COUNTING DIVISORS WITH PRESCRIBED SINGULARITIES

BY

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ABSTRACT. Given a family of divisors $\{D_s\}$ in a family of smooth varieties $\{Y_s\}$ and a sequence of integers $m_1, \ldots, m_t$, we study the scheme parametrizing the points $(s, y_1, \ldots, y_t)$ such that y_i is a (possibly infinitely near) m_i-fold point of D_s. We obtain a general formula which yields, as special cases, the formula of de Jonquières and other classical results of Enumerative Geometry. We also study the questions of finiteness and the multiplicities of the solutions.

1. Introduction. The aim of this work is to obtain formulas for the number of divisors in a family which possess points with specified multiplicities. For instance, if the family is given by the hyperplane sections of a smooth subvariety Y of a projective space, we wish to count the hyperplanes satisfying specified contact conditions with Y.

About a century ago de Jonquières published his Mémoire [10], exhibiting a formula for the number of plane curves of a given degree having prescribed contacts with a fixed plane curve. The formula itself has generated much interest (cf. Enriques and Chisini [3], MacDonald [16], [17] and Mattuck [19]). Shortly afterwards, the flourishing school of Enumerative Geometry produced a wealth of results on contacts of lines and planes with surfaces in 3-space and other enumerative problems (cf. Zeuthen and Pieri [27]).

The next generation of algebraic geometers somewhat changed the emphasis, from the counting of singularities per se, to the discovery of certain invariants definable in terms of those singularities. Thus, one finds in C. Segre [24, p. 75] the definition of the genus of a curve in terms of the invariant $v - 2n$, where v is the number of double points of a g_n^1 on the curve. The Zeuthen-Segre invariant (Enriques [4, p. 167]) and the very definition of the canonical system [4, p. 49] are further examples of the "new" trend.

Possessing now as we do, a well-developed intersection theory, it is natural to try and go back to the origins, and vindicate (to today's taste and sense of rigor) those classical formulas. In fact, Hilbert's 15th problem calls for " . . . the actual carrying out of the process of elimination in the case of equations of special form in such a way that the degree of the final equations and the multiplicity of their solutions may be foreseen" [8, p. 464]. We hope this work will provide a step in this direction. For a survey on recent developments on the subject, we refer to Kleiman [12], [13].

The paper is organized as follows. In order to study singular divisors in a family $\{D_s\}$, it is convenient to look at the pairs (y, s) such that y is a singular point of D_s. This leads to our definition of incidence correspondences in §2. The case where the ambient spaces Y_s are curves is treated in §3. Here is also handled the case of arbitrary dimension but with just one assigned multiplicity. In §4 we apply the machinery developed so far to retrieve the formula for the degree of the dual variety (4.2), the Zeuthen-Segre invariant (4.3) and the formula of de Jonquières (4.4). In §5 we derive formulas for contacts of lines with a surface in 3-space. In §§6 and 7 we introduce a step-by-step procedure to handle the case of arbitrary relative dimension. Roughly, to get a hold on the divisors that possess, say, one triple and another double point, we first restrict to the subfamily of divisors with a triple point. Then we construct a new family by taking (virtual) transforms at each assigned triple point, and finally we search for the double points in the new family. In §8 we obtain formulas that apply to the counting of bitangent and tritangent planes of a surface in 3-space. We explain how to deal with contacts of a hypersurface with higher-dimensional linear spaces and give an example in 4-space (8.4). We also show how to get formulas for the number of curves (embedded in a family of surfaces) displaying specified coincidences of tangents at a singularity. This yields, as special cases, the number of cusps of a general net on a surface (8.6.3), as well as the cuspnodes of a general web (8.6.4). The last two paragraphs are devoted to the questions of finiteness and multiplicity of the solutions.

As a general proviso, all schemes are assumed to be quasiprojective over an algebraically closed field k, and points are k-rational. The intersection theory used here is that of Fulton [5]. If X is a scheme and $Z \subseteq X$ is closed, we denote by $[Z]$ the class of Z in the Chow group $A.(X)$. We denote by $\int z$ the degree of a zero cycle z of a projective scheme.

2. Incidence correspondences. Suppose we are given a diagram of maps of schemes,

$$\underline{W} \subset \underline{X} \supset \underline{D}$$
$$\downarrow$$
$$\underline{S}$$

We think of $\underline{X} \to \underline{S}$ as a family of ambient spaces, and of $\underline{W}$ and $\underline{D}$ as the total spaces of families of subschemes $\{\underline{W}_s\}_{s \in \underline{S}}$, $\{\underline{D}_s\}_{s \in \underline{S}}$. We call a subscheme of $\underline{S}$ the *incidence correspondence* of $\underline{W}$ in $\underline{D}$ if it has the following universal property: a map $T \to \underline{S}$ factors through it iff $\underline{W}_T \subset \underline{D}_T$ holds (as subschemes of the pullback $\underline{X}_T = \underline{X}_S \times T$).

Proposition (2.2) will give a criterion for the existence of incidence correspondences. It is instructive, however, to look first at some of the examples that motivate the whole business.

(2.1) EXAMPLES. (1) Let Y be a smooth, projective variety. Let $\Delta \subset Y \times Y$ be the diagonal subvariety. Given a positive integer m, let $m\Delta$ denote the subscheme of

$Y \times Y$ with ideal $I(m\Delta) = I(\Delta)^m$. Let S be (the parameter space of) a linear system on Y with universal divisor $D \subset Y \times S$. Put $\underline{S} = Y \times S$, $\underline{X} = Y \times \underline{S}$ and take $\underline{W}$ and $\underline{D}$ to be the pullbacks of $m\Delta$ and D to $\underline{X}$. Then the incidence correspondence of $\underline{W}$ in $\underline{D}$ consists of the pairs (y, s) such that y is an (at least) m-fold point of D_s.

(2) Suppose now dim $Y = 1$. Put $Y^t = Y \times \cdots \times Y$ (t-times). Let Δ_{0i} denote the diagonal $\{(y_0, \ldots, y_t) \in Y \times Y^t|_{y_0} = y_i\}$. Given a sequence of positive integers $\underline{m} := m_1, \ldots, m_t$, let $\underline{m}\Delta$ stand for the positive divisor

$$\underline{m}\Delta = m_1\Delta_{01} + \cdots + m_t\Delta_{0t}.$$

Observe that the fibre of $\underline{m}\Delta$ over $(y_1, \ldots, y_t) \in Y^t$ is the divisor $m_1 y_1 + \cdots + m_t y_t$ in Y. Let S and D be as in (1). Finally, put $\underline{S} = Y^t \times S$, $\underline{X} = Y \times \underline{S}$, and take $\underline{W}$ and $\underline{D}$ to be the pullbacks of $\underline{m}\Delta$ and D to $\underline{X}$. In the present situation, the incidence correspondence of $\underline{W}$ in $\underline{D}$ consists of the points $(y_1, \ldots, y_t, s) \in \underline{S}$ such that $D_s \geqslant \sum m_i y_i$ holds. Roughly, the classical formula of de Jonquières counts the number of such points (if finite and provided $t + \dim S = \sum m_i$). We retrieve the formula by computing the class of the incidence correspondence in the Chow ring $A(\underline{S})$.

(3) Let S be the Grassmann variety of lines in $\mathbf{P}^n$ and let $P \to S$ be the universal $\mathbf{P}^1$-bundle. Put $\underline{S} = P \times_S \cdots \times_S P$ (t-times) and put $\underline{X} = P \times_S \underline{S}$. Now let $\underline{m}\Delta \subset \underline{X}$ denote the sum of multiples $\sum m_i \Delta_{0i}$ of the relative diagonals. Given a hypersurface $H \subset \mathbf{P}^n$, set $D = P \cap (H \times S)$, then take $\underline{D}$ to be the pullback of D to $\underline{X}$. The incidence correpondence of $\underline{m}\Delta$ in $\underline{D}$ now consists of the points $(y_1, \ldots, y_t, l)$ in $\mathbf{P}^n \times \cdots \times \mathbf{P}^n \times S$ such that either the line l is contained in H or H cuts out in l a divisor $\geqslant \sum m_i y_i$. This is certainly an appropriate setup for counting lines satisfying prescribed contact conditions with a hypersurface.

(2.2) PROPOSITION. *Let $f: \underline{X} \to \underline{S}$ be a map of schemes, let $\underline{W} \subset \underline{X}$ be a closed subscheme, and let $\underline{D} \subset \underline{X}$ be the scheme of zeros of a section of an invertible $\mathcal{O}_X$-module L. Suppose $f_*(L \otimes \mathcal{O}_W)$ is locally free and that its formation commutes with base change (e.g. if $\underline{W}$ is flat and proper$/\underline{S}$ and $R^1 f_*(L \otimes \mathcal{O}_W) = 0$ holds). Then the incidence correspondence of $\underline{W}$ in $\underline{D}$ exists and is equal to the scheme of zeros of a section of the locally free $\mathcal{O}_S$-module $f_*(\mathcal{O}_W \otimes L)$.*

PROOF. There is a natural diagram of maps of $\mathcal{O}_X$-modules,

$$\begin{array}{c} \mathcal{O}_X \\ \downarrow \quad \nwarrow a \\ 0 \to L \otimes I(\underline{W}) \to L \to L \otimes \mathcal{O}_W \to 0, \end{array}$$

(2.2.1)

where the vertical arrow is the section defining $\underline{D}$.

Now it is clear that, for each map $T \to \underline{S}$, $\underline{D}_T \supseteq \underline{W}_T$ holds if and only if $a_T = 0$. The assertion now follows from Altman and Kleiman [1, Proposition (2.3)]. $\square$

3. **Relative dimension 1.** In order to apply (2.2) to the preceding examples, we start by working out their main common features.

(3.1) Fix a smooth map $f: X \to S$. Let D denote the scheme of zeros of a section of an invertible $\mathcal{O}_X$-module L. Let $\underline{m} = m_1, \ldots, m_t$ be a sequence of nonnegative

integers. *Unless stated otherwise, we will assume* $t = 1$ *if the relative dimension of f is* > 1. We denote the t-fold cartesian product of X over S by X^t. The projections from X^t onto (resp. omitting) the ith factor will be denoted by $p_{t,i}$ (resp. $p_{t,i}$). The diagonal subscheme of X^2 over S is denoted by Δ or $\Delta_{X/S}$. We denote by $m\Delta$ the subscheme with ideal $I(m\Delta) = I(\Delta)^m$. The pullback of Δ to X^t via the projection onto the i, j factors is denoted by Δ_{ij}.

