THE DROR-WHITEHEAD THEOREM IN PRO-HOMOTOPY AND SHAPE THEORIES

BY

S. SINGH

ABSTRACT. Many analogues of the classical Whitehead theorem from homotopy theory are now available in pro-homotopy and shape theories. E. Dror has significantly extended the homology version of the Whitehead theorem from the well-known simply connected case to the more general, for instance, nilpotent case. We prove a full analogue of Dror's theorems in pro-homotopy and shape theories. More specifically, suppose $f: \underline{X} \rightarrow \underline{Y}$ is a morphism in the pro-homotopy category of pointed and connected topological spaces which induces isomorphisms of the integral homology pro-groups. Then f induces isomorphisms of the homotopy pro-groups, for instance, when $\underline{X}$ and $\underline{Y}$ are simple, nilpotent, complete, or $\underline{H}$ -objects; these notions are well known in homotopy theory and we have naturally extended them to pro-homotopy and shape theories.

0. Introduction. The purpose of this paper is to establish an analogue of Dror's generalization of the Whitehead theorem (see [DR]), stated below as the Dror-Whitehead theorem, in the context of the pro-homotopy and shape theories. We shall elaborate on these matters in the next few paragraphs.

THE DROR-WHITEHEAD THEOREM. *Suppose a map $f: X \rightarrow Y$ induces isomorphisms of all the homology groups of spaces X and Y with integral coefficients. Then f induces isomorphisms of all the homotopy groups if and only if $\Gamma_\omega \pi_* f$ is an epimorphism, $\Gamma'_\omega \pi_* f$ is a monomorphism, and $\Gamma \pi_* f$ is a monomorphism.*

The functors Γ_ω , Γ'_ω , and Γ are defined by considering the action of the fundamental group on the homotopy groups; see [DR] or §2 of this paper. An important class of spaces to which the Dror-Whitehead theorem applies is nilpotent spaces; see [DR], [HI], [BK] or §3 of this paper.

Theorem (4.1.3) of this paper is our extension of the Dror-Whitehead theorem to pro-homotopy and shape theories; also, see Theorems (4.1.1), (4.1.2), (4.1.3), (4.5.1), and Corollary (4.3). Our Corollary (4.3) extends a theorem of Raussen [RA] in the same manner as Dror extends the Whitehead theorem. A parallel development of pivotal ingredients of "pro-algebra" is provided by [SI₁] which extends the work of Stallings [ST] and Dror [DR] concerning algebra; and our entire program is a natural extension of Dror's work.

As a concluding remark, we may add that many analogues of the various versions of the classical Whitehead theorem have been studied in pro-homotopy

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and shape theories; for instance, see [AM], [DS], [EH], [MA], [MO], [MR], [RA] where many other references and related discussions may also be found.

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1. Notation and terminology.

(1.1) *Category theoretic conventions.* A diagram in a category is said to commute if every square or a triangle in the diagram commutes whenever appropriate. All categories will be denoted by script letters. By a map we mean a morphism in a category of interest which will be clear from the context. Suppose $F: \mathcal{D} \rightarrow \mathcal{E}$ is a functor between the categories $\mathcal{D}$ and $\mathcal{E}$. If $f: X \rightarrow Y$ is a map in $\mathcal{D}$, we denote the corresponding map by $Ff: FX \rightarrow FY$, rather than $F(f): F(X) \rightarrow F(Y)$, i.e., we omit the cumbersome round brackets whenever convenient. The zero object in any category, whenever it exists, will be denoted by 0. For any category $\mathcal{D}$ we let $\text{pro-}\mathcal{D}$ denote the well-known pro-category constructed by Grothendieck [GR]. We shall assume familiarity with the construction of $\text{pro-}\mathcal{D}$ and related matters; see, for instance, [AM], [DS], [MA] for a relevant discussion.

(1.2) *Other conventions.* Our main reference concerning homotopy theory is [WH]. We let $\mathbb{Z}$ denote the additive group of integers. The basepoints will often be suppressed. For other related references and terminology, one may also consult [SI₁]. All the homology groups of spaces are the singular homology groups.

2. Algebraic preliminaries.

(2.1) *Group actions.* Let G and π be groups. We say G is a (left) π -group if there exists a homomorphism $\eta: \pi \rightarrow \text{Aut } G$ into the group of automorphisms of G , or, equivalently, we say G is a (left) π -group if there exists a map $\alpha: \pi \times G \rightarrow G$ satisfying $e \cdot g = g$, $(xy) \cdot g = x \cdot (y \cdot g)$, and $x \cdot (gh) = (x \cdot g)(x \cdot h)$, where e is the identity of π , x, y belong to π , g, h belong to G , and $\alpha(x, g)$ is denoted by $x \cdot g$. The phrase “ G is a π -group” will be replaced by “ G is a π -module” only when G is abelian. Let $\mathcal{G}\mathcal{A}$ denote the category whose objects are the group actions, α 's as above, and whose morphisms are pairs $(\phi, \psi): \alpha \rightarrow \alpha'$ such that $\phi: \pi \times G \rightarrow \pi' \times G'$, $\psi: G \rightarrow G'$, $\psi\alpha = \alpha'\phi$, where α and α' are two objects of $\mathcal{G}\mathcal{A}$.

