthant of $\mathbf{R}^d$ we shall call it a positive zonoid. The countably-additive, vector-valued, nonatomic measures giving rise to positive zonoids are the nonnegative ones.

In [4] it is shown that zonoids in $\mathbf{R}^d$ are precisely the class of weak* $(\sigma(l^\infty, l^1))$ continuous linear images in $\mathbf{R}^d$ of the unit ball, $\square_\infty$, of l^∞ . Since $\square_\infty$ is a $\sigma(l^\infty, l^1)$ continuous linear image of the positive unit ball, $\square_\infty^+ = [0, 1]^{\aleph_0}$, all zonoids are $\sigma(l^\infty, l^1)$ continuous linear images of $\square_\infty^+$. This is reasonable for $\square_\infty^+$ is an "infinite-dimensional zonoid". If $\mu = \{\mu_n: n \in N\}$ is a sequence of mutually singular, countably-additive, nonatomic probability measures on a measurable space (X, Σ) , then μ may be considered a nonnegative, l^∞ -valued, nonatomic vector measure by setting $\mu(A) = (\mu_n(A))$. Then $\mathcal{R}(\mu)$ is easily verified to be $\square_\infty^+$. If $\tau: l^\infty \rightarrow \mathbf{R}^d$ is $\sigma(l^\infty, l^1)$ continuous and linear then $\tau \circ \mu$ is a finite-dimensional, nonatomic, countably-additive measure on (X, Σ) with range $\tau(\square_\infty^+)$. Note that τ may be described by a matrix $\{t_{ij}: 1 \leq i \leq d, 1 \leq j < \infty\}$ so that $\tau((x_n))$ has its i th coordinate given by $\sum_{j=1}^\infty t_{ij}x_j$. For each $1 \leq i \leq d$, $\{t_{ij}: 1 \leq j < \infty\} \in l^1$. It may be assumed that for each j in N there is an i with $t_{ij} \neq 0$. The zonoid $\tau(\square_\infty^+)$ is a positive zonoid iff $\{t_{ij}: 1 \leq j < \infty\} \in l^1_+$ for each $1 \leq i \leq d$. The fact that $\tau \circ \mu$ is nonatomic is a consequence of the fact that countable convex combinations of nonatomic measures (countably additive or finitely additive) are nonatomic. The countable additivity of $\tau \circ \mu$ is established similarly.

PROPOSITION 4-1. *Let μ be a finitely-additive map from the Boolean algebra $\mathfrak{B}$ into the locally convex space E with dual E' . Let $l \circ \mu$ be a nonatomic element of $BA(\mathfrak{B})$ for all $l \in E'$. The $\sigma(E, E')$ closure of $\mathfrak{R}(\mu)$ is convex and $\sigma(E, E')$ bounded.*

PROOF. Boundedness of $\mathfrak{R}(\mu)$ is immediate. Let C be the closed convex hull of $\mathfrak{R}(\mu)$ for $\sigma(E, E')$. For any $\{l_1, \dots, l_n\} \subset E'$ consider $(l_1, \dots, l_n) \circ \mu: \mathfrak{B} \rightarrow R^n$. Let $\mathfrak{R}(l_1, \dots, l_n)$ be the set of $e \in E$ such that $(l_1(e), \dots, l_n(e))$ is in the closure of the range of $(l_1, \dots, l_n) \circ \mu$. $\mathfrak{R}(l_1, \dots, l_n)$ is $\sigma(E, E')$ closed and convex by Lemma 1-1, hence contains C . The intersection D , of all $\mathfrak{R}(l_1, \dots, l_n)$ as $\{l_1, \dots, l_n\}$ ranges over finite subsets of E' , contains C . If $e \in D$ there is, for any $\{l_1, \dots, l_n\} \subset E'$ and $\varepsilon > 0$, an $A \in \mathfrak{B}$ with $|l_i(\mu(A)) - l_i(e)| < \varepsilon$ for $i = 1, \dots, n$. Thus, D is in the closure of $\mathfrak{R}(\mu)$ for $\sigma(E, E')$. Hence $D = C$, which establishes the proposition. $\square$