We are interested in the set of singular points of the fibres of $D \to S$. Since each fibre D_s is (locally) defined by one equation in X_s, a point is of multiplicity $\geqslant m$ on D_s if and only if the local equation of D_s lies in the mth power of the maximal ideal there. (If $D_s = X_s$, each point of X_s is considered to be of multiplicity $\geqslant m$ for every m.) In order to globalize this observation, as well as to treat the case of several assigned multiplicities, we are lead to consider the subscheme $\underline{m}\Delta \subset X \times_S X^t$ defined by the ideal

$$I(\underline{m}D) = I(\Delta_{01})^{m_1} \ldots I(\Delta_{0t})^{m_t} \quad \text{(product)}.$$

(3.2) LEMMA. $\underline{m}\Delta$ *is finite and flat over X^t.*

PROOF. The subscheme $m\Delta$ of X^2 is flat over X (via, say, $p_{2,1}$). Indeed, this assertion is trivial for $m = 1$. For $m \geqslant 2$, we consider the exact sequence

$$(3.2.1) \qquad\qquad 0 \to I^{m-1}/I^m \to \mathcal{O}_{m\Delta} \to \mathcal{O}_{(m-1)\Delta} \to 0$$

where I is short for $I(\Delta)$. Because X is smooth$/S$, each I^{m-1}/I^m is a locally free $\mathcal{O}_X$-module (in fact isomorphic to the symmetric power $S_{m-1}\Omega^1_{X/S}$). Thus $m\Delta$ is flat over X as claimed. For $t \geqslant 2$ (hence rel dim = 1), the proof is easy and will be omitted. $\square$

(3.3) We define the *$\underline{m}$-Jacobian of D* as the incidence correspondence of $\underline{m}\Delta$ in $p_{t+1,1}^{-1}D$. It exists by (2.2), because $\underline{m}\Delta$ is finite and flat over X^t. It will be denoted by $J(\underline{m}; D)$. The *$\underline{m}$-contact sheaf of L* is the $\mathcal{O}_{X^t}$-module

$$(3.3.1) \qquad\qquad \mathcal{E}_{X/S}(\underline{m}; L) = (p_{t+1,\hat{1}})(\mathcal{O}_{m\Delta} \otimes p_{t+1,1}^* L).$$

(This is a "secant bundle" in the sense of Schwarzenberger [23].) It will also be written simply $\mathcal{E}_X(\underline{m}; L)$ or $\mathcal{E}(\underline{m})$ if no confusion is likely. The *$\underline{m}$-contact section* is the section of $\mathcal{E}_{X/S}(\underline{m}; L)$ which is the adjoint (= direct image via $p_{t+1,\hat{1}}$) of the composition

$$\mathcal{O} \xrightarrow{p_{t+1,1}^* s} p_{t+1,1}^* L \xrightarrow{r} L \otimes \mathcal{O}_{\underline{m}\Delta}.$$

Here s is the section of L defining D and r is induced by the restriction $\mathcal{O} \twoheadrightarrow \mathcal{O}_{\underline{m}\Delta}$. We say D is *$\underline{m}$-regular* if the Koszul complex of the $\underline{m}$-contact section is exact. If S is Cohen-Macaulay, then D is $\underline{m}$-regular iff $J(\underline{m}; D)$ is of the right codimension rank $\mathcal{E}(\underline{m})$

$$\left(= \sum_{i=1}^{t} \binom{m_i + n - 1}{n} \right),$$

where n = rel dim X/S (cf. (3.5) below)).

(3.4) PROPOSITION. (1) $J(\underline{m}; D)$ *is the scheme of zeros of the $\underline{m}$-contact section.*

(2) *Suppose D is $\underline{m}$-regular. Then $J(\underline{m}; D)$ repesents the top Chern class of* $\mathcal{E}_{X/S}(\underline{m}; L)$ *(in any decent intersection theory).*

PROOF. (1) The assertion follows from (2.2).

(2) The assertion is well known for nonsingular S [6, p. 153]. For the general case, we sketch a proof in the lemma below, for lack of reference.

(3.4.1) LEMMA. *Let X be a quasiprojective scheme. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank ρ. Let s be a section of $\mathcal{E}$ and let Z denote the scheme of zeros of s. If s is regular (i.e., the Koszul complex $\dot{\wedge}\mathcal{E}^* \to \mathcal{O}_Z$ is exact) then $[Z] = c_\rho(\mathcal{E}) \cap [X]$ in $A.(X)$.*

PROOF. We proceed by induction on the rank of $\mathcal{E}$. When $\rho = 1$, then Z is a Cartier divisor, $\mathcal{E} = \mathcal{O}_X(Z)$ and the assertion is proven in Fulton [5, p. 160]. Suppose $\rho \geqslant 2$ and consider $p: \mathbf{P}(\mathcal{E}) \to X$. Now $p^*: A.(X) \to A.(\mathbf{P}(\mathcal{E}))$ is injective, according to Verdier [26, Assertion 6.9, p. 170]. Thus we may assume $\mathcal{E}$ fits into an exact sequence $\mathcal{E}' \hookrightarrow \mathcal{E} \twoheadrightarrow \mathcal{E}''$, where $\mathcal{E}''$ is an invertible $\mathcal{O}_X$-module. We get a section s'' of $\mathcal{E}''$ which is easily seen to be regular. Denoting by Z'' the scheme of zeros of s'', we have $[Z''] = c_1(\mathcal{E}'') \cap [X]$. Let $i: Z'' \subset X$ be the inclusion map. Clearly, i^*s factors through a section s' of $i^*\mathcal{E}'$. It can easily be seen that s' is also regular and that its scheme of zeros is just Z. Hence, we have, by induction,

$$[Z] = c_{\rho-1}(i^*\mathcal{E}') \cap [Z''] \quad \text{in } A.(Z'').$$

We push forward to $A.(X)$ and use the projection formula. Now standard properties of Chern classes yield the assertion. $\square$

Next we compute the class of the $\underline{m}$-contact sheaf in the Grothendieck ring $K X'$. We start with the case $t = 1$ (and arbitrary relative dimension).

(3.5) PROPOSITION. *We have the formula*

$$\mathcal{E}_{X/S}(m; L) = L \sum_0^{m-1} S_i(\Omega^1_{X/S}) \quad \text{in } K X.$$

($S_i = $ ith symmetric power.) In particular,

$$\text{rank } \mathcal{E}_{X/S}(m; L) = \binom{m + n - 1}{n},$$

where $n = $ rel dim X/S.

PROOF. We argue as in the proof of (3.2).

Tensoring the sequence (3.2.1) by $p_{2,1}^* L$ and applying $(p_{2,1})_*$ yields the exact sequence

$$0 \to L \otimes S_{m-1}(\Omega^1_{X/S}) \to \mathcal{E}(m) \to \mathcal{E}(m-1) \to 0,$$

where $\mathcal{E}(m)$ is short for $\mathcal{E}_{X/S}(m; L)$. The formula now follows by induction on m. $\square$

(3.5.1) REMARK. $\mathcal{E}(m) := \mathcal{E}_{X/S}(m; L)$ is equal to **EGA**'s sheaf of principal parts of order $m - 1$ [EGA IV$_4$, 16.7.1.2].

Here is what $\mathcal{E}(m)$ looks like, locally. Suppose $X = \operatorname{Spec} B$, $S = \operatorname{Spec} A$, $L = \mathcal{O}_X$, $\Omega_{B/A} = B\,dx_1 \oplus \cdots \oplus B\,dx_n$. Now $\mathcal{E}(m)$ corresponds to the quotient $B \otimes_A B/I^m$, where I denotes the diagonal ideal of $B \otimes_A B$ generated by all $b \otimes 1 - 1 \otimes b$ (b in B), and the B-module structure is induced by $b_0(b_1 \otimes b_2) = (b_0 b_1) \otimes b_2$. Recall that $\Omega_{B/A} = I/I^2$ and that the universal A-derivation of B is given by $db = \overline{b \otimes 1 - 1 \otimes b}$ (class mod I^2). $\mathcal{E}(m)$ splits in the direct sum of the I^i/I^{i+1} ($i = 0, \ldots, m-1$), and each of these is free with basis $dx_1^{i_1} \ldots dx_n^{i_n}$, where the exponents range over all sequences of nonnegative integers which add up to i. Denote the dual basis by $\partial_{i_1 \ldots i_n}$. Then, for each b in B, the class of $1 \otimes b$ mod I^m is equal to

$$\sum (\partial_{i_1 \ldots i_n} b)\,dx_1^{i_1} \ldots dx_n^{i_n}.$$

Let F be a local equation of D. Then the m-contact section has the coordinates $\partial_{i_1 \ldots i_n} F$. Therefore, $J(m; D)$ is defined locally by the $\binom{m+n-1}{n}$ equations $\partial_{i_1 \ldots i_n} F = 0$.

(3.6) THEOREM. *We have, in the Grothedieck ring $K^{\cdot}(X')$, the formula*

$$\mathcal{E}_{X/S}(\underline{m}; L) = \sum_{i=1}^{t} \left(p_{t,i}^* \mathcal{E}_{X/S}(m_i; L)\right)\left(-\sum_{h>i} m_h \Delta_{hi}\right).$$

(The last parenthesis stand for the twisting by the ideal of a divisor.)

PROOF. We assume $t \geqslant 2$. Write for short $q_j = p_{t+1,j}$ ($j = 0, \ldots, t$), $p_1 = p_{t,i}$, and similarly for q_j, p_i. Consider the closed subscheme W of $X \times_S X'$ with ideal

$$I(W) = I(m_2 \Delta_{02}) \cdots I(m_t \Delta_{0t}) \quad \text{(products)}.$$

Thus, $I(\underline{m}\Delta) = I(m_1 \Delta_{01})I(W)$ holds. Because $I(W)$ is invertible, we get an exact sequence

$$0 \to \mathcal{O}_{m_1 \Delta_{01}} \otimes I(W) \to \mathcal{O}_{m\Delta} \to \mathcal{O}_W \to 0.$$

Tensoring it with $q_0^* L$ and applying $(q_0)_*$ yields the exact sequence (3.6.1)

$$0 \to (q_0)_*(\mathcal{O}_{m_1 \Delta_{01}} \otimes I(W) \otimes q_0^* L) \to \mathcal{E}_{X/S}(\underline{m}; L) \to (q_0)_*(\mathcal{O}_W \otimes q_0^* L) \to 0$$

$$\|\text{(definition)} \qquad\qquad\qquad\qquad\qquad\qquad \|$$
$$A \qquad\qquad\qquad\qquad\qquad\qquad\qquad p_1^* \mathcal{E}_{X/S}(\underline{m}'; L)$$

where $\underline{m}'$ denotes the sequence $m_2, \ldots, m_t$. The latter equality holds by the Principle of Exchange (or flat base change). Here is the relevant diagram:

$$
\begin{array}{ccc}
W \subset X \times_S X' & \overset{q_0}{\to} & X' \\
\downarrow q_1 & & \downarrow p_1 \\
\underline{m}'\Delta \subset X \times_S X'^{t-1} & \overset{p_0}{\to} & X'^{t-1} \\
\swarrow p_0 & & \\
X & &
\end{array}
$$

By induction on t, it follows that $p_1^* \mathscr{E}_{X/S}(\underline{m}'; L)$ is the sum of the $(t-1)$-last terms of the proposed formula.