(2.1.0) *Pro-group actions.* Consider the category $\text{pro-}\mathcal{G}\mathcal{A}$ (cf. [AM], [GR]). Let $\mathcal{G}$ and $\text{pro-}\mathcal{G}$ denote the categories of groups and pro-groups, respectively. Let $\underline{\pi} = (\pi_\lambda, r_{\lambda\lambda'}, \Lambda)$ and $\underline{G} = (G_\lambda, p_{\lambda\lambda'}, \Lambda)$ be two pro-groups. We say $\underline{G}$ is a $\underline{\pi}$ -(pro-group) if for each λ we have an action $\alpha_\lambda: \pi_\lambda \times G_\lambda \rightarrow G_\lambda$ of π_λ on G_λ such that $\underline{\alpha}: \underline{\pi} \times \underline{G} \rightarrow \underline{G}$ is a special morphism in $\text{pro-}\mathcal{G}$; see [MA] for a definition of a special morphism and many other related matters concerning $\text{pro-}\mathcal{G}$. If G_λ is an abelian group, we shall substitute “ $\underline{G}$ is a $\underline{\pi}$ -(pro-module)” for “ $\underline{G}$ is a $\underline{\pi}$ -(pro-group)”.

(2.1.1) *π -lower central series of G .* Let G be a π -group. Put $\Gamma_1 G = G$. Define $\Gamma_2 G$ to be the normal π -subgroup of G generated by elements of the form $(x \cdot g)g^{-1}$ for x in π and g in G . Define $\Gamma_\alpha G = \Gamma_2 \Gamma_{\alpha-1} G$ if α is not a limit ordinal and define $\Gamma_\beta G = \bigcap_{\alpha < \beta} \Gamma_\alpha G$ when β is a limit ordinal. This process defines a *lower central series*

$$(2.1.2) \quad \Gamma_s G: \cdots \subset \Gamma_{\alpha+1} G \subset \Gamma_\alpha G \subset \cdots \subset \Gamma_2 G \subset \Gamma_1 G = G$$

of G which depends on the action of π on G .

If $\underline{G}$ is a π -(pro-group), we have a pro-(lower central series)

$$(2.1.3) \quad \Gamma_s \underline{G}: \cdots \subset \Gamma_{\alpha+1} \underline{G} \subset \Gamma_\alpha \underline{G} \subset \cdots \subset \Gamma_2 \underline{G} \subset \Gamma_1 \underline{G} = \underline{G}$$

of $\underline{G}$ which depends on the action of π on $\underline{G}$.

(2.1.4) *The π -completions of $\underline{G}$.* Let β be a limit ordinal. The π -completion $\hat{\Gamma}_\beta \underline{G}$ of $\underline{G}$ up to β is the inverse limit of the inverse system

$$(2.1.5) \quad \cdots \rightarrow \underline{G}/\Gamma_\alpha \underline{G} \rightarrow \cdots \rightarrow \underline{G}/\Gamma_3 \underline{G} \rightarrow \underline{G}/\Gamma_2 \underline{G} \rightarrow 0$$

where $\alpha < \beta$. The quotient maps of the form $\underline{G} \rightarrow \underline{G}/\Gamma_\alpha \underline{G}$ determine a map $i: \underline{G} \rightarrow \hat{\Gamma}_\beta \underline{G}$ whose kernel is $\Gamma_\beta \underline{G}$ and whose cokernel is denoted by $\Gamma'_\beta \underline{G} = \hat{\Gamma}_\beta \underline{G}/i\underline{G}$. We have an exact sequence

$$(2.1.6) \quad 0 \rightarrow \Gamma_\beta \underline{G} \rightarrow \underline{G} \xrightarrow{i} \hat{\Gamma}_\beta \underline{G} \rightarrow \Gamma'_\beta \underline{G} \rightarrow 0$$

where $\Gamma'_\beta \underline{G}$ is a pointed set if $\underline{G}$ is nonabelian.

Given a π -(pro-group) $\underline{G}$. The π -completion $\hat{\Gamma}_\beta \underline{G} = (\hat{\Gamma}_\beta G_\lambda, \hat{\Gamma}_\beta p_{\lambda\lambda'}, \Lambda)$ of $\underline{G}$ up to a limit ordinal β is defined as the (component-wise) inverse limit of

$$(2.1.7) \quad \cdots \rightarrow \underline{G}/\Gamma_\alpha \underline{G} \rightarrow \cdots \rightarrow \underline{G}/\Gamma_3 \underline{G} \rightarrow \underline{G}/\Gamma_2 \underline{G} \rightarrow 0$$

where $\alpha < \beta$. We have an exact sequence

$$(2.1.8) \quad 0 \rightarrow \Gamma_\beta \underline{G} \rightarrow \underline{G} \xrightarrow{i} \hat{\Gamma}_\beta \underline{G} \rightarrow \hat{\Gamma}'_\beta \underline{G} \rightarrow 0$$

where $\hat{\Gamma}'_\beta \underline{G}$ is, in general, a (pointed) pro-set. It is easy to see that $\Gamma_\beta \underline{G}$ is the kernel of i in $\text{pro-}\mathcal{G}$ and $\hat{\Gamma}'_\beta \underline{G}$ is the cokernel of i if each G_λ is abelian.

