UNIFORM PARTITIONS OF AN INTERVAL

BY

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ABSTRACT. Let $\{x_n\}$ be a sequence of numbers in $[0, 1]$; for each n let $u_0(n), \dots, u_n(n)$ be the lengths of the intervals resulting from partitioning of $[0, 1]$ by $\{x_1, x_2, \dots, x_n\}$. For $p > 1$ put $A^{(p)}(n) = (n+1)^{p-1} \sum_0^n [u_j(n)]^p$; the paper investigates the behavior of $A^{(p)}(n)$ as $n \rightarrow \infty$ for various sequences $\{x_n\}$. **THEOREM 1.** If $x_n = n\theta \pmod{1}$ for an irrational $\theta > 0$, then $\liminf A^{(p)}(n) < \infty$. However $\limsup A^{(p)} < \infty$ if and only if the partial quotients of θ are bounded (in the continued fraction expansion of θ). **THEOREM 2** gives the exact values for $\liminf$ and $\limsup$ when $\theta = \frac{1}{2}(1 + \sqrt{5})$. **THEOREM 3.** If x_n 's are random variables, uniformly distributed on $[0, 1]$, then $\lim A^{(p)}(n) = \Gamma(p+1)$ almost surely.

1. Introduction. Let $x_1, x_2, \dots$ be an infinite sequence of points between 0 and 1. For each n the points $x_1, x_2, \dots, x_n$ partition the interval $[0, 1]$ into $n+1$ subintervals. Extensive studies have been made of irregularities of such partitions by considering the quantity D_n , called discrepancy, defined by

$$D_n = \sup_{0 \leq \alpha < \beta \leq 1} \left| \frac{1}{n} \sum_{j=1}^n \chi_{(\alpha, \beta)}(x_j) - (\beta - \alpha) \right|,$$

where $\chi_E(\cdot)$ is the characteristic function of a set E . (See [6].) In this paper we propose to study the problem by introducing a different measure of uniformity defined as follows. For each n , let $x_1(n), x_2(n), \dots, x_n(n)$ be the points $x_1, x_2, \dots, x_n$ arranged in nondecreasing order, let $x_0(n) \equiv 0$, $x_{n+1}(n) \equiv 1$, $u_j(n) = x_{j+1}(n) - x_j(n)$ ($j = 0, 1, \dots, n$), and for each $p > 1$ consider

$$(1) \quad A^{(p)}(n) = (n+1)^{p-1} \sum_{j=0}^n [u_j(n)]^p.$$

The closer to 1 the value of $A^{(p)}(n)$ is, the more uniform is the partition (if the points $x_1, x_2, \dots, x_n$ divide $[0, 1]$ into $n+1$ equal parts then $A^{(p)}(n) = 1$). In this paper we investigate $\lim_n A^{(p)}(n)$ for various sequences $x_1, x_2, \dots$. We begin with the classical case $x_n = n\theta \pmod{1}$ for some irrational $\theta > 0$. It turns out that the limiting behavior of $A^{(p)}(n)$ strongly depends on the arithmetic character of θ . We have the following theorem.

THEOREM 1. Let $x_n = n\theta \pmod{1}$ for an irrational $\theta > 0$ and let $A^{(p)}(n)$ be defined by (1). We have for $p > 1$:

I. $\liminf_n A^{(p)}(n) < \infty$;

II. $\limsup_n A^{(p)}(n) < \infty$ if and only if the partial quotients of the continued fraction expansion of θ are bounded.

Received by the editors June 23, 1980 and, in revised form, September 29, 1980.

AMS (MOS) subject classifications (1970). Primary 10K30.

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0002-9947/81/0000-0507/\$03.50

For certain values of θ both $\limsup A^{(p)}(n)$ and $\liminf A^{(p)}(n)$ can be evaluated as the following theorem shows:

THEOREM 2. Let $\theta = \frac{1}{2}(1 + \sqrt{5})$ so that $\theta^2 = \theta + 1$. Let

$$\psi(t) = 5^{-p/2}(\theta + t)^{p-1} [t(\theta - 1)^p + (1 - t)\theta^p + t + \theta - 1].$$

Let $A^{(p)}(n)$ be obtained from the sequence $n\theta \pmod{1}$ as in (1). We have

$$\liminf_n A^{(p)}(n) = \psi(0) = \psi(1) = 5^{-p/2}(\theta^{2p-1} + \theta^{p-2}),$$

$$\limsup_n A^{(p)}(n) = \psi(t_0),$$

where

$$t_0 = (1 - p^{-1})(\theta^p - (\theta - 1)^p - 1) / (\theta^p + \theta - 1).$$

Special cases of Theorems 1 and 2 ($p = 3$) appear in [2]. We discuss next the behavior of $A^{(p)}(n)$ in case the sequence $\{x_n\}$ is chosen at random.