Now it remains to identify A in (3.6.1) with $p_1^* \mathscr{E}_{X/S}(m_1; L)(-\Sigma_{h>1} m_h \Delta_{hi})$. For this, we observe that $X \times_S X^t$ is equal to the fiber product $X^t \times_{X^{t-1}} X^t$, where we regard X^t as a scheme over X^{t-1} via $p_{\hat{1}}$. Further, with this identification in mind, the projections onto the 1st and 2nd factors are equal to $q_{\hat{0}}$ and $q_{\hat{1}}$ (see diagram). Moreover Δ_{01} is precisely $\Delta_{X^t/X^{t-1}}$. Thus, applying (3.5) to $X^t \to X^{t-1}$ and $p_1^* L \otimes I(W)$ (in place of $X \to S$ and L, respectively), we get

$$A = \mathscr{E}_{X^t/X^{t-1}}(m_1; p_1^* L \otimes I(W)) = p_1^* L \otimes I(W) \sum_0^{m_1-1} S_i(\Omega^1_{X^t/X^{t-1}}).$$

Observe that $p_{\hat{1}}: X^t \to X^{t-1}$ is the pullback of $X \to S$:

$$
\begin{array}{ccc}
X^t & \xrightarrow{p_1} & X \\
p_{\hat{1}} \downarrow & & \downarrow \\
X^{t-1} & \to & S.
\end{array}
$$

Finally, since formation of relative differentials and of symmetric powers of a locally free module commutes with base change, we see that A is indeed equal to the 1st term of our formula. This completes the proof of the theorem. $\square$

The recursive relation stated below will be needed later for the proof of the formula of de Jonquières.

(3.6.2) PROPOSITION. *The contact sheaves satisfy the relation*

$$\mathscr{E}_{(\underline{m}, u+1; L)} = \mathscr{E}_{(\underline{m}, u; L)} + p_{t+1}^* \big(L(\Omega^1_{X/S})^u \big) \big(-\sum m_h \Delta_{h, t+1} \big) \quad in \ K(X^{t+1}).$$

PROOF. Consider the fibre square

$$
\begin{array}{ccc}
X \times_S X^{t+1} & \overset{j}{\hookrightarrow} & X \times_S X^{t+2} \\
\uparrow q_{\hat{0}}' & & \downarrow q_{\hat{0}} \\
X^{t+1} & \overset{i}{\hookrightarrow} & X^{t+2} \\
(x_1, \dots, x_{t+1}) & \mapsto & (x_{t+1}, x_1, \dots, x_t, x_{t+1}).
\end{array}
$$

where the horizontal maps are diagonal embeddings. Set $W = (1, \underline{m}, u)\Delta$, that is, the subscheme of $X \times_S X^{t+2}$ with ideal $I(\Delta_{01}) I(\Delta_{02})^{m_1} \dots I(\Delta_{0t+1})^{m_t} I(\Delta_{0, t+2})^u$. Since W is flat over X^{t+2}, we have the formula

$$I(j^{-1}(W)) = j^* I(W).$$

By the construction of j, we have

$$j^* I(W) = I(\Delta_{0, t+1})^{u+1} I(\Delta_{01})^{m_1} \dots I(\Delta_{0t})^{m_t}.$$

Thus, $j^{-1}(W)$ is just $(\underline{m}, u + 1)\Delta$. Consequently, we may write

$$\mathcal{E}_{X/S}(\underline{m}, u + 1; L) = (q_0')_* (\mathcal{O}_{(\underline{m}, u+1)\Delta} \otimes (q_0')^* L) \quad \text{(by definition)}$$

$$= (q_0')_* (j^*(\mathcal{O}_W \otimes q_0^* L))$$

$$= i^*(q_0')_* (\mathcal{O}_W \otimes q_0^* L) \quad \text{(by the Principle of Exchange)}$$

$$= i^* \mathcal{E}_{X/S}(1, \underline{m}, u; L)$$

$$= \left[i^* q_1^* L\left(-u\Delta_{t+2,1} - \sum m_h \Delta_{t+2, h+1}\right)\right.$$

$$\left. + \sum q_{t+1}^* \mathcal{E}_l + q_{t+2}^* \mathcal{E}_{X/S}(u; L)\right] \quad \text{in } K^\cdot(X^{t+1})$$

(by (3.6)), where $\mathcal{E}_l$ is short for

$$\mathcal{E}_{X/S}(m_l; L)\left(-u\Delta_{t+2}, l + 1 - \sum_{h > l} m_h \Delta_{h+1}, l + 1\right).$$

By definition of i, we have

$$q_l i = \begin{cases} p_{t+1} & \text{for } l = 1 \text{ or } t + 2, \\ p_{l-1} & \text{for } 2 \leqslant l \leqslant t + 1. \end{cases}$$

We also have

$$i^* I(\Delta_{h+1, l+1}) = \begin{cases} p_{t+1}^* \Omega_{X/S}^1 & \text{for } (h + 1, l + 1) = (t + 2, 1), \\ I(\Delta_{h, l}) & \text{for } 1 \leqslant l < h \leqslant t + 1. \end{cases}$$

Therefore we get

$$\mathcal{E}_{X/S}(\underline{m}, u + 1; L) = p_{t+1}^* L\left(\Omega_{X/S}^1\right)^u\left(-\sum m_h \Delta_{t+1, h}\right)$$

$$+ \sum p_l^* \mathcal{E}_{X/S}(m_l; L)\left(-\sum_{h > l} m_h' \Delta_{hl}\right),$$

where we put $m_h' = m_h$ for $h \leqslant t$ and $m_{t+1}' = u$. By (3.6) the last term in the expression above is precisely $\mathcal{E}(\underline{m}, u; L)$ in $K^\cdot(X^{t+1})$, thus completing the proof. $\square$

4. Applications. Let Y, S and D be as in (2.1)(1). Let M be an invertible $\mathcal{O}_Y$-module associated to the linear system S. Put $X = Y \times S$ and $L = M \otimes \mathcal{O}_S(1)$. There is a section of L defining the universal divisor $D \subset X$. Now, if the codimension of $J(\underline{m}; D)$ in X^t is equal to the rank of $\mathcal{E}_{X/S}(\underline{m}; L)$, we have

$$[J(\underline{m}; D)] = c_{\text{top}} \mathcal{E}_{X/S}(\underline{m}; L).$$

(4.1) PROPOSITION. *Preserve the above notation. If we also assume $J(\underline{m}; D)$ finite then the degree of its zero cycle is the integer $|\underline{m}; M|$ defined by*

$$(4.1.1) \qquad\qquad |\underline{m}; M| = \int c_{tn} \mathcal{E}_Y(\underline{m}; M),$$

where we put $n = \dim Y$.

PROOF. The degree may be computed by first pushing forward the zero cycle from $A(X^t)$ to $A(Y^t)$. Since X^t is the trivial projective bundle $Y^t \times S$, the assertion

follows from standard properties of Chern classes and the fact that

$$\mathcal{E}_{X/S}(\underline{m}; L) = \mathcal{E}_Y(\underline{m}; M) \otimes \mathcal{O}_S(1). \quad \square$$

(4.2) *Degree of the dual variety*. If S is induced by the hyperplane sections of $Y \subset \mathbf{P}^N$, then

(4.2.1)
$$|2; M| = c_n \mathcal{E}_Y(2; M) = c_n(M + M\Omega_Y^1)$$

is the degree of the dual variety of Y (up to the inseparable degree of the dual map [11, p. 250]).

(4.3) *The Zeuthen-Segre invariant*. Suppose Y is a surface. Set $\chi := c_2(\Omega_Y^1)$, $K := c_1(\Omega_Y^1)$ and (by abuse) $M := c_1(M)$. Suppose dim $S = 1$. Then the formula for $|2; M|$ is equivalent to that expressing the classical Zeuthen-Segre invariant I in terms of χ. Indeed, set

$\beta := $ # base points,

$\gamma := $ # singular members,

$g := $ (arithmetic) genus of a member of S.

Then I is defined classically by

$I := \gamma - \beta - 4g$ (cf. Baker [2, p. 185]).

On the other hand, we may compute

$$\begin{aligned} \gamma &= |2; M| \quad \text{(by (4.1))} \\ &= \chi + 2KM + 3M^2 \quad \text{(by (4.2.1))} \\ &= \chi + 2(KM + M^2) + M^2 = \chi + 4(g - 1) + \beta. \end{aligned}$$

(The assertion $\beta = M^2$ holds because the base points are the zeros of the map $\mathcal{O}_Y^{\oplus 2} \to M$ defined by a choice of 2 members of S.)

Thus, we retrieve the well-known formula $I + 4 = \chi$ (cf. Iversen [9, p. 974]).

(4.4) *The formula of de Jonquières*. Suppose now Y is a smooth projective curve of genus g and S is a linear system associated to an injective $\mathcal{O}_Y$-module M of degree d. The $\underline{m}$-Jacobian $J(\underline{m}; D)$ presently parametrizes the points $(y_1, \ldots, y_t, s)$ $\in Y^t \times S$ such that $D_s \geqslant \sum m_i y_i$. Suppose $t + r = \sum m_i \leqslant d$. If $J(\underline{m}; D)$ is finite, then the formula of de Jonquières expresses the degree of its zero cycle. Setting $v = d - \sum m_i$, we have

(4.4.1)
$$\int [J] = \frac{(\prod m_i)}{v!} \sum_{h=0}^{t} (t + v - h)! h! \binom{g}{h} \sigma_h(m_1 - 1, \ldots, m_t - 1),$$

where $\sigma_0 = 1$, and $\sigma_1, \ldots, \sigma_t$ denote the elementary symmetric functions in t variables.

Recent proofs of the formula used some deep numerical relations in the Jacobian of Y (cf. Mattuck [19]). The rest of this paragraph is devoted to the proof of (4.4.1).

By (4.1), we have $\int [J] = |\underline{m}; M|$. Denote by $[\underline{m}]$ the r.h.s. of (4.4.1). We claim that

(4.4.2)
$$|\underline{m}; M| = [\underline{m}].$$

To prove the latter equality (and hence (4.4.1)), we will need the following.

(4.4.3) LEMMA. *There are formulas*:

(1) $|\underline{m}, 0; M| = 0$,

(2) $|\underline{m}, 1; M| = (d - \Sigma\, m_i)|\underline{m}; M|$,

(3) $|1, \ldots, 1; M| = (d!)/(d - t)!$,

(4)

$$|\underline{m}, u + 1; M| = |\underline{m}, u; M| + (d + u(2g - 2))|\underline{m}; M(-uy)|$$

$$- \Sigma\, m_i|m_1, \ldots, m_i + u, \ldots, m_t; M|,$$

where y is a point of Y and $M(-uy) := M \otimes I(y)^u$.

