INTEGRALGEOMETRIC PROPERTIES OF CAPACITIES

BY

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ABSTRACT. Let m and n be positive integers, $0 < m < n$, and C_K and C_H the usual potential-theoretic capacities on R^n corresponding to lower semicontinuous kernels K and H on $R^n \times R^n$ with $H(x, y) = K(x, y)|x - y|^{n-m} > 1$ for $|x - y| < 1$. We consider relations between the capacities $C_K(E)$ and $C_H(E \cap A)$ when $E \subset R^n$ and A varies over the m -dimensional affine subspaces of R^n . For example, we prove that if E is compact, $C_K(E) \leq c \int C_H(E \cap A) d\lambda_{n,m}A$ where $\lambda_{n,m}$ is a rigidly invariant measure and c is a positive constant depending only on n and m .

1. Introduction. Suppose that E is a Borel set in R^n with $0 < \mathcal{H}^s(E) < \infty$, where $\mathcal{H}^s$ is the s -dimensional Hausdorff measure. It was shown in [MP] that if m is an integer, $0 < m < n$, then, in the case $s \leq m$, the Hausdorff dimension $\dim p(E)$ equals s for almost all orthogonal projections $p: R^n \rightarrow R^m$, and, in the case $s > n - m$, $\dim(E \cap V) = s + m - n$ with $\mathcal{H}^{s+m-n}(E \cap V) < \infty$ for almost all m planes V through almost all points of E . There are examples which show that in general the statements $\dim p(E) = s$ and $\dim(E \cap V) = s + m - n$ cannot be replaced by $\mathcal{H}^s(p(E)) > 0$ and $\mathcal{H}^{s+m-n}(E \cap V) > 0$, respectively; see [M, 5.6 and 6.6]. However, if one uses capacities in place of Hausdorff measures, one can say more. If C_s is the Riesz capacity defined via the kernel $|x - y|^{-s}$, then, in the case $0 < s < m$, $C_s(E) > 0$ implies $C_s(p(E)) > 0$ for almost all orthogonal projections $p: R^n \rightarrow R^m$. Moreover, if F is compact, then

$$\int C_s(p(F))^{-1} d\theta_{n,m}^* \leq c C_s(F)^{-1}$$

where $\theta_{n,m}^*$ is the orthogonally invariant measure on the space of all orthogonal projections $R^n \rightarrow R^m$ and c is a constant depending on n , m and s . These results were proved in [MP, 5.1–2], and Kaufman also considered capacities of projections in [K]. In 4.11 we show that also

$$C_s(F) \leq c \int C_s(p(F)) d\theta_{n,m}^*.$$

The main portion of this paper is devoted to the study of the capacities of the intersections of E with m planes. We shall show that if $s > n - m$ and $C_s(E) > 0$, then $C_{s+m-n}(E \cap V) > 0$ for almost all m planes V through C_s almost all points of E , and if F is compact

$$C_s(F) \leq c \int C_{s+m-n}(F \cap V) d\lambda_{n,m}V,$$

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where $\lambda_{n,m}$ is the rigidly invariant measure on the space of all m -dimensional affine subspaces of R^n and c depends only on n and m (see 4.6–8). In fact, we prove more general results by replacing the kernels $|x - y|^{-s}$ and $|x - y|^{-(s+m-n)}$ by general lower semicontinuous kernels $K(x, y)$ and $K(x, y)|x - y|^{n-m}$, respectively.

To prove these results we need to define the slices of a Radon measure μ on m planes and to derive some integral relations between these slices and μ . This will be done in §3.

In §5 we make some remarks on the structure of purely unrectifiable subsets of R^n . For example, it follows from Theorem 5.1 that if $E \subset R^n$ is purely $(\mathcal{H}^{n-1}, n-1)$ unrectifiable with $\mathcal{H}^{n-1}(E) < \infty$, then from C_{n-1}^* almost all points of $R^n \setminus E$ projects radially into a set of $\mathcal{H}^{n-1}$ measure zero, where C_{n-1}^* is the outer capacity corresponding to C_{n-1} . This generalizes a result of Marstrand [M, §8].

2. Preliminaries.

2.1. *Notation and terminology.* We shall use the notation and terminology of [F]. In the whole paper m and n will be integers with $0 < m < n$. Radon measure always means a nonnegative (outer) Radon measure. If μ is a Radon measure on R^n , so are $\mu \llcorner A: B \mapsto \mu(A \cap B)$ for any $A \subset R^n$ and, if the support of μ , $\text{spt } \mu$, is compact, $f_\# \mu: B \mapsto \mu(f^{-1}(B))$ for any continuous $f: R^n \rightarrow R^n$ [F, 2.2.17]. If μ is absolutely continuous with respect to ν , we denote $\mu \ll \nu$.

We let $G(n, m)$ be the Grassmannian manifold of m -dimensional linear subspaces of R^n . There is a unique Radon measure $\gamma_{n,m}$ on $G(n, m)$ which has a total mass one and which is invariant under orthogonal transformations of R^n [F, 2.7.16(6)]. The following lemma was proved in [MP, 2.6]:

2.2. LEMMA. *There is a constant c depending only on n and m such that for $x \in R^n$ and $\delta > 0$,*

$$\gamma_{n,m} \{ V: \text{dist}(x, V) \leq \delta \} \leq c \delta^{n-m} |x|^{m-n}.$$

2.3. *The space of affine subspaces.* We shall denote by $A(n, m)$ the space of all m -dimensional affine subspaces of R^n . Each $A \in A(n, m)$ has a unique representation

$$A = V_a, \quad V \in G(n, m), a \in V^\perp,$$

where $V^\perp$ is the orthogonal complement of V and $V_a = V + \{a\}$ is the m plane through a parallel to V . We let $\lambda_{n,m}$ be the standard Radon measure on $A(n, m)$ which is invariant under the isometries of R^n . It follows from [F, 2.7.16(7)] and Fubini's theorem [F, 2.6.2] that

$$\int f d\lambda_{n,m} = \int \int_{V^\perp} f(V_a) d\mathcal{H}^{n-m} a d\gamma_{n,m} V$$

for any nonnegative Borel function f on $A(n, m)$.

