THE COMPLEXIFICATION AND DIFFERENTIAL STRUCTURE OF A LOCALLY COMPACT GROUP

BY

KELLY MCKENNON

ABSTRACT. The concept of a complexification of a locally compact group is defined and its connections with the differential structure developed. To provide an interpretation in terms of irreducible representations of separable, Type I groups, a duality theorem and Bochner theorem are presented.

The Pontryagin-Van Kampen duality theorem states that any locally compact Abelian group may be regarded as the set of continuous homomorphisms on a dual LCA group X into the circle group T . Thus G is always a subgroup of the group G_C of continuous homomorphisms of X into the multiplicative group of nonzero complex numbers. It is not difficult to see that G_C is the direct product of G and the group G_C^+ of continuous homomorphisms of X into the multiplicative group of positive real numbers. The natural logarithm converts G_C^+ into the linear space of continuous real homomorphisms of X , which space in turn is transformed via the map $f \rightarrow e^{-if}$ into the Lie algebra Λ of continuous one-parameter subgroups of G (see [11, 24.33 et seq.]). Thus the "complexification" G_C may be viewed as a product of G with its Lie algebra Λ . Since the component of the identity contains all the one-parameter subgroups, it is convenient to deal only with connected groups when studying the complexification.

The situation for a compact group is similar, though it must be couched in terms of the algebra $T(G)$ of trigonometric polynomials (or representative functions) on G and relies on Tannaka's duality theorem [13]. The dual of $T(G)$ may be regarded as a product M of simple finite-dimensional von Neumann algebras and Tannaka's theorem states that the unitary elements of this product which are in the spectrum of the multiplicative algebra $T(G)$ may be identified with G . Thus G is a subset of the spectrum G_C of $T(G)$. If G_C^+ denotes the elements of G_C which are positive in M , then G_C^+ is, in general, not a group, but it is still true that G_C is a group and is the direct product of G with G_C^+ . The function $i \ln$ (viewed as a function on the product M of von Neumann algebras) maps G_C^+ onto the algebra of derivations of $T(G)$ and the map $x \rightarrow e^{ix}$ transform the Lie algebra of derivations onto the Lie algebra of one-parameter subgroups of G (see [6] for Lie groups and [18] for the general case).

It is the business of the sequel to investigate to what extent the idea of a "complexification" and its connection with the differential structure extends to a

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general locally compact group. The task is complicated by the fact that a perfectly satisfactory generalization of the Pontryagin-Van Kampen theory does not seem to be available for general locally compact groups, although there have been many attempts. A number of these, closely connected with the left regular representation of G , have been unified by Takesaki [23] in a theory based on Hopf-von Neumann algebras. We have chosen here to clad our ideas in the trappings of the group W^* -algebras, and so will have recourse to the theory of Walter [25].

We begin in §1 with some technical matters allowing us to speak of "unbounded" elements of a W^* -algebra. In §2 the group complexification G_C is introduced and its basic structure determined. §3 presents the relation G_C has to the group Lie algebra. The unboundedness of the positive elements of G_C (the elements of R_C^+ , for instance, are the exponential functions on R) causes a problem in considering G_C as a group, which is discussed in §4. The remainder of the paper is concerned with the complexification relative to various representations—particularly the irreducible ones. The principal tools here are a Bochner-type theorem (§7) and a duality theorem (§8).

Since the reader may find the symbols employed in this paper somewhat formidable in number, a list is provided immediately preceding the bibliography. A number x^+ (x^-) after a listing means that that particular symbol is first found just after (before) the number (x) in the text.

1. Background. Let W be a W^* -algebra with identity ι , adjoint $\sim$, and multiplication $*$. We shall view W as a family of linear functionals on its predual W_* . The positive cones of W and W_* will be denoted by W^+ and W_*^+ , respectively. For $\omega \in W$ and $F \in W_*$ we define $\omega * F$, $F * \omega \in W_*$ by letting $\omega * F(\theta) \equiv F(\omega \sim * \theta)$ and $F * \omega(\theta) \equiv F(\theta * \omega \sim)$ for all $\theta \in W$.

By a *resolution of a projection* π_∞ in W we shall mean a map π from the family B of Borel subsets of a locally compact Hausdorff space $X(\pi)$ into the set of projections in W^+ such that:

- (1a) $\pi_\phi = 0$, $\pi_{X(\pi)} = \pi_\infty$;
- (1b) $\pi_Y \neq 0$ ($\forall Y \subset X(\pi)$: Y is nonvoid and open);
- (1c) $\pi_{Y \cap Z} = \pi_Y \pi_Z$ ($\forall Y, Z \in B$);
- (1d) $\pi_{Y \cup Z} = \pi_Y + \pi_Z$ ($\forall Y, Z \in B$: $Y \cap Z = \emptyset$);
- (1e) $\mu(\pi, F)|B \ni Y \rightarrow \pi_Y(F)$ is a regular complex measure for all $F \in W_*$.

By a *resolution of a partial unitary element* ν_∞ of W , we shall mean a map ν from the Borel σ -algebra B of a locally compact space $X(\nu)$ into the partial unitary elements of W such that $\nu_\infty = \nu_X$ and $\nu \sim * \nu$ is a resolution of the projection $\nu_\infty \sim * \nu_\infty$ (or, equivalently, $\nu * \nu \sim$ is a resolution of the projection $\nu_\infty * \nu_\infty \sim$).

Let $R = R(W)$ be the family of all resolutions ν of partial unitary elements such that $X(\nu)$ is a closed subset of the set $\mathbf{R}^+$ of positive real numbers, and write R^+ for the set of resolutions of projections in R . For $\nu \in R$, let $W_*^+[\nu]$ be the order ideal consisting of all $F \in W_*^+$ such that

$$\int_{X(\nu)} x^2 d\mu(\nu \sim * \nu, F)(x) < \infty; \quad (2)$$

it is a consequence of the spectral theory that $W_*^+[\nu]$ is dense in W_*^+ . Thus the linear span $W_*[\nu]$ of $W_*^+[\nu]$ is dense in W_* . For each $\nu \in R$ define $\nu|W_*[\nu] \rightarrow C$ by

$$\nu(F) \equiv \int_{X(\nu)} x \, d\mu(\nu \sim * \nu, \nu_{X(\nu)} \sim * F)(x) \quad (\forall F \in W_*[\nu]). \quad (3)$$

The existence of (3) is justified by the fact that if $F \in W_*^+[\nu]$ and S is any W^* -representation of W on a Hilbert space H with a vector $v \in H$ satisfying $\langle S_\omega v, v \rangle = \omega(F)$ for all $\omega \in W$, the spectral theory guarantees the existence of a unique positive selfadjoint operator

$$A = \int x \, dS_{\nu \sim * \nu}(x), \quad (4)$$

and thus (2) becomes $\langle Av, Av \rangle < \infty$, which implies that

$$\int_{X(\nu)} x \, d\mu(\nu \sim * \nu, \nu_{X(\nu)} \sim * F)(x) = \langle Av, S_{\nu_{X(\nu)}} v \rangle$$

is well defined.

Define $\overline{W}$ to be the set of all $\nu \sim$ such that $\nu \in R$, and write $\text{dom}(\nu \sim)$ for its domain $W_*[\nu]$. Let $\nu \in R$ be arbitrary, and consider any W^* -representation S of W on a Hilbert space H . Let A satisfy (4) and write ν_∞ for $\nu_{X(\nu)}$. Then $S_{\nu_\infty} \circ A$ is a closed linear operator on H and we define

$$S_{\nu \sim} \equiv S_{\nu_\infty} \circ A.$$

Note further that, if $S(W)'$ denotes the commutant of $S(W)$, then

$$S_{\nu \sim} \circ V = V \circ S_{\nu \sim} \quad (\forall V \in S(W)'). \quad (5)$$

On the other hand, if R is a closed linear operator with dense domain, it can be written uniquely as a product $U \circ A$ where A is positive selfadjoint, U is a partial unitary operator, and R , A and U have identical initial projections [7, XII 7.7]; if $R \circ V = V \circ R$ as well for all $V \in S(W)'$, then the same holds for A and U so $U \in S(W)$ and $A = \int x \, dP$, where P is a resolution of the initial projection of A in the W^* -algebra $S(W)$. Thus if

$$W = W(S) \oplus \ker(S) \quad (6)$$

is the canonical decomposition of W into a direct sum of ideals,

$$(\exists! \nu \in R(W(S))) \quad S_{\nu \sim} = R. \quad (7)$$

When S is faithful on W , it follows from (7) that its extension to $\overline{W}$ is also faithful; hence we can define

$$\theta * \omega \equiv S^{-1}(S_\theta \circ S_\omega) \quad (\forall \theta \in W, \omega \in \overline{W}) \quad (8)$$

and a simple approximation argument shows that the definition is independent of the particular faithful S employed.

For $\omega \in \overline{W}$ and $F \in \text{dom}(\omega)$, we define

$$\omega * F|W \ni \theta \rightarrow \omega \sim * \theta(F).$$

We leave it to the reader to verify that, if S is a W^* -representation on H , then $(\forall \omega, \theta \in \overline{W})$,

$$S_\omega + S_\theta \text{ is densely defined if } \text{dom}(\omega) \cap \text{dom}(\theta) \text{ is dense in } W_*, \quad (9)$$

and

$$S_\omega \circ S_\theta \text{ is densely defined if } \{\theta \sim * F: F \in \text{dom}(\theta)\} \cap \text{dom}(\omega) \text{ is dense in } W_*, \quad (10)$$

and the converse holds if $\omega, \theta \in \overline{W(S)}$. Thus, for $\omega, \theta \in \overline{W}$, $\omega + \theta$ is defined and satisfies

$$(\omega + \theta)(F) = \theta(F) + \omega(F) \quad (\forall F \in \text{dom}(\theta) \cap \text{dom}(\omega))$$

whenever $\text{dom}(\theta) \cap \text{dom}(\omega)$ is dense in W_* , and

$$S_{\omega + \theta} = S_\omega + S_\theta. \quad (11)$$

Likewise $\omega * \theta$ is defined and satisfies

$$\omega * \theta(F) = \omega(\theta \sim * F) \quad (\forall F \in \text{dom}(\theta))$$

whenever $\{\theta \sim * F: F \in \text{dom}(\theta)\} \cap \text{dom}(\omega)$ is dense in W_* , and

$$S_{\omega * \theta} = S_\omega \circ S_\theta. \quad (12)$$

The adjoint can analogously be extended to $\overline{W}$ such that

$$S_{\omega^-} = S_\omega^- \quad (\forall \omega \in \overline{W}). \quad (13)$$

If $\omega \in \overline{W}$ is selfadjoint and f is any continuous complex-valued function on the spectrum of ω , one can define (where S is any faithful W^* -representation)

$$f(\omega) \equiv S^{-1}(f(S_\omega)).$$

We close the present section by noting that, if $\psi|W \rightarrow A$ is any homomorphism of W^* -algebras, ψ has a canonical extension (also denoted ψ) of $\overline{W}$ into $\overline{A}$ such that

$$f(\psi(\omega)) = \psi(f(\omega)) \quad (14)$$

where $\omega \in \overline{W}$ is selfadjoint and f is any continuous complex-valued function on the spectrum of ω .