THEOREM 3. Let $X_1, X_2, \dots$ be a sequence of independent random variables uniformly distributed on $[0, 1]$, and let $A^{(p)}(n)$ be the random variable defined by (1). Then $\lim_n A^{(p)}(n) = \Gamma(p + 1)$ almost surely.

We introduce the following definition.

DEFINITION. Let $\{x_n\}$ be a sequence of numbers in an interval $[0, 1]$ and let $A^{(p)}(n)$ be given by (1). We say that this sequence p -partitions $[0, 1]$ if $\lim_n A^{(p)}(n)$ exists.

COROLLARY. For every $p > 1$ there is a sequence which p -partitions $[0, 1]$.

This is immediate from Theorem 3. We now proceed with the proofs of the theorems.

2. Proof of Theorems 1 and 2. We summarize first the basic facts about the distribution of the points $\{\theta\}, \{2\theta\}, \dots, \{n\theta\}$ in $[0, 1]$. (Here $\{t\} = t \pmod{1}$.) For the details and references see [7]. Let n be fixed, let $1 \leq a_n \leq n$ be such that $\{a_n\theta\}$ is the smallest among $\{\theta\}, \{2\theta\}, \dots, \{n\theta\}$ and let $1 \leq b_n \leq n$ be such that $\{b_n\theta\}$ is the largest. Set $\alpha_n = \{a_n\theta\}$, $\beta_n = 1 - \{b_n\theta\}$. The interval $[0, 1]$ is divided by $\{\theta\}, \{2\theta\}, \dots, \{n\theta\}$ into $n + 1$ subintervals as follows: $n + 1 - a_n$ of them are of length α_n , $a_n + b_n - (n + 1)$ of them are of length $\alpha_n + \beta_n$ and $n + 1 - b_n$ have length β_n . The fact that $n + 1 \leq a_n + b_n$ can be deduced from the definitions of a_n and b_n . Thus with this notation,

$$(2) \quad A^{(p)}(n) = (n + 1)^{p-1} [(n + 1 - a_n)\alpha_n^p + (a_n + b_n - n - 1)(\alpha_n + \beta_n)^p + (n + 1 - b_n)\beta_n^p].$$

One can actually find a_n and b_n in terms of the continued fraction expansion of θ :

$$\theta = [d_0; d_1, d_2, \dots] = d_0 + \frac{1}{d_1 + \frac{1}{d_2 + \dots}}.$$

As usual, we set $q_{-1} = 0, p_{-1} = 1, q_0 = 1, p_0 = d_0, q_{k+1} = d_{k+1}q_k + q_{k-1}, p_{k+1} = d_{k+1}p_k + p_{k-1}, \delta_k = (-1)^k(q_k\theta - p_k) > 0$. Given n , to find a_n and α_n express n as

$$(3) \quad n = q_{2m} + rq_{2m+1} + s, \quad 0 \leq r < d_{2m+2}, \quad 0 \leq s < q_{2m+1},$$

so that $q_{2m} \leq n < q_{2m+2}$. Then

$$(4) \quad a_n = q_{2m} + rq_{2m+1}, \quad \alpha_n = \delta_{2m} - r\delta_{2m+1}.$$

To find b_n and β_n we express n as

$$(5) \quad n = q_{2m-1} + uq_{2m} + v, \quad 0 \leq u < d_{2m+1}, \quad 0 \leq v < q_{2m},$$

so that $q_{2m-1} \leq n < q_{2m+1}$. We have

$$(6) \quad b_n = q_{2m-1} + uq_{2m}, \quad \beta_n = \delta_{2m-1} - u\delta_{2m}.$$

The following are standard facts about continued fractions (see [4]):

$$(7) \quad d_{k+1} = [\delta_{k-1}/\delta_k], \quad \delta_{k+1} = \delta_{k-1} - d_{k+1}\delta_k, \\ d_{k+2}/q_{k+2} < \delta_k < 1/q_{k+1}.$$