2.4. *Differentiation of measures.* We review the facts from the theory of the relative differentiation of measures which will be needed in the sequel. Let $V \in G(n, k)$ and let μ be a Radon measure on R^n . We define for $x \in V$ the lower

and upper derivatives of $\mu \ll \mathcal{H}^k \ll V$ with respect to $\mathcal{H}^k \ll V$ by

$$\underline{D}(\mu, V, x) = \liminf_{r \downarrow 0} \alpha(k)^{-1} r^{-k} \mu(B(x, r) \cap V),$$

$$\overline{D}(\mu, V, x) = \limsup_{r \downarrow 0} \alpha(k)^{-1} r^{-k} \mu(B(x, r) \cap V),$$

where $B(x, r)$ is the closed ball with centre x and radius r and $\alpha(k)$ is the Lebesgue measure of the unit ball in R^k . If these two limits agree, we define the derivative

$$D(\mu, V, x) = \lim_{r \downarrow 0} \alpha(k)^{-1} r^{-k} \mu(B(x, r) \cap V).$$

It follows from [F, 2.9.5] combined with [F, 2.8.7–8] that

$$D(\mu, V, x) < \infty \quad \text{for } \mathcal{H}^k \text{ a.a. } x \in V,$$

and from [F, 2.9.7] that for any $\mathcal{H}^k$ measurable set $B \subset V$,

$$\int_B D(\mu, V, x) d\mathcal{H}^k x \leq \mu(B).$$

The equality holds if $\mu \ll \mathcal{H}^k \ll V$, and then it follows $D(\mu, V, x) > 0$ for μ a.a. $x \in V$. Moreover, using [F, 2.9.15] one sees that $\mu \ll \mathcal{H}^k \ll V$ if and only if

$$\underline{D}(\mu, V, x) < \infty \quad \text{for } \mu \text{ a.a. } x \in V.$$

2.5. Some Borel functions. For $\emptyset \neq E \subset R^n$ and $\delta > 0$, we set

$$E(\delta) = \{x: \text{dist}(x, E) < \delta\}.$$

Suppose that μ is a Radon measure on R^n with compact support, $W \in G(n, k)$, f is a nonnegative Borel function on R^n and α is a real number. Then the following functions are Borel functions:

$$x \mapsto \underline{D}(\mu, W, x), \quad x \in W, \quad x \mapsto \overline{D}(\mu, W, x), \quad x \in W,$$

$$A \mapsto \liminf_{\delta \downarrow 0} \delta^\alpha \int_{A(\delta)} f d\mu, \quad A \in A(n, m),$$

$$A \mapsto \limsup_{\delta \downarrow 0} \delta^\alpha \int_{A(\delta)} f d\mu, \quad A \in A(n, m),$$

$$(x, V) \mapsto \liminf_{\delta \downarrow 0} \delta^\alpha \int_{V_x(\delta)} f d\mu, \quad (x, V) \in R^n \times G(n, m),$$

$$(x, V) \mapsto \limsup_{\delta \downarrow 0} \delta^\alpha \int_{V_x(\delta)} f d\mu, \quad (x, V) \in R^n \times G(n, m).$$

The proofs of these facts are rather standard, and we briefly consider only

$$F(A) = \liminf_{\delta \downarrow 0} \delta^\alpha \int_{A(\delta)} f d\mu.$$

The others can be dealt with similarly. First one verifies

$$F(A) = \liminf_{\delta \downarrow 0} \delta^\alpha \int_{A(\delta)^0} f d\mu$$

where $A(\delta)^0$ is the interior of $A(\delta)$. Using

$$\int_{A(\delta)^0} f d\mu = \lim_{\epsilon \uparrow \delta} \int_{A(\epsilon)^0} f d\mu,$$

and the compactness of $\text{spt } \mu$, one then shows that $A \mapsto \int_{A(\delta)^0} f d\mu$ is lower semicontinuous. Finally it follows from the facts that δ^α is continuous and $\int_{A(\delta)^0} f d\mu$ is nondecreasing with respect to δ that δ may be restricted to run through the positive rationals. This implies that F is a Borel function.

3. Slicing of measures. We shall use the theory of differentiation of measures to define the slices of a Radon measure of R^n on affine subspaces of R^n . Our method is somewhat similar to those which Federer [F, 4.3] and Almgren [A, I.3] have used to slice currents and varifolds.

3.1. Definition of slices. Let μ be a Radon measure on R^n with compact support. For any nonnegative Borel function f on R^n with $\int f d\mu < \infty$ we define a Radon measure ν_f by

$$\nu_f(B) = \int_B f d\mu.$$

Let $V \in G(n, m)$ and let $\pi_V: R^n \rightarrow V^\perp$ be the orthogonal projection. First we fix $\varphi \in C^+(R^n)$, the space of nonnegative continuous functions on R^n . Using 2.4 we differentiate $\pi_{V\#}\nu_\varphi$ with respect to $\mathcal{H}^{n-m} \llcorner V^\perp$ and obtain the existence of

$$\mu_{V,a}(\varphi) = D(\pi_{V\#}\nu_\varphi, V^\perp, a) = \lim_{\delta \downarrow 0} \alpha(n-m)^{-1} \delta^{m-n} \int_{V_a(\delta)} \varphi d\mu < \infty \quad (3.2)$$

for $\mathcal{H}^{n-m}$ a.a. $x \in V^\perp$.

Let then D be a countable subset of $C^+(R^n)$ which is dense in $C^+(R^n)$ with respect to the uniform convergence. Then for $\mathcal{H}^{n-m}$ a.a. $a \in V^\perp$, $\mu_{V,a}(\varphi)$ is defined for all $\varphi \in D$, and it follows immediately that for every such a , $\mu_{V,a}(\varphi)$ is defined by (3.2) for all $\varphi \in C^+(R^n)$. Thus for $\mathcal{H}^{n-m}$ a.a. $a \in V^\perp$ we may use Riesz's representation theorem [F, 2.5.13–14] to extend $\mu_{V,a}$ to a Radon measure on R^n . Whenever $\mu_{V,a}$ is defined, we set

$$\mu_{V,x} = \mu_{V,a} \text{ for } x \in V_a \text{ and } \mu_A = \mu_{V,a} \text{ for } A = V_a \in A(n, m).$$

3.3. LEMMA. (1) $\text{spt } \mu_A \subset A \cap \text{spt } \mu$ whenever μ_A is defined.

(2) The set P of those $A \in A(n, m)$ for which μ_A is defined is a Borel set and $\lambda_{n,m}(A(n, m) \sim P) = 0$.

(3) The set Q of all pairs $(x, V) \in R^n \times G(n, m)$ for which $\mu_{V,x}$ is defined is a Borel set. If $\pi_V \# \mu \ll \mathcal{H}^{n-m} \llcorner V^\perp$ for $\gamma_{n,m}$ a.a. $V \in G(n, m)$, then $\mu \times \gamma_{n,m}(R^n \times G(n, m) \sim Q) = 0$ and $\mu_{V,x}(R^n) > 0$ for $\mu \times \gamma_{n,m}$ a.a. $(x, V) \in R^n \times G(n, m)$.

