THE SECOND CONJUGATE ALGEBRA OF THE FOURIER ALGEBRA OF A LOCALLY COMPACT GROUP

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ABSTRACT. Let G be a locally compact group and let $VN(G)$ denote the von Neumann algebra generated by the left translations of G on $L_2(G)$. Then $VN(G)^*$, when regarded as the second conjugate space of the Fourier algebra of G , is a Banach algebra with the Arens product. We prove among other things that when G is amenable, $VN(G)^*$ is neither commutative nor semisimple unless G is finite. We study in detail the class of maximal regular left ideals in $VN(G)^*$. We also show that if G_1 and G_2 are discrete groups, then G_1 and G_2 are isomorphic if and only if $VN(G_1)^*$ and $VN(G_2)^*$ are isometric order isomorphic.

1. Introduction. Let G be a locally compact group. Then $A(G)$, the *Fourier algebra* of G , is the linear subspace of $C_0(G)$ (bounded continuous complex-valued functions on G vanishing at infinity) consisting of all functions of the form $(h * \tilde{k})^\sim$ where $h, k \in L_2(G)$, $\tilde{g}(x) = g(x^{-1})$ and $\tilde{k} = k(x^{-1})$. If $f \in L_2(G)$ and $g \in G$, define $\lambda(g)f(t) = f(gt^{-1})$, $t \in G$. Let $VN(G)$ denote the closure in the weak operator topology of the linear span of $\{\lambda(g); g \in G\}$ in $\mathfrak{B}(L_2(G))$, the algebra of bounded linear operators from $L_2(G)$ into $L_2(G)$. Then $\phi = (h * \tilde{k})^\sim$ in $A(G)$ can be regarded as an ultraweakly continuous linear functional on the von Neumann algebra $VN(G)$ defined by $\phi(T) = (Th, k)$ (inner product in $L_2(G)$) for each $T \in VN(G)$. Furthermore, each ultraweakly continuous linear functional on $VN(G)$ is of this form [6, pp. 210 and 218]. In particular, $A(G)$ can be identified as the unique predual of $VN(G)$. Also, $A(G)$ with pointwise multiplication and the norm $\|\phi\| = \sup\{|\phi(x)|; x \in VN(G) \text{ and } \|x\| \leq 1\}$ is a commutative Banach algebra with spectrum G [6, p. 222].

It is the purpose of this paper to study the Banach algebra $VN(G)^*$, the second conjugate algebra of $A(G)$ equipped with the Arens product [1]. We prove, in §3, that $VN(G)^*$ is semisimple if and only if G is finite. We also prove that $A(G)$ is an ideal in $VN(G)^*$ if and only if G is discrete. When G is amenable, then the Fourier-Stieltjes algebra of G is a quotient of $VN(G)^*$. We characterize, in §4, the regular maximal left ideals in $VN(G)^*$. We show that any such ideal must contain $I_g = \{\phi \in A(G); \phi(g) = 0\}$ for some $g \in G$. Furthermore, if G is amenable, then every regular maximal left ideal in $VN(G)^*$ contains a unique I_g , $g \in G$, if and

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only if G is compact. We also show that G is discrete if and only if each regular left ideal in $VN(G)^*$ either contains $A(G)$ or must be the kernel of some $\lambda(g) \in VN(G)$, $g \in G$. In §5 this result is used to prove the following isomorphism theorem for discrete groups: if G_1 and G_2 are discrete groups, then the Banach algebras $VN(G_1)^*$ and $VN(G_2)^*$ are isometric order isomorphic if and only if G_1 and G_2 are isomorphic.

Most of our results in §§3 and 4 are known and proved by Civin and Yood [4] and Civin [3] when G is an abelian locally compact group. In this case, $A(G)$ is isometric isomorphic to $L_1(\hat{G})$ via the Fourier transform. Our proofs depend heavily on some deep analysis of the Fourier algebra $A(G)$ and various subspaces of $VN(G)$ of P. Eymard [6] and E. Granirer [7]–[9].

2. Preliminaries. Let E be a linear space, and ϕ be a linear functional on E , then the value of ϕ at an element x in E will be written as $\phi(x)$ or $\langle \phi, x \rangle$.

Throughout this paper, G denotes a locally compact group with a fixed left Haar measure. Let $C(G)$ denote the Banach space of bounded continuous complex-valued functions on G with the supremum norm. Then G is *amenable* if there exists a positive linear functional ϕ on $C(G)$ of norm one such that $\phi({}_af) = \phi(f)$ for each $a \in G$ and each $f \in C(G)$, where ${}_af(t) = f(at)$ for each $t \in G$. Amenable groups include all solvable groups and all compact groups. However, the free group on two generators is not amenable.

Let $P(G)$ denote the subspace of $C(G)$ consisting of all continuous positive definite functions on G , and let $B(G)$ be its linear span. Then $B(G)$ can be identified with the dual of $C^*(G)$, the group C^* -algebra of G (see [6, p. 192]). Also, $B(G)$ with pointwise multiplication and the dual norm is a commutative Banach algebra called the *Fourier-Stieltjes algebra* of G [6, Proposition 2.16].