2. The complexification of a group. Let G be a locally compact group, e its identity, λ a left Haar measure on G , $L^1 = L^1(G, \lambda)$ the Banach $*$ -algebra of λ -integrable functions, $\mathcal{C} = \mathcal{C}(G)$ the enveloping C^* -algebra of G , $W = W(G)$ the enveloping W^* -algebra of C , $C = C(G)$ the algebra of continuous complex-valued functions on G , $P = P(G)$ the cone of positive definite elements of $\mathcal{C}$, and $B = B(G)$ the linear span of P in C . Then $B(G)$ is the Fourier-Stieltjes algebra of G and may be associated with W_* via a canonical map $F|B \rightarrow W_*$ satisfying

$$h(F_f) = \int_G hf \, d\lambda \quad (\forall h \in L^1, f \in B) \quad (15)$$

(where we view $L^1 \subset C \subset W$). For $\omega \in \overline{W}$ and $F_f \in W_*[\omega]$, we define

$$\langle \omega, f \rangle \equiv \omega(F_f). \quad (16)$$

Let γ be the canonical monomorphism of G into the group of unitary elements of W —then

$$\langle \gamma(x), f \rangle = f(x) \quad (\forall x \in G, f \in B). \quad (17)$$

We shall frequently use the fact that the continuous unitary representations of G and C^* -algebra representations of C are, respectively, just the representations $V \circ \gamma$ and $V|_C$, where V runs through the class of W^* -algebra representations of W (here a C^* -algebra representation implies norm-continuity and a W^* -algebra representation means weak*-continuity) [8, 12.1.5 and 13.9.3].

To each $f \in P(G)$ is associated a unique (up to unitary equivalence) cyclic W^* -algebra representation T^f of W on a Hilbert space $H(f)$ satisfying

$$\langle T_\omega^f v_f, v_f \rangle_f = \langle \omega, f \rangle \quad (\forall \omega \in W) \quad (18)$$

where $\langle \cdot, \cdot \rangle_f$ and v_n are the inner product and cyclic vector, respectively, in $H(f)$. For f and h in P , the tensor product unitary representation $T^f \circ \gamma \otimes T^h \circ \gamma$ of G is connected with a unique W^* -representation $T^{f,h}$ of W such that

$$T_{\gamma(x)}^f \otimes T_{\gamma(x)}^h = T_{\gamma(x)}^{f,h} \quad (\forall x \in G). \quad (19)$$

By the *complexification* G_C of G we shall mean the set of all $\omega \in \bar{W}$ such that

$$T_\omega^f \otimes T_\omega^h = T_\omega^{f,h} \quad (\forall f, h \in P). \quad (20)$$

It trivially follows from (19) that the image G_γ of G by γ is a subset of G_C . We write G_C^+ for $G_C \cap \bar{W}^+$. In dealing with G_C^+ it is convenient to introduce the set $\Lambda = \Lambda(G)$ of all $\alpha \in \bar{W}$ such that

$$T_\alpha^f \otimes I + I \otimes T_\alpha^h = T_\alpha^{f,h} \quad (\forall f, h \in P) \quad (21)$$

(where I always represents the identity operator) and

$$\alpha^\sim = -\alpha. \quad (22)$$

PROPOSITION 1. For $\alpha \in \Lambda$ and $z \in C$, $\exp(z\alpha)$ is in G_C .

PROOF. Let $f, h \in P$ be arbitrary. That $T_{\exp(z\alpha)}^{f,h} = \exp z T_\alpha^{f,h}$ is trivial. That

$$T_{\exp(z\alpha)}^f \otimes T_{\exp(z\alpha)}^h = \exp z (T_\alpha^f \otimes I + I \otimes T_\alpha^f) \quad (23)$$

follows from an elementary rearrangement of power series. Thus (21) and (23) imply

$$T_{\exp(z\alpha)}^f \otimes T_{\exp(z\alpha)}^h = T_{\exp(z\alpha)}^{f,h}. \quad \text{Q.E.D.}$$

PROPOSITION 2. The map $\Lambda \ni \alpha \rightarrow \exp(-i\alpha)$ is a bijection onto G_C^+ .

PROOF. That the map under consideration is injective is a consequence of the spectral theory. In view of (22) and Proposition 1, it remains only to show the map is surjective.

Let $\pi \in G_C^+$ and $f, h \in P$ be arbitrary. For each $n \in \mathbb{N}$, define $f_n|_{\mathbb{R}} \ni x \rightarrow \max\{n^{-1}, \min(n, x)\}$ and choose $\pi_n \in \bar{W}^+$ such that $f_n(\pi) = \pi_n$. Then the series

$$\ln(n\iota) + \sum_{m=1}^{\infty} (-1)^{m+1} m^{-1} n^{-m} (\pi_n - n\iota)^m$$

converges absolutely to $\ln \pi_n$. A simple exercise with power series now leads to

$$\ln(T_{\pi_n}^f \otimes T_{\pi_n}^h) = (\ln T_{\pi_n}^f \otimes I) + (I \otimes \ln T_{\pi_n}^h). \quad (24)$$

Since the sequence $T_{\pi_n}^f$ ($T_{\pi_n}^h$, resp.) converges to T_{π}^f (T_{π}^h , resp.) on a dense subset of $H(f)$ ($H(h)$, resp.), (24) implies

$$\ln(T_{\pi}^f \otimes T_{\pi}^h) = (\ln T_{\pi}^f \otimes I) + (I \otimes \ln T_{\pi}^h). \quad (25)$$

But π is in G_C^+ so, if $\alpha \equiv -i \ln \pi$, (25) implies that (21) holds. That (22) holds follows from the spectral theory. Hence $\alpha \in \Lambda$ and $\exp(-i\alpha) = \pi$. Q.E.D.

PROPOSITION 3. *For $\omega \in G_C$, both ω^{-1} and $\omega^{\sim}$ are in G_C . If $\omega, \beta \in G_C$ and $\omega * \beta$ is defined in $\bar{W}$, then $\omega * \beta$ is in G_C .*

PROOF. Let $f, h \in P$ be arbitrary. Then

$$T_{\omega^{-1}}^f \otimes T_{\omega^{-1}}^h = (T_{\omega}^f \otimes T_{\omega}^h)^{-1} = (T_{\omega}^{f,h})^{-1} = T_{\omega^{-1}}^{f,h},$$

so ω^{-1} is in G_C . That $\omega^{\sim}$ is in G_C is proved analogously. If $\omega, \beta \in G_C$ and $\omega * \beta$ is defined, then

$$T_{\omega * \beta}^f \otimes T_{\omega * \beta}^h = (T_{\omega}^f \otimes T_{\omega}^h) \circ (T_{\beta}^f \otimes T_{\beta}^h) = T_{\omega}^{f,h} \circ T_{\beta}^{f,h} = T_{\omega * \beta}^{f,h},$$

so $\omega * \beta$ is in G_C . Q.E.D.

THEOREM 1. *The product $G_{\gamma} * G_C^+$ is direct and equals G_C .*

PROOF. That $G_{\gamma} * G_C^+ \subset G_C$ follows from Proposition 3. That $G_{\gamma} * G_C^+$ is direct follows from the uniqueness of polar decomposition of invertible operators.

Let $\omega \in G_C$ be arbitrary and write $\nu * \pi$ for its polar decomposition, $\nu \in W$ unitary and $\pi \in \bar{W}^+$. Proposition 3 implies that $\omega^{\sim}$ is in G_C and that $\omega^{\sim} * \omega = \pi^2$ is in G_C^+ . Proposition 2 yields $\alpha \in \Lambda$ such that $\exp(-i\alpha) = \pi^2$. Proposition 1 now implies that $\exp(-i\alpha/2) = \pi$ and $\exp(i\alpha/2) = \pi^{-1}$ are in G_C^+ . Hence, by Proposition 3 again $\nu = \omega * \pi^{-1}$ is in G_C . But Walter's duality theorem [25, Theorem 1] states that G_{γ} is precisely the family of unitary elements of G_C ; hence $\omega \in G_{\gamma} * G_C^+$. Q.E.D.

3. The complexification and the group Lie algebra. Thus far we have omitted discussing if or when G_C is a group. This question is connected with the differential structure on G , which we shall discuss in the present section. We shall denote by T the direct sum of all representations T^f , $f \in P$, and by $H(T)$ the representation space.

Let $\Gamma = \Gamma(G)$ be the family of continuous one-parameter subgroups of G . For $\alpha \in \Lambda$, the map $s^{\alpha} | \mathbf{R} \ni t \rightarrow \gamma^{-1}(\exp(t\alpha))$ is evidently a one-parameter subgroup of G .

PROPOSITION 4. *The map $\Lambda \ni \alpha \rightarrow s^{\alpha}$ is a bijection onto Γ .*

PROOF. It follows from spectral theory that there is a dense subspace M of H on which all the operators $T_{\exp(t\alpha)}$ are defined and such that

$$\lim_{t \rightarrow 0} T_{\exp(t\alpha)} = I \text{ pointwise on } M. \quad (26)$$

Since the operators $T_{\exp(t\alpha)}$ are all unitary, the convergence (26) holds pointwise on all of $H(T)$. Since γ is a homeomorphism [25, Theorem 1] when W carries the weak-* topology $\sigma(W, W_*)$, it follows that $\lim_{t \rightarrow 0} \gamma^{-1}(\exp(t\alpha)) = e$ in G . Hence $s^\alpha \in \Gamma$ for each $\alpha \in \Lambda$.

The formula

$$\lim_{t \rightarrow r} (t - r)^{-1} (T_{\exp(t\alpha)} - T_{\exp(r\alpha)}(v)) = T_\alpha \circ T_{\exp(r\alpha)}(v) \quad (27)$$

is valid for all $r \in \mathbf{R}$ and v in the domain of T_α [22, 13.35]. When $r = 0$ (27) implies that $\Lambda \ni \alpha \rightarrow s^\alpha$ is injective.

Now suppose that $s \in \Gamma$ is arbitrary. Then the map $\mathbf{R} \ni t \rightarrow T_{\gamma(s(t))}$ is a continuous, one-parameter subgroup of unitary operators so, by Stone's theorem [22, 13.37], there exists a closed skew operator V on $H(T)$ such that $T_{\gamma(s(t))} = \exp(tV)$ for all $t \in \mathbf{R}$. Since V evidently commutes with each element of $T(W)'$, we have $V = T_\alpha$ for some $\alpha \in \overline{W}$ satisfying (22). Since $\lim_{t \rightarrow 0} t^{-1}[\exp(tV)(v) - v] = V(v)$ for all v in a dense subspace of $H(T)$, the product rule shows that (21) holds. Hence $\alpha \in \Lambda$ and $s = s^\alpha$. Q.E.D.