(2.2) *The maximal π -perfect subgroup of $\underline{G}$.* A π' -group H is π' -perfect if $\Gamma_2 H = H$. Every π -group $\underline{G}$ contains a unique maximal π -perfect subgroup $\Gamma \underline{G}$ which is the π -subgroup of $\underline{G}$ generated by the family of π -perfect subgroups. Observe that $\Gamma \underline{G} \subset \Gamma_\beta \underline{G}$ for any ordinal β . Also, Γ can be viewed as a functor from $\mathcal{G} \mathcal{Q}$ into $\mathcal{G}$.

Given a π -(pro-group) $\underline{G}$. The pro-group $\Gamma \underline{G} = (\Gamma G_\lambda, \Gamma p_{\lambda\lambda'}, \Lambda)$ will be called the maximal π -perfect pro-subgroup of $\underline{G}$.

(2.3) *The functoriality.* Observe that Γ_s , Γ_β , Γ'_β , and $\hat{\Gamma}_\beta$ are functors defined on $\mathcal{G} \mathcal{Q}$: Our constructions $\Gamma_s \underline{G}$, $\Gamma_\beta \underline{G}$, $\Gamma'_\beta \underline{G}$, and $\hat{\Gamma}_\beta \underline{G}$ are the pro-functors defined on $\text{pro-}\mathcal{G} \mathcal{Q}$ which are the natural extensions of these functors.

3. Nilpotent spaces and related matters.

(3.1) *The action of $\pi_1 X$.* All spaces considered in the sequel are pointed and the basepoint is often suppressed. It is well known that the fundamental group $\pi_1 X$ of a space X acts on the homotopy group $\pi_i X$ where $i = 1, 2, \dots$; and furthermore, this action is natural (cf. [WH]). By " $\pi_i X$ is a $\pi_1 X$ -group" we shall always mean this natural action and we often leave the map, $\pi_1 X \times \pi_i X \rightarrow \pi_i X$, determining this action unlabelled.

(3.1.1) *The case when $i = 1$.* The group $\pi_1 X$ is considered as a $\pi_1 X$ -group, and this action is by conjugation (cf. [WH]); in this case, one obtains the classical lower central series $\Gamma_s \pi_1 X$.

(3.1.2) *The case when $i \geq 2$.* Since $\pi_i X$ is abelian, we often say " $\pi_i X$ is as $\pi_1 X$ -module" or we regard $\pi_i X$ as a module over the integral group ring $\mathbb{Z}\pi_1 X$. The filtration of $\Gamma_s \pi_i X$ is induced by the powers, $(I\pi_1 X)^\alpha$, of the augmentation ideal $I\pi_1 X$ of $\mathbb{Z}\pi_1 X$ for finite α and this can be appropriately interpreted for infinite α .

(3.2) *A pro-homotopy category.* Let $\mathcal{H}\mathcal{T}_0$ denote the homotopy categories of pointed and connected topological spaces. We are interested in the category $\mathcal{C} = \text{pro-}\mathcal{H}\mathcal{T}_0$. Let $\underline{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an object of $\mathcal{C}$. Now, $\pi_i \underline{X}$ for $1 \leq i < \infty$ is a $\pi_1 \underline{X}$ -(pro-group).

(3.2.1) *The case when $i = 1$.* We obtain the pro-(lower central series) $\Gamma_s \pi_1 \underline{X} = (\Gamma_s \pi_1 X_\lambda, \Gamma_s p_{\lambda\lambda'}, \Lambda)$. Observe that the sequence (2.1.8) and $\Gamma \pi_1 \underline{X}$ (see (2.2)) are meaningfully defined in this setting.

(3.2.2) *The case when $i \geq 2$.* The (abelian) pro-group $\pi_i \underline{X}$ is a $\pi_1 \underline{X}$ -module and we obtain the lower central series $\Gamma_s \pi_i \underline{X}$, $\Gamma \pi_i \underline{X}$, and the exact sequence (2.1.8).

Observe that $\Gamma_s \pi_i \underline{X}$, $\Gamma \pi_i \underline{X}$, $\Gamma_\beta \pi_i \underline{X}$, $\Gamma'_\beta \pi_i \underline{X}$, and $\hat{\Gamma}_\beta \pi_i \underline{X}$ are functors for $i \geq 1$; hence, they are invariants of the isomorphism class of $\underline{X}$ in $\mathcal{C}$.