PROOF. (1) is obvious by (3.2). To prove (2) let D be the countable dense subset of $C^+(R^n)$ which was used in 3.1. For $\varphi \in D$ the functions

$$\underline{D}_\varphi: A \mapsto \liminf_{\delta \downarrow 0} \alpha(n-m)^{-1} \delta^{m-n} \int_{A(\delta)} \varphi d\mu,$$

$$\overline{D}_\varphi: A \mapsto \limsup_{\delta \downarrow 0} \alpha(n-m)^{-1} \delta^{m-n} \int_{A(\delta)} \varphi d\mu$$

are Borel functions on $A(n, m)$ by 2.5. Hence

$$P = \bigcap_{\varphi \in D} \{A: \underline{D}_{\varphi}(A) = \overline{D}_{\varphi}(A) < \infty\}$$

is a Borel set. Since $\mathcal{H}^{n-m}\{a \in V^{\perp}: V_a \notin P\} = 0$ for all $V \in G(n, m)$, we obtain $\lambda_{n,m}(A(n, m) \sim P) = 0$.

To prove (3) we observe that the mapping $F: (x, V) \mapsto V_{\pi_V(x)}$ of $R^n \times G(n, m)$ onto $A(n, m)$ is continuous and $Q = F^{-1}(P)$. Thus Q is a Borel set.

Suppose then that for $\gamma_{n,m}$ a.a. $V \in G(n, m)$, $\pi_V \# \mu \ll \mathcal{H}^{n-m} \llcorner V^{\perp}$, which means that $\mu(E) = 0$ whenever $\mathcal{H}^{n-m}(\pi_V(E)) = 0$. Since $\lambda_{n,m}(A(n, m) \sim P) = 0$, we have

$$\begin{aligned} \mathcal{H}^{n-m}(\pi_V\{x: V_{\pi_V(x)} \in A(n, m) \sim P\}) \\ = \mathcal{H}^{n-m}\{a \in V^{\perp}: V_a \in A(n, m) \sim P\} = 0 \end{aligned}$$

for $\gamma_{n,m}$ a.a. $V \in G(n, m)$. Hence, by the absolute continuity,

$$\mu\{x: (x, V) \in R^n \times G(n, m) \sim Q\} = \mu\{x: V_{\pi_V(x)} \in A(n, m) \sim P\} = 0$$

for $\gamma_{n,m}$ a.a. $V \in G(n, m)$, and Fubini's theorem yields

$$\mu \times \gamma_{n,m}(R^n \times G(n, m) \sim Q) = 0.$$

Finally, if $\pi_V \# \mu \ll \mathcal{H}^{n-m} \llcorner V^{\perp}$, we have by 2.4, $\mu_{V,a}(R^n) = D(\pi_V \# \mu, V^{\perp}, a) > 0$ for $\pi_V \# \mu$ a.a. $a \in V^{\perp}$, and then $\mu_{V,x}(R^n) > 0$ for μ a.a. $x \in R^n$. Fubini's theorem implies $\mu_{V,x}(R^n) > 0$ for $\mu \times \gamma_{n,m}$ a.a. $(x, V) \in R^n \times G(n, m)$.

3.4. LEMMA. *Let f be a nonnegative Borel function on R^n with $\int f d\mu < \infty$ and let $V \in G(n, m)$.*

(1) *For $\mathcal{H}^{n-m}$ a.a. $a \in V^{\perp}$,*

$$\int f d\mu_{V,a} = \lim_{\delta \downarrow 0} \alpha(n-m)^{-1} \delta^{m-n} \int_{V_a(\delta)} f d\mu.$$

(2) *The function $a \mapsto \int f d\mu_{V,a}$ is $\mathcal{H}^{n-m}$ measurable on $V^{\perp}$.*

(3) *$\int_{V^{\perp}} \int f d\mu_{V,a} d\mathcal{H}^{n-m}a \leq \int f d\mu$.*

(4) *If $\pi_V \# \mu \ll \mathcal{H}^{n-m} \llcorner V^{\perp}$, then*

$$\int_{V^{\perp}} \int f d\mu_{V,a} d\mathcal{H}^{n-m}a = \int f d\mu.$$

(5) *The function $A \mapsto \int f d\mu_A$ is $\lambda_{n,m}$ measurable on $A(n, m)$.*

PROOF. The set $P_V = \{a \in V^{\perp}: V_a \in P\}$ is a Borel set by Lemma 3.3(2) and $\mathcal{H}^{n-m}(V^{\perp} \sim P_V) = 0$ by 3.1.

Let g be a nonnegative lower semicontinuous function on R^n . Then there is a nondecreasing sequence (φ_i) of continuous functions with $\lim \varphi_i = g$. For $a \in P_V$ by (3.2),

$$\int g d\mu_{V,a} = \lim_{i \rightarrow \infty} \int \varphi_i d\mu_{V,a} = \lim_{i \rightarrow \infty} \lim_{\delta \downarrow 0} \alpha(n-m)^{-1} \delta^{m-n} \int_{V_a(\delta)} \varphi_i d\mu.$$

Therefore $a \mapsto \int g \, d\mu_{V,a}$ is a Borel function by 2.5, and using the monotone convergence theorem and 2.4 we get

$$\begin{aligned} \int_{V^\perp} \int g \, d\mu_{V,a} \, d\mathfrak{K}^{n-m}a &= \int_{V^\perp} \lim_{i \rightarrow \infty} D(\pi_{V\#} \nu_{\mathfrak{K}}, V^\perp, a) \, d\mathfrak{K}^{n-m}a \\ &= \lim_{i \rightarrow \infty} \int_{V^\perp} D(\pi_{V\#} \nu_{\mathfrak{K}}, V^\perp, a) \, d\mathfrak{K}^{n-m}a \leq \lim_{i \rightarrow \infty} \pi_{V\#} \nu_{\mathfrak{K}}(V^\perp) \\ &= \lim_{i \rightarrow \infty} \int \varphi_i \, d\mu = \int g \, d\mu. \end{aligned}$$

Since $\int f \, d\mu < \infty$ there are sequences (ψ_i) of continuous functions and (g_i) of lower semicontinuous functions such that

$$|f - \psi_i| \leq g_i \quad \text{and} \quad \lim_{i \rightarrow \infty} \int g_i \, d\mu = 0.$$

Then $\int_{V^\perp} \int g_i \, d\mu_{V,a} \, d\mathfrak{K}^{n-m}a \leq \int g_i \, d\mu \rightarrow 0$; hence for a subsequence, which we may assume to be the whole sequence,

$$\lim_{i \rightarrow \infty} \int g_i \, d\mu_{V,a} = 0 \quad \text{for } \mathfrak{K}^{n-m} \text{ a.a. } a \in V^\perp. \quad (6)$$

We also have by 2.5 and 2.4,

$$\begin{aligned} \int \lim_{\delta \downarrow 0} \alpha(n-m)^{-1} \delta^{m-n} \int_{V_a(\delta)} |f - \psi_i| \, d\mu \, d\mathfrak{K}^{n-m}a \\ \leq \int D(\pi_{V\#} \nu_{\mathfrak{K}}, V^\perp, a) \, d\mathfrak{K}^{n-m}a \leq \int g_i \, d\mu \rightarrow 0. \end{aligned}$$