Since the range of each $s \in \Gamma$ is connected, it is contained in the component of e in G . Thus, in view of Propositions 2 and 4, we lose little when studying G_c by considering only connected groups.

Throughout the sequel, G will always be assumed to be connected.

Combining ideas of Mackey [16], Riss [21], and Bruhat [5], Boseck and Czichowski [3], [4] have constructed a theory of differentiability for G based on Γ .

Let E be a complete, locally convex, continuous G -module with dual E' . A vector $v \in E$ is said to be *differentiable* if

$$D_s v \equiv \sigma(E, E') - \lim_{t \rightarrow 0} \frac{s(t) \cdot v - v}{t} \text{ exists in } E \quad (\forall s \in \Gamma); \quad (28)$$

the set of all $v \in E$ such that $D_s D_r \dots D_q v$ is differentiable for every finite subset $\{s, r, \dots, q\}$ of Γ will be denoted E_∞ . An important subset $E_{\infty,1}$ consists of those vectors $v \in E_\infty$ such that the stationary subgroup $\{x \in G: x \cdot v = 0\}$ contains a compact normal subgroup S_0 of G such that G/S_0 is a Lie group. The Boseck-Gårding theorem [4, Satz 3.1] states that

(29) $E_{\infty,1}$ is dense in E .

If E contains a topologically G -cyclic vector $v \in E_{\infty,1}$, then the normality of S_0 implies its containment in the annihilator $G(E)$ of E , which yields

(30) $G/G(E)$ is a Lie group.

Boseck's work is based on Yamabe's theorem [28] that there exists a directed family $\mathcal{G}$ of compact normal subgroups such that each quotient G/N , $N \in \mathcal{G}$, is a Lie group and

(31) G is topologically isomorphic to $\text{proj} \lim_{N \in \mathcal{G}} G/N$.

Lashof [15] used (31) to construct a group Lie algebra for G , which he defined as the inductive (or inverse) limit of the Lie algebras of the Lie groups G/N . Boseck

showed that Lashof's Lie algebra is isomorphic with Γ [4, Hilfsatz 2.2] when one defines

$$\begin{aligned} r \cdot s | \mathbf{R} \ni t &\rightarrow s(rt), & s + w | \mathbf{R} \ni t &\rightarrow \lim_{n \rightarrow \infty} \left(s\left(\frac{t}{n}\right) w\left(\frac{t}{n}\right) \right)^n, \\ [s, w] | R \ni t^2 &\rightarrow \lim_{n \rightarrow \infty} \left(s\left(\frac{-t}{n}\right) w\left(\frac{-t}{n}\right) s\left(\frac{t}{n}\right) w\left(\frac{t}{n}\right) \right)^{n^2}, \\ [s, w] | R \ni -t^2 &\rightarrow [w, s](t^2). \end{aligned} \quad (32)$$

Proposition 4 and (27) imply that $H(T)_\infty$ is an invariant subset of the domain of each operator T_α , $\alpha \in \Lambda$. Thus sums and products of elements of Λ are well defined, and evidently it is a Lie algebra. It is not difficult to see that the map $\Lambda \ni \alpha \rightarrow s^\alpha \in \Gamma$ is in fact a Lie algebra isomorphism. Hence we do not essentially conflict with Lashof in calling Λ the *group Lie algebra*. If $g|G_1 \rightarrow G_2$ is a continuous homomorphism of connected groups, the map $\Gamma(G_1) \ni s \rightarrow g \circ s \in \Gamma(G_2)$ induces a Lie algebra homomorphism of $\Lambda(G_1)$ into $\Lambda(G_2)$. When g is an epimorphism and G is a Lie group, a simple dimension argument shows that the Lie algebra homomorphism is surjective as well. Our next task is to show this still holds for general connected G_1 , which is modest generalization of [19, 4.15].

LEMMA 1. *Let $\mathfrak{D}$ be a directed family of Lie groups and suppose that, for all $H, S \in \mathfrak{D}$ with $H > S$, there is given a continuous epimorphism $\epsilon_{H,S}$ of H on S . If $E \subset \mathfrak{D}$ and $s \in \Gamma(\prod_{H \in E} H)$, say that (E, s) satisfies property P if, for any finite subset F of E , there exist $H \in \mathfrak{D}$ and $r \in \Gamma(H)$ such that $H > S$ and $\epsilon_{H,S} \circ r = s_S$ for all $S \in F$. Then, if a pair (E, s) does satisfy property P, there exists $q \in \Gamma(\text{proj } \lim_{H \in D} H)$ such that $q(t)_H = s(t)_H$ for all $t \in \mathbf{R}$ and $H \in E$.*

PROOF. For each $H \in \mathfrak{D}$, let $\Lambda(H)$ be the Lie algebra of H . Well-order $\mathfrak{D}$ such that F is an initial section $I(E)$ and write $<$ for this well-ordering. Suppose that $(I(A), q)$ satisfies property P (where $I(A)$ is an initial section determined by some $A \in \mathfrak{D}$) and $q(t)_H = s(t)_H$ for all $t \in R$ and $H \in E \cap I(A)$. If $A \in E$, let $q(t)_A \equiv s(t)_A$ for all $t \in R$ —if $A \notin E$, proceed as follows: let Φ be the family of all finite subsets of $I(E)$ and, for each $F \in \Phi$, define $\mathfrak{U}(F)$ to be the union of all sets

$$d\epsilon_{D,A}(\{\theta \in \Lambda(D: \exp_B(d\epsilon_{D,B}(t\theta)) = q(t)_B \ (\forall B \in \Phi, t \in \mathbf{R})\})$$

where $D \in \mathfrak{D}$ satisfies $D > S$ for all $S \in F \cup \{A\}$ (here $d\epsilon_{D,A}$ is the differential of $\epsilon_{D,A}$ and $\exp_B$ the exponential from the Lie algebra $\Lambda(B)$ into B). Since $\mathfrak{D}$ is directed and $(I(A), q)$ satisfies property P, it follows that each $\mathfrak{U}(F)$ is nonvoid. Further, since each $\mathfrak{U}(F)$ is an affine subset of the *finite-dimensional* space $\Lambda(A)$, there exists at least one element η in the intersection of them all. Let $q(t)_A \equiv \exp_A(t\eta)$ for all $t \in \mathbf{R}$, and regard now q as an element of $\Gamma(\prod_{H \in I(A) \cup \{A\}} H)$. Evidently $(I(A) \cup \{A\}, q)$ satisfies property P. It now follows from the Principle of Transfinite Induction that q can be extended to an element (which we also denote by q) of $\Gamma(\prod_{H \in D} H)$ such that $(\mathfrak{D}, q)$ has property P. That q is in $\Gamma(\text{proj } \lim_{H \in \mathfrak{D}} H)$ is clear. Q.E.D.

THEOREM 2. *For the locally compact groups G and H , let $w|G \rightarrow H$ be a continuous homomorphism and $s \in \Gamma(H)$ satisfy $s(\mathbf{R}) \subset w(G)$. Then there exists $r \in \Gamma(G)$ such that $w \circ r = s$.*

PROOF. The component H_e of the identity in H contains $s(\mathbf{R})$ and is topologically isomorphic to $\text{proj} \lim_{K \in K(H)} H_c/K$ where $K(H)$ is the family of normal compact subgroups K of H such that H_c/K is a Lie group. If $\pi_K|H_c \rightarrow H_c/K$ is the quotient map for each $K \in K(H)$, then the canonical homomorphisms

$$\rho_K|G_c/[G_c \cap \text{Ker}(\pi_K \circ w)] \rightarrow H_c/K$$

are injective and continuous—hence the domain of each ρ_K is a Lie group [14, VIII.1.1], and each $\rho_K^{-1} \circ \pi_K \circ s$ is in $\Gamma(G_c/[G_c \cap \text{Ker}(\pi_K \circ w)])$ [14, X.4.1]. Let $\mathfrak{D} \equiv \{G_c/S : S \text{ is a closed normal subgroup of } G_c \text{ and } G_c/S \text{ a Lie group}\}$ be a directed set under the natural direction, let

$$\mathfrak{E} \equiv \{G_c/[G_c \cap \text{Ker}(\pi_K \circ w)] : K \in K(H)\},$$

and define $\bar{r} \in \Gamma(\prod_{W \in \mathfrak{E}} W)$ by letting

$$\bar{r}(t)_{G_c/[G_c \cap \text{Ker}(\pi_K \circ w)]} \equiv \rho_K^{-1} \circ \pi_K \circ s$$

for each $K \in K(H)$. Evidently, $(\mathfrak{E}, \bar{r})$ satisfies property P of the preceding lemma. Thus $\bar{r}$ has an extension (which we also write as $\bar{r}$) to an element of $\Gamma(\text{proj} \lim_{S \in \mathfrak{D}} G_c/S)$. If $v|G_c \rightarrow \text{proj} \lim_{S \in \mathfrak{D}} G_c/S$ is the canonical topological isomorphism, then the map $r \equiv v^{-1} \circ \bar{r}$ is in $\Gamma(G)$ and satisfies $w \circ r = s$. Q.E.D.

4. The complexification as a group. In view of Proposition 3, G is a group precisely when $\omega * \nu$ exists in $\bar{W}$ for all $\omega, \nu \in G_c$. Theorem 1 implies that this is the case when $\omega * \nu$ exists in $\bar{W}$ for all $\omega, \nu \in G_c^+$. Thus one is led to seek a dense subspace of $H(T)$ common to the domains of all the operators T_ω , $\omega \in G_c^+$. Since each $\omega \in G_c^+$ is of the form $\exp(-i\alpha)$ for $\alpha \in \Lambda$, the problem is to find a G -invariant subspace $H(T)_e$ of $H(T)$ such that, for each $\alpha \in \Lambda$,

$$\sum_{n=0}^{\infty} \left(\frac{-i}{n!} \right)^n \alpha^n v \text{ converges to an element of } H(T)_e \quad (\forall v \in H(T)_e). \quad (33)$$

Nelson's theorem [26, 4.4.5.7] on the density of analytic vectors for continuous Banach G -modules combined with (30) can be employed to find a G -invariant dense subspace $H(T)_\omega$ of $H(T)$ such that (33) holds for each, but only for each α in a neighborhood of 0 in Λ (where Λ bears its finest locally convex topology). This gives at best a local group structure. At this time, we can give only a partial solution to the problem.

We shall call G *entire* if there exists a subspace L_e^1 of $L^1(G)$ such that

$$\sum_{n=0}^{\infty} \frac{1}{n!} z^n D_s^n f \in L_e^1 \quad (\forall z \in C, s \in \Gamma, f \in L_e^1) \quad (34)$$

where the convergence is absolute. (Here $L^1(G)$ is viewed as a left G -module with action $x \cdot f|G \ni y \rightarrow f(x^{-1}y)$.)