(3.2.3) *Nilpotent objects in $\mathcal{C}$.* An object $\underline{X}$ in $\mathcal{C}$ is *nilpotent in $\mathcal{C}$* if for each $i \geq 1$ there exists an integer $k = k(i)$ such that $\Gamma_k \pi_i \underline{X} \approx 0$ in $\text{pro-}\mathcal{G}$. An object $\underline{X}$ in $\mathcal{C}$ is called *simple in $\mathcal{C}$* if $\Gamma_2 \pi_i \underline{X} \approx 0$ in $\text{pro-}\mathcal{G}$ for all $i \geq 1$.

(3.2.4) *Complete objects in $\mathcal{C}$.* An object $\underline{X}$ is called *complete in $\mathcal{C}$* if the natural map $\pi_i \underline{X} \rightarrow \hat{\Gamma}_\omega \pi_i \underline{X}$ is an isomorphism in $\text{pro-}\mathcal{G}$ for $i \geq 1$.

(3.3) *Some preliminary lemmas.* We assume familiarity with the matters related to the “singular functor” and the “geometric realization functor”; see, for instance, [AM], [BK], [WH]. Let $\underline{f}: \underline{X} \rightarrow \underline{Y}$ be a special map in $\mathcal{C}$. We are only interested in the maps induced by $\underline{f}$ on the homotopy and homology pro-groups; hence, we shall assume without loss of generality that Postnikov decompositions and related constructions can be appropriately applied; see (3.3.2) below and [AM, Chapter 1]. The following extends Lemma (6.1) of [DR].

(3.3.1) **LEMMA.** *In addition, suppose $\pi_i \underline{f}: \pi_i \underline{X} \rightarrow \pi_i \underline{Y}$ is an isomorphism for $0 \leq i \leq (n-1)$, and suppose that $H_i \underline{f}: H_i \underline{X} \rightarrow H_i \underline{Y}$ is an isomorphism for $i = n$ and an epimorphism for $i = n+1$. Then the following conclusions hold:*

(C₁) *In the case $n = 1$, $H_i \underline{f}: H_i \underline{X} \rightarrow H_i \underline{Y}$ is an isomorphism for $i = n$ and an epimorphism for $i = n+1$ (this conclusion merely restates the hypothesis).*

(C_n; $n \geq 2$) *In the case $n \geq 2$, the map $H_j(\pi_1 \underline{X}; \pi_n \underline{X}) \rightarrow H_j(\pi_1 \underline{Y}; \pi_n \underline{Y})$ induced by $\underline{f}$ is an isomorphism for $j = 0$ and an epimorphism for $j = 1$.*

(3.3.2) *Some preliminary discussions.* For each CW-complex X the coskeleton, $P_n X$, is a functor defined on the homotopy category of CW-complexes [AM], [PO]. The following is a characterization of $P_n X$: The homotopy groups of $P_n X$, $n \geq 1$, vanish in dimension $> n$ and the canonical map $X \rightarrow P_n X$ is universal with respect to maps of X into CW-complexes with vanishing homotopy groups in dimensions $> n$; see [AM]. Observe that Artin and Mazur [AM] denote $P_n X$ by $\text{cos } k_{n+1} X$.

(3.3.3) *Proof of Lemma (3.3.1): A sketch.* Consider the system $K(\pi_n \underline{X}, n) \rightarrow P_n \underline{X} \rightarrow P_{n-1} \underline{X}$ of fibrations of the n th stage Postnikov system, i.e., for each λ in Λ , we have a fibration $K(\pi_n X_\lambda, n) \rightarrow P_n X_\lambda \rightarrow P_{n-1} X_\lambda$. Consider the pro-(spectral sequence)

$$(3.3.4) \quad \underline{E}_{p,q}^2 = \{ {}_\lambda E_{p,q}^2 = H_p(P_{n-1} X_\lambda; \{ H_q K(\pi_n X_\lambda, n) \}) \Rightarrow H_{p+q} P_n X_\lambda \}_\lambda,$$

where the bracket around $H_q K(\pi_n X_\lambda, n)$ signifies that the homology is with “twisted coefficients”, i.e., $H_q K(\pi_n X_\lambda, n)$ is considered as a $\pi_1 P_{n-1} X_\lambda$ -module. It follows that for each λ in Λ , the sequence

$$(3.3.5) \quad \begin{cases} H_{n+2} P_n X_\lambda \rightarrow H_{n+2} P_{n-1} X_\lambda \rightarrow H_1(\pi_1 X_\lambda; \pi_n X_\lambda) \rightarrow H_{n+1} P_n X_\lambda \\ \rightarrow H_{n+1} P_{n-1} X_\lambda \rightarrow H_0(\pi_1 X_\lambda; \pi_n X_\lambda) \rightarrow H_n X_\lambda \rightarrow H_n P_{n-1} X_\lambda \rightarrow 0 \end{cases}$$

is exact.

