FINITE SUBLATTICES OF A FREE LATTICE

BY

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ABSTRACT. Every finite semidistributive lattice satisfying Whitman's condition is isomorphic to a sublattice of a free lattice.

Introduction. The aim of this paper is to show that a finite semidistributive lattice satisfying Whitman's condition can be embedded in a free lattice. This confirms a conjecture of Bjarni Jónsson, and indeed our proof will follow the line of approach originally suggested by him in unpublished notes around 1960. This approach was later described in Jónsson and Nation [15], to which the reader is referred for a more complete discussion of the background material and related work than will be given here.

Let us recall some relevant definitions and results. A finite sublattice of a free lattice satisfies Whitman's condition [23]

(W) $ab \leq c + d$ iff $a \leq c + d$ or $b \leq c + d$ or $ab \leq c$ or $ab \leq d$
and the semidistributive laws introduced by Jónsson [12]

$(SD_{\vee}) u = a + b = a + c$ implies $u = a + bc$,

$(SD_{\wedge}) u = ab = ac$ implies $u = a(b + c)$.

As in [15], we shall refer to a finite lattice satisfying these three conditions as an S -lattice.

We will often use the following (equivalent) form of the semidistributive laws [14].

$(SD_{\vee}) u = \sum a_i = \sum b_j$ implies $u = \sum_i \sum_j a_i b_j$,

$(SD_{\wedge}) u = \prod a_i = \prod b_j$ implies $u = \prod_i \prod_j (a_i + b_j)$.

Let $J(L)$ denote the set of nonzero join-irreducible elements in a finite lattice L . Every element $p \in J(L)$ has a unique lower cover, which we will denote by p_* . If $p_* \in J(L)$, let $p_{**} = (p_*)_*$. Dually, $M(L)$ denotes the set of nonunit meet-irreducible elements of L , and for $y \in M(L)$, $y^* > y$. In a finite semidistributive lattice there is a bijection between $J(L)$ and $M(L)$,

$$p \leftrightarrow \kappa(p) \equiv \sum \{x \in L: x > p_* \text{ and } x \nlessdot p\}.$$

(In fact, A. Day has shown that this characterizes finite semidistributive lattices [4].) Now $px = p_*$ iff $x > p_*$ and $x \nlessdot p$, and, by $(SD_{\wedge})$, $p\kappa(p) = p_*$; thus $\kappa(p)$ is the largest element in L with this property. Repeatedly we will use the following observations.

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LEMMA 1.1. *If L is a finite semidistributive lattice and $p \in J(L)$, then*

- (i) $p + \kappa(p) = \kappa(p)^*$,
- (ii) $x \leq \kappa(p)$ iff $p_* + x \not\leq p$.

For a finite semidistributive lattice L , we define binary relations A and B on $J(L)$ as follows.

$$\begin{aligned} p A q & \text{ if } q < p \leq \kappa(q)^*, \\ p B q & \text{ if } p \neq q, q_* \leq \kappa(p), q \not\leq \kappa(p). \end{aligned}$$

For technical purposes these relations are further subdivided.

- $p A_1 q$ if $p A q$ and $p\kappa(q) > q_*$,
- $p A_2 q$ if $p A q$ and $p\kappa(q) = q_*$,
- $p B_1 q$ if $p B q$ and $q_* \not\leq p$,
- $p B_2 q$ if $p B q$ and $q_* < p_*$.

Note that by Lemma 1.1(ii) we have $p B q$ iff $p \neq q$, $p \leq p_* + q_*$, $p \leq p_* + q$. It follows that if $p B q$, then $p_* \not\leq q$ (whence $p \not\leq q$) and $q \not\leq p$. Moreover, if $p B q$, then $p \leq \kappa(q)$. For otherwise we would have $p + q = p_* + q$ and $p + q = p + q_*$, whence by $(SD_{\vee})$ $p + q = p_* + q_* + pq = p_* + q_*$ (as p and q must be incomparable), while since $q_* \leq \kappa(p)$ we have $p_* + q_* \not\leq p$, a contradiction. Thus the drawings of Figure 1 accurately represent these relations insofar as the joins and meets of the labeled elements are concerned.