PROPOSITION 5. *If G is a product of entire groups N and K , then G is entire.*

PROOF. If $f \in L^1_\epsilon(N)$ and $h \in L^1_\epsilon(R)$, then $f \otimes h|N \times K \ni (x, y) \rightarrow f(x)h(y)$ is in $L^1(G)$. If s is in $\Gamma(G)$, then there exist $m \in \Gamma(N)$ and $r \in \Gamma(K)$ such that $s(t) = (m(t), r(t))$ and so

$$D_s^n(f \otimes h) = \sum_{k=0}^n \binom{n}{k} D_m^k f \otimes D_r^{n-k} h.$$

Thus

$$\sum_{n=0}^{\infty} \frac{1}{n!} \|z^n D_s^n(f \otimes h)\|_1 \leq \left(\sum_{i=1}^{\infty} \frac{1}{i!} \|z^i D_m^i f\|_1 \right) \left(\sum_{j=1}^{\infty} \frac{1}{j!} \|z^j D_r^j h\|_1 \right) < \infty$$

and

$$\sum_{n=0}^{\infty} \frac{1}{n!} z^n D_s^n(f \otimes h) = \left(\sum_{i=1}^{\infty} \frac{1}{i!} z^i D_m^i f \right) \otimes \left(\sum_{j=1}^{\infty} \frac{1}{j!} z^j D_r^j h \right).$$

Hence we may let $L^1_\epsilon(G)$ be the linear span of $\{f \otimes h: f \in L^1_\epsilon(N), h \in L^1_\epsilon(R)\}$. Q.E.D.

COROLLARY 1. If G is a SIN group (i.e. G possesses a basis of neighborhoods invariant under inner automorphisms), then G is entire. In particular, Abelian and compact groups are entire.

PROOF. Connected SIN groups are products of compact groups and copies of $\mathbf{R}$. If G is compact, we may take the space of trigonometric polynomials for L^1_ϵ (see [18]). If $G = \mathbf{R}$, we may take the linear span of all functions

$$\mathbf{R} \ni t \rightarrow \exp(-b(t-a)^2) \quad \text{where } b > 0 \text{ and } a \in \mathbf{R}. \quad \text{Q.E.D.}$$

PROPOSITION 6. If N is a closed normal subgroup of an entire group G , then G/N is entire.

PROOF. Let $p|G \rightarrow G/N$ be the canonical quotient epimorphism. Equip N and G/N with left Haar measure ν and θ , respectively, such that, if

$$f^0|G/N \ni p(x) \rightarrow \int_N f(xy) d\nu(y) \quad (\forall f \in L^1(G)),$$

then

$$\int_{G/N} f^0 d\theta = \int_G f d\lambda \quad (35a)$$

and $L^1(G)|f \rightarrow f^0$ is an epimorphism onto $L^1(G/N)$ [12, 28.54]. Let $s \in \Gamma$ and $f \in L^1_\epsilon(G)$ be arbitrary. Then, for each $h \in L^\infty(G/N)$,

$$\begin{aligned} \left| \int_{G/N} \left(\frac{p \circ s(t) \cdot f^0 - f^0}{t} - (D^s f)^0 \right) \cdot h d\theta \right| &= \left| \int_G \left(\frac{s(t) \cdot f - f}{t} - D^s f \right) \cdot h \circ p d\lambda \right| \\ &\leq \left\| \frac{s(t) \cdot f - f}{t} - D^s f \right\|_1 \cdot \|h\|_\infty \end{aligned} \quad (35b)$$

which implies that $f^0 \in L^1(G/N)_\infty$ and $D_{p \circ s} f^0 = (D^s f)^0$.

For $z \in \mathbb{C}$, we have by (35a)

$$\sum_{n=0}^{\infty} \frac{1}{n!} \|z^n D_p^n \circ_s f^0\|_1 = \sum_{n=0}^{\infty} \frac{1}{n!} \|z^n D_s^n f\|_1 < \infty$$

and

$$\sum_{n=0}^{\infty} \frac{1}{n!} z^n D_p^n \circ_s f^0 = \left(\sum_{n=0}^{\infty} \frac{1}{n!} z^n D_s^n f \right)^0.$$

Thus if we define $L_e^1(G/N)$ to be $\{f^0: f \in L_e^1(G)\}$, it follows from Theorem 2 that (34) holds. That $L_e^1(G/N)$ is dense in $L^1(G/N)$ follows from the fact that the map $L^1(G) \ni f \rightarrow f^0 \in L^1(G/N)$ is a continuous epimorphism. Q.E.D.

A partial converse to Proposition 6 holds in that, if G/N is entire, then evidently (36) (34) holds when $L_e^1 \equiv \{f \circ p: f \in L_e^1(G/N)\}$.

Boseck [4] and Lashof [15] showed that many statements hold for G precisely when they hold for all the groups G/N of (31). Proposition 6 and (36) imply that

(37) G is entire iff G/N is entire ($\forall N \in \mathcal{G}$) where $G \cong \text{proj } \lim_{N \in \mathcal{G}} G/N$ is in (31).

PROPOSITION 7. *If G is entire, then $G_{\mathbb{C}}$ is a group.*

PROOF. The set $M \equiv \{T_f v: f \in L_e^1, v \in H(T)\}$ is a dense subset of $H(T)$. For each $\alpha \in \Lambda, f \in L_e^1, v \in H$, and $z \in \mathbb{C}$ we have (where $s = s^\alpha$)

$$\sum_{n=0}^{\infty} \left| \frac{z^n}{n!} T_\alpha^n(T_f v) \right| = \sum_{n=0}^{\infty} \left| \frac{z^n}{n!} T_{D_s^n f} v \right| \leq \sum_{n=0}^{\infty} \left\| \frac{z^n}{n!} D_s^n f \right\|_1 |v| < \infty$$

so $T_f v$ is the domain of $T_{\exp(z\alpha)}$. Further,

$$T_{\exp(z\alpha)}(T_f v) = T_{\sum_{n=1}^{\infty} (z^n/n!) D_s^n f}(v),$$

so by Proposition 1, M is $G_{\mathbb{C}}$ -invariant. Hence $T_\beta \circ T_\omega$ is well defined for each $\beta, \omega \in G_{\mathbb{C}}$; thus Proposition 3 implies that $G_{\mathbb{C}}$ is a group. Q.E.D.

5. The complexification and the regular representation. If V is any faithful continuous unitary representation of G , and S is the canonical "extension" to $\overline{W}(G)$, then

(38) $S|_{G_{\mathbb{C}}}$ is faithful.

To see this note, that by Theorem 1, (38) holds if S is faithful on $G_{\mathbb{C}}^+$ which, by Proposition 2, holds when S is faithful on Λ . Since $V_{s^\alpha(t)} = S_{\exp(t\alpha)}$ for all $\alpha \in \Lambda$ and $t \in \mathbb{R}$, it follows from Proposition 4 and the faithfulness of V that S is faithful on Λ .

In particular, $G_{\mathbb{C}}$ is faithfully represented on $L^2(G)$ by the left regular representation R of W . It is possible to base the theory of $G_{\mathbb{C}}$ on $R(W)$ rather than W , rewriting (20) as

$$T_\omega^f \otimes T_\omega^h = T_\omega^{f,h} \quad (\forall f, h \in P \cap A(G))$$

where $A(G)$ is the Fourier algebra. The only major change would be use of the Eymard duality theorem [10, 3.34] instead of Walter's theorem.

Another faithful representation of G , the one to which the sequel is directed, is the product of all continuous, irreducible, unitary, representations, one from each unitary equivalence class.

6. The dual of G . Since (irreducible representation) duality theory has only been successfully developed for "tame" groups, *throughout the sequel G will be assumed to be a separable, Type I, connected, locally compact group.*

Recall that a W^* -representation S of W is *primary* (or *factorial*) if $S(W) \cap S(W)' = CI$. To say that a separable group G is of Type I, is equivalent to each of the following statements [8, 9.1]:

(39a) each primary W^* -representation of W is a multiple of an irreducible representation;

(39b) irreducible W^* -representations of W with common kernel are unitarily equivalent;

(39c) if S is an irreducible W^* -representation of W , then $S(C)$ contains all compact operators.

Let $\Sigma = \Sigma(G)$ be the family of minimal central projections in W^+ . If π is the identity of the kernel of an irreducible W^* -representation of W , then $(\iota - \pi) * W$ is isomorphic to the algebra of all continuous linear operators on the representation space, so $\iota - \pi$ is in Σ . Conversely, if τ is in Σ , and S is a faithful W^* -representation of $\tau * W$, then S is primary so (39a) implies that $\iota - \tau$ is the identity of the kernel of an irreducible representation. Hence

(40) $\{\pi * W: \iota - \pi \in \Sigma\}$ is the set of primitive ideals of W (a *primitive ideal* being the kernel of an irreducible representation).

The dual of a locally compact group is "classically" considered to be the family of unitary equivalence classes of continuous, irreducible, unitary representations of G ; or, what amounts to the same thing, the family of unitary equivalence classes of irreducible W^* -representations of W . This being a rather cumbersome structure, we adduce (39b) and (40) to justify calling Σ the *dual* of G .

The family $\mathcal{J}$ of closed two-sided ideals in $C(G)$ can be employed to describe the *Jacobson topology* on Σ :

$$\{\pi \in \Sigma: \pi * a \neq 0 (\exists a \in J)\} \quad (J \in \mathcal{J}) \quad (41)$$

is a listing of the open sets. The σ -algebra $\mathfrak{B}$ of Borel sets generated by the Jacobson topology makes Σ a standard Borel space (i.e., isomorphic to $[0, 1]$ as a Borel or measurable space) [8, 4.b.1]. We write $\mathfrak{M}(\Sigma)$ for the set of all σ -finite Borel measures on Σ .

Let $X = X(G)$ be the family of extreme points of the convex set $\{f \in P: f(e) \leq 1\}$. When G is Abelian, X is the character group of G ; in the general case X is the set of representative functions of the continuous irreducible representations of G . Thus, to each $f \in X$ corresponds a unique element $\pi_f \in \Sigma$ such that

$$(42) T^f|_{\pi_f * W} \text{ is an isomorphism onto } \mathcal{L}(H(f))$$

(where $\mathcal{L}(H(f))$ is the von Neumann algebra of all bounded linear operators on $H(f)$). For $\pi \equiv \pi_f$ and $\omega \in \pi * W^+$, we define

$$\text{tr}_\pi(\omega) \equiv \text{Tr}(T_\omega^f) \quad (43)$$

where Tr is the trace on $\mathcal{L}(H(f))^+$. The function tr_π is independent of the $f \in X$ (satisfying $\pi = \pi_f$) employed in its construction. The Banach algebra of trace-class operators in $\mathcal{L}(H(f))$ transfers to $\pi * W$:

$$W_{\pi,1} \equiv \{ \omega \in \pi * W : \|\omega\|_{\pi,1} \equiv \text{tr}_\pi(\omega \sim * \omega)^{1/2} < \infty \} \quad (44)$$

and tr_π extends uniquely to a linear functional on $W_{\pi,1}$.