Observe that for each λ in Λ , $H_{n+1} X_\lambda \rightarrow H_{n+1} P_n X_\lambda$ is an epimorphism; hence, $H_{n+1} \underline{X} \rightarrow H_{n+1} P_n \underline{X}$ is an epimorphism. It follows that $H_{n+1} P_n \underline{X} \rightarrow H_{n+1} P_n \underline{Y}$ is an epimorphism since $H_{n+1} \underline{X} \rightarrow H_{n+1} \underline{Y}$ is an epimorphism.

Since (3.3.5) is exact, the corresponding sequences of pro-groups for $\underline{X}$ and $\underline{Y}$ are exact. Form a diagram whose rows are these exact sequences of pro-groups for $\underline{X}$ and $\underline{Y}$ and whose vertical maps are induced by $\underline{f}$. Our proof is finished by the “Five Lemma”; see (3.6.6). $\square$

(3.4) LEMMA. Suppose $\underline{f}: \underline{X} \rightarrow \underline{Y}$ satisfies the hypotheses of Lemma (3.3.1). Then the following conclusion holds:

($\hat{C}_n$; $n \geq 1$) The induced map $\hat{\Gamma}_\omega \pi_n \underline{f}: \hat{\Gamma}_\omega \pi_n \underline{X} \rightarrow \hat{\Gamma}_\omega \pi_n \underline{Y}$ is an isomorphism.

PROOF. Observe that $n \geq 1$ is some fixed integer. We assume (C_1) or (C_n ; $n \geq 2$) holds: In either case, we conclude that the map $\pi_n \underline{X} / \Gamma_i \pi_n \underline{X} \rightarrow \pi_n \underline{Y} / \Gamma_i \pi_n \underline{Y}$, induced by $\underline{f}$ is an isomorphism. This follows from Theorem (3.3.1) or Theorem (4.2) of [SI₁], respectively. This suffices to prove the result. $\square$

(3.4.1) REMARK. In the setting described above, for each $n \geq 1$, the conclusion (C_n) implies ($\hat{C}_n$).

(3.5) LEMMA. Suppose $\underline{f}: \underline{X} \rightarrow \underline{Y}$ satisfies the hypotheses of Lemma (3.3.1). Then the following assertions are equivalent:

(a) The map $\pi_n \underline{f}$ is an isomorphism.

(b) $\Gamma_\omega \pi_n \underline{f}$ is an epimorphism, $\Gamma'_\omega \pi_n \underline{f}$ is a monomorphism, and $\hat{\Gamma}_\omega \pi_n \underline{f}$ is a monomorphism.

PROOF. It suffices to show that (b) implies (a). Consider the commutative diagram

$$(3.5.1) \quad \begin{array}{ccccccccc} 0 & \rightarrow & \Gamma_\omega \pi_n \underline{X} & \rightarrow & \pi_n \underline{X} & \rightarrow & \hat{\Gamma}_\omega \pi_n \underline{X} & \rightarrow & \Gamma'_\omega \pi_n \underline{X} & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \Gamma_\omega \pi_n \underline{Y} & \rightarrow & \pi_n \underline{Y} & \rightarrow & \hat{\Gamma}_\omega \pi_n \underline{Y} & \rightarrow & \Gamma'_\omega \pi_n \underline{Y} & \rightarrow & 0. \end{array}$$

The unlabelled maps $\Gamma_\omega \pi_n \underline{f}$, $\hat{\Gamma}_\omega \pi_n \underline{f}$, and $\Gamma'_\omega \pi_n \underline{f}$ are an epimorphism, an isomorphism, and a monomorphism, respectively. It follows from a “Five Lemma” (this is discussed in (3.6)) that $\pi_n \underline{f}$ is an epimorphism. In the next few paragraphs, we shall show that $\pi_n \underline{f}$ is a monomorphism.

It is easy to see that $\pi_n \underline{X} / \Gamma_\omega \pi_n \underline{X} \rightarrow \pi_n \underline{Y} / \Gamma_\omega \pi_n \underline{Y}$ is an epimorphism since $\pi_n \underline{X} \rightarrow \pi_n \underline{Y}$ is an epimorphism. It follows from (3.5.1) that both $\pi_n \underline{X} / \Gamma_\omega \pi_n \underline{X} \rightarrow \hat{\Gamma}_\omega \pi_n \underline{X}$ and $\pi_n \underline{Y} / \Gamma_\omega \pi_n \underline{Y} \rightarrow \hat{\Gamma}_\omega \pi_n \underline{Y}$ are monomorphisms, and $\hat{\Gamma}_\omega \pi_n \underline{f}$ is an isomorphism by Lemma (3.4). It is now easy to see that the map $\pi_n \underline{X} / \Gamma_\omega \pi_n \underline{X} \rightarrow \pi_n \underline{Y} / \Gamma_\omega \pi_n \underline{Y}$ is a monomorphism; hence, it is an isomorphism. By Lemma (3.3.2) of

[SI₁] and the argument given above at the limit ordinals, the induced map $\pi_n \underline{X} / \Gamma_\alpha \pi_n \underline{X} \rightarrow \pi_n \underline{Y} / \Gamma_\alpha \pi_n \underline{Y}$ is an isomorphism for any ordinal α .