We are now ready to prove Theorem 1. Set $x_n = n\theta \pmod{1}, p > 1$ and let

$$A(n) = A^{(p)}(n) = (n+1)^{p-1} \sum_{j=0}^n [u_j(n)]^p$$

be given as in (1). To show I we will show that $A(q_{2m} + q_{2m+1} - 1)$ is bounded. From the discussion above and (2) we see that for $n = q_{2m} + q_{2m+1} - 1$ the following hold:

$$a_n = q_{2m}, \quad b_n = q_{2m+1}, \quad \alpha_n = \delta_{2m}, \quad \beta_n = \delta_{2m+1},$$

and using (7) we get

$$A(n) = (q_{2m} + q_{2m+1})^{p-1} (q_{2m}\delta_{2m+1}^p + q_{2m+1}\delta_{2m}^p) \\ \leq (q_{2m} + q_{2m+1})^{p-1} (q_{2m}q_{2m+2}^{-p} + q_{2m+1}q_{2m+1}^{-p}) \\ = O(1) \quad (m \rightarrow \infty).$$

Hence I follows. To show II assume that all the partial quotients d_k of θ are bounded by D , say. We wish to show that $A(n)$ is bounded. We claim that a number $c > 0$ can be chosen such that the following three inequalities hold:

$$(8) \quad c^{-1} < \delta_{k-1}/\delta_k < c, \quad c^{-1} < a_n/b_n < c, \quad c^{-1} < \alpha_n/\beta_n < c.$$

The first inequality holds for some $c > 0$ because $[\delta_{k-1}/\delta_k] = d_{k+1}$ and we are assuming that d 's are bounded. Suppose next that $q_{2m} \leq n < q_{2m+1}$ for some m so that $a_n = q_{2m}$ (see (4)). Let u and v be determined by (5) so that

$$\frac{a_n}{b_n} = \frac{q_{2m}}{uq_{2m} + q_{2m-1}} \leq \frac{q_{2m}}{q_{2m-1}} = \frac{d_{2m}q_{2m-1} + q_{2m-2}}{q_{2m-1}} \\ \leq d_{2m} + 1 \leq D + 1.$$

On the other hand,

$$\frac{a_n}{b_n} \geq \frac{q_{2m}}{d_{2m+1}q_{2m} + q_{2m-1}} \geq \frac{1}{d_{2m+1} + 1} \geq \frac{1}{D + 1}.$$

If n satisfies $q_{2m-1} \leq n < q_{2m}$ then the analysis is based on (5) and (3) and leads to the same conclusion. As for the ratio of α_n and β_n we have the following: For $q_{2m} \leq n < q_{2m+1}$ and u given by (5),

$$1 \leq \frac{\alpha_n}{\beta_n} = \frac{\delta_{2m}}{\delta_{2m-1} - u\delta_{2m}} \leq \frac{\delta_{2m}}{\delta_{2m+1}} < d_{2m+2} + 1 \leq D + 1.$$

For $q_{2m-1} \leq n < q_{2m}$ and r given by (3),

$$1 \leq \frac{\beta_n}{\alpha_n} = \frac{\delta_{2m-1}}{\delta_{2m-2} - r\delta_{2m-1}} \leq \frac{\delta_{2m-1}}{\delta_{2m}} \leq d_{2m+1} + 1 \leq D + 1.$$

This establishes (8). To show now that $A(n)$ is bounded, we take (2) and bound all the terms by $a_n\beta_n$ using (8):

$$\begin{aligned} A(n) &\leq (a_n + b_n)^{p-1} [b_n\alpha_n^p + (b_n - 1)(\alpha_n + \beta_n)^p + a_n\beta_n^p] \\ &\leq M(a_n\beta_n)^p \end{aligned}$$

where M depends only on the constant c from (8) (and hence on $\max d_k$). Since $a_n\beta_n + b_n\alpha_n = 1$, the first part of II follows. We next show the converse of II, that is, if the partial quotients are unbounded then $\limsup A(n) = \infty$.