We define the set $\text{tm}(\Sigma)$ of *temperate measurable operator fields on Σ* to be the set of all $\omega \in \prod_{\pi \in \Sigma} W_{\pi,1}$ such that the function $\Sigma \ni \pi \rightarrow \text{tr}_\pi(\beta * \omega_\pi)$ is Borel measurable for each $\beta \in C$. Denote by $m(\Sigma)$ the set of all $\theta \in \prod_{\pi \in \Sigma} \pi * W$ such that $\Sigma \ni \pi \rightarrow \text{tr}_\pi(\theta_\pi * \omega_\pi)$ is Borel measurable for each $\omega \in \text{tm}(\Sigma)$, and let $m^\infty(\Sigma)$ be the set of all $\omega \in m(\Sigma)$ such that

$$\|\omega\|_\infty \equiv \sup \{ \|\omega_\pi\|_W : \pi \in \Sigma \} < \infty. \quad (45)$$

Then $m^\infty(\Sigma)$ is evidently a C^* -algebra.

The *Fourier-Stieltjes transformation* $\hat{\cdot} : W \rightarrow \prod_{\pi \in \Sigma} \pi * W$ is defined by

$$\hat{\omega}(\pi) \equiv \omega * \pi \quad (\forall \omega \in W, \pi \in \Sigma). \quad (46)$$

PROPOSITION 8. *The Fourier-Stieltjes transformation is a C^* -algebra homomorphism of W onto $m^\infty(\Sigma)$. If $z \equiv \sup_{\pi \in \Sigma} \pi$, then the restriction of $\hat{\cdot}$ to $z * W$ is an isomorphism onto $m^\infty(\Sigma)$.*

PROOF. Let $\omega \in W$ and $\theta \in \text{tm}(\Sigma)$ be arbitrary. Then ω is the $\sigma(W, W_*)$ -limit of a sequence ω_n in C . Since each of the functions $\Sigma \ni \pi \rightarrow \text{tr}_\pi(\omega_n * \theta_\pi)$ is Borel measurable, so is $\Sigma \ni \pi \rightarrow \text{tr}_\pi(\omega * \theta_\pi)$. Hence $\hat{\omega} \in m^\infty(\Sigma)$. That $\hat{\cdot}$ is a $*$ -homomorphism is trivial, as is the fact that $\hat{\cdot} : z * W \rightarrow m^\infty(\Sigma)$ is a monomorphism.

Let $\theta \in m^\infty(\Sigma)$ be arbitrary. It is a consequence of [8, 8.4.1] that, for each $\mu \in \mathfrak{M}(\Sigma)$,

$$A(\mu) \equiv \{ \chi \in W : \|\chi\|_W \leq \|\theta\|_\infty, \hat{\chi} = \theta \mu\text{-a.e.} \}$$

is nonvoid. The partial ordering $\ll$ on $\mathfrak{M}(\Sigma)$ induced by “absolute continuity” is a direction. The net $\{A(\mu)\}_{\mu \in \mathfrak{M}(\Sigma)}$ consists of $\sigma(W, W_*)$ -compact, nonvoid subsets of W satisfying $A(\mu) \subset A(\nu)$ whenever $\nu \ll \mu$. Thus $\bigcap_{\mu \in \mathfrak{M}(\Sigma)} A(\mu)$ contains an element ω . Evidently $\hat{\omega} = \theta$, whence follows that $(z * \omega)^\wedge = \theta$, which proves that $\hat{\cdot} : z * W \rightarrow m^\infty(\Sigma)$ is surjective. Q.E.D.

7. Bochner’s theorem. One of the most useful tools in exploiting the Fourier-Stieltjes transform on Abelian groups is Bochner’s theorem [12, 33.3], which identifies $B(G)$ with the measure algebra of the dual group. Our next goal, attained in Theorem 3 below, is to establish such a theorem for a tame group. The first step is an improvement of a result of J. Tomiyama [8, 4.2.5].

PROPOSITION 9. *Let A be a Type I C^* -algebra and $\{V^j\}_{j \in \underline{n}}$ a finite set of pairwise inequivalent nonzero irreducible $*$ -representations of A on Hilbert space. For each representation V^j , let T_j be a compact operator on the representation space $\mathfrak{H}_j$ of V^j . Then there exists $f \in A$ such that $V_f^j = T_j$ for all $j \in \underline{n}$.*

PROOF. Proposition 9.1 holds for $n = 1$ by Sakai’s theorem [8, 9.5.9]. Suppose that it has been established for $n - 1$, for all Type I C^* -algebras.

Claim 1. There is a permutation θ of $\underline{n} \equiv \{1, 2, \dots, n\}$ such that $\text{Ker } V^{\theta(j)} \not\subset \text{Ker } V^{\theta(n)}$ for all $j \in \underline{n-1}$. Suppose $k \in \underline{n}$ is maximal with the property that there exists an injection $\tau|_{\underline{k}} \rightarrow \underline{n}$ satisfying $\text{Ker } V^{\tau(j)} \not\subset \text{Ker } V^{\tau(k)}$ for all $j \in \underline{k-1}$, and assume that $k \neq n$. Let $m \in n \setminus \tau(k)$ arbitrary. For each $j \in \underline{k-1}$, let $\eta(j) \equiv \tau(j)$. If $\text{Ker } V^m \not\subset \text{Ker } V^{\tau(k)}$, let $\eta(k) \equiv m$ and $\eta(k+1) = \tau(k)$, which would contradict the maximality of k . If $\text{Ker } V^m \subset \text{Ker } V^{\tau(k)}$, then clearly $\text{Ker } V^{\tau(j)} \not\subset \text{Ker } V^m$ for all $j \in \underline{k-1}$, and let $\eta(k) \equiv \tau(k)$ and $\eta(k+1) \equiv m$; since primitive ideals are distinct for Type I algebras [8, 4.3.7], we have $\text{Ker } V^{\tau(k)} \not\subset V^m$, which contradicts the maximality of k .

Let θ be as in the statement of Claim 1, and let $J \equiv \{f \in A: V_f^{\theta(n)} \text{ is compact}\}$, which is a nonzero ideal of A and thus Type I.

Claim 2. For each $k \in \underline{n-1}$, $V^{\theta(k)}|_J \neq 0$. We may assume $n \geq 2$. Let T be a compact nonzero projection in $\mathfrak{L}(H(V^{\theta(k)}))$. By our inductive hypothesis, there exist $p, q \in A$ such that $V_p^{\theta(j)} = 0$ for all $j \in \underline{n-1} \setminus \{k\}$, $V_p^{\theta(k)} = T = V_q^{\theta(k)}$, and $V_q^{\theta(n)} = 0$. Clearly, $V_{pq}^{\theta(k)} = T \neq 0$ and $pq \in J$.

Claim 3. For $j, k \in \underline{n-1}$ with $j \neq k$, $V^{\theta(j)}|_J$ and $V^{\theta(k)}|_J$ are inequivalent. We may assume $n > 2$. Assume Claim 3 is false and let U be the unitary operator such that $V_g^{\theta(j)} = U \circ V_g^{\theta(k)} \circ U^{-1}$ for all $g \in J$. By hypothesis there exists $h \in A$ such that $V_h^{\theta(j)} \neq U \circ V_h^{\theta(k)} \circ U^{-1}$. Hence $V_h^{\theta(j)} \circ K \neq U \circ V_h^{\theta(k)} \circ U^{-1} \circ K$ for some compact operator K . By our inductive hypothesis, there exists $g \in A$ such that $V_g^{\theta(j)} = K$ and $V_g^{\theta(n)} = 0$. Thus hg is in J and

$$\begin{aligned} V_{hg}^{\theta(j)} &= V_h^{\theta(j)} \circ K \neq U \circ V_h^{\theta(k)} \circ U^{-1} \circ V_g^{\theta(j)} \\ &= U \circ V_h^{\theta(k)} \circ U^{-1} \circ U \circ V_g^{\theta(k)} \circ U^{-1} = U \circ V_{hg}^{\theta(k)} \circ U^{-1}, \end{aligned}$$

which is absurd.

It follows from Claim 2 and [8, 2.11.2] that $V^{\theta(k)}|_J$ is nonzero irreducible for each $k \in \underline{n-1}$. Thus, our inductive hypothesis and Claim 3 yield some $h \in J$ such that $V_h^{\theta(k)} = T_{\theta(k)}$ for all $k \in \underline{n-1}$. Let L be the ideal $\bigcap_{k \in \underline{n-1}} \text{Ker } V^{\theta(k)}$. By Claim 1 and [8, 2.11.4], we have $L \not\subset \text{Ker } V^{\theta(n)}$. Hence $V^{\theta(n)}|_L$ is nonzero irreducible and so, since L is Type I, $V^{\theta(n)}(L)$ contains all compact operators on the representation space of $V^{\theta(n)}$. Choose $g \in L$ such that $V_g^{\theta(n)} = T_{\theta(n)} - V_h^{\theta(n)}$. Let $f \equiv g + h$. Q.E.D.

Each space $W_{\pi,1}$, $\pi \in \Sigma$, has norm $\|\cdot\|_{\pi,1}|_{W_{\pi,1}} \ni \omega \rightarrow \text{tr}_{\pi}(\omega^{\sim} * \omega)^{1/2}$ which makes (47) $W_{\pi,1}$ linearly isometric to the predual of $\pi * W$ where the isomorphism $E_{\pi}|_{W_{\pi,1}} \rightarrow W_*$ is given by

$$\eta(E_{\pi}(\omega)) = \text{tr}_{\pi}(\eta * \omega) \quad (\forall \eta \in W, \omega \in W_{\pi,1}). \quad (48)$$

Let $\text{tm}_{00}(\Sigma)$ be the set of all $\theta \in \text{tm}(\Sigma)$ such that $\theta_{\pi} \neq 0$ only for a finite set of π , and let $\text{tm}_1(\Sigma)$ be the set of all $\theta \in \text{tm}(\Sigma)$ such that

$$\|\theta\| \equiv \sum_{\pi \in \Sigma} \|\theta_{\pi}\|_{\pi,1} < \infty.$$

Then $\text{tm}_1(\Sigma)$ is a Banach *-algebra in which $\text{tm}_{00}(\Sigma)$ is a dense ideal. For each $\theta \in \text{tm}_1(\Sigma)$ and $\pi \in \Sigma$, (43) implies

$$\|E_{\pi}(\theta_{\pi})\|_{W_*} \leq \|\theta_{\pi}\|_{\pi,1} \quad (49)$$

so the series $\sum_{\pi \in \Sigma} E_{\pi}(\theta_{\pi})$ converges in W_{*} . We define $\#|_{\text{tm}_1(\Sigma)} \rightarrow B(G)$ by appealing to (15) to find the unique $\theta^{\#} \in B$ such that

$$F_{\theta^{\#}} = \sum_{\pi \in \Sigma} E_{\pi}(\theta_{\pi}). \quad (50)$$

If B is equipped with the norm $\|\cdot\|_B$ induced from W_{*} by F , then it is evident from (49) and (50) that $\#$ is norm nonincreasing.