(5) $|\underline{m}; M| = \pi\underline{m}; M|$, *where $\pi\underline{m}$ denote a permutation of the m_i.*

PROOF OF (4.4.2). Replacing $\underline{m}$ by $\underline{m}, 1, \ldots, 1$, we may assume $v = 0$, because of (4.4.3)(2). If $\underline{m} = 1, \ldots, 1$ (t times), then $[\underline{m}] = t!$. This is just fine by (4.4.3)(3), because $d = t$ (as $v = 0$). Now it is clear that (4.4.3)(3)–(4.4.3)(5) completely determine $|\underline{m}; M|$. So, to finish the proof, it suffices to verify that $[\underline{m}]$ satisfies the following recursive relation:

$$[\underline{m}, u + 1] = [\underline{m}, u, 1] + (d + u(2g - 2))[\underline{m}, 1]$$

$$- \Sigma\, m_i[m_1, \ldots, m_i + u, \ldots, m_t, 1]$$

with $1 + u + \Sigma\, m_i = d$. The verification offers no surprises and will be omitted. $\square$

PROOF OF (4.4.3). Clearly, (2) implies (3). Also, (2) is a consequence of (1) and (4).

To prove (5), let π_0 denote the permutation of the coordinates of $Y \times Y'$ such that $p_0\pi_0 = p_0$ and $p_i\pi_0 = p_{\pi i}$ for $i = 1, \ldots, t$. We clearly have that $\pi_0(\Delta_{01}) = \Delta_{0, \pi i}$, and therefore, $\pi_0^{-1}(\underline{m}\Delta) = (\pi\underline{m})\Delta$, where $\pi\underline{m} := m_{\pi 1}, \ldots, m_{\pi t}$. With a slight abuse of notation we may write, by definition of contact sheaves,

$$\pi^*\mathcal{E}_Y(\underline{m}; M) = \pi^*(p_{\hat{0}})_*(p_0^* M \otimes \mathcal{O}_{\underline{m}\Delta}) = (p_{\hat{0}})_*\pi_0^*(p_0^* M \otimes \mathcal{O}_{\underline{m}\Delta})$$

$$= (p_{\hat{0}})_*(p_0^* M \otimes \pi_0^*\mathcal{O}_{\underline{m}\Delta}) = \mathcal{E}_Y(\pi\underline{m}; M).$$

As $\pi^* = \pi_*^{-1}$ holds (because π is an automorphism of Y'), we get

$$\int c_t\mathcal{E}_Y(\underline{m}; M) = \int \pi_*\pi^* c_t\mathcal{E}_Y(\underline{m}; M) = \int \pi_* c_t\mathcal{E}_Y(\pi\underline{m}; M) = \int c_t\mathcal{E}_Y(\pi\underline{m}; M),$$

thus proving (5).

For the remaining assertions, set, for short, $\mathcal{E}(u) := \mathcal{E}_Y(\underline{m}, u; M)$. To prove (1), notice that $\mathcal{E}(0) = p^*\mathcal{E}_Y(\underline{m}; M)$, where $p: Y'^{+1} \to Y'$ omits the last coordinate. (1) now follows because the r.h.s. of this last equality has zero $(t + 1)$-Chern class.

To prove (4), we recall from (3.6.2) the following formula:

$$\mathcal{E}(u + 1) = \mathcal{E}(u) + \left(q_{t+1}^*(M\Omega_Y^{\otimes u})\right)\left(- \sum_{h < t} m_h\Delta_{h, t+1}\right).$$

Let y and $\underline{h}$ denote, respectively, the embeddings of Y' onto $Y' \times y$, and $\Delta_{h, t+1}$ in Y'^{+1}. We compute Chern classes modulo algebraic equivalence. Thus $c_1(M\Omega_Y^{\otimes u}) = (d + u(2g - 2))y$. Using the above formula to compute $c_{t+1}\mathcal{E}(u + 1)$, we get

$$|\underline{m}, u + 1; M| = |\underline{m}, u; M| + (d + u(2g - 2))\int c_t\underline{y}^*\mathcal{E}(u) - \sum m_h\int c_t\underline{h}^*\mathcal{E}(u).$$

Using (3.6) together with the formulas

$$(q_{t+1}\underline{y})(Y') = \{y\}, \qquad \underline{y}^*(\Delta_{t+1,i}) = q_i^*(y),$$

$$\underline{y}^*(\Delta_{h,i}) = \Delta_{h,i} \quad (i < h \le t), \qquad q_i\underline{y} = q_i \quad (1 \le i \le t),$$

we get

$$\underline{y}^*\mathcal{E}(u) = \mathcal{O}_{Y'}^{\oplus u} + \sum q_i^*\mathcal{E}_Y(m_i; M(-uy))\left(-\sum_{h>i} m_h\Delta_{hi}\right) \quad \text{in } K(Y').$$

Hence,

$$\int c_t \, \underline{y}^*\mathcal{E}(u) = |\underline{m}; M(-uy)|.$$

Similarly, we get, for $h = t$,

$$\underline{t}^*\mathcal{E}(u) = q_t^*\mathcal{E}_Y(u; M) + q_t^*\mathcal{E}_Y(m_t; M)\Omega_Y^{\otimes u}$$

$$+ \sum_1^{t-1} q_i^*\mathcal{E}_Y(m_i; M)\left(-u\Delta_{t,i} - m_t\Delta_{t,i} - \sum_{i+1}^{t-1} m_j\Delta_{ji}\right).$$

However, the 1st two terms in the r.h.s. adds up precisely to

$$q_t^*\mathcal{E}_Y(u + m_t; M)!!!$$

Hence, we have proven

$$\underline{t}^*\mathcal{E}(u) = \mathcal{E}_Y(m_1, \ldots, m_{t-1}, m_t + u; M).$$

Finally, with the help of permutations on Y^t and Y^{t+1} (cf. proof of (5)) we get, for any $h = 1, \ldots, t$,

$$\underline{h}^*\mathcal{E}(u) = \mathcal{E}_Y(m_1, \ldots, m_h + u, \ldots, m_t; M),$$

thus completing the proof of (4.4.3). $\square$

5. Lines with prescribed contact with a hypersurface.

Fix a projective space $\mathbf{P}^r = \mathbf{P}(V)$. Let S denote the Grassmann variety of lines in $\mathbf{P}^r$. Let Q denote the universal 2-quotient of V_S. Thus $X := \mathbf{P}(Q) \to S$ is the universal family of lines. Let $H \subset \mathbf{P}^r$ be a hypersurface of degree d, and let $s_H \in H^0(\mathbf{P}^r, \mathcal{O}(d))$ be a defining equation. Now s_H gives rise to a section of $L := \mathcal{O}_X(d)$ whose scheme of zeros $D \subset X$ is just the intersection $X \cap (H \times S) \subset \mathbf{P}^r \times S$. We say *a line l and the hypersurface H have $\underline{m}$-contact with each other* if $l \cap H$ contains a divisor of l of the form $\sum m_i x_i$. The scheme $J(\underline{m}; D)$ presently parametrizes the pairs $((x_1, \ldots, x_t), l) \in H \times \cdots \times H \times S$ such that $l \cap H \ge \sum m_i x_i$. Thus, one may retrieve classical formulas for the number of lines satisfying prescribed contact conditions with a hypersurface by computing the appropriate $c.\mathcal{E}_{X/S}(\underline{m}; \mathcal{O}_X(d))$. Let us make a few explicit calculations for a surface in $\mathbf{P}^3$.

Set $\lambda = c_1(Q)$ and set $\pi = c_2 Q$. Thus, λ (resp. π) is the class of the "condition" that a line meet a fixed other line (resp. that a line pass through a fixed point). There are formulas:

$\lambda^4 = 2$ (= # of lines meeting 4 others),
$\pi^2 = 1$ (= # of lines through 2 points . . .),
$\lambda^2\pi = 1$ (= # of lines meeting 2 others and passing through a point).

Set $\theta = c_1 \mathcal{O}_X(1)$. We need the Chern class of $\Omega := \Omega^1_{X/S}$. In view of the canonical sequence on $X = \mathbf{P}(Q)$, $0 \to \Omega(1) \to Q_X \to \mathcal{O}_X(1) \to 0$, we can compute $w := c_1\Omega$ as

$$w = \lambda - 2\theta.$$

We also have the formulas $\theta^2 = \lambda\theta - \pi$, $\theta^3 = (\lambda^2 - \pi)\theta - \lambda\pi$, $\theta^4 = 0$. (The 1st holds by Grothendieck's construction of Chern classes; the 2nd then follows.)

(5.1) *Five-fold contact with a surface of degree d ($t = 1$, $m = 5$).* The number we are after is the degree of the zero cycle

$$J(5; D) = c_5\mathcal{E}_{X/S}(5; \mathcal{O}_X(d)).$$

By (3.5), this is the term of top degree in $(1 + d\theta)(1 + d\theta + w)\ldots(1 + d\theta + 4w)$. Using, as needed, the relations listed above, and pushing down to S, one finds the zero cycle

$$d\left[24\lambda^4 + (50d - 192)\lambda^4 + (35d^2 - 300d + 576)(\lambda^4 - \pi\lambda^2)\right]$$

which has degree $5d(7d^2 - 40d + 48)$ (cf. Baker [**2**, Formula (4), p. 90]).

(5.2) *Doubly inflexional lines ($t = 2$; $m_1 = m_2 = 3$).* Their number is one-half the degree of the zero cycle

$$z := c_6\mathcal{E}_{X/S}(3, 3; \mathcal{O}_X(d)) = (d\theta_1 - 3\Delta)\left[(d - 2)\theta_1 - 3\Delta + \lambda\right]$$
$$\cdot \left[(d - 4)\theta_1 - 3\Delta + 2\lambda\right] \cdot d\theta_2\left[(d - 2)\theta_2 + \lambda\right]\left[(d - 4)\theta_2 + 2\lambda\right].$$

(The indices mean pullback to $X \times_S X$ via the corresponding projection.)

Using the formulas $\Delta^2 = -c_1(\Omega) = -w\Delta = (2\theta - \lambda)\Delta$, $p_{2*}(\Delta) = p_{2*}(\theta_1) = 1$ and $p_{2*}(1) = 0$, we may compute $p_{2*}(z)$. We retrieve the number

$$d(d - 4)(d - 5)(d^3 + 3d^2 + 29d - 60)$$

(cf. Baker [**2**, Formula (5), p. 91], or Schubert [**22**, Formula (19), p. 236]).

(5.3) *Tritangent lines ($t = 3$, $m_1 = m_2 = m_3 = 2$) meeting a general line.* One has to calculate the degree of

$$z := \lambda c_6\mathcal{E}(2, 2, 2; \mathcal{O}_X(d)) = \lambda \prod_{i=1}^{3} \left(d\theta_i - 2\sum_i\right)\left((d - 2)\theta_i + \lambda - 2\sum_i\right),$$

where $\Sigma_i := \Sigma_{h>i} \Delta_{hi}$. We get 6 times the number

$$\tfrac{1}{3}d(d - 3)(d - 4)(d - 5)(d^2 + 3d - 2)$$

(cf. Schubert [**22**, Formula (15), p. 236]).