There are two cases: either $\{d_{2m}\}$ is unbounded or $\{d_{2m+1}\}$ is unbounded. We present the arguments in the first case only, the second is completely analogous. For each m let $y_m = [d_{2m+2}/2] - 2$, so that $y_m > 0$ for infinitely many m 's and $\limsup y_m = \infty$. Let

$$\begin{aligned} n &= n_m = q_{2m} + (y_m + 1)q_{2m+1} - 1 \\ &= q_{2m+1} + (q_{2m} + y_m q_{2m+1} - 1). \end{aligned}$$

For those m 's for which $y_m > 1$ we have from (3)–(6),

$$\begin{aligned} a &= q_{2m} + y_m q_{2m+1}, \quad \alpha = \delta_{2m} - y_m \delta_{2m+1} > y_m \delta_{2m+1}, \\ b &= q_{2m+1}, \quad \beta = \delta_{2m+1}, \quad a + b = n + 1. \end{aligned}$$

Thus, substituting in (2),

$$\begin{aligned} A(n_m) &= (q_{2m} + (y_m + 1)q_{2m+1})^{p-1} (q_{2m+1}\alpha^p + (q_{2m} + y_m q_{2m+1})\beta^p) \\ &\geq (y_m q_{2m+1})^{p-1} (q_{2m+1} y_m^p \delta_{2m+1}^p) = y_m^{2p-1} q_{2m+1}^p \delta_{2m+1}^p. \end{aligned}$$

It follows from (7) that

$$\begin{aligned} (q_{2m+1}\delta_{2m+1})^p &\geq \left[\frac{q_{2m+1}d_{2m+3}}{d_{2m+3}q_{2m+2} + q_{2m+1}} \right]^p \geq \left[\frac{q_{2m+1}}{q_{2m+2} + q_{2m+1}} \right]^p \\ &= \left[\frac{q_{2m+1}}{d_{2m+2}q_{2m+1} + q_{2m} + q_{2m+1}} \right]^p > (d_{2m+2} + 3)^{-p}. \end{aligned}$$

Thus $A(n_m) \geq y_m^{2p-1} (d_{2m+2} + 3)^{-p}$, so $\limsup A(n) = \infty$.

This completes the proof of Theorem 1 and we take up Theorem 2. For $\theta = \frac{1}{2}(1 + \sqrt{5})$ all partial quotients are equal to 1 and we have

$$\begin{aligned} (9) \quad q_{-1} &= 0, \quad q_0 = 1, \quad q_{k+1} = q_k + q_{k-1}, \\ p_{-1} &= 1, \quad p_0 = 1, \quad p_{k+1} = p_k + p_{k-1}, \\ \delta_{k+1} &= \delta_{k-1} - \delta_k, \quad q_{k-1} = 5^{-1/2}(\theta^k - (-1)^k \theta^{-k}), \quad \delta_k = \theta^{-(k+1)}, \end{aligned}$$

so that

$$(10) \quad \lim_k (q_k/\theta^k) = \theta/\sqrt{5}.$$

Let $0 \leq t \leq 1$ be given, and suppose $0 < t_k < 1$ are such that $t_k q_{2k-1} - 1$ is a positive integer and $t_k \rightarrow t$. We will show that if $n_k = q_{2k} + t_k q_{2k-1} - 1$ then

$$(11) \quad \lim_k A(n_k) = 5^{-p/2}(\theta + t)^{p-1} [t(\theta - 1)^p + (1 - t)\theta^p + t + \theta - 1] \\ = \psi(t).$$

Similarly, if $0 < s_k < 1$ is such that $s_k q_{2k} - 1$ is a positive integer and $s_k \rightarrow t$, then with $n_k = q_{2k+1} + s_k q_{2k} - 1$,

$$(12) \quad \lim_k A(n_k) = \psi(t).$$

To show (11) we have from (3)–(6):

$$a = q_{2k}, \quad \alpha = \delta_{2k}, \quad b = q_{2k-1}, \quad \beta = \delta_{2k-1}, \quad n + 1 - a = t_k q_{2k-1}, \\ a + b - (n + 1) = (1 - t_k) q_{2k-1}, \quad n + 1 - b = q_{2k} + (t_k - 1) q_{2k-1}.$$

Substituting these values into (2) we get

$$A(n_k) = (q_{2k} + t_k q_{2k-1})^{p-1} [t_k q_{2k} \delta_{2k}^p + (1 - t_k) q_{2k-1} (\delta_{2k} + \delta_{2k-1})^p \\ + (q_{2k} + (t_k - 1) q_{2k-1}) \delta_{2k-1}^p].$$

Substituting values for δ 's from (9) we obtain

$$A(n_k) = (q_{2k} + t_k q_{2k-1})^{p-1} [t_k q_{2k-1} \theta^{-(2kp+p)} + (1 - t_k) q_{2k-1} (1 + \theta^{-1})^p \theta^{-2kp} \\ + (q_{2k} + (t_k - 1) q_{2k-1}) \theta^{-2kp}].$$