PROPOSITION 4-2. *Let Z be a zonoid and $\mathfrak{B}$ a Boolean algebra with a nontrivial nonatomic measure. There is a finitely-additive, nonatomic measure ν on $\mathfrak{B}$ such that $\overline{\mathfrak{R}(\nu)} = Z$.*

PROOF. Let τ be a $\sigma(l^\infty, l^1)$ continuous function so that $\tau(\square Z_\infty^+) = Z$. Let $\{\mu_n: n \in N\}$ be a sequence of mutually singular, nonatomic, finitely-additive probability measures on $\mathfrak{B}$ given by Corollary 1-2-3. Let μ be the l^∞ -valued $\sigma(l^\infty, l^1)$ nonatomic measure on $\mathfrak{B}$ given by $\mu(A) = (\mu_n(A))$. By Lemma 4-1 $\mathfrak{R}(\mu)$ is $\sigma(l^\infty, l^1)$ dense in $\square_\infty^+$. Let $\nu = \tau \circ \mu$. It is immediate that $\mathfrak{R}(\nu)$ is dense in Z (the topology on Z being that induced by its linear span so $Z^0 \neq \emptyset$). This establishes the proposition. $\square$

PROPOSITION 4-3. *Let Z be a zonoid and $\mathfrak{B}$ an F -algebra. There is a nonatomic ν on $\mathfrak{B}$ such that $\mathfrak{R}(\nu)$ is $Z^0 \cup \{\nu(\emptyset), \nu(X)\}$.*

PROOF. Use Corollary 3-2-2 to construct a sequence $\{\mu_n: n \in N\}$ of mutually singular, nonatomic, finitely-additive probabilities on $\mathfrak{B}$ such that these all have identical null sets. Construct the nonatomic l^∞ -valued measure μ as in the proof of Proposition 4-2. Then, $\mathfrak{R}(\mu)$ is $\sigma(l^\infty, l^1)$ dense in $\square_\infty^+$. The only sequence $\{x_n: n \in N\}$ in $\mathfrak{R}(\mu)$ with $x_n = 0$ for some n is 0, and the only sequence in $\mathfrak{R}(\mu)$ with $x_n = 1$ for some n is 1. Let τ be a $\sigma(l^\infty, l^1)$ continuous linear function with $\tau(\square_\infty^+) = Z$. Let $\nu = \tau \circ \mu$. Since $\mathfrak{R}(\nu)$ is dense in Z by Proposition 4-2 and is convex by Proposition 2-4, it includes $Z^0 \cup \{\nu(\emptyset), \nu(X)\}$ (which is convex). If $0 \in \partial Z$ let F be the minimal face of Z containing 0. $\tau^{-1}(F)$ is a $\sigma(l^\infty, l^1)$ closed face of $\square_\infty^+$ which must be proper since $\tau(1) = \nu(X)$, and since the face of Z determined by 0 and $\nu(X)$ is Z and not $F \subset \partial Z$. There are no elements of $\mathfrak{R}(\nu)$ in $F \setminus \{0\}$, for if there were there would be a sequence $(\mu_n(A)) \in \tau^{-1}(F)$. By [2, Lemma 3 (6)], $\tau^{-1}(F)$ is of the form $\{(x_n): x_n = 0 \text{ for all } n \in J\}$ for some nonempty subset J of N . If $\mu(A) \in \tau^{-1}(F)$ then $\mu_n(A) = 0$ for some $n \in N$, hence $\mu_n(A) = 0$ for all $n \in N$, hence $\mu(A) = 0$, hence $\nu(A) = 0$. Thus, $\{0\} = F \cap \mathfrak{R}(\nu)$. Similarly, if F' is the smallest face of Z containing $\nu(N)$ then $\mathfrak{R}(\nu) \cap F' = \{\nu(N)\}$. Now consider $x \in \partial Z$ such that the smallest face F of Z containing x does not contain either 0 or $\nu(X)$ (which is the case if $\{0, \nu(X)\} \subset Z^0$). $\tau^{-1}(F)$ is a