Let $P_\rho(G)$ denote the closure of $P(G) \cap C_{00}(G)$ in the compact open topology, where $C_{00}(G)$ is the set of all functions in $C(G)$ with compact support, and let $B_\rho(G)$ denote the linear span of $P_\rho(G)$. Then $B_\rho(G)$ is a closed ideal in $B(G)$ and $B_\rho(G)$ is precisely the dual of $C_\rho^*(G)$ [6, Propositions 1.21 and 2.16], where $C_\rho^*(G)$ is the norm closure of $\{\lambda(f); f \in L_1(G)\}$ in $\mathcal{B}(L_2(G))$, and $\lambda(f)(h) = f * h$ for $h \in L_2(G)$. As is known [10, p. 61], $B_\rho(G) = B(G)$ if and only if G is amenable.

The Fourier algebra $A(G)$ as defined in §1 is the closed linear span of $P(G) \cap C_{00}(G)$. It is a closed ideal in $B(G)$ (see [6, p. 208]). Also $A(G) \subseteq B_\rho(G)$.

There is a natural module action of $A(G)$ on $VN(G)$ given by $\langle \phi \cdot x, \gamma \rangle = \langle \phi\gamma, x \rangle$ for each $\phi, \gamma \in A(G)$ and each $x \in VN(G)$. Recently E. Granirer [7, p. 373] has defined the subspace $UCB(\hat{G})$ of $VN(G)$ as the norm closure of $A(G) \cdot VN(G)$. Then $UCB(\hat{G})$ is a C^* -subalgebra of $VN(G)$ [8, Proposition 2(a₁)] and elements in $UCB(\hat{G})$ are called *uniformly continuous functionals* on $A(G)$. It follows from [8, Propositions 1 and 3] and [9, Theorem 3] that $UCB(\hat{G}) = VN(G)$ if and only if G is compact.

A positive linear functional $m \in VN(G)^*$ of norm one is called a *topological invariant mean* on $VN(G)$ if $\langle m, \phi \cdot x \rangle = \langle m, x \rangle$ for each $x \in VN(G)$ and each $\phi \in P(G) \cap A(G)$ such that $\phi(e) = 1$ where e is the identity of G . As known [16, Theorem 4], $VN(G)$ always has a topological invariant mean.

A subset X of $VN(G)$ is *topologically invariant* if $\phi \cdot x \in X$ for each $\phi \in A(G)$ and each $x \in X$. If X is a topological invariant linear subspace of $VN(G)$, we say that X is *topologically introverted* if for each $m \in VN(G)^*$ and each $x \in X$, the functional $\gamma \rightarrow \langle m, \gamma \cdot x \rangle$ on $A(G)$, denoted by $m \odot x$, also defines an element in X . Examples of topologically introverted subspaces of $VN(G)$ include $UCB(\hat{G})$ and $C^*_\rho(G)$ (see [13, Proposition 5.2]). Also, any weak*-closed topologically invariant subspace of $VN(G)^*$ is topologically introverted [13, Lemma 5.1].

3. The Banach algebra $VN(G)^*$. In [1] Arens shows that given a Banach algebra B , it is possible to define a multiplication on B^{**} which extends multiplication on B . In case $B = A(G)$, $m, n \in VN(G)^*$, the Arens product $m \odot n$ is defined by the formula: $\langle m \odot n, x \rangle = \langle m, n \odot x \rangle$ for each $x \in VN(G)$, where $\langle n \odot x, \phi \rangle = \langle n, \phi \cdot x \rangle$, $\phi \in A(G)$. This same formula certainly makes sense when $VN(G)$ is replaced by a topologically invariant and introverted subspace X of $VN(G)$. The following observations will be useful in the sequel.

LEMMA 3.1. (i) $A(G)$ is in the centre of $VN(G)^*$.

(ii) For each $n \in VN(G)^*$, the map $m \rightarrow m \odot n$ from $VN(G)^*$ into $VN(G)^*$ is weakly*-weak* continuous.

(iii) Any weak*-closed right ideal in $VN(G)^*$ is an ideal.

(iv) If $g \in G$, then the functional on $VN(G)^*$ defined by $m \rightarrow \langle m, \lambda(g) \rangle$ is multiplicative.

PROOF. Both (ii) and (iii) are immediate. Also (iii) follows from (i) and [13, Lemma 5.1]. To prove (iv), we simply observe that if $n \in VN(G)^*$, then $n \odot \lambda(g) = \langle n, \lambda(g) \rangle \lambda(g)$. Hence $\langle m \odot n, \lambda(g) \rangle = \langle m, \lambda(g) \rangle \langle n, \lambda(g) \rangle$ for any $m \in VN(G)^*$.

PROPOSITION 3.2. (a) $VN(G)^*$ has a right identity if and only if G is amenable.

(b) $VN(G)^*$ has a left identity if and only if G is compact.

PROOF. (a) is a direct consequence of Leptin's theorem [14] and Proposition 7 in [2, p. 146].