6. Arbitrary relative dimension. Let Y be a smooth variety and let $D \subset Y$ be a positive divisor. We say that D *has a singularity of type $\underline{m}$* if the following holds: first require that D have a point y_1 of multiplicity $\geqslant m_1$; next let Y_1 denote the blowup to Y at y_1, let E_1 denote the exceptional divisor and let D_1 denote the total transform of D; then require that the effective divisor $D_1 - m_1E_1$ have a point y_2 of multiplicity $\geqslant m_2$, and so on. The sequence $(y_1, y_2, \ldots)$ thus constructed is called a *singularity of type $\underline{m}$ of D*.

Of course, if $\dim Y = 1$, it is clear that D has a singularity of type $\underline{m}$ iff $D \geqslant \Sigma m_i y_i$ holds.

(6.1) Let $f: X \to S$, $D \subset X$ and L be as in (3.1). We no longer impose restrictions on the relative dimension. Our goal is to construct an S-scheme $J(\underline{m}; D)$ whose

fibre over any point s in S consists of the sequences of singularities of type $\underline{m}$ of the fibre D_s.

We first define inductively a sequence of maps

$$p_{t,1}, p_{t,2} \colon X\{t\} \to X\{t-1\}.$$

We set $X\{0\} := S$, $X\{1\} := X$ and $p_{1,1} = p_{1,2} := f$. For $t \geqslant 2$, we view $X\{t-1\}$ as a scheme/$X\{t-2\}$ via $p_{t-1,1}$, and we let

$$b_t \colon X\{t\} \to X\{t-1\} \times_{X\{t-2\}} X\{t-1\}$$

denote the blowup of the diagonal. Then we set $p_{t,i} = p_i b_t$, where p_i denotes the projection onto $X\{t-1\}$. Note that $p_{t,i}$ is smooth [**EGA IV$_4$**, 19.4.8]. We denote the exceptional divisor by E_t (or E if no confusion is likely) and the mth power of its ideal by $\mathcal{O}_t(m)$.

For each $\mathcal{O}_X$-module $\mathcal{Q}$, we set

$$\mathcal{Q}(m_1) := \mathcal{O}_2(m_1) \otimes p_{2,2}^* \mathcal{Q}$$

and define inductively, for $t \geqslant 2$,

(6.1.1) $$\mathcal{Q}(\underline{m}) := \mathcal{O}_{t+1}(m_t) \otimes p_{t+1,2}^* \mathcal{Q}(m_1, \ldots, m_{t-1}).$$

$J(\underline{m}; D)$ will be a subscheme of $X\{t\}$, defined by the following step-by-step procedure.

For $t = 1$, $m = m_1$, the subscheme $J(m; D) \subset X$ was already constructed in (3.3). The result below will enable us to go on assigning more singular points, one at a time.

(6.2) PROPOSITION. *$J(m; D)$ is also equal to the incidence correspondence of mE in $p_{2,2}^{-1}D$.*

$$mE \subset X\{2\} \supset p_{2,2}^{-1}D$$

$$p_{2,1} \swarrow \qquad \searrow p_{2,2}$$

$$J(m; D) \subset X \qquad X \supset D$$

PROOF. The diagram (2.2.1) now reads

(6.2.1)
$$\mathcal{O}_{X\{2\}} \xrightarrow{\quad a \quad}$$
$$\downarrow$$
$$0 \to L(m) \to p_{2,2}^* L \to p_{2,2}^* L \otimes \mathcal{O}_{mE} \to 0.$$

Using the formulas

$$\left(R^i p_{2,1}\right)_* L(m) = \left(R^i p_1\right)_* \left(p_2^* L \otimes I(\Delta)^m\right),$$

$$\left(R^i p_{2,1}\right)_* \left(p_{2,2}^* L \otimes \mathcal{O}_{mE}\right) = \left(R^i p_1\right)_* \left(p_2^* L \otimes \mathcal{O}_{m\Delta}\right)$$

(see Manin [**18**, p. 63]), one shows that mE is flat/X, that $(R^1 p_{2,1})_* (p_{2,2}^* L \otimes \mathcal{O}_{mE}) = 0$ and that the adjoint of a in X coincides with the m-contact section. This proves the proposition. $\square$

(6.3) Continuing with our program of constructing $J(\underline{m}; D)$, notice that, because mE is flat/X, the sequence (6.2.1) remains exact after restriction over $J := J(m; D)$. Hence, the section of $p_{2,2}^* L$ that defines $p_{2,2}^{-1}D$ yields a section s_m of

$L(m)_J$. We call the scheme of zeros of s_m the *m-virtual transform* of D, and denote it by $D(m)$.

At this stage, we replace $D \subset X \to S$ by $D(m) \subset X\{2\}_J \to J$ and define, by induction,

$$(6.3.1) \qquad\qquad J(\underline{m}; D) := J(m_2, \ldots, m_t; D(m_1)).$$

We leave to the reader the verification that (6.3.1) agrees with (3.3).

7. The generic class of $J(\underline{m}; D)$. Preserve the notation of §6. Denote by $\underline{m}_i$ the truncated sequence $m_1, \ldots, m_i$.

(7.1) We say D is *$\underline{m}$-regular* if, for each $i = 1, \ldots, t$, the Koszul complex of the m_i-contact section defining $J(\underline{m}_i; D)$ in $J(\underline{m}_{i-1}; D) \times_{X\{i-1\}} X\{i\}$ is exact.

If S is Cohen-Macaulay, then D is $\underline{m}$-regular iff *each $J(\underline{m}_i; D)$ is either empty or of the right codimension*

$$\sum_{j=1}^{i} \binom{m_j + n - 1}{n}$$

in $X\{i\}$.

We define inductively the *$\underline{m}$-contact virtual sheaf* of L by

$$(7.1.1) \quad \mathcal{E}_{X/S}(\underline{m}; L) := \mathcal{E}_{X\{t\}/X\{t-1\}}(m_t; L(\underline{m}_t)) + p_{t,1}^* \mathcal{E}_{X/S}(\underline{m}_t; L) \quad \text{in } K^{\cdot}(X\{t\}),$$

where the first term on the r.h.s. is the (class of the) m_t-contact sheaf of $L(\underline{m}_t)$ (6.1.1).

(7.2) LEMMA. *We have the formula, in $K^{\cdot}(X\{t\})$,*

$$\mathcal{E}_{X/S}(\underline{m}; L) = \mathcal{E}_{X\{2\}/X}(m_2, \ldots, m_t; L(m_1)) + (p_{t,1}^* \ldots p_{2,1}^*) \mathcal{E}_{X/S}(m_1; L).$$

PROOF. The assertion follows immediately from the definitions. Just observe that, replacing $X \to S$ by $\tilde{X} := X\{2\} \to \tilde{S} := X$, we have $\tilde{X}\{t-1\} = X\{t\}$. $\square$

(7.3) THEOREM. *Suppose D is $\underline{m}$-regular. Then the class of $J(\underline{m}; D)$ in $A.(X\{t\})$ is the Poincaré dual of the top Chern class of the $\underline{m}$-contact sheaf of L. In symbols,*

$$[J(\underline{m}; D)] = c_{\text{top}}(\mathcal{E}_{X/S}(\underline{m}; L)) \cap [X\{t\}].$$

PROOF. We have, to start with, the equality

$$[J(m_1; D)] = c_{\text{top}}(\mathcal{E}_{X/S}(m_1; L)) \cap [X] \quad \text{in } A.(x),$$

in view of (3.4)(2).

Set $\underline{m}' = m_2, \ldots, m_t$, replace $X \to S$ by

$$X' := X\{2\} \times_X J(m_1; D) \to S' := J(m_1; D),$$

D by $D' := D(m_1)$ (6.3), and L by $L' := L(m_1)_{X'}$. By construction, we have $J(\underline{m}; D) = J(\underline{m}'; D)$. By induction, the class of the latter in $A.(X'\{t-1\})$ is dual to the top Chern class of $\mathcal{E}_{X'/S'}(\underline{m}'; L)$. Now $X'\{t-1\}$ is obviously equal to $X\{t\} \times_X S'$. Since $X\{t\}$ is smooth (hence flat)/X, the operations of pulling back the homology class of a subscheme and taking the homology class of the pullback

of that subscheme are interchangeable. Thus, setting, for short, $p = p_{2,1} \ldots p_{t,1}$, we may write, in $A.(X\{t\})$,

$$[X'\{t-1\}] = p^*[J(m_1; D)]$$
$$= p^* c_{\text{top}}(\mathcal{E}_{X/S}(m_1; L)) \cap [X\{\tau\}].$$

Now, denoting by i the inclusion $X'\{t-1\} \subset X\{t\}$, we have

$$[J(\underline{m}; \mathbf{D})] = i_*[J(\underline{m}'; \mathbf{D}(m_1))]$$
$$= i_*(c_{\text{top}} \mathcal{E}_{X'/S'}(\underline{m}': L') \cap [X'\{t-1\}]) \quad \text{(introduction)}$$
$$= c_{\text{top}} \mathcal{E}_{X\{2\}/X}(\underline{m}'; L(m_1)) \cap i_*[X'\{t-1\}] \quad \text{(projection formula)}$$
$$= c_{\text{top}}(\mathcal{E}_{X\{2\}/X}(\underline{m}': L(m_1)) + p^*\mathcal{E}_{X/S}(m_1; L)) \cap [X\{t\}]$$
$$= c_{\text{top}}(\mathcal{E}_{X/S}(\underline{m}; L)) \cap [X\{t\}] \quad \text{(by (7.2)).} \quad \square$$

(7.4) REMARK. It would be interesting to find explicitly the universal polynomial $p(m_1, \ldots, m_t; c_1, \ldots, c_n; \lambda)$ expressing the push down of $c_{\text{top}}(\mathcal{E}_{X/S}(\underline{m}; L)) \cap [X\{t\}]$ to $A.(X)$ in terms of $\underline{m}$ and of the Chern classes of $\Omega^1_{X/S}$ and L. De Jonquières did it for X a curve. For $t = 1$ and X a curve or a surface, A. Lascoux related $p(m; c.; \lambda)$ to Thom polynomials [15]. We will retrieve several classical formulas of Enumerative Geometry by computing the approrpiate $p(\underline{m}; c., \lambda)$

8. More examples.

Return to the notation of §4. We restate (4.1).

(8.1) PROPOSITION. *Let Y be a smooth projective variety of dimension n. Let S be a linear system associated to an invertible $\mathcal{O}_Y$-module M. Let D denote its universal divisor. Suppose D is $\underline{m}$-regular. If*

$$\dim S + tn = \sum \binom{m_i + n - 1}{n}$$

then $J(\underline{m}; D)$ is finite and the degree $|\underline{m}; M|$ of its zero cycle is $|\underline{m}; M| = \int c_{tn}\mathcal{E}_Y(\underline{m}; M)$.

PROOF. See the proof of (4.1). $\square$

(8.1.1) PROBLEM. Is $|\underline{m}; M| = |\pi\underline{m}; M|$ for $n > 1$? (See (4.4.3)(5).)

In the first two examples below we take Y to be a surface. We freely use the notation of (4.3).