$\sigma(l^\infty, l^1)$ closed face of $\square_\infty^+$. Hence there are, by [2, Lemma 3 (6)], subsets $J_1 \subset J_2$ of N so that (x_n) in $\square_\infty^+$ is in $\pi^{-1}(F)$, $x_n = 0$ if $n \notin J_2$, and $x_n = 1$ if $n \in J_1$. It must be the case that $J_2 \neq N$ and $J_1 \neq \emptyset$. It is impossible to have an $A \in \mathfrak{B}$ so that $\mu_n(A) = 0$ for $n \notin J_2$ and $\mu_n(A) = 1$ if $n \in J_1$. Thus, $\mathfrak{R}(\mu) \cap \tau^{-1}(F) = \emptyset$. Thus, $\mathfrak{R}(\nu) \cap F = \emptyset$. This suffices to establish the proposition. $\square$

REMARK. It immediately follows that if $\mathfrak{B}$ is any algebra supporting a nonatomic measure, there is a nonatomic vector-valued measure ν on $\mathfrak{B}$ whose range is dense in Z yet contains no boundary points of Z other than 0 and $\nu(X)$.

In the proof of Proposition 4-3 it was observed that if F is a face of the zonoid Z then $\tau^{-1}(F)$ is a $\sigma(l^\infty, l^1)$ closed face $\square_\infty^+$, hence of the form

$$\square_{A(F), B(F)}^+ = \{(x_n): \chi_{A(F)} \leq (x_n) \leq \chi_{B(F)}\}$$

by [2, Lemma 3 (6)], where $A(F) \subset B(F)$ are subsets of N . If F_1 and F_2 are faces of Z then

$$A(F_1 \cap F_2) = A(F_1) \cup A(F_2) \quad \text{and} \quad B(F_1 \cap F_2) = B(F_1) \cap B(F_2).$$

F_1 and F_2 are disjoint iff $A(F_1) \setminus B(F_2) \neq \emptyset$ or $A(F_2) \setminus B(F_1) \neq \emptyset$ so $A(F_1 \cap F_2) \subsetneq B(F_1 \cap F_2)$. When $F_1 \neq F_2$ are zero dimensional (i.e. extreme points) then F_1 and F_2 are disjoint.

If one wishes to find a nonatomic ν so that $\mathfrak{R}(\nu) \cap F \neq \emptyset$ for a given face F of Z with $\nu = \tau \circ \mu$ as in the proof of Proposition 4-3, it is necessary to guarantee that μ_n for $n \in A(F)$ be simultaneously disjoint with μ_n for $n \notin B(F)$ so that there is an $A \in \mathfrak{B}$ so that $\mu_n(A) = 1$ for $n \in A(F)$ and $\mu_n(A) = 0$ if $n \notin B(F)$.

PROPOSITION 4-4. *Let $\mathfrak{B}$ be an algebra with a nonzero nonatomic measure. Let Z be a zonoid. Let $\{F_1, \dots, F_n\}$ be faces of Z . There is a nonatomic vector-valued measure ν with $\mathfrak{R}(\nu)$ dense in Z with $\mathfrak{R}(\nu) \cap F_j \neq \emptyset$ for $j = 1, \dots, n$ and so that there are at most finitely many faces F of Z with $\mathfrak{R}(\nu) \cap F \neq \emptyset$.*