Hence going once more to a subsequence without changing the notation, we may assume

$$\lim_{i \rightarrow \infty} \lim_{\delta \downarrow 0} \alpha(n-m)^{-1} \delta^{m-n} \int_{V_a(\delta)} |f - \psi_i| \, d\mu = 0 \quad \text{for } \mathfrak{K}^{n-m} \text{ a.a. } a \in V^\perp. \quad (7)$$

Let R be the set of those $a \in P_V$ for which (6) and (7) hold and for which

$$\lim_{\delta \downarrow 0} \alpha(n-m)^{-1} \delta^{m-n} \int_{V_a(\delta)} f \, d\mu = D(\pi_{V\#} \nu_{\mathfrak{K}}, V^\perp, a) < \infty.$$

Then by 2.5 and 2.4, R is a Borel set, $\mathfrak{K}^{n-m}(V^\perp \sim R) = 0$ and for $a \in R$ and for all i ,

$$\begin{aligned} \lim_{\delta \downarrow 0} \alpha(n-m)^{-1} \delta^{m-n} \int_{V_a(\delta)} f \, d\mu &= \lim_{\delta \downarrow 0} \alpha(n-m)^{-1} \delta^{m-n} \int_{V_a(\delta)} \psi_i \, d\mu \\ &\quad + \lim_{\delta \downarrow 0} \alpha(n-m)^{-1} \delta^{m-n} \int_{V_a(\delta)} (f - \psi_i) \, d\mu \\ &= \int \psi_i \, d\mu_{V,a} + \lim_{\delta \downarrow 0} \alpha(n-m)^{-1} \delta^{m-n} \int_{V_a(\delta)} (f - \psi_i) \, d\mu \\ &\rightarrow \int f \, d\mu_{V,a} \quad \text{as } i \rightarrow \infty. \end{aligned}$$

Hence for $a \in R$,

$$\lim_{\delta \downarrow 0} \alpha(n-m)^{-1} \delta^{m-n} \int_{V_a(\delta)} f d\mu = \int f d\mu_{V,a}.$$

The left-hand side is a Borel function by 2.5, whence $a \mapsto \int f d\mu_{V,a}$ is $\mathcal{H}^{n-m}$ measurable. This proves (1) and (2). (3) and (4) follow from (1) and 2.4 when applied to the measure $\pi_{V\#} \nu_f$.

Essentially the same argument which was used to prove (1) gives for $\lambda_{n,m}$ a.a. $A \in \mathcal{A}(n, m)$,

$$\int f d\mu_A = \lim_{\delta \downarrow 0} \alpha(n-m)^{-1} \delta^{m-n} \int_{A(\delta)} f d\mu.$$

The only difference is that one now performs integrations also over $G(n, m)$. (5) follows then from 2.5.

The following lemma, which is a generalization of [M, Lemma 13], gives a sufficient condition for $\pi_{V\#} \mu$ to be absolutely continuous with respect to $\mathcal{H}^{n-m} \llcorner V^\perp$ for $\gamma_{n,m}$ a.a. $V \in G(n, m)$. An immediate corollary is the fact that if $C_{n-m}(E) > 0$ or $\dim E > n-m$ then $\mathcal{H}^{n-m}(\pi_V(E)) > 0$ for $\gamma_{n,m}$ a.a. $V \in G(n, m)$. More general results involving multiplicities of the projections were derived in [MP, §4].

3.5. LEMMA. *If $\int |x-y|^{m-n} d\mu y < \infty$ for μ a.a. $x \in R^n$, then $\pi_{V\#} \mu \ll \mathcal{H}^{n-m} \llcorner V^\perp$ for $\gamma_{n,m}$ a.a. $V \in G(n, m)$.*

PROOF. By Fatou's lemma, Fubini's theorem and Lemma 2.2, we obtain for μ a.a. $x \in R^n$,

$$\begin{aligned} & \int \underline{D}(\pi_{V\#} \mu, V^\perp, \pi_V(x)) d\gamma_{n,m} V \\ & \leq \liminf_{\delta \downarrow 0} \alpha(n-m)^{-1} \delta^{m-n} \int \mu\{y: \text{dist}(y, V_x) \leq \delta\} d\gamma_{n,m} V \\ & = \liminf_{\delta \downarrow 0} \alpha(n-m)^{-1} \delta^{m-n} \int \gamma_{n,m}\{V: \text{dist}(y, V_x) \leq \delta\} d\mu y \\ & \leq c\alpha(n-m)^{-1} \int |x-y|^{m-n} d\mu y < \infty. \end{aligned}$$

Hence for $\gamma_{n,m}$ a.a. $V \in G(n, m)$,

$$\underline{D}(\pi_{V\#} \mu, V^\perp, \pi_V(x)) < \infty \quad \text{for } \mu \text{ a.a. } x \in R^n,$$

and therefore by the definition of $\pi_{V\#} \mu$,

$$\underline{D}(\pi_{V\#} \mu, V^\perp, a) < \infty \quad \text{for } \pi_{V\#} \mu \text{ a.a. } a \in V^\perp,$$

which according to 2.4 means $\pi_{V\#} \mu \ll \mathcal{H}^{n-m} \llcorner V^\perp$.

Observe in the following lemma that the μ exceptional set is independent of the Borel function f . This will be needed later on.

3.6. LEMMA. Suppose that $\int |x - y|^{m-n} d\mu y < \infty$ for μ a.a. $x \in R^n$. There are a constant c depending only on n and m , and $B \subset R^n$ such that $\mu(R^n \sim B) = 0$ and that for every nonnegative Borel function f on R^n ,

$$\int \int f d\mu_{V,x} d\gamma_{n,m} V \leq c \int f(y) |x - y|^{m-n} d\mu y \quad \text{for } x \in B.$$

PROOF. We reduce the proof to the case where f is continuous by first approximating f from above with lower semicontinuous functions and then approximating these lower semicontinuous functions from below by continuous functions. When f is continuous, (3.2) holds with φ replaced by f and V_a replaced by V_x for $(x, V) \in Q$, where Q is the Borel set of 3.3(3). Let B be the set of all $x \in R^n$ for which $\gamma_{n,m}\{V: (x, V) \in R^n \times G(n, m) \sim Q\} = 0$. Since

$$\mu \times \gamma_{n,m}(R^n \times G(n, m) \sim Q) = 0$$

by Lemmas 3.5 and 3.3(3), Fubini's theorem gives $\mu(R^n \sim B) = 0$. For $x \in B$ we use (3.2), Fatou's Lemma, Fubini's theorem and Lemma 2.2, and let g_δ be the characteristic function of the set $\{(y, V): \text{dist}(y, V_x) \leq \delta\}$ to obtain

$$\begin{aligned} \iint f d\mu_{V,x} d\gamma_{n,m} V &\leq \liminf_{\delta \downarrow 0} \alpha(n-m)^{-1} \delta^{m-n} \int \int_{V_x(\delta)} f d\mu d\gamma_{n,m} V \\ &= \liminf_{\delta \downarrow 0} \alpha(n-m)^{-1} \delta^{m-n} \int \int f(y) g_\delta(y, V) d\mu y d\gamma_{n,m} V \\ &= \liminf_{\delta \downarrow 0} \alpha(n-m)^{-1} \delta^{m-n} \int f(y) \gamma_{n,m}\{V: \text{dist}(y, V_x) \leq \delta\} d\mu y \\ &\leq \alpha(n-m)^{-1} c \int f(y) |x - y|^{m-n} d\mu y. \end{aligned}$$