Finally, let $C = A \cup B$, i.e., $p C q$ if $p A q$ or $p B q$.

By a *cycle* in a finite semidistributive lattice, we mean a sequence $\langle p_0, p_1, \dots, p_n \rangle$ with $n \geq 1$ of join-irreducible elements such that $p_i C p_{i+1}$ for $0 \leq i < n$, and $p_n C p_0$. A *minimal cycle* means one of minimal length in L . In particular, a minimal cycle has the property that $p_i C p_j$ only if $j = i + 1$.

Our approach is based on the following result, which combines Theorems 2.1, 6.4, and 9.3 of [15].

THEOREM 1.2. *A finite lattice is embeddable in a free lattice iff it is an S -lattice containing no cycle.*

What we will show is that no S -lattice contains a cycle, so that every S -lattice is isomorphic to a sublattice of a free lattice.

For the sake of completeness, let us sketch the proof of the relevant direction of Theorem 1.2, which shows that an S -lattice not containing a cycle is in fact projective. These arguments were all contained in Jónsson's original notes. The details may be found in [15].

For $U, V \subseteq L$, we write $V \ll U$ if for every $v \in V$ there exists $u \in U$ with $v \leq u$. We let $D_0(L)$ be the set of all join-prime elements of L , and for $k \in \omega$ we let $D_{k+1}(L)$ be the set of all $a \in L$ such that whenever $a < \sum U$ for some $U \subseteq L$ and $a \not\leq u$ for all $u \in U$, then there exists $V \subseteq D_k(L)$ such that $V \ll U$ and $a \leq \sum V$. If $k \leq m$, then $D_k(L) \subseteq D_m(L)$, and we let $D(L) = \bigcup_{k \in \omega} D_k(L)$. Define $\gg$, $D'_k(L)$ and $D'(L)$ dually.

FIGURE 1

First, we want to show that if a finite lattice L satisfies (W) and $D(L) = L = D'(L)$, then L is projective. (The converse is also true; see [6, 9, 15 or 18], and cf. [3].) Let $f: K \rightarrow L$ be a homomorphism. Since L is finite, we can easily find a monotonic transversal $g_0: L \rightarrow K$ (i.e., $a \leq b$ implies $g_0(a) \leq g_0(b)$, and $fg_0(a) = a$ for all $a, b \in L$). Inductively we define for all $a \in L$,

$$g_{k+1}(a) = g_k(a) \prod \left(\sum g_k(U): U \subseteq D_k(L) \text{ and } a < \sum U \right).$$

It is then not hard to show that if $a \in D_k(L)$ and $m > k$, then $g_m(a) = g_k(a)$. Since L is finite and we are assuming that $D(L) = L$, we have $D_n(L) = L$ for sufficiently large n , whence $g_m = g_n$ for all $m \geq n$. Let $h_0 = g_n$, and check that h_0 is a join-preserving transversal.

If we dualize the above construction, beginning with h_0 and using the fact that $D'(L) = L$, we obtain a meet-preserving transversal h . However, since h_0 was join-preserving, we can use (W) to show inductively that each h_k ($k \geq 0$), and hence h , is also join-preserving. Leaving these calculations to the reader, we conclude that h is the desired embedding of L into K , and L is projective.

Next we must show that if L is finite, semidistributive and $D(L) \neq L$, then L contains a cycle. First note that if $\emptyset \neq U \subseteq D_k(L)$, then $\sum U \in D_{k+1}(L)$. Thus if $D(L) \neq L$, some join-irreducible element of L is not in $D(L)$. The existence of a cycle is then a consequence of the following claim and the finiteness of L .