From (10) it follows then that

$$\lim_k A(n_k) = 5^{-p/2}(\theta + t)^{p-1} [t/\theta^p + (1 - t)(1 + \theta^{-1})^p + \theta + t - 1],$$

which implies (11) since $1/\theta = \theta - 1$ and $1 + 1/\theta = \theta$. Equation (12) follows pretty much the same way. Let now $\{n_j\}$ be such that $A(n_j)$ converges to ξ , say. Clearly n_j belongs infinitely often to an interval of the form $[q_{2k+1}, q_{2k+1})$ or infinitely often to an interval of the form $[q_{2k+1}, q_{2k+2})$. In the first case $n_j = q_{2k} + t_k q_{2k-1} - 1$ for some $0 < t_k < 1$ and $k = k(j)$, depending on j ; in the second case $n_j = q_{2k+1} + s_k q_{2k} - 1$, $0 < s_k < 1$, $k = k(j)$. By taking subsequences, if needed, we may assume that t_k (or s_k) converges to t , say. Thus in both cases $\lim_j A(n_j) = \xi = \psi(t)$ for some $0 \leq t \leq 1$. Hence $\limsup A(n)$ and $\liminf A(n)$ are, respectively, the maximum and the minimum of $\psi(t)$ for $0 \leq t \leq 1$. By direct calculation we can obtain that

$$(13) \quad \psi(0) = \psi(1) = 5^{-p/2}(\theta^{2p-1} + \theta^{p-2}).$$

The simplification is based on the fact that $\theta^2 = \theta + 1$. Also

$$(14) \quad 5^{p/2} \psi'(t) = (d/dt)(\theta + t)^{p-1} [t(-\theta^p + (\theta - 1)^p + 1) + \theta^p + \theta - 1] \\ = (d/dt)(\theta + t)^{p-1} [Et + f] \\ = (\theta + t)^{p-2} [(p - 1)(Et + F) + E(\theta + t)]; \\ E = -\theta^p + (\theta - 1)^p + 1, \quad F = \theta^p + \theta - 1.$$

Solving the equation $\psi'(t) = 0$ gives the only solution between 0 and 1:

$$t_0 = (1 - p^{-1})(\theta^p - (\theta - 1)^p - 1) / (\theta^p + \theta - 1).$$

To finish the proof we will show that $\psi'(0) > 0$, which is certainly sufficient since $\psi(0) = \psi(1)$. Now, from (14),

$$\begin{aligned} 5^{p/2}\psi'(0) &= \theta^{p-2}[(p-1)(\theta^p + \theta - 1) + \theta(-\theta^p + (\theta - 1)^p + 1)] \\ &= \theta^{p-2}f(p). \end{aligned}$$

Thus it is enough to show that $f(p) > 0$ for $p > 1$. Direct calculation gives $f(1) = 0$ and

$$\begin{aligned} f'(p) &= (p-1)\theta^p \log \theta + \theta^p + \theta - 1 \\ &\quad + \theta[-\theta^p \log \theta + (\theta - 1)^p \log(\theta - 1)] \\ &= \theta^p[(p-1)\log \theta + 1 - \theta \log \theta] - (\log \theta)/\theta^{p-1} + \theta - 1. \end{aligned}$$

Since $1 - \theta \log \theta = 0.221 \dots$, $f'(p)$ is an increasing function for $p > 1$. Also,

$$f'(1) = \theta(1 - \theta \log \theta) - \log \theta + \theta - 1 = 0.495 \dots$$

so that $f(p)$ is positive for $p > 1$. The proof of Theorem 2 is now complete.

3. Proof of Theorem 3. Since p is going to be fixed throughout, we will write A_n for $A^{(p)}(n)$. The basic tool to be used is the martingale convergence theorem: Let F_n be an increasing sequence of σ -fields, Z_n a random variable measurable with respect to F_n . If $E(Z_{n+1}|F_n) = Z_n$ and $\sup_n E(|Z_n|) < \infty$, then the sequence Z_n converges almost surely. $E(Z|F)$ is the conditional expectation of Z relative to F . (See J. L. Doob [1] for the details.) We let $(X_1^{(n)}, \dots, X_n^{(n)})$ be the order statistic of size n , i.e. the values of $X_1, X_2, \dots, X_n$ arranged in increasing order, put $X_0^{(n)} \equiv 0$, $X_{n+1}^{(n)} \equiv 1$ and introduce random variables $U_j(n) = X_{j+1}^{(n)} - X_j^{(n)}$ so that once again