PROOF. Let $\{C_1, \dots, C_m\}$ be the atoms of the finite algebra generated by $\{A(F_j): 1 \leq j \leq n\}$ in 2^N . Let $\{A_1, \dots, A_m\}$ be the atoms of a finite subalgebra $\mathcal{Q}$ of $\mathfrak{B}$ so that each A_i receives strictly positive measure under some nonatomic measure on $\mathfrak{B}$. Select for each $n \in C_j$ a nonatomic probability μ_n on $\mathfrak{B}$ such that $\mu_n(A_j) = 1$ so that $\{\mu_n: n \in C_j\}$ are mutually singular and have identical null sets. Let $\mu = (\mu_n)$ and let $\nu = \tau \circ \mu$. Since μ has mutually singular probabilities as components, $\mathfrak{R}(\nu)$ is dense in Z . For any j , $A(F_j)$ is a union of a subset of $\{C_1, \dots, C_m\}$, say $\{C_{j_1}, \dots, C_{j_l}\}$. If $n \in A(F_j)$ then $\mu_n(A_{j_1} \cup \dots \cup A_{j_l}) = 1$. If $n \notin B(F_j)$ then $n \notin A(F_j)$ so $\mu_n(A_{j_1} \cup \dots \cup A_{j_l}) = 0$. Thus, $\{\mu_n: n \in A(F_j)\}$ are mutually disjoint with $\{\mu_n: n \notin B(F_j)\}$. Thus, $\mathfrak{R}(\nu) \cap F_j \neq \emptyset$. Now suppose that F is a face of Z with $\mathfrak{R}(\nu) \cap F \neq \emptyset$. Let $A \in \mathfrak{B}$ with $\nu(A) \in F$ so that $(\mu_n(A)) \in \tau^{-1}(F)$. If $n \in C_j$ and $\mu_n(A) = 1$ then $\mu_n(A \cap A_j) = 1$ so $\mu_m(A \cap A_j) = 1$ for all $m \in C_j$, hence $\mu_m(A) = 1$ for all $m \in C_j$. Thus, $C_j \subset A(F)$ if $C_j \cap A(F) \neq \emptyset$. That is, $A(F)$ is a union of the C_j 's, hence is in $\mathcal{Q}$. Similarly the complement of $B(F)$ is in $\mathcal{Q}$. Since F is determined by $A(F)$ and $B(F)$ and since there are only finitely many $A(F) \subset B(F)$ in $\mathcal{Q}$, only finitely many faces F of Z have $F \cap \mathfrak{R}(\nu) \neq \emptyset$. $\square$

REMARK. If $r = 2^n$ an upper bound for the number of faces F of Z meeting $\mathcal{R}(\nu)$ is

$$\sum_{j=0}^r \left[C(r, j) \sum_{k=0}^j C(j, k) \right] \leq (2^r)^2.$$

PROPOSITION 4-5. Let $\mathcal{B}$ be an infinite F -algebra. Let Z be a zonoid. There is a nonatomic ν on $\mathcal{B}$ with $\mathcal{R}(\nu) = Z$.

PROOF. Let μ_0 be a nonatomic probability $\mathcal{B}$. The measure algebra $\mathcal{B}_{\mu_0}$ is complete. Let $\{A_n\}$ be a partition of $\mathcal{B}_{\mu_0}$ with nonempty elements. Let μ_n be the probability measure on $\mathcal{B}_{\mu_0}$ obtained by restricting μ_0 (considered as a probability on $\mathcal{B}_{\mu_0}$) to A_n and normalizing. The sequence $\{\mu_n: n \in N\}$ consists of mutually disjoint probabilities. If $J \subset N$ then $\mu_n(\cup \{A_m: m \in A(F)\}) = 1$ and $\mu_n(\cup \{A_m: m \notin J\}) = 0$ if $n \in J$. Let $\nu = \tau \circ \mu$ be defined on $\mathcal{B}_{\mu_0}$, hence on $\mathcal{B}$ in the usual fashion. Let F be a face of Z . We have $\mu(\cup \{A_m: m \in A(F)\}) = \chi_{A(F)} \in \tau^{-1}(F)$ so $\nu(\cup \{A_m: m \in A(F)\}) \in F$. Thus, $\mathcal{R}(\nu) \cap F \neq \emptyset$ for all faces F of Z . In particular, $\mathcal{R}(\nu)$ contains all extreme points of Z . Since $\mathcal{R}(\nu)$ is convex, $Z = \mathcal{R}(\nu)$.

□

If μ is a finitely-additive nonatomic measure on a Boolean algebra then $\overline{\mathcal{R}(\mu)}$ is a zonoid Z . If F is a face of Z then F is the intersection of a supporting hyperplane H with Z and H may be considered as a supporting hyperplane to $\mathcal{R}(\mu)$. Conversely if H is a supporting hyperplane to $\mathcal{R}(\mu)$ it is a supporting hyperplane to Z . If $\mathcal{R}(\mu)$ is convex its faces are the traces on $\mathcal{R}(\mu)$ of the faces of Z .