If $p \in J(L) - D(L)$, then there exists $q \in J(L) - D(L)$ with $p \leq q$. For since $p \notin D(L)$, there must exist $U \subseteq L$ such that $p \leq \sum U$ but $p \not\leq u$ for all $u \in U$, and for every $V \ll U$ such that $V \subseteq D(L)$, $p \not\leq \sum V$. Since $p \leq \sum U$, we have $\sum U \not\leq \kappa(p)$, whence $u_0 \not\leq \kappa(p)$ for some $u_0 \in U$. Choose $y \leq u_0$ minimal with respect to the property $y \not\leq \kappa(p)$. Clearly $y \in J(L)$ and $p \leq y$; if $y \notin D(L)$ we may take $q = y$, and the desired conclusion holds.

Otherwise, $y \in D(L)$ and $p \leq p_* + y$. Choose a minimal element $z \leq p_*$ subject to the condition $p \leq y + z$. Then $z \notin D(L)$, for otherwise since $z < p \leq \sum U$, we would have either $z \leq u_1$ for some $u_1 \in U$, or else there exists $W \ll U$ such that $W \subseteq D(L)$ and $z \leq \sum W$. Since also $y \in D(L)$ and $y \leq u_0$, either case leads to a contradiction. Thus $z \notin D(L)$, and some canonical joinand (see [14]) q of z is not in $D(L)$. Now by Lemma 1.1(ii), the remaining canonical joinands of z (if any) lie below $\kappa(q)$, and by the minimality of z we also have $y \leq \kappa(q)$. With this information, it is not hard to check that $p \leq q$.

From the above arguments we may conclude that if L is an S -lattice which is not projective, then either L or L^d , the dual of L , contains a cycle. However, a result of Alan Day [4] (cf. [5, 15, 19]) shows that for a finite semidistributive lattice, $D(L) = L$ iff $D'(L) = L$. We will use a more technical version of Day's theorem, from [19], which allows us to transform any cycle into a dual cycle with the roles of A and B interchanged.

LEMMA 1.3. *Let L be a finite semidistributive lattice, and $p, q \in J(L)$.*

(i) *If $p \leq q$, then $\kappa(p) \leq \kappa(q)$.*

(ii) *If $p \leq q$, then $\kappa(p) \leq \kappa(q)$.*

Thus L contains a cycle iff L^d does.

PROOF. (i) If $p \leq q$, then $\kappa(p) \neq \kappa(q)$ since κ is bijective, and $p \leq \kappa(p)^* \kappa(q)^*$, so $\kappa(p) \not\leq \kappa(p)^* \kappa(q)^*$. On the other hand, we have $\kappa(p) \geq \kappa(p)^* \kappa(q)$, for otherwise $\kappa(p) + p = \kappa(p)^* = \kappa(p) + \kappa(p)^* \kappa(q)$, whence by $(SD_\vee)$, $\kappa(p)^* = \kappa(p) + p\kappa(q) \leq \kappa(p) + p_* = \kappa(p)$, a contradiction. Thus $\kappa(p) \leq \kappa(q)$.

(ii) Let $p \leq q$. Then $\kappa(p) \geq q_*$ and $\kappa(p) \not\leq q$, so $\kappa(p) \leq \kappa(q)$. Because κ is bijective, $\kappa(p) \neq \kappa(q)$. Thus $\kappa(q) > \kappa(p) \geq q_*$, as desired.

Combining Lemma 1.3 with the previous arguments, we have proved the direction of Theorem 1.2 which we will be using: if L is an S -lattice which is not projective, then L contains a cycle. At this point it would seem appropriate to indicate to the reader our general plan for showing that no S -lattice contains a

cycle. First of all, cycles can exist in finite semidistributive lattices failing (W), e.g., as in Figure 2 (from [15]).

FIGURE 2. $p_0 A p_1 A p_2 B p_3 B p_0$

However, there are certain configurations which cannot exist in a finite lattice satisfying (W). We will develop two types of these excluded configurations in §2, and use them repeatedly in the rest of our arguments.

Suppose that $p_0 C p_1 C \dots C p_n = p_0$ is a minimal cycle in an S-lattice. Whenever $p_i A p_{i+1}$, then $p_i > p_{i+1}$, so of course $p_{i*} > p_{i+1*}$. Some of our lemmas will state that under the right circumstances, if $p_i B p_{i+1} B \dots B p_j A p_{j+1}$, then $p_{i*} > p_{j+1*}$. Now clearly these circumstances cannot always persist, for then (with appropriate indexing) we could obtain $p_{0*} > p_{j+1*} > p_{k+1*} > \dots > p_{0*}$, a contradiction.