PROPOSITION 10. *The map $\#|_{\text{tm}_1(\Sigma)} \rightarrow B(G)$ is a linear isometry onto a closed subspace of $B(G)$.*

PROOF. The only nontrivial thing to show is that $\#$ is an isometry, and this will follow once we have established that the restriction of $\#$ to $\text{tm}_{00}(\Sigma)$ is an isometry. Let $\theta \in \text{tm}_{00}(\Sigma)$ be arbitrary and choose a finite subset Ω of Σ such that $\pi \in \Omega$ whenever $\theta_{\pi} \neq 0$. For each $\pi \in \Sigma$, $T_{\theta_{\pi}}$ is a trace-class operator on $\pi(H(T))$ so there exists a compact operator K_{π} on $\pi(H(T))$ of unit norm such that

$$\text{Tr}(K_{\pi} \circ T_{\theta_{\pi}}) = \|\theta_{\pi}\|_{\pi,1}. \quad (51)$$

Since G is Type I, $C(G)$ is of Type I and it follows from (39) and Proposition 9 that there exists some $\omega \in C$ such that $T_{\pi * \omega} = K_{\pi}$ for all $\pi \in \Omega$. Since each K_{π} has norm 1, we have

$$\left\| \omega * \sum_{\pi \in \Omega} \pi \right\|_W = 1. \quad (52)$$

From (51) and (50) it follows that

$$\left\langle \omega * \sum_{\pi \in \Omega} \pi, \theta^{\#} \right\rangle = \sum_{\pi \in \Omega} \text{tr}_{\pi}(\omega * \pi * \theta_{\pi}) = \sum_{\pi \in \Omega} \text{Tr}(K_{\pi} \circ T_{\theta_{\pi}}) = \sum_{\pi \in \Omega} \|\theta_{\pi}\|_{\pi,1}. \quad (53)$$

But (52) and (53) together imply that $\|\theta^{\#}\|_B > \|\theta\|$. Hence $\|\theta^{\#}\|_B = \|\theta\|$. Q.E.D.

LEMMA 2. *For $\pi \in \Sigma$ and $\theta \in W_{\pi,1}^{+}$, let $f \equiv \theta^{\#}$. Then the representation T^f is primary.*

PROOF. Since $\pi * W$ is simple, the representation $T^f|_{\pi * W}$ is an isomorphism. Similarly, if τ is a minimal projection of $W_{\pi,1}$ and $h = \tau^{\#}$, then $T^h|_{\pi * W}$ is an isomorphism. Thus by [8, 5.3.1(ii)], $T^f|_{\pi * W}$ and $T^h|_{\pi * W}$ are quasi-equivalent. Since T^h is irreducible, T^f is primary. Q.E.D.

LEMMA 3. *Let V be a W^* -representation of W on a Hilbert space $H(V)$, and suppose that $\int_{\Sigma} V^{\pi} d\mu(\pi)$ is a disintegration of V such that each V^{π} ($\pi \in \Sigma$) is quasi-equivalent to the W^* -representation of W associated with π . Then the disintegration is central (i.e. $V(W) \cap V(W)'$ consists precisely of the diagonalizable operators).*

PROOF. Let $w \in W$ satisfy $V_w \in V(W)'$. Then V_w^{π} is in $V^{\pi}(W)'$ for μ -almost all $\pi \in \Sigma$. Since V^{π} is primary for each $\pi \in \Sigma$, we have $V^{\pi}(W)' \cap V^{\pi}(W) = \mathbb{C}I$; consequently V_w is diagonalizable.

Now let T be an arbitrary (bounded) diagonalizable operator on $H(V)$, and let $\int_{\Sigma} T^{\pi} d\mu(\pi)$ be its decomposition. For each $\pi \in \Sigma$, let $H(\pi)_r$ be a subspace of the representation space $H(\pi)$ of V^{π} such that the restriction of V^{π} to $H(\pi)_r$ is irreducible.

The restriction T' of T to $H(V)_r \equiv \int_{\Sigma} H(\pi)_r d\mu(\pi)$ is obviously diagonalizable. Since the restriction V' of V to $H(V)_r$ is a direct integral of irreducible representations, it follows from [8, Théorème 8.6.5] that T' is of the form V'_{η} for some $\eta \in W$. Let $\pi \in \Sigma$ be arbitrary and write V'^{π} for the restriction of V' to $H(\pi)_r$. Then V'^{π} is a scalar multiple k_{π} of the identity. Since R_{η} is also k_{π} times the identity for any representation R of W equivalent to V'^{π} , it follows that V'_{η} must be k_{π} times the identity. But T^{π} being a scalar multiple of the identity whose restriction to $H(\pi)_r$ is V'^{π} , it follows that $T^{\pi} = V'_{\eta}$. Consequently, $T = V'_{\eta}$. That T is in $V(W)'$ is trivial. Q.E.D.

By an *operator-measure* $\theta\mu$ we shall mean an equivalence class of ordered pairs $(\theta, \mu) \in \text{tm}(\Sigma)^+ \times \mathfrak{M}(\Sigma)^+$, equivalence being defined by

$$\theta\mu \sim \eta\nu \Leftrightarrow \theta = \eta \cdot \frac{d\nu}{d\mu} \quad \mu\text{-a.e.} \quad (54)$$

and μ and ν are equivalent measures. The set of all such $\theta\mu$ shall be written $\mathfrak{OM}(\Sigma)$. For $\theta\mu \in \mathfrak{OM}(\Sigma)$, define

$$\|\theta\mu\|_{\mathfrak{OM}} \equiv \int_{\Sigma} \|\theta_{\pi}\|_{\pi,1} d\mu(\pi) \quad (55)$$

and write $\mathfrak{OM}_{\text{bd}}(\Sigma)$ for the set of all $\theta\mu$ such that $\|\theta\mu\|_{\mathfrak{OM}} < \infty$. The transformation $\cdot^{\vee} : \mathfrak{OM}_{\text{bd}}(\Sigma) \rightarrow P(G)$ is defined by

$$(\theta\mu)^{\vee} | G \ni x \rightarrow \int_{\Sigma} \text{tr}_{\pi}(\gamma(x) * \theta_{\pi}) d\mu(\pi) \quad (\forall \theta\mu \in \mathfrak{OM}_{\text{bd}}(\Sigma)), \quad (56)$$

so that, as well,

$$\langle \omega, (\theta\mu)^{\vee} \rangle = \int_{\Sigma} \text{tr}_{\pi}(\omega * \theta_{\pi}) d\mu(\pi) \quad (\forall \omega \in W). \quad (57)$$

If $\theta \in \text{tm}_1(\Sigma)^+$ and κ is counting measure on $\{\pi \in \Sigma : \theta_{\pi} \neq 0\}$, then

$$\theta^{\#} = (\theta\kappa)^{\vee}. \quad (58)$$

THEOREM 3. *The map $\cdot^{\vee} : \mathfrak{OM}_{\text{bd}}(\Sigma) \rightarrow P(G)$ is a bijection and, for each $\theta\mu \in \mathfrak{OM}_{\text{bd}}(\Sigma)$,*

$$(i) \ \|(\theta\mu)^{\vee}\|_B = \|\theta\mu\|_{\mathfrak{OM}}.$$

PROOF. Let $f \in P$ be arbitrary. The representation T^f is unitarily equivalent to a direct integral $\int_{\Sigma} S^{\pi} d\mu(\pi)$ where μ is a measure in $\mathfrak{M}(\Sigma)^+$ and each S^{π} is a primary representation quasi-equivalent to the irreducible W^* -representation of W associated with π . For each $\pi \in \Sigma$, write $H(\pi)$ for the representation space of S^{π} . Let w be the cyclic vector of $\int_{\Sigma} H(\pi) d\mu(\pi)$ corresponding to $v_f \in H(f)$ by the unitary equivalence. By modification of a μ -negligible set, if necessary, we may assume, that each w_{π} is cyclic in H_{π} .

Consider $\pi \in \Sigma$. Since G is separable and S cyclic, $H(\pi)$ is the direct sum $\bigoplus_{n=1}^{\infty} H_{\pi,n}$ of irreducible subspaces. Let $\sum_{n=1}^{\infty} w_{\pi,n}$ be the corresponding decomposition of w_{π} . Then each operator

$$w_{\pi,n} \otimes w_{\pi,n}^* | H_{\pi,n} \ni v \rightarrow \langle w_{\pi,n}, v \rangle_{H(\pi)} w_{\pi,n}$$

is a positive operator of trace $|w_{\pi,n}|^2$, so there exists $\eta_{\pi,n} \in W_{\pi,1}$ such that

$$\|\eta_{\pi,n}\|_{\pi,1} = |w_{\pi,n}|^2, \quad S_{\eta_{\pi,n}}^{\pi} = w_{\pi,n} \otimes w_{\pi,n}^*. \quad (59)$$

Since

$$\sum_{n=1}^{\infty} \|\eta_{\pi,n}\|_{\pi,1} = \sum_{n=1}^{\infty} |w_{\pi,n}|^2 = |w_{\pi}|^2 < \infty, \quad (60)$$

the series $\sum_{n=1}^{\infty} \eta_{\pi,n}$ converges to an element η_{π} of $W_{\pi,1}$. We have now defined an element η of $\text{tm}(\Sigma)$ satisfying, for all $\omega \in W$,

$$\begin{aligned} \langle \omega, f \rangle &= \langle T_{\omega}^f v_f, v_f \rangle_{H(f)} = \int_{\Sigma} \langle S_{\omega}^{\pi} w_{\pi}, w_{\pi} \rangle d\mu(\pi) \\ &= \int_{\Sigma} \sum_{n=1}^{\infty} \text{Tr}(S_{\omega}^{\pi} \circ (w_{\pi,n} \otimes w_{\pi,n}^*)) d\mu(\pi) \\ &= \int_{\Sigma} \sum_{n=1}^{\infty} \text{tr}_{\pi}(\omega * \eta_{\pi,n}) d\mu(\pi) = \int_{\Sigma} \text{tr}_{\pi}(\omega * \eta_{\pi}) d\mu(\pi) = \langle \omega, (\eta\mu)^{\vee} \rangle. \end{aligned} \quad (61)$$

Applying (61) when $\omega = \iota$ yields (i).

Suppose $\theta\nu \in \mathfrak{D}\mathfrak{M}_{\text{bd}}(\Sigma)$ also satisfies $(\theta\nu)^{\vee} = f$. We may (and shall) assume

(62) $\mu = \nu$ if μ and ν are equivalent.