4. Energies and capacities.

4.1. Definitions. In the following K will be a nonnegative lower semicontinuous function on $R^n \times R^n$ (which may have value ∞). The K -energy of a Radon measure μ on R^n is

$$I_K(\mu) = \int \int K(x, y) d\mu x d\mu y.$$

The K -capacity of a compact set $F \subset R^n$ is defined by

$$C_K(F) = \sup I_K(\mu)^{-1}$$

where the supremum is taken over all Radon measures μ on R^n with $\text{spt } \mu \subset F$ and $\mu(F) = 1$. For arbitrary $E \subset R^n$ we set

$$C_K(E) = \sup\{C_K(F): F \text{ compact} \subset E\}.$$

If for some $s > 0$, $K(x, y) = |x - y|^{-s}$ for $x \neq y$ and $K(x, x) = \infty$, we denote C_s instead of C_K .

Let $P \subset A(n, m)$ and $Q \subset R^n \times G(n, m)$ be the Borel sets where μ_A and $\mu_{V,x}$ are defined, respectively (recall Lemma 3.3).

4.2. LEMMA. Let μ be a Radon measure on R^n with compact support. Then the functions

$$\begin{aligned} A &\mapsto I_K(\mu_A), & A &\in P, \\ (x, V) &\mapsto I_K(\mu_{V,x}), & (x, V) &\in Q, \\ (x, V) &\mapsto \int K(x, y) d\mu_{V,x}y, & (x, V) &\in Q, \end{aligned}$$

are Borel functions.

PROOF. By the monotone convergence theorem we reduce the proof to the case where K is continuous with compact support. It follows from the Stone-Weierstrass approximation theorem that there is a sequence (K_i) converging uniformly to K such that each K_i is a finite sum of functions of the form $(x, y) \mapsto \varphi(x)\psi(y)$ where φ and ψ are continuous functions on R^n with compact support. Since $\mu_A(R^n) < \infty$ for $A \in P$ and $\mu_{V,x}(R^n) < \infty$ for $(x, V) \in Q$, uniform convergence implies convergence for μ_A - and $\mu_{V,x}$ -integrals. Hence we may assume that

$$K(x, y) = \varphi(x)\psi(y) \quad \text{for } x, y \in R^n$$

where φ and ψ are continuous. Then

$$\begin{aligned} I_K(\mu_A) &= \int \varphi d\mu_A \int \psi d\mu_A \quad \text{for } A \in P, \\ I_K(\mu_{V,x}) &= \int \varphi d\mu_{V,x} \int \psi d\mu_{V,x} \quad \text{for } (x, V) \in Q, \end{aligned}$$

and

$$\int K(x, y) d\mu_{V,x}y = \varphi(x) \int \psi d\mu_{V,x} \quad \text{for } (x, V) \in Q,$$

and the result follows from (3.2) and 2.5.

4.3. LEMMA. Let f be a real-valued function on $R^k \times R^1$ such that the function $x \mapsto f(x, t)$ is $\mathcal{H}^k$ measurable for $t \in R^1$ and the function $t \mapsto f(x, t)$ is nonincreasing and left continuous for $x \in R^k$. Then f is $\mathcal{H}^{k+1}$ measurable.

PROOF. Let Q be the set of rational numbers. For $\alpha \in R^1$ and $r \in Q$, set

$$E_\alpha = \{(x, t): f(x, t) \geq \alpha\}, \quad E_{\alpha,r} = \{x: f(x, r) \geq \alpha\}.$$

Then $E_{\alpha,r}$ is $\mathcal{H}^k$ measurable and

$$E_\alpha = \bigcap_{i=1}^{\infty} \bigcup_{r \in Q} \{(x, t): x \in E_{\alpha,r}\} \cap \{(x, t): r < t < r + 1/i\}.$$

Hence E_α is $\mathcal{H}^{k+1}$ measurable.

From now on we shall assume that for some positive constant b ,

$$K(x, y) \geq b|x - y|^{m-n} \quad \text{for } (x, y) \in R^n \times R^n, |x - y| \leq 1. \quad (4.4)$$

We define another lower semicontinuous kernel H by

$$\begin{aligned} H(x, y) &= K(x, y)|x - y|^{n-m} \quad \text{for } x \neq y, \\ H(x, x) &= \liminf_{\substack{(y,z) \rightarrow (x,x) \\ y \neq z}} H(y, z). \end{aligned}$$

We first derive an integralgeometric inequality for energy-integrals.

4.5. THEOREM. *There is a constant c depending only on n and m such that for any Radon measure μ on R^n with compact support,*

$$\int I_H(\mu_A) d\lambda_{n,m}A \leq cI_K(\mu).$$

PROOF. We may assume $I_K(\mu) < \infty$. Then by (4.4), $\int |x - y|^{m-n} d\mu y < \infty$ for μ a.a. $x \in R^n$.

Let P be the Borel set of Lemma 3.3(2). Then $\lambda_{n,m}(A(n, m) \sim P) = 0$ and $A \mapsto I_H(\mu_A)$ is a Borel function on P by Lemma 4.2. We shall use the formula

$$\int f d\nu = \int_0^\infty \nu\{x: f(x) \geq t\} dt$$

for the ν -integral of a nonnegative ν measurable function f . We denote for $0 \leq t < \infty$, $V \in G(n, m)$,

$$E_{V,t} = \left\{ x \in R^n: \int H(x, y) d\mu_{V,x}y \geq t \right\}.$$

Then $E_{V,t}$ is a Borel set by Lemma 4.2. Recalling that $\text{spt } \mu_{V,a} \subset V_a$ and $\mu_{V,x} = \mu_{V,a}$ if $x \in V_a$, we have for $V_a \in P$,

$$\begin{aligned} I_H(\mu_{V,a}) &= \int_0^\infty \mu_{V,a} \left\{ x: \int H(x, y) d\mu_{V,a}y \geq t \right\} dt \\ &= \int_0^\infty \mu_{V,a} \left\{ x: \int H(x, y) d\mu_{V,x}y \geq t \right\} dt \\ &= \int_0^\infty \mu_{V,a}(E_{V,t}) dt. \end{aligned}$$