$$A_n = (n+1)^{p-1} \sum_{j=0}^n [U_j(n)]^p$$

is a random variable. We take our σ -fields to be $F_n = F(U_0, U_1, \dots, U_n)$, the σ fields generated by the random variables $U_0, U_1, \dots, U_n$ and consider the random variable

$$Z_n = A_n + \sum_{j=1}^{n-1} [A_j - E(A_{j+1}|F_j)].$$

We will show the following:

- (a) $\{Z_n\}$ is a martingale relative to $\{F_n\}$.
- (b) $\sum_{n=1}^{\infty} E(|A_n - E(A_{n+1}|F_n)|) < \infty$.
- (c) $\lim_n E(|A_n|) = \Gamma(p+1)$.

Note that (b), (c) imply $\limsup E(|Z_n|) < \infty$ so that Z_n converges a.e. by the martingale convergence theorem. In addition, (b) shows $\sum [A_n - E(A_{n+1}|F_n)]$ converges a.e. and thus A_n converges a.e. The limit will be identified later. The proof of (a) is straightforward:

$$\begin{aligned} Z_{n+1} &= Z_n + A_{n+1} - A_n + A_n - E(A_{n+1}|F_n) \\ &= Z_n + A_{n+1} - E(A_{n+1}|F_n) \end{aligned}$$

so that $E(Z_{n+1}|F_n) = Z_n$.

Before we take up (b) we recall facts regarding random variables $U_0(n), U_1(n), \dots, U_n(n)$ (see [5, Chapter 9]). Since $U_0(n) + \dots + U_n(n) \equiv 1$, the U 's are certainly not independent, but if we delete one of them, the remaining n are "uniformly" distributed on the simplex

$$T_n = \{(t_1, t_2, \dots, t_n): t_j \geq 0, t_1 + t_2 + \dots + t_n \leq 1\};$$

more precisely, the joint density function of the remaining n is given by

$$f_n(t_1, t_2, \dots, t_n) = \begin{cases} n! & \text{if } (t_1, t_2, \dots, t_n) \in T_n, \\ 0 & \text{otherwise.} \end{cases}$$

Thus for any $\alpha > 0$ and any i ,

$$(15) \quad E([U_i(n)]^\alpha) = n! \int_{T_n} x_i^\alpha dx_1 dx_2 \dots dx_n = \frac{n! \Gamma(\alpha + 1)}{\Gamma(n + \alpha + 1)}.$$

Similarly for any $\alpha > 0, \beta > 0, i \neq j$,

$$(16) \quad E([U_i(n)]^\alpha [U_j(n)]^\beta) = n! \int_{T_n} x_i^\alpha x_j^\beta dx_1 \dots dx_n \\ = n! \Gamma(\alpha + 1) \Gamma(\beta + 1) / \Gamma(n + \alpha + \beta + 1).$$

The values of these integrals can be either evaluated directly or looked up in [3]. Notice that the right-hand side of both of the above formulas is independent of i and j .

To prove (b) we use the Cauchy-Schwarz inequality $E(|W|) \leq [E(W^2)]^{1/2}$ with $W = E(A_{n+1}|F_n) - A_n$ and show that for some constant $c(p)$, depending only on p , we have

$$E([E(A_{n+1}|F_n) - A_n]^2) \leq c(p)n^{-3}.$$

This will certainly prove (b) since $\sum n^{-3/2}$ converges. We derive now the formula for $E(A_{n+1}|F_n)$. Since X_{n+1} is uniformly distributed on $[0, 1]$ and independent of $U_0, \dots, U_n$ we have