Fortunately, however, in those situations where $p_i B p_{i+1} B \dots B p_j A p_{j+1}$ and $p_{i*} \not> p_{j+1*}$, one of two things occurs. Observe that for every p_i in our minimal cycle, by virtue of $p_{i-1} C p_i$, p_{i*} is meet reducible, while $p_i C p_{i+1}$ implies $p_{i*} \neq 0$ (see Figure 1). Therefore $p_{i*} \in J(L)$, and p_{i**} exists. In most cases, from $p_i B p_{i+1} B \dots B p_j A p_{j+1}$ we can conclude that $p_{i*} > p_{j+1**}$, whence $p_{i**} \geq p_{j+1**}$. In the remaining case, we find that $p_j A p_{j+1}$ is a single A_2 sandwiched

between B 's, and that this section of the cycle behaves enough like a sequence of all B 's to enable us to use our arguments at the next occurrence of an A in the cycle. (Here we will employ the notion of a B -type sequence, which will be defined in §2.)

Our modified arguments enable us to obtain $p_{0**} > p_{j+1**} > p_{k+1**} > \cdots > p_{0**}$ (with appropriate indexing) for any minimal cycle in an S -lattice. Moreover, one of the inequalities will be strict (and thus lead to a contradiction) if our cycle contains any A_1 or any two consecutive A 's. By the duality induced by Lemma 1.3, neither can our cycle contain two consecutive B 's. Thus the A 's and B 's alternate, in which case we can show that one of the inequalities will be strict if any of the B 's is a B_2 . So we are left only to consider cycles of the form $p_0 B_1 p_1 A_2 p_2 B_1 p_3 \cdots A_2 p_n = p_0$. This type of cycle is excluded by a separate argument, which will complete the proof.

Of course, projectivity and related concepts for lattices have been extensively studied. Several of these ideas which are distinct for general lattices coalesce in the finite case. Combining what is already known with the present result, we obtain the following list of characterizations of finite projective lattices.

THEOREM 1.4. *For any finite lattice L , the following conditions are equivalent.*

- (i) L is a sublattice of a free lattice.
- (ii) L is projective.
- (iii) L is semidistributive and satisfies (W) .
- (iv) L does not contain any of the lattices L_1 – L_8 from Figure 3 as a sublattice.
- (v) L is a bounded homomorphic image of a free lattice and satisfies (W) .
- (vi) L is transferable.
- (vii) L is sharply transferable.

The equivalence of (i), (ii) and (v) was found by R. McKenzie [18]; generalizations to infinite lattices were given by R. Freese, B. Jónsson, A. Kostinsky and the author [6, 15, 16]. The equivalence of (iii) and (iv) is due to R. Antonius, B. Davey, W. Poguntke and I. Rival [1, 2]. The equivalence of (vi) and (vii) is due to C. Platt [20], while the equivalence of (i) and (vii) was shown by H. Gaskill, G. Grätzer and C. Platt [8, 9] (see also [10, 17, 19]).

Also, two important special cases of Jónsson's conjecture were previously known to be true. I. Rival and B. Sands [21] proved that a planar S -lattice is always projective, while J. Ježek and V. Slavík [11] showed that a subdirectly irreducible S -lattice is always a sublattice of a free lattice. Ježek and Slavík in fact gave a complete description of all subdirectly irreducible S -lattices.

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2. Configuration lemmas. In this section we will develop some configurations which cannot exist in an S -lattice, to be used later in showing that no cycle exists. We begin by isolating some useful properties about a string of B 's in a minimal cycle.