For $\gamma_{n,m}$ a.a. $V \in G(n, m)$ the function $a \mapsto \mu_{V,a}(E_{V,t})$ is $\mathcal{H}^{n-m}$ measurable on $V^\perp$ for $0 \leq t < \infty$ by Lemma 3.4(2), and for such V we may apply Lemma 4.3 with $f(a, t) = \mu_{V,a}(E_{V,t})$. Integrating over $V^\perp$ we get by Fubini's theorem and Lemma 3.4(3),

$$\begin{aligned} \int_{V^\perp} I_H(\mu_{V,a}) d\mathcal{H}^{n-m}a &= \int_{V^\perp} \int_0^\infty \mu_{V,a}(E_{V,t}) dt d\mathcal{H}^{n-m}a \\ &= \int_0^\infty \int_{V^\perp} \mu_{V,a}(E_{V,t}) d\mathcal{H}^{n-m}a dt \leq \int_0^\infty \mu(E_{V,t}) dt. \end{aligned}$$

Finally we integrate over $G(n, m)$, use Fubini's theorem, which is justified because the set of all $(x, V, t) \in R^n \times G(n, m) \times R^1$ for which $\int H(x, y) d\mu_{V,x}y \geq t$ is a Borel set, and apply Lemma 3.6 to obtain

$$\begin{aligned}
 \int I_H(\mu_A) d\lambda_{n,m}A &\leq \int \int_0^\infty \mu(E_{V,t}) dt d\gamma_{n,m}V \\
 &= \int_0^\infty \int \mu(E_{V,t}) d\gamma_{n,m}V dt \\
 &= \int_0^\infty \int \gamma_{n,m} \left\{ V: \int H(x, y) d\mu_{V,x}y \geq t \right\} d\mu x dt \\
 &= \int \int_0^\infty \gamma_{n,m} \left\{ V: \int H(x, y) d\mu_{V,x}y \geq t \right\} dt d\mu x \\
 &= \int \int \int H(x, y) d\mu_{V,x}y d\gamma_{n,m}V d\mu x \\
 &\leq c \int \int K(x, y) d\mu y d\mu x = cI_K(\mu).
 \end{aligned}$$

4.6. THEOREM. *There is a constant c depending only on n and m such that for any compact set $F \subset R^n$,*

$$C_K(F) \leq c \int C_H(F \cap A) d\lambda_{n,m}A.$$

PROOF. The function $A \mapsto C_H(F \cap A)$ is upper semicontinuous on $A(n, m)$. To see this suppose $C_H(F \cap A_0) < \alpha$, $A_0 \in A(n, m)$. Then there is an open set G such that $F \cap A_0 \subset G$ and $C_H(G) < \alpha$ (see [FB, Lemma 2.3.4]). Since F is compact there is a neighborhood U of A_0 in $A(n, m)$ such that $F \cap A \subset G$ for $A \in U$. Hence $C_H(F \cap A) \leq C_H(G) < \alpha$ for $A \in U$.

We may assume $C_K(F) > 0$. Let $\varepsilon > 0$ and let μ be a Radon measure such that $\text{spt } \mu \subset F$, $\mu(F) = 1$ and $I_K(\mu) \leq C_K(F)^{-1} + \varepsilon$. Since $I_K(\mu) < \infty$ (4.4) implies $\int |x - y|^{m-n} d\mu y < \infty$ for μ a.a. $x \in R^n$. Let R be the set of all $A \in A(n, m)$ for which $\mu_A(R^n) > 0$. We define $\nu_A = \mu_A(R^n)^{-1} \mu_A$ for $A \in R$. Then $\text{spt } \nu_A \subset F \cap A$ and $\nu_A(F \cap A) = 1$.

By Lemmas 3.5 and 3.4(4) we have

$$\int \mu_A(R^n) d\lambda_{n,m}A = \mu(R^n) = 1.$$

Since by Theorem 4.5, $I_H(\nu_A) = \mu_A(R^n)^{-2} I_H(\mu_A) < \infty$ for $\lambda_{n,m}$ a.a. $A \in R$, we get from Hölder's inequality and Theorem 4.5,

$$\begin{aligned}
 1 &= \left(\int \mu_A(R^n) d\lambda_{n,m}A \right)^2 = \left(\int_R \mu_A(R^n) I_H(\nu_A)^{1/2} I_H(\nu_A)^{-1/2} d\lambda_{n,m}A \right)^2 \\
 &\leq \left(\int_R \mu_A(R^n)^2 I_H(\nu_A) d\lambda_{n,m}A \right) \left(\int_R I_H(\nu_A)^{-1} d\lambda_{n,m}A \right) \\
 &= \left(\int_R I_H(\mu_A) d\lambda_{n,m}A \right) \left(\int_R I_H(\nu_A)^{-1} d\lambda_{n,m}A \right) \\
 &\leq cI_K(\mu) \int C_H(F \cap A) d\lambda_{n,m}A \\
 &\leq c(C_K(F)^{-1} + \varepsilon) \int C_H(F \cap A) d\lambda_{n,m}A.
 \end{aligned}$$

Letting $\varepsilon \rightarrow 0$ we get the desired result.

4.7. REMARK. It is clear that the inequality of Theorem 4.6 holds for arbitrary subsets of R^n if the integral is replaced by the lower integral.

4.8. THEOREM. If $E \subset R^n$ and $C_K(E) > 0$, then there is $B \subset E$ such that $C_K(E \sim B) = 0$ and for $x \in B$,

$$C_H(E \cap V_x) > 0 \quad \text{for } \gamma_{n,m} \text{ a.a. } V \in G(n, m).$$

PROOF. Suppose this is false. Then there is a compact set $F \subset E$ such that $C_K(F) > 0$ and $\gamma_{n,m}\{V: C_H(E \cap V_x) = 0\} > 0$ for $x \in F$, and we can find a Radon measure μ such that $\text{spt } \mu \subset F$, $\mu(F) = 1$ and $I_K(\mu) < \infty$. By (4.4) and Lemmas 3.5 and 3.3(3), $\mu_{V,x}(R^n) > 0$ for $\mu \times \gamma_{n,m}$ a.a. $(x, V) \in R^n \times G(n, m)$. Since $\text{spt } \mu_{V,x} \subset F \cap V_x$ and $F \subset E$, we have $I_H(\mu_{V,x}) = \infty$ whenever $C_H(E \cap V_x) = 0$ and $\mu_{V,x}(R^n) > 0$. Therefore

$$\gamma_{n,m}\{V: I_H(\mu_{V,x}) = \infty\} > 0 \quad \text{for } \mu \text{ a.a. } x \in F.$$

Letting f be the characteristic function of the Borel set $\{(x, V): I_H(\mu_{V,x}) = \infty\}$ (cf. Lemma 4.2), we obtain from Fubini's theorem,

$$0 < \int \int f d\gamma_{n,m} d\mu = \int \int f d\mu d\gamma_{n,m} = \int \mu\{x: I_H(\mu_{V,x}) = \infty\} d\gamma_{n,m} V.$$