(8.2) *Bitangent planes of a surface in $\mathbf{P}^3$.* Assume $\dim S = 2$. Then the formula for $\int [J(2, 2; D)]$ is $[\int (\chi + 2KM + 3M^2)]^2 - \int (7\chi + 6K^2 + 39KM + 42M^2)$. This is derived from (8.1) and (3.5) by pushing down $c_4\mathcal{E}_Y(2, 2; M)$ from $A(Y\{2\})$ to $A(Y \times Y)$ and then to $A(Y)$. The main ingredients of the computation are:

(a) The formula

$$\Omega^1_{Y\{2\}/Y} = p^*_{2,2}\Omega^1 \otimes \mathcal{O}(E) + \mathcal{O}(-E) - \mathcal{O} \quad \text{in } K(Y\{2\})$$

for the class of the cotangent sheaf of $Y\{2\}/Y$ (via $p_{2,1}$, we recall). This formula follows from the two standard exact sequences,

$$b^*_2\Omega^1_{Y\times Y/Y} \subset \Omega^1_{Y\{2\}/Y} \to \Omega^1_{Y\{2\}/Y\times Y} = j_*\Omega^1_{E/\Delta},$$

and

$$\Omega_{E/\Delta} \subset \Omega_Y(-1) \twoheadrightarrow \mathcal{O}_E,$$

where $j\colon E \hookrightarrow Y\{2\}$ is the inclusion of the exceptional divisor. (It is worth recalling the well-known facts of a blowup such as b_2: first, $E = \mathbf{P}(\Omega_Y^1)$; second, the tautological ample sheaf $\mathcal{O}_E(1)$ on E is equal to $j^*\mathcal{O}_{Y\{2\}}(-E)$.)

(b) The intersection relations

$$E^i = j_*((-e)^{i-1}) \quad \text{in } A(Y\{2\})$$

where we put $e := c_1\mathcal{O}_E(1)$. We also have $e^2 = Ke - \chi$, $(b_2 j)_*(1) = 0$ and $(b_2 j)_*(e) = 1_{A(Y)}$.

In particular, if Y is a surface in $\mathbf{P}^3$ of degree d and S is a net of plane sections, we have

$$(8.2.1) \qquad \int \chi = (d^2 - 4d + 6)d,$$

$$\int K^2 = (d-4)^2 d; \quad \int M^2 = d; \quad \int KM = d(d-4).$$

Substituting these in the formula above, we get

$$d(d-1)(d-2)(d^3 - d^2 + d - 12),$$

the expression which is twice the number of bitangent planes through a general point (cf. Baker [2, p. 153]).

If Y is the projective plane and S is a net of curves of degree d, the number $|2, 2;\ M|$ is $9d^4 - 36d^3 + 12d^2 + 81d - 66$. This is twice the degree of the "... conditions nécessaires pour qu'une courbe plane puisse avoir deux points doubles" (cf. S. Roberts [20, p. 276]).

(8.3) *Tritangent planes of a surface in* $\mathbf{P}^3$. Assume dim $S = 3$. A rather lengthy calculation yields the following expression for $|2, 2, 2;\ M|$:

$$\left[\int (\chi + 2KM + 3M^2)\right]^3 - 14\left[\int (\chi + 2KM + 3M^2)\right]^2$$

$$- \int (\chi + 2KM + 3M^2)\int (7\chi + 18K^2 + 84M^2 + 89KM)$$

$$+ \int (138\chi + 376K^2 + 1380M^2 + 1576KM).$$

If Y is $\mathbf{P}^2$ and S is a general web of cubics, so that we have $\int \chi = 3$, $\int M^2 = \int K^2 = 9$, $\int MK = -9$, the formula gives the number $6 \cdot 15$. One can check directly that 15 is precisely the number of triangles containing 6 general points.

If Y is a surface in $\mathbf{P}^3$ of degree d and S is the complete system of plane sections, substituting in the values for χ, K^2, etc. computed in (8.2.1), we get

$$d^9 - 6d^8 + 15d^7 - 59d^6 + 204d^5 - 339d^4 + 770d^3 - 2056d^2 + 1920d,$$

which is 6 times the classical formula for the number of tritangent planes (cf. Salmon [21, Formula (vi), p. 292]). We recall that the factor $6 = 3!$ comes in

because $J(2, 2, 2; D)$ accounts for the divisor together with an ordered triplet of singular points. It can be checked that most surfaces of degree $\geqslant 3$ possess exactly the prescribed number of tritangent planes, though this does not follow from our general criterion (see (9.1) below).

(8.4) *Contacts of higher-dimensional linear spaces with a hypersurface*. The difficulties with the explicit computations of formulas increase rapidly. Conceptually, however, this is a special case of the situation for divisors with specified singularities on a smooth family. One takes $X \to S$ to be the universal family of r-subspaces in a fixed projective n-space $\mathbf{P}^n$, or its restriction to a suitable subvariety of the Grassmannian of r-spaces in $\mathbf{P}^n$. Each hypersurface H in $\mathbf{P}^n$ of degree d induces a subscheme $D_H \subset X$, given by the zeros of a section of $L := \mathcal{O}_X(d)$. The fibre of D_H over $[r]$ in S is the intersection $H \cap [r]$. In this situation, we can again play the game of successive blowing-ups and compute the generic class of $J(\underline{m}; D_H)$.

The reader is invited to carry out the computations leading to the formula below.

(8.4.1) *Number of 2-planes in $\mathbf{P}^4$ meeting a line and cutting a general hypersurface of degree $d \geqslant 4$ in a curve with one triple point and another double point*:

$$20d^7 - 161d^6 + 510d^5 - 1{,}617d^4 + 6{,}176d^3 - 12{,}916d^2 + 10{,}080d.$$

That is, the degree of the zero cycle $c_9(\mathcal{E}_{X/S}(3, 2; \mathcal{O}_X(d)))\sigma$, where X/S is the universal family of 2-planes in $\mathbf{P}^4$ and σ denotes the special Schubert cycle of 2-planes meeting a line.

(8.5) *Degree of the condition that a surface in $\mathbf{P}^3$ acquire 2 double points*. Take $Y := \mathbf{P}^3$, $M := \mathcal{O}_{\mathbf{P}^3}(d)$, $S :=$ complete system of surfaces of degree d. For $d \geqslant 3$ we know (9.1)(3) D is (2, 2)-regular. The subvariety of S parametrizing the surfaces with (at least) 2 (possibly infinitely near) double points is integral and of codimension 2. Its degree is, for char $k \neq 2$,

$$\frac{1}{2} \int c_6 \mathcal{E}_Y(2, 2; M) = 2(d - 1)^2(d - 2)(4d^3 - 8d^2 + 8d - 25),$$

in agreement with S. Roberts [20, top of p. 277].

(8.6) *Curves with specified coincidences of tangents at a singularity*. Let $f: X \to S$ be proper and smooth of relative dimension 2. Let $D \subset X$ be the scheme of zeros of a section of an invertible $\mathcal{O}_X$-module L.

We have defined, for each positive integer m, a closed subscheme $J := J(m; D)$ of X, which parametrizes the points x of X such that the fibre D_s of D over $s = f(x)$ contains x as an m-fold point. "In general", there should be m distinct tangent directions to D_s at x. Now, for each sequence $\underline{n} := n_1, \ldots, n_t$ of positive integers such that $n_1 + \cdots + n_t \leqslant m$, one may ask for the generic homology class of the set of x for which there are tangent directions $\tau_1, \ldots, \tau_t$ at x (in the surface X_s) such that n_1 of the tangents to D_s at x coincide with τ_1, n_2 with τ_2, etc. We discuss here the general setup, then "compute" the generic homology class we sought, and finally compute it explicitly in a few cases.

Consider the m-virtual (projectivized) tangent cone of D, i.e., the intersection $T_m D = D(m) \cap E \subset X\{2\}$ of the m-virtual transform of D (6.3) with the exceptional divisor E. Thus, $T_m D$ is the scheme of zeros of the restriction of the section

s_m of $L(m)_J$ (6.3) to $C := E_J$. Notice that, for each x in J, the fibre E_x is the projective line of tangent directions of the surface X_s at x (where $s = f(x)$). If x is m-fold, not $(m + 1)$-fold in D_s, then $(T_m D)_x$ is the positive divisor of the tangent directions of D_s at x. If x is $(m + 1)$-fold in D_s then $(T_m D)_x = E_s$.

So we are back to the happy situation where we have got a family of curves (in fact projective lines) $C \to J$, together with the scheme of zeros of an invertible $\mathcal{O}_C$-module. We call $J(\underline{n}; T_m D)$ *the scheme of tangent coincidences of type n of D*. We say D is $(m; \underline{n})$-*regular* if D is m-regular and $T_m D$ is $\underline{n}$-regular.

Applying (7.3) twice, we get, whenever D is $(m; \underline{n})$-regular,

$$(8.6.1) \quad [J(\underline{n}; T_m D)] = c_\nu(\mathcal{E}_{X/S}(m; L) + \mathcal{E}_{E/X}(\underline{n}; L(m)_E)) \quad \text{in } A.(E'),$$

where $\nu := \binom{m+1}{2} + \Sigma\, n_i$.

Suppose now D is the universal divisor of a linear system S on a smooth, projective surface Y associated to the invertible $\mathcal{O}_Y$-module M. Set $\mathbf{P} = \mathbf{P}(\Omega^1_Y)$. Suppose

$$\dim S = \binom{m+1}{2} - 2 - t + \sum_1^t n_i.$$

If D is $(m; \underline{n})$-regular then $J(\underline{n}; T_m D)$ is finite and the degree of its zero cycle is the degree of

$$(8.6.2) \qquad\qquad c_{t+2}(\mathcal{E}_Y(m; M) + M\mathcal{E}_{\mathbf{P}/Y}(\underline{n}; \mathcal{O}_P(m))).$$

This follows from (8.6.1), recalling that now $L = M \otimes \mathcal{O}_S(1)$, $X = Y \times S$ and $E = \mathbf{P} \times S$.

(8.6.3) *Cusps of a general net on a surface.* Taking $m = 2$ and $\underline{n} = 2$ in the above formula and pushing down to $A(Y)$, we find $2(6M^2 + 6MK + K^2 + \chi)$ (cf. Lascoux [15, p. 151]).

(8.6.4) *Cuspnodes of a general web on a surface.* We start with our surface Y and a general web S (= 3-dimensional linear system). Construct $Z := J(2; D) \subset X = S \times Y$. Then look at the restriction $B \to Z$ of $X\{2\} = S \times Y\{2\}$ over Z, together with the 2-virtual transform D' of the universal divisor of S. Finally, take $T := J(2; T_2 D')$. This gives us the cuspidal points of the fibres of D' over Z, which is what we were after. Computing the class of T and pushing it down from $S \times \mathbf{P}$ to $\mathbf{P} := \mathbf{P}(\Omega^1_{Y\{2\}/Y})$, then to $Y\{2\}$, then to $Y \times Y$ and finally to Y, we find, for the number of cuspnodes, the formula

$$2\left\{\left[\int (\chi + 2KM + 3M^2)\right]^2 + \int (K^2 + 4KM + 3M^2) \int (\chi + 2KM + 3M^2)\right.$$

$$\left. - \int (12\chi + 36K^2 + 144KM + 120M^2)\right\}.$$

Computing for $Y = \mathbf{P}^2$ and $M = \mathcal{O}_{\mathbf{P}^2}(3)$ one finds zero. This can be checked directly by analyzing the possible degenerations of a cubic. The only ones with a double point *and* a cusp are the unions of double lines with another line. But these form a family of dimension 4, which can, therefore, be safely avoided by a general web of plane cubics.