FIGURE 3

LEMMA 2.1. If $\langle q_0, \dots, q_k \rangle$ with $k \geq 0$ is a subsequence of a minimal cycle in an S -lattice such that $q_j B q_{j+1}$ for $0 \leq j < k$, then

- (i) $q_{j-1*} \not\leq q_j$ for $1 \leq j \leq k$,
- (ii) $\sum_{i < j} q_i \leq \kappa(q_j)$ for $1 \leq j \leq k$,
- (iii) $q_{j+1*} \leq \kappa(q_j)$ for $0 \leq j \leq k-1$,
- (iv) $q_{j+i*} \not\leq \kappa(q_j)$ for $i \geq 2$ and $0 \leq j \leq k-i$.

PROOF. (i) and (iii) are immediate from the relations $q_{j-1} B q_j$ and $q_j B q_{j+1}$.

For (ii), recall that $q_i B q_{i+1}$ implies $q_i \leq \kappa(q_{i+1})$, and moreover $\kappa(q_i) A^d \kappa(q_{i+1})$, whence $\kappa(q_i) \leq \kappa(q_{i+1})$. It follows that, for $i < j$, $q_i \leq \kappa(q_{i+1}) \leq \kappa(q_j)$ as claimed.

For (iv), first note that $q_{j+1} \leq q_{j+1*} + q_{j+2}$, and $q_{j+1} \not\leq \kappa(q_j)$ while $q_{j+1*} \leq \kappa(q_j)$, so we must have $q_{j+2} \not\leq \kappa(q_j)$. If $q_{j+2*} \leq \kappa(q_j)$, we would have $q_j B q_{j+2}$, in contradiction to the minimality of our cycle. Thus $q_{j+2*} \not\leq \kappa(q_j)$. For $i > 2$, we have $q_{j+i*} \not\leq \kappa(q_{j+1})$ by induction, and $\kappa(q_j) \leq \kappa(q_{j+1})$ since $\kappa(q_j) A^d \kappa(q_{j+1})$, so that $q_{j+i*} \not\leq \kappa(q_j)$, as desired.

The proof of (iv) above included the first use of a simple observation which will appear often in our arguments. If p and q are distinct elements from a minimal cycle and q is not the successor of p in the cycle, then $p B q$ does not hold; therefore $q \not\leq \kappa(p)$ implies $q_* \not\leq \kappa(p)$. We shall refer to this argument as the *free star principle*.

We wish to generalize the situation where $q_0 B \dots B q_k$. Let $p_0 C \dots C p_n C p_0$ be a minimal cycle in an S -lattice, and let $\langle q_0, \dots, q_k \rangle$ be a subsequence of $\langle p_0, \dots, p_n \rangle$ with $k \geq 0$. (Thus the q_j 's are in their correct order from the cycle, but q_j and q_{j+1} need not be consecutive elements in the cycle.) We say that $\langle q_0, \dots, q_k \rangle$ is a *B-type sequence* if conditions (i)–(iv) of Lemma 2.1 are satisfied. The difference between a *B-type sequence* and a sequence of *B*'s (i.e., $q_j B q_{j+1}$) is that we do not require $q_{j+1} \not\leq \kappa(q_j)$ for a *B-type sequence*. This notion will play a crucial role in our proof.

The first configuration we consider which cannot exist in a finite S -lattice comes from [15, Lemma 7.4].

LEMMA 2.2. *A finite lattice L satisfying (W) cannot contain elements a, a_0, a_1, b, b_0 such that the following conditions hold.*

- (i) a and b are join-reducible.
- (ii) $a \leq a_1 < a_0$.
- (iii) $b \leq b_0$.
- (iv) $a \not\leq b_0$.
- (v) $b \not\leq a_0$.
- (vi) $a_0 \not\leq a_1 + b_0$.
- (vii) $b_0 \not\leq a_1 + b$.

A variation of this lemma will also prove useful.

LEMMA 2.3. *A finite lattice L satisfying (W) cannot contain elements a, a_0, a_1, b, b_0 such that*

- (i)' $a \in J(L)$ and $a \leq a_* + b$, and b is join-reducible,
- and (ii)–(vii) of Lemma 2.2 hold.*

These configurations are illustrated in Figure 4.