PROPOSITION 4-6. If μ is a nonatomic measure on a Boolean algebra $\mathcal{B}$ and H is a supporting hyperplane to $\mathcal{R}(\mu)$ then $H \cap \mathcal{R}(\mu)$ is $\mu(\mathcal{F}_H \cap \mathcal{G}_H)$ where $\mathcal{F}_H$ is a filter and $\mathcal{G}_H$ is an ideal on $\mathcal{B}$. Furthermore, $H \cap \mathcal{R}(\mu)$ is a zonoid or a translate of a zonoid.

PROOF. Let l be a linear functional so that H is the affine variety $\{l = c\}$ so that $\mathcal{R}(\mu) \subset \{l \leq c\}$. Then $c = \|l \circ \mu\|$ and $l(\mu(A)) \leq c$ for $A \in \mathcal{B}$. If $\mu(A) \in H$ then $\mu(A) = c$. Let $\mathcal{C} = \{A: l(\mu(A)) = c\}$. If $\{A_1, A_2\} \subset \mathcal{C}$ then

$$2c = l(\mu(A_1)) + l(\mu(A_2)) = l(\mu(A_1 \cup A_2)) + l(\mu(A_1 \cap A_2)) \leq 2c.$$

Consequently, $\{A_1 \cup A_2, A_1 \cap A_2\} \subset \mathcal{C}$. Thus, $\mathcal{C}$ must be of the form $\mathcal{F}_H \cap \mathcal{G}_H$ for some filter $\mathcal{F}_H$ and ideal $\mathcal{G}_H$ in $\mathcal{B}$.

Choose $\{A_n\}$, a decreasing sequence in $\mathcal{C}$, and $\{B_n\}$, an increasing sequence in $\mathcal{C}$, with $A_n \subset B_n$ for all n and with $\overline{H \cap \mathcal{R}(\mu)}$ equal to the closure of $\bigcup_{n=1}^{\infty} \{\mu(C): A_n \subset C \subset B_n\}$. By passage to a subsequence assume that $\mu(A_n)$ converges to x_0 . Then $\{\mu(C): A_n \subset C \subset B_n\} - \mu(A_n)$ are zonoids which converge to $\overline{H \cap \mathcal{R}(\mu)} - x_0$ in the Hausdorff metric. Results of Bolker [4] show that $\overline{H \cap \mathcal{R}(\mu)} - x_0$ is a zonoid. This establishes the proposition. □

REMARKS ADDED IN PROOF. (1) Peter Loeb in *A combinatorial analog of Lyapounov's theorem for infinitesimally generated atomic vector measures*, Proc. Amer. Math. Soc. 39 (1973), 585–586, establishes a result which has been used by Donald Brown to establish Lemma 1-1.

(2) Domenico Candeloro and Anna Martellotti Sacchetti in *Sul rango di una massa vettoriale*, Atti. Sem. Mat. Fis. Univ. Modena **28** (1979), 102–111, establish Theorem 2-2 for $\mathfrak{B} = 2^X$ using Halmos' proof.

(3) F. Dashiell, A. Hager and M. Henriksen, in *Order-Cauchy completions of rings and vector lattices of continuous functions*, Canad. J. Math. **32** (1980), 657–685, have introduced the notion of a quasi- F -space X which, for compact X , is one for which $C(X)$ is sequentially order-Cauchy complete. If X is the Stone space of a Boolean algebra $\mathfrak{B}$ then $\mathfrak{B}$ should be called a quasi- F -algebra iff X is a quasi- F -space. This is the case iff whenever $\{A_n\}$ is an increasing sequence and $\{B_n\}$ is a decreasing sequence in $\mathfrak{B}$ with $A_n \subset B_n$ for all n and for which $\{B_n \setminus A_n\}$ decreases to $\emptyset$ there is a $C \in \mathfrak{B}$ with $A_n \subset C \subset B_n$ for all n . In the results of this article in which F -algebras appear one may use quasi- F -algebras instead. This is clear from remarks after Corollary 2-2-1. It is useful to note that it follows from Corollary 4.7 of Dashiell, Hager and Henriksen that, just as for F -spaces, the support of any Radon measure or a compact quasi- F -space is Stonian.

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