Hence there is a set $G \subset G(n, m)$ such that $\gamma_{n,m}(G) > 0$ and $\mu\{x: I_H(\mu_{V,x}) = \infty\} > 0$ for $V \in G$. Since $\pi_{V\#} \mu \ll \mathcal{H}^{n-m} \llcorner V^\perp$ for $\gamma_{n,m}$ a.a. $V \in G(n, m)$, this gives

$$\mathcal{H}^{n-m}\{a \in V^\perp: I_H(\mu_{V,a}) = \infty\} > 0 \quad \text{for } \gamma_{n,m} \text{ a.a. } V \in G.$$

Integrating and using Theorem 4.5 we get a contradiction:

$$\infty = \int \int_{V^\perp} I_H(\mu_{V,a}) d\mathcal{H}^{n-m} a d\gamma_{n,m} V \leq c I_K(\mu) < \infty.$$

In the case $K(x, y) = |x - y|^{m-n}$ we have the following

4.9. THEOREM. If $E \subset R^n$ and $C_{n-m}(E) > 0$, then there is $B \subset E$ such that $C_{n-m}(E \sim B) = 0$, and for $x \in B$, $E \cap V_x$ is uncountable for $\gamma_{n,m}$ a.a. $V \in G(n, m)$.

PROOF. If this is false, there is a compact set $F \subset E$ such that $C_{n-m}(F) > 0$ and

$$\gamma_{n,m}\{V: E \cap V_x \text{ is at most countable}\} > 0 \quad \text{for } x \in F, \quad (1)$$

and we can find a Radon measure μ such that $\text{spt } \mu \subset F$, $\mu(F) = 1$ and $I_K(\mu) < \infty$ where $K(x, y) = |x - y|^{m-n}$. We define for $0 < r < \infty$

$$H_r(x, y) = 1, \text{ if } |x - y| < r,$$

$$H_r(x, y) = 0, \text{ if } |x - y| \geq r,$$

$$H(x, y) = 1, \text{ if } x = y,$$

$$H(x, y) = 0, \text{ if } x \neq y,$$

$$K_r(x, y) = H_r(x, y)|x - y|^{m-n}.$$

Then the functions H_r and K_r are lower semicontinuous and $H_r \downarrow H$ as $r \downarrow 0$.

Whenever μ_A is defined, Lebesgue's bounded convergence theorem gives

$$\int \mu_A\{x\} d\mu_A x = \int \int H(x, y) d\mu_A y d\mu_A x = \lim_{r \downarrow 0} \int \int K_r(x, y) d\mu_A y d\mu_A x = \lim_{r \downarrow 0} I_{K_r}(\mu_A),$$

and we obtain by Theorem 4.5 and Fatou's lemma

$$\begin{aligned} \int \int \mu_A \{x\} d\mu_A x d\lambda_{n,m} A &\leq \liminf_{r \downarrow 0} \int I_{H_r}(\mu_A) d\lambda_{n,m} A \\ &\leq c \liminf_{r \downarrow 0} I_K(\mu) = 0 \end{aligned}$$

because $I_K(\mu) < \infty$. Hence for $\lambda_{n,m}$ a.a. $A \in \mathcal{A}(n, m)$, $\int \mu_A \{x\} d\mu_A = 0$ which means $\mu_A \{x\} = 0$ for all $x \in R^n$. Since $\pi_{V^\#} \mu \ll \mathcal{H}^{n-m} \ll V^\perp$ for $\gamma_{n,m}$ a.a. $V \in G(n, m)$ by Lemma 3.5, it follows that for $\mu \times \gamma_{n,m}$ a.a. $(x, V) \in R^n \times G(n, m)$, $\mu_{V,x} \{y\} = 0$ for all $y \in R^n$. Using Lemma 3.3(3) we find for $\mu \times \gamma_{n,m}$ a.a. $(x, V) \in R^n \times G(n, m)$, $\mu_{V,x}(R^n) > 0$ and $\mu_{V,x} \{y\} = 0$ for $y \in R^n$, whence $\text{spt } \mu_{V,x}$ is uncountable. This contradicts $\text{spt } \mu_{V,x} \subset E \cap V_x$ and (1).

4.10. REMARKS. Suppose that $s > n - m$ and E is $\mathcal{H}^s$ measurable with $0 < \mathcal{H}^s(E) < \infty$. Then one can use Theorem 4.5 and the well-known relations between Hausdorff measure and capacity to prove $\dim(E \cap V_x) \geq s + m - n$ for $\mathcal{H}^s \times \gamma_{n,m}$ a.a. $(x, V) \in E \times G(n, m)$. This is Lemma 6.4 of [MP].

I do not know whether there are general results similar to 4.5–4.8 in the opposite direction. Ohtsuka has considered product sets in [O].

As a rather immediate consequence of [MP, Lemma 5.1] and Hölder's inequality we can give an inequality analogous to 4.6 for the Riesz capacities of the orthogonal projections. Here $O^*(n, m)$ is the space of all orthogonal projections $R^n \rightarrow R^m$ and $\theta_{n,m}^*$ is the orthogonally invariant measure on $O^*(n, m)$ of total mass one (see [F, 1.7.4 and 2.7.16]).

4.11. THEOREM. For $0 < s < m$ there is a constant c depending only on n, m and s such that for any compact set $F \subset R^n$,

$$C_s(F) \leq c \int C_s(p(F)) d\theta_{n,m}^* p \leq c C_s(F).$$

PROOF. The right-hand inequality follows from $C_s(p(F)) \leq C_s(F)$ (see [L, Theorem 2.9, p. 158]). To prove the left-hand inequality we may assume $C_s(F) > 0$. By [MP, 5.1],

$$\int C_s(p(F))^{-1} d\theta_{n,m}^* p \leq c C_s(F)^{-1}.$$

Hölder's inequality gives

$$\begin{aligned} 1 &= \int C_s(p(F))^{1/2} C_s(p(F))^{-1/2} d\theta_{n,m}^* p \\ &\leq \left(\int C_s(p(F)) d\theta_{n,m}^* p \right)^{1/2} \left(\int C_s(p(F))^{-1} d\theta_{n,m}^* p \right)^{1/2}, \end{aligned}$$

whence

$$C_s(F) \leq c \left(\int C_s(p(F))^{-1} d\theta_{n,m}^* p \right)^{-1} \leq c \int C_s(p(F)) d\theta_{n,m}^* p.$$

4.12. REMARK. The method of [MP] does not seem to give a similar inequality for general kernels K . However, in some special cases it can be modified, for example if $K(x, y) = \sup\{-\log|x - y|, 0\}$.