(b) If G is compact, then $A(G) = B(G)$. Hence $A(G)$ has an identity which is also the identity for $VN(G)^*$. Conversely, if G is not compact, then $UCB(\hat{G})$ is a proper C^* -subalgebra of $VN(G)$ by Granirer's results [8, Proposition 1] and [9, Theorem 3]. Hence there exists $m \in VN(G)^*$ such that $m \neq 0$ and $m(y) = 0$ for all $y \in UCB(\hat{G})$. Then $m \odot x = 0$ for each $x \in VN(G)$. Consequently $n \odot m = 0$ for each $n \in VN(G)^*$. In particular $VN(G)^*$ cannot have a left identity.

If G is finite, then $A(G) = VN(G)^*$. In particular, $VN(G)^*$ is commutative. The following shows that the converse of this also holds when G is amenable.

PROPOSITION 3.3. Let G be amenable. Then $VN(G)^*$ is commutative if and only if G is finite.

PROOF. If $VN(G)^*$ is commutative, then $VN(G)$ has a unique topological invariant mean (see [13, Theorem 5.6] and [7, Proposition 5]). Hence G is discrete by [16, Theorem 11]. Also since G is amenable, it follows from [13, Theorem 5.6]

and [8, Proposition 3] that $VN(G) = UCB(\hat{G})$. In particular, G is finite by [8, Proposition 1 and Theorem 4].

Let $\text{Rad}(VN(G)^*)$ denote the radical of $VN(G)^*$.

THEOREM 3.4. *The Banach algebra $VN(G)^*$ is semisimple if and only if G is finite.*

PROOF. If G is finite, then $A(G) = VN(G)^*$ which is semisimple.

If $VN(G)^*$ is semisimple, we first show that $VN(G)$ must have a unique topological invariant mean. In particular G is discrete [16, Theorem 11]. Indeed, if m_1, m_2 are distinct topological invariant means on $VN(G)$, let $J = \{m \in VN(G)^*; \langle m, \lambda(e) \rangle = 0 \text{ and } m \odot \phi = m \text{ for each state } \phi \text{ in } A(G)\}$. Then $m_1 - m_2$ is a nonzero element in J . J is a right ideal of $VN(G)^*$. In fact, if $m \in J$, $n \in VN(G)^*$, then $\langle m \odot n, \lambda(e) \rangle = \langle m, \lambda(e) \rangle \langle n, \lambda(e) \rangle = 0$ by Lemma 3.1(iv). Also if ϕ is a state in $A(G)$, then $(m \odot n) \odot \phi = (m \odot \phi) \odot n = m \odot n$ by Lemma 3.1(i). A similar argument shows that J is a left ideal. Also $J^2 = \{0\}$. In particular, J is nil and must be included in $\text{Rad}(VN(G)^*)$ (see Corollary in [11, p. 9]). Hence $VN(G)^*$ is not semisimple.

If G is not finite, then $UCB(\hat{G})$ is a proper C^* -subalgebra of $VN(G)$ ([8, Proposition 1] and [9, Theorem 3]). Let

$$K = \{m \in VN(G)^*; \langle m, x \rangle = 0 \text{ for each } x \in UCB(\hat{G})\}.$$

Then $K \neq 0$. Also, an application of Lemma 3.1(i) and (ii) shows that K is a left ideal and $K^2 = \{0\}$. In particular $K \subseteq \text{Rad}(VN(G)^*)$ and hence $VN(G)^*$ is not semisimple.

Let $K_\rho = \{m \in VN(G)^*; \langle m, \lambda(f) \rangle = 0 \text{ for each } f \in L_1(G)\}$.

THEOREM 3.5. *K_ρ is a weak*-closed two-sided ideal in $VN(G)^*$ containing $\text{Rad}(VN(G)^*)$. The quotient Banach algebra $VN(G)^*/K_\rho$ is isometrically isomorphic to $B_\rho(G)$. Also $K_\rho = \text{Rad}(VN(G)^*)$ if and only if G is discrete.*

PROOF. Suppose $n \in VN(G)^*$, then $n \odot x \in C_\rho^*(G)$ for each $x \in C_\rho^*(G)$ by [13, Proposition 5.2]. Hence $m \odot n \in K_\rho$ whenever $m \in K_\rho$. Since K_ρ , being the intersection of weak*-closed sets, is weak*-closed, it follows from Lemma 3.1(iii) that K_ρ is an ideal.

Since $K_\rho = C_\rho^*(G)^\perp$, the annihilator of $C_\rho^*(G)$, there is a natural linear isometry π from the quotient space $VN(G)^*/K_\rho$ onto $C_\rho^*(G)^*$. A simple computation shows that π is even an algebra isomorphism when $C_\rho^*(G)^*$ is equipped with the induced Arens product. Since the Arens product on $C_\rho^*(G)^* = B_\rho(G)$ is precisely the pointwise multiplication, [13, Proposition 5.3], the second assertion follows.

Finally, since $B_\rho(G)$ is semisimple, K_ρ must contain $\text{Rad}(VN(G)^*)$.

If G is discrete, then $C_\rho^*(G) = UCB(\hat{G})$ [8, Proposition 3]. Hence if $m \in K_\rho$, then Lemma 3.1(i) and (ii) implies that $n \odot m = 0$ for all $n \in VN(G)^*$. In particular, K_ρ is nil, and hence must be included in $\text{Rad}(VN(G)^*)$.