Computing for Y = surface in $\mathbf{P}^3$ of degree d and $M = \mathcal{O}_Y(1)$, subsituting in the values for χ, etc. from (8.2.1), we get, lo and behold, precisely the number $4d(d - 2)(d - 3)(d^3 + 3d - 16)$ of Salmon [21, Formula (4), p. 292].

9. Conditions for $\underline{m}$-regularity. Fix a smooth, surjective and proper map $g\colon Y \to Z$ and let M be an invertible $\mathcal{O}_Y$-module. Suppose $V := g_* M$ is locally free. Set $S := \mathbf{P}(V^*)$ and $X := S \times_Z Y = \mathbf{P}(V_Y^*)$. Let $f\colon X \to S$ be the pullback of g. The natural map of $\mathcal{O}_Y$-modules $V_Y \to M$ gives rise to a section s of $L := \mathcal{O}_X(1) \otimes M$ just as in §4. Again we denote by D the scheme of zeros of s in X and call it *the universal divisor* of M.

We perform the constructions of (6.1) with $Y \to Z$ in place of $X \to S$. Denote by $q_{t,1}, q_{t,2}$ the maps $Y\{t\} \to Y\{t - 1\}$ of (6.1). Clearly, the maps $p_{t,i}\colon X\{t\} \to X\{t - 1\}$ are just the pullbacks of $q_{t,i}$, and each $X\{t\}$ is now the projective bundle $\mathbf{P}(V_{Y\{t\}}^*)$.

For the statement of the theorem, we recall (6.1.1) that $M(\underline{m})$ stands for the successive twistings of M by powers of the ideals of the exceptional divisors. Thus, $M(m_1) = (q_{2,2}^* M) \otimes \mathcal{O}_2(m_1)$, etc . . . .

(9.1) THEOREM. (1) *Suppose* $(R^1 q_{t+1,1})_* M(\underline{m}) = 0$. *Then* $J(\underline{m}; D)$ *is a projective* $Y\{t\}$-*subbundle of* $X\{t\} = \mathbf{P}(V_{Y\{t\}}^*)$ *of the right codimension*

$$\left(= \sum \binom{m_i + n - 1}{n}, \ n := \mathrm{rel\ dim}\ Y/Z \right)$$

and D *is* $\underline{m}$-*regular.*

(2) $(R^1 q_{t+1,1})_* M(\underline{m}) = 0$ *provided that either*
(a) $\underline{m}$ *satisfies the relaxed proximity inequalities*

$$(9.1.1) \qquad m_{i-1} \geqslant m_i + \cdots + m_t - 1 \quad \textit{for}\ i = 2, \ldots, t,$$

and M *is a sufficiently high multiple of some* $\mathcal{O}_Y$-*module* N *ample*$/Z$; *or*
(b) *the relative dimension of* Y/Z *is* 1 *and*

$$\mathrm{degree}\ M_{Y(z)} \geqslant 2(\mathrm{genus}\ Y(z)) - 2 + \sum m_i$$

for all z *in* Z.

(3) *If* $Y \to Z$ *is a projective bundle (e.g.* $Z = \mathrm{Spec}\ k$, $Y = \mathbf{P}^n$) *and* M *is the* rth *power of the canonical ample line bundle then* D *is* $\underline{m}$ *regular provided* $r > (\sum m_i) - 1$ *holds and* $\underline{m}$ *satisfies* (9.1.1).

(9.2) LEMMA. *Suppose* $(R^1 q_{2,1})_* M(m) = 0$ *for some positive integer* m. *Then*:
(1) $R^1 g_* M = 0$;
(2) $V := g_* M$ *and* $V_m := (q_{2,1})_* M(m)$ *are locally free and there is a canonical exact sequence*

$$(9.2.1) \qquad 0 \to V_m \to g^* V \to \mathcal{E}_{Y/Z}(m; M) \to 0$$

such that the dual surjection $g^* V^* \to V_m^*$ *yields the identification*

$$J(m; D) = \mathbf{P}(V_m^*) \subset \mathbf{P}(g^* V^*) = X;$$

(3) *the* m-*virtual transform* (6.3) *of the universal divisor of* M *is equal to the universal divisor of* $M(m)$.

Grant the lemma. We proceed to prove (1) of the theorem. Set $\underline{m}_i := m_1, \ldots, m_{i-1}$. Applying (1) of the lemma to $q_{t,1}$, $M(\underline{m}_t)$ and m_t in place of g, M and m, we get that $(R^1 q_{t,1})_* M(\underline{m}_t) = 0$. Repeating the process, we get down to $(R^1 q_{2,1})_* M(m_1) = 0$. Hence $J(m_1; D)$ and $\mathbf{P}(V_{m_1}^*)$ are equal. Now replace g, M and $\underline{m}$ by $q_{2,1}$, $M(m_1)$ and $m_2, \ldots, m_t$. We have proven, by induction on t, that $J(\underline{m}; D)$ is equal to the $Y\{t\}$-projective subbundle $\mathbf{P}(V_{\underline{m}}^*) \subset X\{t\}$, where $V_{\underline{m}} := (q_{t+1,1})_* M(\underline{m})$ is locally free and fits into the exact sequence

$$0 \to V_{\underline{m}} \to q_{t,1}^* V_{\underline{m}_t} \to \mathcal{E}_{Y\{t\}/Y\{t-1\}}(m_t; M(\underline{m}_t)) \to 0.$$

This completes the proof of (1) of the theorem.

PROOF OF LEMMA (9.2). We apply $(q_{2,1})_*$ to the canonical exact sequence

$$0 \to M(m) \to q_{2,2}^* M \to q_{2,2}^* M \otimes \mathcal{O}_{mE} \to 0.$$

We get the exact sequence (9.2.1), together with the formula

$$\left(R^1 q_{2,1}\right)_* q_{2,2}^* M = 0.$$

However, the left-hand side is equal to $(R^1 q_2)_* q_1^* M$. By flat base change, the latter is just $g^*(R^1 g_* M)$. Since g is faithfully flat, we get $R^1 g_* M = 0$.

$$
\begin{array}{ccccc}
 & q_{2,1} & Y\{2\} & q_{2,2} & \\
 & \swarrow & & \searrow & \\
Y & \overset{q_1}{\leftarrow} & Y \times_Z Y & \overset{q_2}{\to} & Y \\
 & \searrow & & \swarrow & \\
 & g & Z & g &
\end{array}
$$

Twisting (9.2.1) by $\mathcal{O}_{\mathbf{P}(V^*)}(1)$ we get the diagram of $\mathcal{O}_X$-modules

$$\mathcal{O}_X$$
$$\downarrow \quad \overset{s}{\searrow}$$
$$0 \to V_m(1) \to V_X(1) \to \mathcal{E}_{X/S}(m; L) \to 0.$$

Here the vertical arrow arises from the universal quotient $u: V_X^* \to \mathcal{O}_X(1)$ by dualizing and twisting. The slant arrow s is just the m-contact section (3.3). It follows that, for any map $T \to X$, $s_T = 0$ holds iff u_T factors through $V_T^* \to (V_m^*)_T$. This proves (2) of the lemma. The remaining assertion follows from the definition (6.3) together with the fact that the formation of $g_* M$ commutes with base change.

The main ingredient of the proof of (2) of the theorem is the following.

(9.3) LEMMA (OF PROXIMITY INEQUALITIES). *Fix a regular scheme* B. *Set* $B_1 := B$, *and for* $i \geqslant 2$ *let* $b_i: B_i + B_{i-1}$ *denote the blowup of* B_{i-1} *at a point* y_{i-1}. *Set* $b := b_2 \ldots b_{t+1}$. *For each* $\mathcal{O}_{B_{t+1}}$-*module* $\mathcal{C}$, *set*

$$\mathcal{C}(\underline{m}) := \mathcal{O}_{B_{t+1}}(m_t) \otimes b_{t+1}^* \mathcal{O}_{B_t}(m_{t-1}) \otimes \cdots \otimes b^* \mathcal{O}_{B_2}(m_1) \otimes \mathcal{C}.$$

Suppose $\underline{m}$ *satisfies the inequalities* (9.2.1). *Then* $(R^1 b)_* \mathcal{O}(\underline{m}) = 0$.

PROOF. The lemma is an immediate consequence of the following assertion. Let $\mathfrak{M}_j$ denote the ideal of a point z_j in B_{t+1}. For each sequence of nonnegative integers $\underline{n} := n_1, \ldots, n_s$, set $\mathfrak{M}^{\underline{n}} := \mathfrak{M}_1^{n_1} \ldots \mathfrak{M}_s^{n_s}$ (product of ideals). Then

(9.3.1) $(R^1 b)_*(\mathfrak{M}^{\underline{n}}(\underline{m})) = 0,$

provided the new sequences $m_1, \ldots, m_t$, $n_1 + \cdots + n_s$ satisfy the inequalities (9.1.1).

We proceed for the proof of (9.3.1) by induction on t. Suppose $t = 1$. For each j such that z_j is not in the exceptional divisior $E := b^{-1}(y_1)$, we have

$$R^i b_*(\mathfrak{M}^n(\underline{m})) = \mathfrak{M}_j^n R^i b_*(\mathfrak{M}^{n_j}(\underline{m}))$$

where $\underline{n_j} := n_1, \ldots, n_{j-1}, 0, n_{j+1}, \ldots, n_s$. Thus, we may assume each z_j lies in E. Without loss of generality, we may assume the z_j are distinct. Now, there is an exact sequence,

(9.3.2) $$0 \to \mathfrak{M}^{n-1}(1) \to \mathfrak{M}^n \to \mathfrak{M}^n \mathcal{O}_E \to 0,$$

where $\underline{1}$ denotes the sequence equal to 1 on each slot where n_j is > 0 and zero otherwise. Tensoring it with $\mathcal{O}_{B_2}(\underline{m})$ and applying b_*, we get the exact sequence

$$R^1 b_*(\mathfrak{M}^{n-1}(\underline{m} + 1)) \to R^1 b_*(\mathfrak{M}^n(\underline{m})) \to R^1 b_*(\mathfrak{M}^n \mathcal{O}_E(\underline{m})).$$

By induction on $\max\{n_1, \ldots, n_s\}$, the 1st term is zero. The last term is just $H^1(E, \mathfrak{M}^n \mathcal{O}_E(\underline{m}))$, which is zero for $m \geq n_1 + \cdots + n_s - 1$ because E is a projective space. This finishes the proof for $t = 1$.