For each $\pi \in \Sigma$, write f_{π} for $(\theta_{\pi})^{\#}$. Let V be the direct integral $\int_{\Sigma} T^{f_{\pi}} d\nu(\pi)$ and denote $\int_{\Sigma} H(f_{\pi}) d\nu(\pi)$ by $H(V)$. Since

$$\begin{aligned} \int_{\Sigma} |v_{f_{\pi}}|^2 d\nu(\pi) &= \int_{\Sigma} \langle T_{\iota}^{f_{\pi}} v_{f_{\pi}}, v_{f_{\pi}} \rangle_{H(f_{\pi})} d\nu(\pi) \\ &= \int_{\Sigma} \text{tr}_{\pi}(\iota * \theta_{\pi}) d\nu(\pi) = \int_{\Sigma} \|\theta_{\pi}\|_{\pi,1} d\nu(\pi) = \|\theta\nu\|_{\mathfrak{M}\mathfrak{D}} < \infty, \end{aligned}$$

the vector $v \equiv \int_{\Sigma} v_{f_{\pi}} d\nu(\pi)$ exists. If M is the orthogonal projection of $H(V)$ onto the minimal V -invariant subspace containing v , then $M = \int M^{\pi} d\nu(\pi)$ where ν -almost every M^{π} is an orthogonal projection onto a $T^{f_{\pi}}$ -invariant subspace [9, II.3.3, Théorème 3]; since $\int_{\Sigma} v_{f_{\pi}} d\nu(\pi) = v = M(v) = \int_{\Sigma} M^{\pi}(v_{f_{\pi}}) d\nu(\pi)$, it follows that $v_{f_{\pi}} = M^{\pi}(v_{f_{\pi}})$ ν -almost everywhere and so, since each $v_{f_{\pi}}$ is cyclic in $H(f_{\pi})$, M is the identity operator. Thus

(63) v is V -cyclic for $H(V)$.

Furthermore, for each $\omega \in W$,

$$\begin{aligned} \langle V_{\omega} v, v \rangle_{H(V)} &= \int_{\Sigma} \langle T_{\omega}^{f_{\pi}} v_{f_{\pi}}, v_{f_{\pi}} \rangle_{H(f_{\pi})} d\nu(\pi) \\ &= \int_{\Sigma} \text{tr}_{\pi}(\omega * \theta_{\pi}) d\nu(\pi) = (\theta\nu)^{\vee}(\omega) = \langle \omega, f \rangle. \end{aligned} \quad (64)$$

It follows from (63) and (64) that V is unitarily equivalent to T^f , and so to $\int_{\Sigma} S^{\pi} d\mu(\pi)$ as well. Lemma 2 implies that each representation T^f , $\pi \in \Sigma$, is primary. From Lemma 3 follows that the diagonal algebras of the given disintegrations of T^f and V are $T^f(W) \cap T^f(W)'$ and $V(W) \cap V(W)'$, respectively. Thus, if $U|H(V) \rightarrow H(f)$ maps v to w and implements the equivalence between V and T^f , U transforms the diagonal algebra of $\int_{\Sigma} V^{\pi} d\nu(\pi)$ into that of $\int_{\Sigma} S^{\pi} d\mu(\pi)$. It follows from (62) and [24, Proposition 8.27] that $\mu = \nu$ and there exist a Borel subset Ω of Σ with ν -negligible complement and a measurable field of unitary operators $U^{\pi}|H(f_{\pi}) \rightarrow H(\pi)$ such that

$$U^{\pi} \circ S_{\omega}^{\pi} \circ (U^{\pi})^{-1} = V_{\omega}^{\pi} \quad (\forall \pi \in \Omega, \omega \in W) \quad (65)$$

and

$$U = \int_{\Omega} U^{\pi} d\nu(\pi). \quad (66)$$

Since $U(v) = w$, it follows from (66) and (65) that $U^{\pi}(v_{\omega}) = w_{\pi}$ for all $\pi \in \Omega$. Thus, for each $\pi \in \Omega$ and $\omega \in W$,

$$\begin{aligned} \text{Tr}(S_{\omega}^{\pi} \circ S_{\eta}^{\pi}) &= \text{Tr}\left(S_{\omega}^{\pi} \circ \sum_{n=1}^{\infty} w_{\pi,n} \otimes w_{\pi,n}^{*}\right) = \langle S_{\omega}^{\pi} w_{\pi}, w_{\pi} \rangle_f \\ &= \langle (U^{\pi})^{-1} \circ V_{\omega}^{\pi} \circ U^{\pi}(w_{\pi}), w_{\pi} \rangle_f = \langle V_{\omega}^{\pi}(v_{\pi}), v_{\pi} \rangle \\ &= \text{Tr}(V_{\omega}^{\pi} \circ v_{\pi} \otimes v_{\pi}^{*}) = \text{tr}_{\pi}(\omega * \theta_{\pi}). \end{aligned}$$

It follows that $\eta = \theta$ ν -almost everywhere on Σ . Q.E.D.

For each $\pi \in \Sigma$, let $\mathfrak{T}_{\pi}(G)$ be the set $\{\theta^{\#}: \theta \in W_{1,\pi}\}$ or, in other words, the closure in B of the linear span of representative functions emanating from the irreducible representation of G associated with π .

COROLLARY 2. *Let $f \in P(G)$ be arbitrary. Then there exist an element h of the Cartesian product $\prod_{\pi \in \Sigma} \mathfrak{T}_{\pi}(G)$ and a measure $\mu \in \mathfrak{M}(\Sigma)^{+}$ such that*

- (i) $\langle \omega, f \rangle = \int_{\Sigma} \langle \omega, h_{\pi} \rangle d\mu(\pi)$, and
- (ii) $\|f\|_B = \int_{\Sigma} \|h_{\pi}\|_B d\mu(\pi)$.

The pair (h, μ) satisfying (i) is unique (h being determined up to a μ -negligible set) provided

- (iii) $\|h_{\pi}\|_B = 1$ for μ -almost all $\pi \in \Sigma$.

PROOF. By Theorem 3, $f = (\theta\mu)^{\sim}$ for some $\theta\mu \in \mathfrak{DM}_{\text{bd}}(\Sigma)$. Letting $h_{\pi} \equiv \theta_{\pi}^{\#}$ for all $\pi \in \Sigma$, we have for all $\omega \in W$,

$$\begin{aligned} \langle \omega, f \rangle &= \langle \omega, (\theta\mu)^{\sim} \rangle = \int_{\Sigma} \text{tr}_{\pi}(\omega * \theta_{\pi}) d\mu(\pi) \\ &= \int_{\Sigma} \langle \omega, \theta_{\pi}^{\#} \rangle d\mu(\pi) = \int_{\Sigma} \langle \omega, h_{\pi} \rangle d\mu(\pi) \end{aligned}$$

which proves (i). If $\omega = \iota$ in (i), we obtain (ii). The uniqueness of (h, μ) subject to (iii) follows from Theorem 3 and (54). Q.E.D.

8. The complexification and the dual. The Fourier-Stieltjes transformation $\gamma|W \rightarrow m^\infty(\Sigma)$ has a canonical extension. For each $\omega \in \overline{W}$, we define

$$\hat{\omega}_\pi \equiv \omega|_{\pi * W_* \cap \text{dom}(\omega)}; \quad (67)$$

then

$$(\overline{\pi * W}) = \{\hat{\omega}_\pi: \omega \in \overline{W}\} \quad (\forall \pi \in \Sigma). \quad (68)$$

We write $\overline{m}(\Sigma)$ for the set of all $\hat{\omega}$, $\omega \in \overline{W}$. Evidently

$$\overline{m}(\Sigma) \cong \overline{m^\infty(\Sigma)}. \quad (69)$$

It follows from (38) and the fact that $G \ni x \rightarrow \gamma(x)^\wedge$ is faithful, that $\wedge$ is also faithful on G_C and Λ : we write $G_C^\wedge$ and $\Lambda^\wedge$ for the respective images by $\wedge$.

It is the business of the sequel to determine how $G_C^\wedge$ may be identified intrinsically in terms of $\overline{m}(\Sigma)$. Theorem 1 implies that it suffices to identify $G_\gamma^\wedge$ and $G_C^{+\wedge}$, and it follows from Proposition 2 that, to find $G_C^{+\wedge}$, it suffices to find $\Lambda^\wedge$, so our task will be done once we have characterized $G_\gamma^\wedge$ and $\Lambda^\wedge$.

In our terminology, Tatsuuma's duality theorem [13, Satz 11.4.2] can be stated as follows. An element $\zeta \in m^\infty(\Sigma)$ is in $G_\gamma^\wedge$ provided each of the following three conditions is satisfied:

(70a) ζ_π is unitary for each $\pi \in \Sigma$;

if R and S are irreducible W^* -representations and V is the W^* -representation of W such that $(R \circ \gamma) \otimes (S \circ \gamma) = V \circ \gamma$, then, where η is the element of $z * W$ satisfying $\hat{\eta} = \zeta$,

(70b) $R_\eta \otimes S_\eta = V_\eta$;

if $\int_\Sigma L^\pi d\mu(\pi)$ is a direct integral of the left regular representation L with each $L^\pi(W)$ quasi-equivalent to $\pi * W$, then

(70c) $\int_\Sigma L_{\eta_\pi}^\pi d\mu(\pi) \neq 0$ if $\eta \neq 0$.

Some such condition as (70c) is necessary to take into the account the "topological" character of Σ (when G is Abelian, Σ is "replaced" by X and (70c) is replaced by the condition that ζ be continuous). The condition (70c) however seems extraneous for a characterization of $G_\gamma^\wedge$ intrinsically in terms of Σ and $m^\infty(\Sigma)$.

Our solution to the problem will be described in terms of *quasi-multipliers*. Let $QM(C)$ be the set of all $\omega \in W$ such that $\alpha * \omega * \beta \in C$ whenever $\alpha, \beta \in C$, and let $qm^\infty(\Sigma)$ be the set of all $\theta \in m^\infty(\Sigma)$ such that $\zeta * \theta * \eta \in C^\wedge$ for all $\zeta, \eta \in C^\wedge$.

It has been shown [1, Theorem 4.1] that, for each $\omega \in QM(C)$ and net f_α in P satisfying $\sigma(W_*, C) - \lim_\alpha F_{f_\alpha} = F_f$ for $f \in P$,

$$\lim_\alpha \langle \omega, f_\alpha \rangle = \langle \omega, f \rangle \quad \text{if} \quad \lim_\alpha \|f_\alpha\| = \|f\|. \quad (71)$$

It is known [1, Corollary 3] that if Ψ is any W^* -homomorphism of W , then

$$\Psi(QM(C)) = \{\omega \in \Psi(W): \alpha * \omega * \beta \in \Psi(C) \ (\forall \alpha, \beta \in \Psi(C))\}. \quad (72)$$

In particular,

$$qm^\infty(\Sigma) = \{\hat{\omega}: \omega \in QM(C)\}. \quad (73)$$

Evidently, (73) implies

$$G_\gamma^* \subset qm^\infty(\Sigma). \quad (74)$$

For $\zeta \in m^\infty(\Sigma)$ and $f \in B(G)$ we define

$$\langle \zeta, f \rangle_\Sigma \equiv \langle \eta, f \rangle \quad (75)$$

where $\zeta = \hat{\eta}$ and $\eta \in z * W$.