Conversely if G is not discrete, then $1 \notin C_\rho^*(G)$ by [13, Corollary 4.3]. Choose $n \in VN(G)^*$ such that $n(1) \neq 0$ and $n(x) = 0$ for all $x \in C_\rho^*(G)$. If $K_\rho = \text{Rad}(VN(G)^*)$, then $n \in \text{Rad}(VN(G)^*)$, which contradicts the second statement of Theorem 3.5 (whose proof does not depend on this result).

Let $C_\delta^*(G)$ denote the C^* -subalgebra of $VN(G)$ generated by the left translations $\{\lambda(g); g \in G\}$ on $L_2(G)$. Then for each $g \in G$, $\phi \in A(G)$, $\phi \cdot \lambda(g) = \phi(g)\lambda(g)$. For $n \in VN(G)^*$, $\phi \in A(G)$ and $g \in G$, we have

$$\langle n \odot \lambda(g), \phi \rangle = \langle n, \phi \cdot \lambda(g) \rangle = \langle n, \lambda(g) \rangle \phi(g) = \langle \langle n, \lambda(g) \rangle \lambda(g), \phi \rangle$$

which shows that $C_\delta^*(G)$ is topologically introverted.

Let $B_\delta(G)$ denote the linear span of $P_\delta(G)$, where $P_\delta(G)$ is the pointwise closure of $A(G) \cap P(G)$. Then $B_\delta(G)$ is a subalgebra of $B(G_d)$, where G_d denotes the group G with the discrete topology. Furthermore, $B_\delta(G)$ can be identified with $C_\delta^*(G)^*$ by the map $\pi(\phi)(f) = \sum \{\phi(t)f(t); t \in G\}$ for each $f \in l_1(G)$ and $\phi \in B_\delta(G)$ (see [6, Proposition 1.21]). Then, as readily checked, $\pi(\phi \cdot \psi) = \pi(\phi) \odot \pi(\psi)$ for all $\phi, \psi \in B_\delta(G)$.

Let $K_\delta = \{m \in VN(G)^*; \langle m, \lambda(g) \rangle = 0 \text{ for all } g \in G\}$. Then K_δ is a weak*-closed two-sided ideal in $VN(G)^*$.

THEOREM 3.6. *$B_\delta(G)$ with pointwise multiplication and the dual norm is a semisimple commutative Banach algebra isometric algebra isomorphic to $VN(G)^*/K_\delta$. In particular K_δ contains $\text{Rad}(VN(G)^*)$. The natural embedding π of $A(G)$ into $VN(G)^*/K_\delta$ is a linear isometry. Furthermore, if G is amenable, then π is onto if and only if G is finite.*

PROOF. That π is a linear isometry follows from the Kaplansky density theorem. If G is finite then clearly π is onto. Conversely, if G is amenable, and π is onto, then $A(G)$ can be identified with $C_\delta^*(G)^*$. Since $C_\delta^*(G)$ has a topological invariant mean [16, Theorem 4], there exists $\phi_0 \in P(G) \cap A(G)$ such that $\phi\phi_0 = \phi_0$ for all $\phi \in P(G)$. In particular G is discrete [16, Proposition 5]. Hence $A(G) = B_\delta(G) = B(G)$ by amenability of G . So G must be finite. Proofs of the remaining assertions are clear.

THEOREM 3.7. *G is discrete if and only if $A(G)$ is an ideal in $VN(G)^*$.*

PROOF. If G is discrete, then $C_\rho^*(G) = UCB(\hat{G})$ [8, Proposition 3]. Hence if $\phi \in A(G)$ and $m \in VN(G)^*$, then

$$\begin{aligned} \langle m \odot \phi, x \rangle &= \langle m, \phi \cdot x \rangle = \langle \psi, \phi \cdot x \rangle \text{ for some } \psi \in B_\rho(G) \\ &= \langle \psi \cdot \phi, x \rangle \end{aligned}$$

for each $x \in VN(G)$. Hence $m \odot \phi = \psi\phi$ which is in $A(G)$. It now follows from Lemma 3.1(i) that $A(G)$ is an ideal in $VN(G)^*$.

Conversely if $A(G)$ is an ideal in $VN(G)^*$, let $\phi_0 \in A(G) \cap P_1(G)$, where $P_1(G) = \{\phi \in P(G); \phi(e) = 1\}$. Let

$$K = \{m \odot \phi_0; m \in VN(G)^*, m \geq 0 \text{ and } \|m\| = 1\}.$$

Then the Banach-Alaoglu Theorem and Lemma 3.1(ii) imply that K is a weakly compact convex subset of $A(G)$. Also, if $\phi \in A(G) \cap P_1(G)$, let $T_\phi(\psi) = \phi\psi$ for each $\psi \in K$. Then $\{T_\phi; \phi \in A(G) \cap P_1(G)\}$ is a commuting family of continuous affine maps from (K, weak) into (K, weak) . An application of the Markov-Kakutani fixed point theorem [5, p. 456] shows that there exists $\psi_0 \in K$ such that

$T_\phi(\psi_0) = \psi_0$ for all $\phi \in A(G) \cap P_1(G)$. In particular ψ_0 is a topological invariant mean on $VN(G)$. By [16, Proposition 5], G must be discrete.