$$\begin{aligned} E(A_{n+1}|U_0(n) = u_0, U_1(n) = u_1, \dots, U_n(n) = u_n) \\ &= (n+2)^{p-1} \sum_{j=0}^n \int_0^{u_j} \left[\sum_{\substack{i=0 \\ i \neq j}}^n u_i^p + t^p + (u_j - t)^p \right] dt \\ &= (n+2)^{p-1} \sum_{j=0}^n \left[\sum_{\substack{i=0 \\ i \neq j}}^n (u_i^p u_j) + \frac{2}{p+1} u_j^{p+1} \right] \\ &= (n+2)^{p-1} \left[\frac{1}{(n+1)^{p-1}} A_n - \sum_{j=0}^n \left(1 - \frac{2}{p+1} \right) u_j^{p+1} \right] \\ &= A_n + \left[\frac{(n+2)^{p-1} - (n+1)^{p-1}}{(n+1)^{p-1}} \right] A_n - (n+2)^{p-1} \frac{p-1}{p+1} \sum_{j=0}^n u_j^{p+1} \\ &= A_n + [(n+2)^{p-1} - (n+1)^{p-1}] \sum_{j=0}^n u_j^p - (n+2)^{p-1} \frac{p-1}{p+1} \sum_{j=0}^n u_j^{p+1}. \end{aligned}$$

Thus

$$(17) \quad E(A_{n+1}|F_n) - A_n = \sum_{j=0}^n [a_n U_j^p(n) - b_n U_j^{p+1}(n)],$$

where

$$a_n = (n+2)^{p-1} - (n+1)^{p-1} = (p-1)n^{p-2}(1+o(1))$$

and

$$b_n = (n+2)^{p-1} \frac{p-1}{p+2} = n^{p-1} \frac{p-1}{p+1} (1+o(1)).$$

In view of remarks after (16) we get (writing U_k for $U_k(n)$)

$$\begin{aligned} E([E(A_{n+1}|F_n) - A_n]^2) &= E\left(\left[\sum_{j=0}^n a_n U_j^p - b_n U_j^{p+1}\right]^2\right) \\ &= nE([a_n U_0^p - b_n U_0^{p+1}]^2) \\ &\quad + n(n+1)E([a_n U_0^p - b_n U_0^{p+1}][a_n U_1^p - b_n U_1^{p+1}]) \\ &= n[a_n^2 E(U_0^{2p}) - 2a_n b_n E(U_0^{2p+1}) + b_n^2 E(U_0^{2p+2})] \\ &\quad + n(n+1)[a_n^2 E(U_0^p U_1^p) - a_n b_n \{E(U_0^p U_1^{p+1}) + E(U_0^{p+1} U_1^p)\} \\ &\quad \quad \quad + b_n^2 E(U_0^{p+1} U_1^{p+1})] \\ &= nP_n + n(n+1)Q_n. \end{aligned}$$

We will show that both nP_n and $n(n+1)Q_n$ are $O(n^{-3})$, the implicit constant depending on p only. Before we do that we need the following estimate:

$$(18) \quad n!/\Gamma(n+2p+1) \leq C_1(p)n^{-2p},$$

$C_1(p)$ depending on p alone. Indeed, using Stirling's formula

$$\log \Gamma(x) = (x - \frac{1}{2})\log x - x + \log \sqrt{2\pi} + o(1)$$

we get

$$\begin{aligned} \log(n!/\Gamma(n+2p+1)) &= \log \Gamma(n+1) - \log \Gamma(n+2p+1) \\ &= (n + \frac{1}{2})\log(n+1) - (n+1) - (n+2p + \frac{1}{2})\log(n+2p+1) \\ &\quad + n+2p+1 + O(1) \\ &= n[\log(n+1) - \log(n+2p+1)] \\ &\quad + \frac{1}{2}[\log(n+1) - \log(n+2p+1)] - 2p\log n + O(1) \end{aligned}$$

Since for any fixed d , $x(\log(x+d) - \log x) \rightarrow d$ ($x \rightarrow \infty$), the result follows. We now estimate nP_n . From the definition of P_n and (15)–(17) we have

$$\begin{aligned} nP_n &= n!n \left[(p-1)^2 n^{2p-4} \frac{\Gamma(2p+1)}{\Gamma(n+2p+1)} - \frac{(p-1)^2}{p+1} n^{2p-3} \right. \\ &\quad \left. \times \frac{\Gamma(2p+2)}{\Gamma(n+2p+2)} + \left(\frac{p-1}{p+1}\right)^2 n^{2p-2} \frac{\Gamma(2p+3)}{\Gamma(n+2p+3)} \right] (1+o(1)). \end{aligned}$$