Observe that for cardinality reasons, there exists an ordinal α such that $\Gamma \pi_n \underline{X} = \Gamma_\alpha \pi_n \underline{X}$. Consider a diagram of pro-groups whose rows are $0 \rightarrow \Gamma \pi_n \underline{X} \rightarrow \pi_n \underline{X} \rightarrow \pi_n \underline{X} / \Gamma \pi_n \underline{X} \rightarrow 0$ and consider a similar one for $\underline{Y}$. The vertical maps of this diagram are induced by f . By the "Five Lemma" (see (3.6.6)) it follows that $\pi_n f$ is a monomorphism; hence, $\pi_n f$ is a bimorphism. This proves $\pi_n f$ is an isomorphism; see [MA]. $\square$

(3.6) *The setting for a "Five Lemma".* Given the following diagram

$$(3.6.1) \quad \begin{array}{ccccccccccc} 0 & \rightarrow & \underline{H} & \xrightarrow{j} & \underline{G} & \xrightarrow{i} & \underline{K} & \xrightarrow{f} & \underline{L} & \rightarrow & 0 \\ & & \downarrow \underline{\alpha} & & \downarrow \underline{\beta} & & \downarrow \underline{\gamma} & & \downarrow \underline{\delta} & & \\ 0 & \rightarrow & \underline{H}' & \xrightarrow{j'} & \underline{G}' & \xrightarrow{i'} & \underline{K}' & \xrightarrow{f'} & \underline{L}' & \rightarrow & 0 \end{array}$$

such that for each λ in Λ we have a commutative diagram

$$(3.6.2) \quad \begin{array}{ccccccccccc} 0 & \rightarrow & H_\lambda & \xrightarrow{j_\lambda} & G_\lambda & \xrightarrow{i_\lambda} & K_\lambda & \xrightarrow{f_\lambda} & L_\lambda & \rightarrow & 0 \\ & & \downarrow \alpha_\lambda & & \downarrow \beta_\lambda & & \downarrow \gamma_\lambda & & \downarrow \delta_\lambda & & \\ 0 & \rightarrow & H'_\lambda & \xrightarrow{j'_\lambda} & G'_\lambda & \xrightarrow{i'_\lambda} & K'_\lambda & \xrightarrow{f'_\lambda} & L'_\lambda & \rightarrow & 0 \end{array}$$

such that each row of (3.6.2) is exact; furthermore, the maps $\underline{\alpha}$, $\underline{\beta}$, and $\underline{\gamma}$ are maps of pro-groups in pro- $\mathcal{G}$, and $\underline{\delta}$ is a map of pro-(pointed sets). More specifically, $L_\lambda = K_\lambda / i_\lambda G_\lambda$ and $L'_\lambda = K'_\lambda / i'_\lambda G'_\lambda$ for each λ in Λ ; and the maps f_λ and f'_λ are the quotient maps. Notation:

$$\begin{aligned} \underline{G} &= (G_\lambda, p_{\lambda\lambda'}, \Lambda), & \underline{G}' &= (G'_\lambda, p'_{\lambda\lambda'}, \Lambda), & \underline{K} &= (K_\lambda, q_{\lambda\lambda'}, \Lambda), \\ \underline{K}' &= (K'_\lambda, q'_{\lambda\lambda'}, \Lambda), & \underline{H} &= (H_\lambda, p_{\lambda\lambda'}, \Lambda), & \underline{H}' &= (H'_\lambda, p'_{\lambda\lambda'}, \Lambda), \\ \underline{L} &= (L_\lambda, r_{\lambda\lambda'}, \Lambda) & \text{and} & \underline{L}' &= (L'_\lambda, r'_{\lambda\lambda'}, \Lambda). \end{aligned}$$

Moreover, for each λ in Λ , the homomorphisms j_λ and j'_λ are inclusion; and, therefore, we have again denoted the restrictions of the homomorphisms $p_{\lambda\lambda'}$ and $p'_{\lambda\lambda'}$ by $p_{\lambda\lambda'}$ and $p'_{\lambda\lambda'}$.

(3.6.3) LEMMA ("A FIVE LEMMA"). *In the setting: If $\underline{\alpha}$ is an epimorphism, $\underline{\gamma}$ is an epimorphism, and $\underline{\delta}$ is a monomorphism, then $\underline{\beta}$ is an epimorphism.*

PROOF. Given an index λ in Λ . (I) Since $\underline{\alpha}$ is an epimorphism, there exists $\lambda' \geq \lambda$ such that the image $\text{Im}(H_{\lambda'} \rightarrow H'_\lambda)$ is contained in the image $\text{Im}(H_\lambda \rightarrow H'_\lambda)$. (II) Given $\lambda' \geq \lambda$ as asserted, choose $\mu \geq \lambda'$ such that the induced map between the kernels, $[\text{Ker}(L_\mu \rightarrow L'_\mu) \rightarrow \text{Ker}(L_{\lambda'} \rightarrow L'_{\lambda'})]$, is the zero map; this follows since $\underline{\delta}$ is a monomorphism. (III) Choose $\nu \geq \mu$ such that $\text{Im}(K'_\nu \rightarrow K'_\mu)$ is contained in $\text{Im}(K_\mu \rightarrow K'_\mu)$.