4. Maximal regular ideals. Let R be a ring. A left ideal I in R is *regular* if there exists $u \in R$ such that $xu - x \in I$ for each $x \in R$. As is well known, any proper regular left ideal in R is contained in a maximal proper regular left ideal in R (see [15, p. 58]). In this section, we shall study in detail the class of maximal proper left ideals in $VN(G)^*$ of a locally compact group G .

If $g \in G$, let $I_g = \{\phi \in A(G); \phi(g) = 0\}$ and let $I_g^\# = \{m \in VN(G)^*; \langle m, \lambda(g) \rangle = 0\}$. Then $I_g^\#$, being the kernel of the nonzero multiplicative linear functional $\lambda(g)$ on $VN(G)^*$ (Lemma 3.1(iv)), is a maximal weak*-closed ideal in $VN(G)^*$ and I_g is weak*-dense in $I_g^\#$. Let $T_g = \{\phi \in A(G); \text{there exist an open set } V \text{ containing } g \text{ and } \phi(x) = 0 \text{ for all } x \in V\}$. Also if V is an open set in G , let

$$S_V = \bigcap \{I_g; g \notin V\}.$$

Then T_g and S_V are also ideals in $A(G)$, and S_V is closed.

LEMMA 4.1. *Let M be a maximal regular left ideal in $VN(G)^*$ and $g \in G$. Then either $M \supseteq T_g$ or there exists an open set V of G containing g such that $M \supseteq S_V$.*

PROOF. If M does not contain T_g , let $\phi \in T_g$ and $\phi \notin M$. There exists an open set V of G containing g such that $\phi \in \bigcap \{I_x; x \in V\}$. It follows from the maximality of M that

$$VN(G)^* = M + VN(G)^* \odot \phi.$$

Let $u \in VN(G)^*$ such that $n \odot u - n \in M$ for all $n \in VN(G)^*$. Then $u = m + p \odot \phi$ for some $p \in VN(G)^*$ and $m \in M$. Let $\psi \in S_V$, then

$$\psi \odot u = \psi \odot (m + p \odot \phi) = \psi \odot m + p \odot \psi \phi.$$

Now $\psi(x) = 0$ for all $x \notin V$ by definition of S_V , and $\phi(x) = 0$ if $x \in V$. Hence $(\psi \phi)(x) = \psi(x)\phi(x) = 0$ for all $x \in G$. Consequently, $\psi \phi = 0$ and so

$$\psi - \psi \odot u = \psi - \psi \odot m \in M.$$

In particular $\psi \in M$. Thus $S_V \subseteq M$.

LEMMA 4.2. *Let $g \in G$. Then I_g is contained in the norm closure of T_g .*

PROOF. This follows directly from Corollary 2(4.11) in [6].

THEOREM 4.3. *Every maximal regular left ideal in $VN(G)^*$ contains I_g for some $g \in G$.*

PROOF. Let M be a maximal regular left ideal in $VN(G)^*$. Then M is closed [15, p. 68]. If $M \not\supseteq I_g$ for any $g \in G$, then by Lemma 4.2, $M \not\supseteq T_g$ for any $g \in G$. By Lemma 4.1, we have for each $g \in G$ an open set $V(g)$ containing g such that $M \not\supseteq S_{V(g)}$. Let J denote the smallest (left) ideal in $A(G)$ containing each $S_{V(g)}$. If J is not dense in $A(G)$, then by the Tauberian property of $A(G)$ [6, p. 223], there exists $g_0 \in G$ such that $J \subseteq I_{g_0}$. Consequently, $\phi(g_0) = 0$ for all $\phi \in J$. However, by [6, Lemma 3.2], there exists $\phi \in A(G)$ such that $\phi(g_0) = 1$ and $\phi \in S_{V(g_0)}$. But

$S_{V(g_0)} \subseteq J$, which is impossible. Hence J is dense in $A(G)$. Since $M \supseteq S_{V(g)}$ for each $g \in G$, it follows that $M \supseteq J$. But M is closed. Hence $M \supseteq A(G) \supseteq I_g$ for all $g \in G$.

COROLLARY 4.4. *Let M be a maximal regular left ideal in $VN(G)^*$. Then either $M \supseteq A(G)$ or there exists a unique $g \in G$ such that $M \supseteq I_g$.*

PROOF. By Theorem 4.3, $M \supseteq I_g$ for some $g \in G$. Since $M \cap A(G)$ is an ideal in $A(G)$ and the ideals I_g are all maximal and distinct in $A(G)$, $A(G) \subseteq M$ if M contains more than one I_g .

COROLLARY 4.5. *Let M be a maximal regular left ideal in $VN(G)^*$. If there exists $x \in VN(G)$ such that $M \subseteq \{m \in VN(G)^*; m(x) = 0\}$, then there exists $g \in G$ such that $M = I_g^\#$.*

PROOF. By Theorem 4.3, there exists $g \in G$ such that $M \supseteq I_g$. So $\langle \phi, x \rangle = 0$ for all $\phi \in I_g$. Since $x \neq 0$, $x = \alpha \lambda(g)$ for some nonzero $\alpha \in \mathbb{C}$. In particular $M = I_g^\#$.