Using the identity $x\Gamma(x) = \Gamma(x+1)$ several times we get

$$nP_n = \frac{n^{2p-3}(p-1)^2 n!}{\Gamma(n+2p+1)} \Gamma(2p+1) \\ \times \left[1 - \frac{2p+1}{p+1} \frac{n}{n+2p+1} + \frac{(2p+1)n^2}{(p+1)(n+2p+1)(n+2p+2)} \right] (1+o(1)),$$

so the estimate $nP_n = O(n^{-3})$ follows from (18). Next we estimate $n(n+1)Q_n$, again using (15)–(17):

$$n(n+1)Q_n = n^2 n! \left[(p-1)^2 n^{2p-4} \frac{\Gamma^2(p+1)}{\Gamma(n+2p+1)} \right. \\ \left. - 2 \frac{(p-1)^2}{p+1} n^{2p-3} \frac{\Gamma(p+1)\Gamma(p+2)}{\Gamma(n+2p+2)} \right. \\ \left. + \left(\frac{p-1}{p+1} \right)^2 n^{2p-2} \frac{\Gamma^2(p+2)}{\Gamma(n+2p+3)} \right] (1+o(1)) \\ = \frac{n! n^{2p-2} (p-1)^2 \Gamma^2(p+1)}{\Gamma(n+2p+1)} \\ \times \left[1 - \frac{2n}{2p+n+1} + \frac{n^2}{(n+2p+1)(n+2p+2)} \right] (1+o(1)).$$

The expression in square brackets is equal to

$$(-n + (2p+1)(2p+2)) / (n+2p+1)(n+2p+2) = O(n^{-1}).$$

Hence it follows from (18) that

$$n(n+1)Q_n = C(p) \frac{n! n^{2p-3}}{\Gamma(n+2p+1)} (1+o(1)) = O(n^{-3}).$$

Thus (b) is proved.

To show (c) we evaluate $E(A_n)$ directly from (15):

$$E(A_n) = (n+1)^{p-1} \sum_{j=0}^n E(U_n^p(j)) = \frac{(n+1)^{p-1} n n! \Gamma(p+1)}{\Gamma(n+p+1)}.$$

We show now that $\gamma_n = (n+1)^{p-1} n n! / \Gamma(n+p+1) \rightarrow 1$ ($n \rightarrow \infty$): Clearly $\gamma_n \sim \beta_n = n^p n! / \Gamma(n+p+1)$, so using Stirling's formula,

$$\log \beta_n = p \log n + \left(n + \frac{1}{2}\right) \log(n+1) - (n+1) + \frac{1}{2} \log(2\pi) \\ - \left(n + p + \frac{1}{2}\right) \log(n+p+1) + n + p + 1 - \frac{1}{2} \log(2\pi) + o(1) \\ = n[\log(n+1) - \log(n+1+p)] + p + o(1).$$

Again, $x[\log(x+d) - \log(x)] \rightarrow d$ ($x \rightarrow \infty$), so $\log \beta_n \rightarrow 0$ ($n \rightarrow \infty$), proving the assertion. Thus $\lim_n E(A_n) = \Gamma(p+1)$. This completes the proof of (a)–(c) and shows that $A_n^{(p)}$ converges almost surely. What remains is the identification of the limit. Since $E(A_n) \rightarrow \Gamma(p+1)$ it is reasonable to expect that $A_n \rightarrow \Gamma(p+1)$ since

the limit should be constant a.e. To establish it rigorously we show that $A_n \rightarrow \Gamma(p+1)$ in probability. This proof is due to Professor Boris Pittel. Let $Y_0, Y_1, \dots$ be a sequence of exponentially distributed independent random variables, so that $P(Y_s < t) = 1 - e^{-t}$. Let $S_n = Y_0 + Y_1 + \dots + Y_n$. It is known that the vectors $(U_0(n), U_1(n), \dots, U_n(n))$ and $(Y_0/S_n, Y_1/S_n, \dots, Y_n/S_n)$ have the same distribution (see [5, p. 242]). Therefore the distributions of A_n and

$$(n+1)^{p-1} \left[\sum_{j=0}^n Y_j^p \right] / S_n^p$$

are also the same. By the strong law of large numbers,

$$\frac{(n+1)^{p-1} \sum_{j=0}^n Y_j^p}{S_n^p} = \frac{(n+1)^{-1} \sum_{j=0}^n Y_j^p}{(S_n/n+1)^p} \rightarrow \frac{E(Y_0^p)}{(E(Y_0))^p} = \Gamma(p+1)$$

almost everywhere, and thus in probability. Hence $A_n^{(p)}$ also converges to $\Gamma(p+1)$ in probability. The proof of Theorem 3 is thus completed.

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