5. On the structure of purely unrectifiable sets. A set $E \subset R^n$ is m rectifiable if $E = f(B)$ for some Lipschitzian map $f: B \rightarrow R^n$ where $B \subset R^m$ is bounded. E is called *purely* $(\mathcal{H}^m, m)$ *unrectifiable* if it contains no m rectifiable subset of positive $\mathcal{H}^m$ measure. If $\mathcal{H}^m(E) < \infty$ and E is purely $(\mathcal{H}^m, m)$ unrectifiable, then according to one of the basic results of geometric measure theory [F, 3.3.15] $\mathcal{H}^m(p(E)) = 0$ for $\theta_{n,m}^*$ a.a. $p \in O^*(n, m)$. If E is a Borel set this means that the integralgeometric measure [F, 2.10.5] $\mathfrak{I}_1^m(E) = 0$.

In [M, §8] Marstrand considered radial projections of purely $(\mathcal{H}^1, 1)$ unrectifiable $\mathcal{H}^1$ measurable plane sets E for which $\mathcal{H}^1(E) < \infty$. He showed that if A is the set of all those points $a \in R^2$ from which the radial projection of E has positive linear measure, that is, $\gamma_{2,1}\{l: (E \sim \{a\}) \cap l_a \neq \emptyset\} > 0$, then $\dim A \leq 1$. He also gave an example of a set E with $\dim A = 1$. But it is not known whether $\mathcal{H}^1(A) = 0$ always or even $\mathcal{H}^1(A) < \infty$.

Here we generalize Marstrand's result to arbitrary dimensions, and we also give more precise information on the exceptional set. However, the above question remains unsolved.

The *outer s -capacity* of $E \subset R^n$ is

$$C_s^*(E) = \inf\{C_s(G): E \subset G, G \text{ is open}\}.$$

For Suslin sets E , $C_s^*(E) = C_s(E)$ [L, Theorem 2.8, p 156]. If $C_s^*(E) = 0$, then $\dim E \leq s$ [L, Theorem 3.13, p. 196].

5.1. THEOREM. Let $E \subset R^n$ with $\mathfrak{I}_1^m(E) = 0$ and let

$$A = \{x \in R^n: \gamma_{n,n-m}\{V: E \cap V_x \neq \emptyset\} > 0\}.$$

Then $C_m^*(A) = 0$, hence $\dim A \leq m$, and A is purely $(\mathcal{H}^m, m)$ unrectifiable.

PROOF. Since $\mathfrak{I}_1^m$ is Borel regular [F, 2.10.1], we may assume that E is a Borel set. We first show that then the set A and

$$B = \{(x, V) \in R^n \times G(n, n-m): E \cap V_x \neq \emptyset\}$$

are Suslin sets. The map $(y, x, V) \mapsto \pi_V(x - y)$ of $R^n \times R^n \times G(n, n-m)$ into R^n is continuous. Hence

$$C = E \times R^n \times G(n, n-m) \cap \{(y, x, V): \pi_V(x - y) = 0\}$$

is a Borel set. If $p: R^n \times R^n \times G(n, n-m) \rightarrow R^n \times G(n, n-m)$, $p(y, x, V) = (x, V)$, is the projection, then $B = p(C)$, and it follows from [F, 2.2.10] that B is a Suslin set. Then

$$A = \{x: \gamma_{n,n-m}\{V: (x, V) \in B\} > 0\}$$

is a Suslin set by [D, VI, 21].

For $F \subset A$ let

$$F_V = \{x \in F: E \cap V_x \neq \emptyset\} \quad \text{for } V \in G(n, n-m).$$

Then $\pi_V(F_V) \subset \pi_V(E)$, and $\mathfrak{I}_1^m(E) = 0$ implies

$$\mathfrak{I}^m(\pi_V(F_V)) = 0 \quad \text{for } \gamma_{n,n-m} \text{ a.a. } V \in G(n, n-m). \quad (1)$$

We shall show that the negation of either one of the assertions yields a Suslin set $F \subset A$ and a Radon measure μ such that

$$\mu(F) > 0 \quad \text{and} \quad \pi_{V\#} \mu \ll \mathfrak{I}^m \llcorner V^\perp \quad \text{for } \gamma_{n,n-m} \text{ a.a. } V \in G(n, n-m). \quad (2)$$

This leads to a contradiction. For (1) and (2) imply $\mu(F_V) = 0$ for $\gamma_{n,n-m}$ a.a. $V \in G(n, n-m)$, while Fubini's theorem gives

$$\int \mu(F_V) d\gamma_{n,n-m} V = \int_F \gamma_{n,n-m} \{V: E \cap V_x \neq \emptyset\} d\mu x > 0.$$

Suppose first that $C_m^*(A) > 0$. Since A is a Suslin set also $C_m(A) > 0$. Hence there are a compact set $F \subset A$ and a Radon measure μ such that $\mu(F) > 0$ and $\int |x - y|^{-m} d\mu y < \infty$ for μ a.a. $x \in R^n$. Then (2) follows from Lemma 3.5.

Suppose then that A is not purely $(\mathfrak{I}^m, m)$ unrectifiable. Then A contains an m rectifiable subset B with $\mathfrak{I}^m(B) > 0$. By [F, 3.2.29] there is a C^1 submanifold M of R^n such that $\mathfrak{I}^m(B \cap M) > 0$. Set $F = A \cap M$. Then F is a Suslin set with $\mathfrak{I}^m(F) > 0$. Let $T_x \in G(n, m)$ be the tangent plane direction of M at $x \in M$ and let

$$J(V, x) = |\det(\pi_V|T_x)|$$

for $V \in G(n, n-m)$, $x \in M$. Then by [F, 3.2.20],

$$\int N(\pi_V|C, y) d\mathfrak{I}^m y = \int_C J(V, x) d\mathfrak{I}^m x \quad (3)$$

for any $\mathfrak{I}^m$ measurable set $C \subset M$, where $N(\pi_V|C, y)$ is the number of points in the set $C \cap \pi_V^{-1}\{y\}$. Since $J(V, x) = 0$ if and only if $\dim(\pi_V(T_x)) < m$, we have for every $x \in M$, $J(V, x) > 0$ for $\gamma_{n,n-m}$ a.a. $V \in G(n, n-m)$. Hence by Fubini's theorem,

$$\begin{aligned} \int \mathfrak{I}^m \{x \in F: J(V, x) = 0\} d\gamma_{n,n-m} V \\ = \int_F \gamma_{n,n-m} \{V: J(V, x) = 0\} d\mathfrak{I}^m x = 0; \end{aligned}$$

thus for $\gamma_{n,n-m}$ a.a. $V \in G(n, n-m)$, $\mathfrak{I}^m \{x \in F: J(V, x) = 0\} = 0$. For every such V (3) implies $\mathfrak{I}^m(\pi_V(C)) > 0$ whenever $C \subset F$ with $\mathfrak{I}^m(C) > 0$. This means that F and $\mu = \mathfrak{I}^m \llcorner F$ satisfy (2).

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