COROLLARY 4.6. *All maximal regular left ideals of $VN(G)^*$ are weak*-dense except those of the form $I_g^\#$, $g \in G$.*

PROOF. Let M be a maximal left ideal of $VN(G)^*$. If the weak*-closure of M is not $VN(G)^*$, then there exists $x \in VN(G)$ such that $\langle x, m \rangle = 0$ for all $m \in M$, and $x \neq 0$. By Corollary 4.5, $M = I_g^\#$ for some $g \in G$.

We need the following simple observation.

LEMMA 4.7. *If G is not discrete, then $\lambda(g) \notin C_p^*(G)$ for any $g \in G$.*

PROOF. If $\lambda(g) \in C_p^*(G)$, then $\lambda(g^{-1}) \notin C_p^*(G)$ by [13, Corollary 4.3]. Hence there exists $m \in VN(G)^*$ such that $m(\lambda(g^{-1})) \neq 0$ and $m(x) = 0$ for each $x \in C_p^*(G)$. Let $n \in VN(G)^*$ be defined by $n(x) = \overline{m(x^*)}$, $x \in VN(G)$. Then $n(\lambda(g)) \neq 0$ and $n(x) = 0$ for each $x \in C_p^*(G)$, which is impossible.

THEOREM 4.8. *G is discrete if and only if each maximal regular left ideal in $VN(G)^*$ either contains $A(G)$ or is of the form $I_g^\#$ for some $g \in G$.*

PROOF. Assume that G is discrete and that M is a maximal regular left ideal in $VN(G)^*$. If M does not contain $A(G)$, then by Corollary 4.4, there exists a unique $g \in G$ such that $I_g \subseteq M$. Since $A(G) \cap M$ is a proper closed ideal in $A(G)$ containing I_g and I_g is maximal in $A(G)$, we have $A(G) \cap M = I_g$. We shall show that $M = I_g^\#$.

Let $\phi_0 \in A(G)$ be such that $\phi_0(g) = 1$ [6, Lemma 3.2]. Then $\phi_0^2 - \phi_0 \in I_g$ and $\phi_0^2 \notin I_g$. In particular $\phi_0^2 \notin M$ and $VN(G)^* = M + VN(G)^* \odot \phi_0$ by maximality of M . Now if $p \in I_g^\#$, then $p = m + n \odot \phi_0$ for some $m \in M$ and $n \in VN(G)^*$. Hence $\phi_0 \odot p = \phi_0 \odot m + \phi_0^2 \odot n$. Since $\phi_0 \odot p \in A(G)$ (by Theorem 3.7) and $\phi_0 \odot p \in I_g^\#$, we have $\phi_0 \odot p \in I_g$. So $\phi_0 \odot p \in M$. Hence $\phi_0^2 \odot n = \phi_0 \odot p - \phi_0 \odot m$ is also in M . However $(\phi_0^2 - \phi_0) \odot n \in I_g^\# \cap A(G) = I_g$ by Theorem 3.7. Hence $\phi_0 \odot n \in M$. Consequently $p \in M$. Since p is arbitrary, we have $I_g^\# \subseteq M$. But $I_g^\#$ is also maximal, hence $I_g^\# = M$.

Conversely if G is not discrete, let $g_0 \in G$ be fixed. If $m \in VN(G)^*$, let $r(m)$ denote the restriction of m to $C_p^*(G)$. Then $r(m)$ may be regarded as an element in $B_p(G)$. Define $h(m) = r(m)(g_0)$. Since the Arens product on $C_p^*(G)$ is precisely the pointwise multiplication on $B_p(G)$ [13, Proposition 5.3], it follows that h is a nonzero continuous multiplicative linear functional on $VN(G)$. Hence $M = h^{-1}(0)$ is a maximal regular left ideal in $VN(G)^*$. M does not contain $A(G)$ since $h(\psi) = 1$ for any $\psi \in A(G)$ such that $\psi(g_0) = 1$. Also $M \neq I_g^\#$ for any $g \in G$. Indeed if $M = I_g^\#$, then $m(\lambda(g)) = 0$ for all $m \in M$. But $\lambda(g) \notin C_p^*(G)$ by Lemma 4.7 and so there exists $m \in VN(G)^*$ such that $m(\lambda(g)) \neq 0$ and $m(x) = 0$ for all $x \in C_p^*(G)$. In particular $r(m)(g_0) = 0$ and so $m \in M$, which is impossible.

THEOREM 4.9. *Let G be an amenable locally compact group. Then the following statements are equivalent.*

- (a) G is compact.
- (b) Each maximal regular left ideal M in $VN(G)^*$ contains I_g for some unique $g \in G$.
- (c) The right identity of $VN(G)^*$ is contained in the left ideal J generated by $A(G)$.