Claim. The image $\text{Im}(G'_\nu \rightarrow G'_\lambda)$ is contained in the image $\text{Im}(G_\lambda \rightarrow G'_\lambda)$.

We next prove this claim by considering the following commutative diagrams:

(3.6.4)

$$\begin{array}{ccccc}
 G_\lambda & \xrightarrow{\quad} & K_\lambda & \xrightarrow{\quad} & L_\lambda \\
 \downarrow & & \downarrow & & \downarrow \\
 G'_\lambda & \xrightarrow{\quad} & K'_\lambda & \xrightarrow{\quad} & L'_\lambda \\
 \uparrow & & \uparrow & & \uparrow \\
 G_\mu & \xrightarrow{\quad} & K_\mu & \xrightarrow{\quad} & L_\mu \\
 \uparrow & & \uparrow & & \uparrow \\
 G'_\mu & \xrightarrow{\quad} & K'_\mu & \xrightarrow{\quad} & L'_\mu \\
 \uparrow & & \uparrow & & \uparrow \\
 G_\nu & \xrightarrow{\quad} & K_\nu & &
 \end{array}$$

and

(3.6.5)

$$\begin{array}{ccc}
 & G_\lambda & \\
 & \downarrow & \\
 H_\lambda & \xrightarrow{\quad} & G_\lambda \\
 \downarrow & & \downarrow \\
 H'_\lambda & \xrightarrow{\quad} & G'_\lambda \\
 \downarrow & & \downarrow \\
 H_\lambda & \xrightarrow{\quad} & G_\lambda \\
 \downarrow & & \downarrow \\
 H'_\lambda & \xrightarrow{\quad} & G'_\lambda
 \end{array}$$

Suppose an element g'_ν of G'_ν goes to an element g'_λ in G'_λ in (3.6.4). By a chase of (3.6.4) along with utilizing (II) and (III), we conclude that there exists an element g_λ in G_λ whose image x'_λ in G'_λ has the property that $(x'_\lambda)^{-1}g'_\lambda$ belongs to H'_λ . Now chase the diagram (3.6.4) and use (I) to conclude that there exists an element h_λ in H_λ whose image g'_λ in G'_λ equals the image $(x'_\lambda)^{-1}g'_\lambda$ in G'_λ of $(x'_\lambda)^{-1}g'_\lambda$. Hence, the elements x'_λ and $(x'_\lambda)^{-1}g'_\lambda$ are in $\text{Im}(G_\lambda \rightarrow G'_\lambda)$. This means $g'_\lambda = x'_\lambda(x'_\lambda)^{-1}g'_\lambda$ is in $\text{Im}(G_\lambda \rightarrow G'_\lambda)$. This proves our claim and suffices to prove that $\underline{\beta}$ is an epimorphism (cf. [MA]). $\square$

(3.6.6) *Remarks on the "Five Lemma".* The category of pro-(abelian groups) is abelian [AM]; hence, one may assume the "Five Lemma" while working in this category. In $\text{pro-}\mathcal{G}$, the various versions of this lemma require some proof: Although we have not studied all the possible versions, we have proved (for lack of better reference concerning these matters) a "Weak Five Lemma" in $\text{pro-}\mathcal{G}$ (see [SI₁]), along with the result of (3.6) which suffices for our applications.

For convenience of reference, we shall summarize the main results of this section in the following section.

4. A summary of the main results.

(4.1) *The setting.* Throughout the following we let $f: \underline{X} \rightarrow \underline{Y}$ denote a map in $\mathcal{C}$. We say f is a *weak pro-homotopy equivalence* if f induces isomorphisms of all homotopy pro-groups of $\underline{X}$ and $\underline{Y}$. All homology is considered with coefficients in $\mathcal{A}$. The map f is called a *pro-homology equivalence* if f induces isomorphisms of all the homology pro-groups of $\underline{X}$ and $\underline{Y}$. We have proved the following results.

(4.1.1) **THEOREM.** *Given a map $f: \underline{X} \rightarrow \underline{Y}$ in $\mathcal{C}$. Suppose (a) $\pi_i f: \pi_i \underline{X} \rightarrow \pi_i \underline{Y}$ is an isomorphism for $0 \leq i \leq (n-1)$, (b) the induced map $H_n f: H_n \underline{X} \rightarrow H_n \underline{Y}$ of the homology pro-groups is an isomorphism, and (c) the induced map $H_{n+1} f: H_{n+1} \underline{X} \rightarrow H_{n+1} \underline{Y}$ is an epimorphism. Then the induced map $\hat{\Gamma}_\omega \pi_n f: \hat{\Gamma}_\omega \pi_n \underline{X} \rightarrow \hat{\Gamma}_\omega \pi_n \underline{Y}$ is an isomorphism.*