THEOREM 4. *A unitary element ζ of $qm^\infty(\Sigma)$ is in G_γ^* if and only if*

$$\langle \zeta, f \rangle_\Sigma \langle \zeta, h \rangle_\Sigma = \int_\Sigma \text{tr}_\pi(\zeta_\pi * \theta_\pi) d\mu(\pi) \quad (\forall f, h \in X) \quad (i)$$

where, for each pair $f, h \in X$, $\theta\mu$ is the unique element of $\mathfrak{DM}(\Sigma)$ such that $(\theta\mu)^\vee = fh$.

PROOF. It follows from (73) that $\zeta = \hat{\eta}$ for some $\eta \in QM(C)$. That (i) holds means

$$\langle \eta, f \rangle \langle \eta, h \rangle = \langle \eta, fh \rangle \quad (\forall f, h \in X). \quad (76)$$

If A is the convex cone generated by X in P , it follows from (76) that

$$\langle \eta, f \rangle \langle \eta, h \rangle = \langle \eta, fh \rangle \quad (\forall f, h \in A). \quad (77)$$

If f and h in P are arbitrary, it follows from the Krein-Milman theorem that there exist nets f_α and h_β in A such that

$$\|f_\alpha\|_B \rightarrow \|f\|_B, \quad \|h_\beta\|_B \rightarrow \|h\|_B \quad (78a)$$

and

$$f_\alpha \rightarrow f, \quad h_\beta \rightarrow h \quad \text{in the topology } \sigma(W_*, C). \quad (78b)$$

It follows from (78a), (78b) and Gelfand's theorem [12, 32.40] that $f_\alpha \rightarrow f$ and $h_\beta \rightarrow h$ in the compact-open topology.

Thus $f_\alpha h_\beta \rightarrow fh$ in the compact open topology so, by Gelfand's theorem again,

$$f_\alpha h_\beta \rightarrow fh \quad \text{in the topology } \sigma(W_*, C). \quad (79)$$

It now follows from (71) and (77) that

$$\langle \eta, f \rangle \langle \eta, h \rangle = \lim_{\alpha, \beta} \langle \eta, f_\alpha \rangle \langle \eta, h_\beta \rangle = \lim_{\alpha, \beta} \langle \eta, f_\alpha h_\beta \rangle = \langle \eta, fh \rangle. \quad (80)$$

From (80) follows

$$\langle \eta, f \rangle \langle \eta, h \rangle = \langle \eta, fh \rangle. \quad (81)$$

Walter's duality theorem and (81) imply that $\eta = \gamma(x)$ for some $x \in G$. Hence $\zeta = \gamma(x)^\wedge$. Q.E.D.

The quasi-strict topology on $qm^\infty(\Sigma)$ is that induced by the seminorms

$$qm^\infty(\Sigma) \ni \theta \rightarrow \|\zeta * \theta * \eta\|_\infty \quad (\zeta, \eta \in C^*). \quad (82)$$

When G is Abelian, $qm^\infty(\Sigma)$ may be identified with the bounded, continuous, complex-valued functions on the dual group X , and, on $\|\cdot\|_\infty$ -bounded subsets of $qm^\infty(\Sigma)$, the quasi-strict topology and the compact-open topology agree under this identification. Since the Fourier-Stieltjes transformation $\hat{\cdot}$ is an isometry when

restricted to C [8, 2.7.3], it follows from (73) that, for $\omega \in QM(C)$ and any net ω_α in $QM(C)$,

$$\omega_\alpha \rightarrow \omega \text{ quasi-strictly} \Leftrightarrow \|\varepsilon * (\omega_\alpha - \omega) * \kappa\|_W \rightarrow 0 \quad (\forall \varepsilon, \kappa \in C). \quad (83)$$

PROPOSITION 11. *The map $\hat{\gamma}|G \ni x \rightarrow \gamma(x) \in qm^\infty(\Sigma)$ is a homeomorphism when $qm^\infty(\Sigma)$ bears the quasi-strict topology.*

PROOF. That $\hat{\gamma}$ is continuous is a consequence of [11, 20.4]. If $\gamma(x_\alpha)$ converges to $\gamma(x)$ in the quasi-strict topology, then (83) and [17, Theorem 7] imply

$$\lim_{\alpha, \beta} \langle \gamma(x_\alpha), f \rangle = \langle \gamma(x), f \rangle \quad (\forall f \in B).$$

That x_α converges to x now follows from Walter's theorem [25, Theorem 1]. Q.E.D.

THEOREM 5. *An element β of $\overline{m}(\Sigma)$ is in $\Lambda^\wedge$ if and only if all the following conditions are satisfied:*

- (i) $-\beta_\pi = \beta_\pi^\sim$ ($\forall \pi \in \Sigma$);
- (ii) $\langle \exp(t\beta), f \rangle_\Sigma \langle \exp(t\beta), h \rangle_\Sigma = \int_\Sigma \text{tr}_\pi(\theta_\pi * \exp(t\beta_\pi)) d\mu(\pi)$ for all $t \in R$ and $f, h \in X$ (where $\theta_\mu \in \mathfrak{D}\mathfrak{M}_{\text{bd}}(\Sigma)$ satisfies $(\theta_\mu)^\wedge = fh$);
- (iii) $\lim_{t \rightarrow 0} \|\zeta * \exp(t\beta) * \eta - \zeta * \eta\|_\infty = 0$ ($\forall \zeta, \eta \in C$).

PROOF. First suppose $\beta = \hat{\alpha}$ for some $\alpha \in \Lambda$. Then (i) follows from (22) and (ii) from Theorem 4 and Proposition 4. That (iii) holds follows from Propositions 4 and 11.

Now suppose that $\beta \in \overline{m}(\Sigma)$ satisfies (i), (ii), and (iii). From (i) follows that each $\exp(t\beta)$ is unitary; by Theorem 4, it is in $G_\gamma^\wedge$. Thus there is a one-parameter subgroup s of G satisfying $\gamma(s(t))^\wedge = \exp(t\beta)$ for all $t \in R$. Condition (iii) just means that $\lim_{t \rightarrow 0} \gamma(s(t))^\wedge = \gamma(e)^\wedge$ in the quasi-strict topology. Thus, Proposition 11 implies that s is continuous at 0. Since s is a homomorphism, it is continuous everywhere and so is an element of Γ . By Proposition 4, there exists $\alpha \in \Lambda$ such that $\gamma(s(t)) = \exp(t\alpha)$ for all $t \in R$. Thus $\exp(t\beta) = \exp(t\alpha)^\wedge$ for all $t \in R$, which implies $\beta = \hat{\alpha}$ by Stone's theorem. Q.E.D.

REFERENCES

1. Charles A. Akemann and Gert K. Pedersen, *Complications of semicontinuity in C^* -algebra theory*, Duke Math. J. **40** (1973), 785–795.
2. S. Bochner, *Vorlesungen über Fouriersche Integrale*, Akademische Verlags Gesellschaft, Leipzig, 1932.
3. H. Boseck and G. Czichowski, *Grundfunktionen und verallgemeinerte Funktionen auf Gruppen*. I, Math. Nachr. **58** (1973), 215–240.
4. H. Boseck, *Über Darstellungen lokal-kompakter topologischer Gruppen*, Math. Nachr. **74** (1976), 233–251.
5. F. Bruhat, *Distributions sur un groupe localement compact et applications à l'étude des représentations de groupes p -adiques*, Bull. Soc. Math. France **89** (1961), 43–75.
6. C. Chevalley, *Theory of Lie groups*, Princeton Univ. Press, Princeton, N. J., 1946.
7. Nelson Dunford and Jacob T. Schwartz, *Linear operators*. Part II, Interscience, New York, 1963.
8. J. Dixmiers, *Les C^* -algèbres et leurs représentations*, Gauthier-Villars, Paris, 1969.
9. ———, *Les algèbres d'opérateurs dans l'espace Hilbertien*, Gauthier-Villars, Paris, 1969.
10. P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France **92** (1964), 181–236.

11. Edwin Hewitt and Kenneth A. Ross, *Abstract harmonic analysis*. Vol. I, Springer-Verlag, Berlin, 1963.
12. ———, *Abstract harmonic analysis*. Vol. II, Springer-Verlag, Berlin, 1970.
13. Herbert Heyer, *Dualität lokalkompakter Gruppen*, Lecture Notes in Math., vol. 150, Springer-Verlag, New York and Berlin, 1970.
14. G. Hochschild, *The structure of Lie groups*, Holden-Day, San Francisco, Calif., 1965.
15. R. Lashof, *Lie algebras of locally compact groups*, Pacific J. Math. **7** (1957), 1145–1162.
16. George W. Mackey, *The Laplace transform for locally compact Abelian groups*, Proc. Nat. Acad. Sci. U.S.A. **34** (1948), 156–162.
17. Kelly McKennon, *The strict topology and the Cauchy structure of the spectrum of a C^* -algebra*, General Topology Appl. **5** (1975), 249–262.
18. ———, *The structure space of the trigonometric polynomials on a compact group*, J. Reine Angew. Math. **307/308** (1979), 166–172.
19. D. Montgomery and L. Zippin, *Topological transformation groups*, Interscience, New York, 1955.
20. D. Raikov, *Positive definite functions on commutative groups with an invariant measure*, Dokl. Akad. Nauk SSSR (N.S.) **28** (1940), 296–300.
21. J. Riss, *Éléments de calcul différentiel et théorie des distributions sur les groupes Abéliens localement compacts*, Acta Math. **89** (1953), 45–105.
22. Walter Rudin, *Functional analysis*, McGraw-Hill, New York, 1973.
23. Masamichi Takesaki, *Duality and von Neumann algebras*, Lecture Notes in Math., vol. 247, Springer-Verlag, Berlin and New York, 1972.
24. ———, *Theory of operator algebras*. I, Springer-Verlag, Berlin.
25. Martin Walter, *W^* -algebras and non-Abelian harmonic analysis*, J. Funct. Anal. **11** (1972), 17–38.
26. Garth Warner, *Harmonic analysis on semi-simple Lie groups*. I, Springer-Verlag, Berlin, 1972.
27. Andre Weil, *L'intégration dans les groupes topologiques et ses applications*, Hermann, Paris, 1941.
28. H. Yamabe, *A generalization of a theorem of Gleason*, Ann. of Math. (2) **50** (1953), 351–365.

DEPARTMENT OF MATHEMATICS, WASHINGTON STATE UNIVERSITY, PULLMAN, WASHINGTON 99164