PROOF. (a) $\Rightarrow$ (b) If G is compact, then $A(G)$ contains the identity of $VN(G)^*$. Hence there exists no proper left ideal of $VN(G)^*$ which contains $A(G)$. Apply now Corollary 4.4.

(b) $\Rightarrow$ (c) If the right identity of $VN(G)^*$ is not contained in J , then there exists a maximal regular left ideal M in $VN(G)^*$ containing J (see [15, pp. 58 and 68]). Hence (b) fails.

(c) $\Rightarrow$ (a) Let θ be the right identity of $VN(G)^*$. If $\theta \in J$, then $\theta = \sum_1^n \phi_i \odot m_i$, $\phi_i \in A(G)$ and $m_i \in VN(G)^*$. Then for each $\psi \in A(G)$ and $q \in VN(G)^*$, we have

$$\begin{aligned} \sum_1^n \psi \phi_i \odot (q \odot m_i - m_i \odot q) &= q \odot \sum_1^n \psi \phi_i \odot m_i - \left(\sum_1^n \psi \phi_i \odot m_i \right) \odot q \\ &= q \odot \psi - \psi \odot q = 0. \end{aligned}$$

Since θ is the weak*-limit of a net in $A(G)$, it follows that

$$\sum_1^n \theta \odot \phi_i \odot (q \odot m_i - m_i \odot q) = 0$$

also. Consequently

$$\begin{aligned} q - \theta \odot q &= \sum_1^n q \odot \phi_i \odot m_i - \sum_1^n \phi_i \odot m_i \odot q \\ &= \sum_1^n \phi_i \odot q \odot m_i - \sum_1^n \phi_i \odot m_i \odot q \\ &= \sum_1^n \phi_i \odot \theta \odot q \odot m_i - \sum_1^n \phi_i \odot \theta \odot m_i \odot q \\ &= 0 \end{aligned}$$

since $A(G)$ is in the centre of $VN(G)^*$. Hence θ is an identity of $VN(G)^*$, which implies that G is compact (Proposition 3.2).

COROLLARY 4.10. *Let G be an amenable locally compact group. Then G is finite if and only if each maximal regular left ideal of $VN(G)^*$ is of the form I_g^* for some $g \in G$.*

PROOF. This follows readily from Theorems 4.8 and 4.9.

5. An isomorphism theorem. We shall prove in this section that if G_1 and G_2 are discrete groups, and if the Banach algebras $VN(G_1)^*$ and $VN(G_2)^*$ are isometric and order isomorphic, then G_1 and G_2 are isomorphic. We begin with some observations.

Given any group G , the second conjugate algebra $VN(G)^{**}$ is again a von Neumann algebra.

LEMMA 5.1. *Let G be a discrete group and u be an invertible element in $VN(G)^{**}$. If u is multiplicative on $VN(G)^*$, then $u = \lambda(g)$ for some $g \in G$.*

PROOF. Let $M = \{m \in VN(G)^*; \langle u, m \rangle = 0\}$. Then M is a regular maximal left ideal in $VN(G)^*$. If $M = I_g^*$, then $u = \alpha\lambda(g)$ for some $\alpha \in \mathbb{C}$. Choose $\phi \in A(G)$ such that $\phi(g) = 1$. Then $\alpha = \langle u, \phi^2 \rangle = \langle u, \phi \rangle \langle u, \phi \rangle = \alpha^2$. So $\alpha = 1$, i.e. $u = \lambda(g)$. Otherwise, using Theorem 4.8, M must contain $A(G)$. Let

$$A(G)^0 = \{f \in VN(G)^{**}; \langle f, \phi \rangle = 0 \text{ for each } \phi \in A(G)\}.$$

Then $A(G)^0$ is a weak*-closed ideal in $VN(G)^{**}$ (see [17, p. 44]). Hence there exists z , a central projection in $VN(G)^{**}$, such that $A(G)^0 = z \cdot VN(G)^{**}$. Then $zu = u$ and $z = z(uu^{-1}) = uu^{-1} = e$, where e is the identity in $VN(G)^{**}$. Hence $A(G)^0 = VN(G)^{**}$, which is impossible.

LEMMA 5.2. *Let W_1, W_2 be von Neumann algebras with identities e_1, e_2 respectively. Let Φ be a linear isometry from W_1 onto W_2 such that $\Phi(e_1) = e_2$. Let G_1 be a group of invertible elements in W_1 . If $\Phi(G_1) = G_2$ is a group of invertible and linearly independent elements in W_2 , then G_1 and G_2 are isomorphic as discrete groups.*

PROOF. We follow an idea of Martin Walter in the proof of Theorem 3 [18, p. 29]. For each $x \in G_1$, let

$$H_x = \{y \in G_1; \phi(xy) = \phi(x)\phi(y)\}, \quad K_x = \{y \in G_1; \phi(xy) = \phi(y)\phi(x)\}.$$

We shall show that for each $x \in G_1$,

- (i) $H_x \cup K_x = G_1$,
- (ii) H_x and K_x are subgroups of G_1 .