(4.1.2) **THEOREM.** *Suppose $f: \underline{X} \rightarrow \underline{Y}$ satisfies the hypotheses of Theorem (4.1.1), and suppose f satisfies the following additional hypotheses:*

- (a) $\Gamma_\omega \pi_n f: \Gamma_\omega \pi_n \underline{X} \rightarrow \Gamma_\omega \pi_n \underline{Y}$ is an epimorphism,
- (b) $\Gamma'_\omega \pi_n f: \Gamma'_\omega \pi_n \underline{X} \rightarrow \Gamma'_\omega \pi_n \underline{Y}$ is a monomorphism, and
- (c) $\Gamma \pi_n f: \Gamma \pi_n \underline{X} \rightarrow \Gamma \pi_n \underline{Y}$ is a monomorphism.

Then $\pi_n f: \pi_n \underline{X} \rightarrow \pi_n \underline{Y}$ is an isomorphism.

(4.1.3) **THEOREM.** *Suppose $f: \underline{X} \rightarrow \underline{Y}$ is a pro-homology equivalence; see (4.1) for terminology. Then the following are equivalent:*

- (a) f is a weak pro-homotopy equivalence.
- (b) For all i , $1 \leq i < \infty$, $\Gamma_\omega \pi_i f$ is an epimorphism, $\Gamma'_\omega \pi_i f$ is a monomorphism, and $\Gamma \pi_i f$ is a monomorphism.

(4.1.4) **REMARK.** These theorems are the exact analogues of Dror's results (see [DR, Theorem (3.1) and Proposition (3.2)]).

(4.2) *Some calculations.* Suppose $\underline{X}$ is an object of $\mathcal{C}$. It is easy to see that for any i , $1 \leq i < \infty$, we have $\Gamma_\omega \pi_i \underline{X} \approx \Gamma'_\omega \pi_i \underline{X} \approx \Gamma \pi_i \underline{X} \approx 0$ in any of the following cases:

- (4.2.1) $\pi_1 \underline{X} \approx 0$;
- (4.2.2) $\underline{X}$ is simple (see (3.2.3) for a definition);
- (4.2.3) $\underline{X}$ is nilpotent (see (3.2.3) for a definition);
- (4.2.4) $\underline{X}$ is complete (see (3.2.4) for a definition).

(4.3) **A COROLLARY OF THEOREM (4.1.3).** *Given a map $f: \underline{X} \rightarrow \underline{Y}$ in $\mathcal{C}$. Suppose any one of the following holds: $\pi_1 \underline{X} \approx \pi_1 \underline{Y} \approx 0$; $\underline{X}$ and $\underline{Y}$ are simple; $\underline{X}$ and $\underline{Y}$ are nilpotent; or $\underline{X}$ and $\underline{Y}$ are complete. Then the following are equivalent:*

- (i) f is a pro-homology equivalence.
- (ii) f is a weak pro-homotopy equivalence.

(4.4) **H -structures in $\mathcal{C}$.** The notion of an H -space or a space with an H -structure is well known in homotopy theory; moreover, Eckmann and Hilton [EH] have discussed an analogous notion of H -structure on objects of suitable categories. In our earlier work [SI₂], we have concretized the notion of H -structure in suitable pro-homotopy categories; indeed, it follows from Theorem (4.2.1) of [SI₂] that an object $\underline{X}$ of $\text{pro-}\mathcal{HC}\mathcal{U}_0$ with an H -structure is simple.

(4.5) *Shape theoretic considerations.* A pointed topological space X is called *s-simple* ("shape simple"), *s-nilpotent*, or *s-complete* if there exists an object $\underline{X}$ of $\text{pro-}\mathcal{H}\mathcal{C}\mathcal{W}_0$ associated with X in the sense of Morita (cf. [MA], [DS]) such that $\underline{X}$ is simple, nilpotent, or complete; see (3.2). We shall be brief; see [SI₂] for a discussion of many related matters. There are many versions of the Whitehead theorem in shape theory, for instance, [MA], [MO], [MR]; moreover, the books [BO], [DS], [ED] contain many other references and related discussions. As a sample, we shall next state an analogue of the Dror-Whitehead theorem in shape theory.

(4.5.1) THEOREM. *Suppose $f: X \rightarrow Y$ is a shape map (or a shaping) of pointed continua of finite fundamental dimension and suppose X and Y are s-simple or, more generally, s-nilpotent. Then the following are equivalent:*

- (a) *f is a shape equivalence.*
- (b) *$\tilde{f}$ induces isomorphisms of all the homotopy pro-groups.*
- (c) *$\tilde{f}$ induces isomorphisms of the homology pro-groups with coefficients in $\mathcal{L}$.*

PROOF. The equivalence of (a) and (b) is well known (cf. [DS]). It follows from Corollary (4.3) that (c) implies (b). This proves the theorem. $\square$

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DEPARTMENT OF MATHEMATICS, THE PENNSYLVANIA STATE UNIVERSITY, ALTOONA, PENNSYLVANIA 16603