Assume $t \geq 2$. Set $b' := b_2 \ldots b_t$. Thus, $b = b' b_{t+1}$. We study the exact sequence (derived from the spectral sequence of composite functors)

$$R^1 b'_* (b_{t+1})_*(A) \to R^1 b_*(A) \to b'_*\left[(R^1 b_{t+1})_* A \right],$$

where we set for short $A := \mathfrak{M}^n(\underline{m})$. By the projection formula, we have

$$(R^1 b_{t+1})_*(A) = \mathcal{O}(\underline{m_t})(R^1 b_{t+1})_*(\mathfrak{M}^n(m_t)).$$

This is zero by the case $t = 1$. Now reorder the z_j so that z_j is in E_{t+1} iff $r \leq j \leq s$. Set $\underline{n}' = n_1, \ldots, n_{r-1}$ and $\underline{n}'' = n_r, \ldots, n_s$. We have the exact sequence,

$$0 \to (\mathfrak{M}^{n'}(1))\mathfrak{M}^{n''-1} \to \mathfrak{M}^n \to \mathfrak{M}^{n''} \mathcal{O}_{E_{t+1}} \to 0.$$

Twist it by $\mathcal{O}_{B_{t+1}}(m_t)$ and apply $(b_{t+1})_*$. By the case $t = 1$, we get a short exact sequence of direct images. Twisting this by $\mathcal{O}(\underline{m_t})$ and applying b'_* finally yields the exact sequence

$$R^1 b'_*(\mathcal{O}(\underline{m_t})b_{t+1*}((\mathfrak{M}^{n'}(m_t + 1))\mathfrak{M}^{n''-1}))$$

$$\to R^1 b'_*(\mathcal{O}(\underline{m_t})b_{t+1*}((\mathfrak{M}^{n'}(m_t))\mathfrak{M}^{n''})) \to 0.$$

The last zero is right because $(b_{t+1})_*(\mathfrak{M}^{n''} \mathcal{O}_{E_{t+1}})$ is supported at one point. Arguing by induction on $\max(\underline{n}'')$, we deduce that the middle term is also zero, thus completing the proof of (9.3).

PROOF OF (2) OF THE THEOREM. Let y be a point of $Y\{t\}$, and let y_i denote its image in $Y\{i\}$ (via $q_{i+1,1} \ldots q_{t+1,1}$). Let B_i denote the fibre of $Y\{i\}$ over y_{i-1}. Thus, we have the sequence of blowings-up

$$B_{t+1} \overset{b_{t+1}}{\to} B_t \overset{b_t}{\to} \ldots \overset{b_2}{\to} B_1 = Y_{y_0},$$

where B_1 is the fibre of $Y \to Z$ containing y_1 and b_{i+1} is the blowup of B_i at y_i. Set $b := b_2 \ldots b_{t+1}$ and set $N_1 := N_{B_1}$.

By the Principle of Exchange, it suffices to prove $H^1(B_{t+1}, N_1^{\otimes \nu}(\underline{m})) = 0$ holds for $\nu \gg 0$. For this, we use the exact sequence, derived from Leray's spectral sequence,

$$H^1\left(B_1, N_1^{\otimes \nu} \otimes b_* \mathcal{O}(\underline{m})\right) \to H^1\left(B_{t+1}, N_1^{\otimes \nu}(\underline{m})\right) \to H^0\left(B_1, N_1^{\otimes \nu} \otimes R^1 b_* \mathcal{O}(\underline{m})\right).$$

By Lemma (9.3), the last term vanishes (for all ν). On the other hand, by ampleness, the 1st term is zero for $\nu \gg 0$. This proves (2)(a). If rel dim Y/Z is 1, then B_1 is a curve, each b_i is the identity map and $\mathcal{O}(\underline{m})$ can be identified with $\mathcal{O}_{B_1}(-m_1 y_1 - \cdots - m_t y_t)$. This proves (2)(b).

PROOF OF (3) OF THE THEOREM. Preserving the notation of the proof of (2), we now have $B_1 = \mathbf{P}^n$ for some n, and $N_1 = \mathcal{O}_{\mathbf{P}^n}(1)$. It can be shown that $b_* \mathcal{O}(\underline{m})$ contains an $\mathcal{O}_{B_1}$-module A of the form $A = \mathfrak{M}_1^{n_1} \ldots \mathfrak{M}_s^{n_s}$, where each $\mathfrak{M}_i$ is the ideal of a point P_i and $\sum n_i \leqslant \sum m_i$ and $\text{supp}(b_* \mathcal{O}(\underline{m})/A)$ is finite. Therefore, $H^1(B_1, N_1^r b_* \mathcal{O}(\underline{m}))$ vanishes provided $H^1(\mathbf{P}^n, A(r)) = 0$. The latter holds iff the natural map $H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(r)) \to H^0(\mathbf{P}^n, \mathcal{O}/A)$ is surjective. The surjectivity amounts to the possibility of finding nonhomogeneous polynomials of degree $< r$ in n variables with arbitrarily given Taylor developments around P_i up to order $n_i - 1$, which can be achieved for $r \geqslant -1 + \sum n_i$. $\square$

PROBLEM (1) Suppose Y is the normalization of a plane curve. Consider the linear system cut out by the plane curves of degree n. When is its universal divisor $\underline{m}$-regular?

(2) Get rid of the artificial condition (9.1.1).

10. Conditions for multiplicity 1. Let Y be a smooth projective variety of dimension n. Let S be an r-dimensional linear system on Y. Set for short $J := J(\underline{m}; D)$ ($D =$ universal divisor of S). Set

$$j := r + tn - \sum \binom{m_i + n - 1}{n}.$$

Let G denote the Grassmann variety of codimension j linear subsystems of S. We have the following.

(10.1) THEOREM. *Suppose J is integral and of the right dimension j. Then there exist integers s, e such that, for every subsystem S' in an open dense subset of G, $J \times_S S'$ has precisely s distinct points and the multiplicity at each is 1 if J/S is separable and is $(\text{char } k)^e$ otherwise.*

PROOF. We use a variation of the theorem on the transversality of a general translate (cf. Kleiman [14]). Consider the diagram

$$
\begin{array}{ccc}
\bar{J} & \to & J \\
q \swarrow \quad \downarrow \bar{p} & & \downarrow p \\
G \leftarrow \bar{S} & \to & S,
\end{array}
$$

where $\bar{S} \to G$ is the universal family and the square is cartesian. Since J is integral and $\bar{S} \to S$ is smooth with integral fibres, therefore $\bar{J}$ is integral. By construction, $q^{-1}(S') = J \times_S S'$ for each S' in G. Also, dim $\bar{J} = $ dim G. Moreover, p is everywhere ramified iff $\bar{p}$ is. Hence, by the lemma on p. 109 of [22], there are integers s

and e such that, for each S' in some open dense subset of G, $q^{-1}(S')$ consists of s distinct points and the multiplicity ($=$ length of the artinian local ring of the fibre) at each is 1 or (char $k)^e$, as asserted.

It remains to produce some manageable criterion for J/S to be separable. In one direction, if S comes from the hyperplane sections of an embedding $Y \subset \mathbf{P}'$, then $J(2; D) \to S$ is the dual map of Katz [11]. He shows that, possibly after replacing the given embedding by its composition with a Veronese embedding, the dual map is generically unramified iff char $k \neq 2$ or dim Y is even. On the other hand, we learned from D. Laksov [secret notes] the following result. We no longer restrict S to be a projective space.

(10.2) PROPOSITION. *Suppose the relative dimension of X/S is 1. Then the ramification locus of $J(m; D)/S$ is equal to $J(m + 1; D)$ if* char $k \nmid m$ *and is* $J(m; D)$ *if* char $k \mid m$.

PROOF. We may assume $X = \operatorname{Spec} B$, $S = \operatorname{Spec} A$, $L = \mathcal{O}_X$ and $\Omega^1_{B/A} = B\,dx$ for some x in B. Let F be a local equation of D. As explained in (3.5.1), there are differential operators $\partial_i : B \to B$ such that $J := J(m; D)$ is cut out by the m equations $\partial_i F = 0$ ($i = 0, \ldots, m - 1$). Furthermore, we have $\partial_0 b = b$ and $db = (\partial_1 b)\,dx$ for all b in B, and $\partial_1 \partial_i = (i + 1)\partial_i$ ([EGA IV$_4$, 16.11.2.2]). Recalling the standard exact sequence [EGA IV$_4$, 16.4.21.1] $N_{J/X} \to (\Omega^1_{X/S})_J \to \Omega^1_{J/S} \to 0$, we see that the ramification locus of J/S is cut out in J by the equations $\partial_1 \partial_i F = 0$ ($i = 0, \ldots, m - 1$). These are equivalent to $\partial_i F = 0$ ($i = 0, \ldots, m - 1$) together with $m\partial_m F = 0$, whence the assertion is proved. $\square$

(10.3) COROLLARY. (1) *If* char $k \mid m_1 \ldots m_t$ *then the ramification locus of* $J(\underline{m}; D)/S$ *is* $J(\underline{m}; D)$. (2) *If* char $k \nmid m_1 \ldots m_t$ *then that ramification locus is contained in the union of the closed subsets* $J(m_1, \ldots, m_i + 1, \ldots, m_t; D)$ *and the diagonals.*

PROOF. The map $J(\underline{m}; D) \to S$ factors as

$$J(\underline{m}; D) \xrightarrow{f_t} \ldots \to J(m_2; D(m_1)) \xrightarrow{f_2} J(m_1; D) \xrightarrow{f_1} S.$$

Now, arguing as in the proof of (4.4.3)(5), if π denotes a permutation of $\{1, \ldots, t\}$, then $J(\underline{m}; D)$ is isomorphic to $J(\pi\underline{m}; D)$ over S, where $\pi\underline{m} := m_{\pi 1}, \ldots, m_{\pi t}$. Thus, if char $k \mid m_i$ for some i, we may as well assume $i = t$. By the proposition, f_t is everywhere ramified, whence so is $f_1 \ldots f_t$.

To prove the 2nd assertion, let x be a point of $J(\underline{m}; D)$ off the above-mentioned union. Thus, $x = (x_1, \ldots, x_t)$, where the x_i are distinct points of X lying over the same point s in S, with $D_s = \sum_1^t m_h x_h + D'$, for some $D' \geqslant 0$ and no $x_i \in D'$. Consequently, each f_i unramified at $(x_1, \ldots, x_i)$, whence $f_1 \ldots f_t$ is unramified at x. $\square$

(10.4) PROBLEM. Conics in char 2 have just one tangent line through a general point; smooth plane cubics have, in general, just 3 inflexional tangent lines. Is the number of bitangent lines to a general quartic in char 2 equal to $14 = (28/2)$ or 7? A similar question applies to example (5.1) in char 5. Is there a general rule for deciding the value of the exponent e in (10.1)?

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