When this is done, then for each $x \in G$, either $H_x = G_1$, or $K_x = G_1$. Let $H = \bigcap \{H_x; x \in G_1\}$ and $K = \bigcap \{K_x; x \in G_1\}$. Then H, K are subgroups of G_1 . Also if $x \in G_1$ and $H_x = G_1$, then by [12, Lemma 6],

$$\Phi(x \cdot y) + \Phi(y \cdot x) = \Phi(x)\Phi(y) + \Phi(y)\Phi(x) \quad (1)$$

for each $y \in G_1$. So $\Phi(y \cdot x) = \Phi(y)\Phi(x)$ and $x \in H$. Similarly if $K_x = G_1$, then $x \in K$. Consequently $H \cup K = G_1$. Hence either $H = G_1$ or $K = G_1$. In each of the two cases, the groups G_1 and G_2 are isomorphic. It remains to prove (i) and (ii).

Let $y \in G_1$. If $xy = yx$, then $\Phi(x)\Phi(y) = \Phi(y)\Phi(x)$ by [12, Theorem 5]. Hence $y \in H_x \cap K_x$ by (1). If $xy \neq yx$ and $y \notin H_x \cup K_x$, then $\Phi(xy), \Phi(yx), \Phi(x)\Phi(y)$

and $\Phi(y)\Phi(x)$ are distinct elements in G_2 . In fact, if $\Phi(x)\Phi(y) = \Phi(y)\Phi(x)$, then applying [12, Theorems 5 and 7] to the operator Φ^{-1} , we get $xy = yx$. Also, $\Phi(yx) \neq \Phi(x)\Phi(y)$. For otherwise $\Phi(xy) = \Phi(y)\Phi(x)$, i.e. $y \in K_x$, by (1). Similarly, $\Phi(yx) \neq \Phi(y)\Phi(x)$. Since distinct elements in G_2 are linearly independent (by assumption), this contradicts (1). Hence (i) holds.

To prove (ii), let $z_i \in W_i$, $i = 1, 2$, be central projections such that Φ is a $*$ -isomorphism from W_1z_1 onto W_2z_2 and a $*$ -anti-isomorphism from $W_1(e_1 - z_1)$ onto $W_2(e_2 - z_2)$ (see [12, Theorem 10]). Then $\Phi(xz_1) = \Phi(x)z_2$ and $\Phi(x(e_1 - z_1)) = \Phi(x)(e_2 - z_2)$ for all $x \in W_1$. Observe that

$$y \in K_x \text{ if and only if } (xy - yx)z_1 = 0 \quad (2)$$

and

$$y \in H_x \text{ if and only if } (xy - yx)(e_1 - z_1) = 0. \quad (3)$$

To prove (2), let $y \in K_x$, then

$$\begin{aligned} \Phi((xy - yx)z_1) &= \Phi(xy - yx)z_2 \\ &= \Phi(y)\Phi(x)z_2 - \Phi(yz_1)\Phi(xz_1) \\ &= 0. \end{aligned}$$

Hence $(xy - yx)z_1 = 0$. Conversely if $(xy - yx)z_1 = 0$ and $y \notin K_x$, then $y \in H_x$ by (i). Hence $(xy - yx)(e_1 - z_1) = 0$. So $xy = yx$. Consequently $y \in K_x$ by [12, Theorem 5] and (1). Statement (3) can be proved similarly. Now if $y_1, y_2 \in K_x$, then

$$x(y_1y_2)z_1 = xy_1z_1y_2 = (y_1x)z_1y_2 = y_1(xy_2)z_1 = (y_1y_2)xz_1$$

using (2). Hence $y_1y_2 \in K_x$ by (2) again. Also if $y \in K_x$, then $(y^{-1}x - xy^{-1})z = y^{-1}[(yx - xy)z]y^{-1} = 0$. Hence $y^{-1} \in K_x$ by (2). Hence K_x is a subgroup of G_1 . Similarly, H_x is also a subgroup of G_x .

THEOREM 5.3. *Let G_1, G_2 be discrete groups. If there exists an order preserving isometric algebra isomorphism from $VN(G_2)^*$ onto $VN(G_1)^*$, then G_1 and G_2 are isomorphic.*

PROOF. We identify $\lambda(x) \in VN(G_i)$ with $x \in G_i$, $i = 1, 2$. Let U be an order preserving isometric isomorphism from $VN(G_2)^*$ onto $VN(G_1)^*$, and $\Phi = U^*$. Let e_i denote the identity of G_i . If $n \in VN(G)^*$, $n \geq 0$, then $\langle n, \Phi(e_1) \rangle = \langle U(n), e_1 \rangle = \|U(n)\| = \|n\| = \langle n, e_2 \rangle$. Hence $\Phi(e_1) = e_2$. By [12, Theorem 7 and Lemma 12], Φ is a Jordan $*$ -isomorphism mapping the set of unitary elements in $VN(G_1)^{**}$ onto $VN(G_2)^{**}$. By Lemma 5.1, Φ maps G_1 onto G_2 . Also, elements in G_2 , being multiplicative on $VN(G)^*$, are linearly independent (see [2, Lemma 17 and Corollary 18, p. 93] and note that their proofs do not depend on the fact that the Banach algebra is commutative). The conclusion now follows from Lemma 5.2.

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