SKETCH OF PROOFS. Suppose that one of these configurations exists in a finite lattice L satisfying (W). First observe that $a_1 = a_0(a_1 + b_0)$ is meet-reducible, and hence join-irreducible, since a lattice satisfying (W) contains no doubly reducible elements. We claim that $a < a_1$. In Lemma 2.2, this follows because a is join-reducible, while $a_1 \in J(L)$. In Lemma 2.3, we have $a_1 = a_0(a_1 + b_0) \not\leq a_{1*} + b$ by (W), while $a \leq a_* + b$, so that again $a \neq a_1$.

Let $a_2 = a_{1*}$ and $b_1 = b_0(a_1 + b)$. The reader can now check that (i)–(vii) hold with a_0, a_1, b_0 replaced by a_1, a_2, b_1 . Therefore by iterating this process we can obtain two infinite descending chains, $\{a_i: i \in \omega\}$ and $\{b_j: j \in \omega\}$, contrary to the finiteness of L . It follows that the configurations cannot exist.

The configurations of Lemmas 2.2 and 2.3 can arise very naturally when we consider sequences of the type $q_0 B \dots B q_k A p$. Thus our most frequent applications of Lemma 2.3 will be in the form of the following lemma, or some variation thereof.

FIGURE 4

LEMMA 2.4. Let $\langle q_0, \dots, q_k, p \rangle$ with $k \geq 1$ be a subsequence of a minimal cycle in an S -lattice, and assume that

- (i) $\langle q_0, \dots, q_k \rangle$ is a B -type sequence.
- (ii) $q_k A_2 p$.
- (iii) $p_{**} \not\leq q_{0*}$.
- (iv) $p B q_0$ does not hold.

Let $j \geq 1$ be chosen minimal with respect to the property $p_* \leq q_{j*}$. Then $\sum_{i < j} q_i \leq \kappa(p)$.

PROOF. First of all, observe that $p_* \not\leq q_{0*}$ by (iii), while $p_* \leq q_{k*}$ by (ii). Therefore j can be chosen as indicated. Supposing that $\sum_{i < j} q_i \not\leq \kappa(p)$, by the free star principle and (iv) we also have $\sum_{i < j} q_{i*} \not\leq \kappa(p)$. Let us apply Lemma 2.3.

We must choose a, a_0, a_1, b, b_0 . Let

$$a = p, \quad b = p_{**} + \sum_{i < j} q_{i*},$$

$$b_0 = p_{**} + \sum_{i < j-1} q_{i*} + q_{j-1}.$$

If $j = k$, let $a_0 = q_k$ and $a_1 = q_{k*}$. Otherwise $j < k$, and we choose a_0 and a_1 as follows. Note that $k - (j - 1) \geq 2$, so that by property 2.1(iv), $q_{k*} \not\leq \kappa(q_{j-1})$. On the other hand, $p_* \leq q_{j*} \leq \kappa(q_{j-1})$ by property 2.1(iii), whence by the free star principle $p \leq \kappa(q_{j-1})$. Pick a_0 minimal in the interval $[p, q_{k*}]$ with respect to the property $a_0 \not\leq \kappa(q_{j-1})$. Since a_0 is clearly join-irreducible in the interval $[p, q_{k*}]$, a_0 covers a unique element a_1 in the interval. Note that in either case, whether $j = k$ or $j < k$, we have $a_1 \leq \kappa(q_{j-1})$. (If $j < k$, this is clear; if $j = k$, use property 2.1(iii).)

Now let us check that the seven conditions of Lemma 2.3 hold.

(i)' $a = p \in J(L)$, and since $\sum_{i < j} q_{i*} \not\leq \kappa(p)$, we have $p \leq p_* + \sum_{i < j} q_{i*} = p_* + b$. To see that b is join-reducible, we show that $b \neq p_{**}$ and $b \neq q_{i*}$ for $i < j$. Now $b \neq p_{**}$, since $q_{j-1*} \leq b$ while $q_{j-1*} \not\leq p_{**}$, as $p_{**} < q_j$ and, by 2.1(i), $q_{j-1*} \not\leq q_j$. Also $b \neq q_{0*}$, since $p_{**} \leq b$ while $p_{**} \not\leq q_{0*}$. If $j = 1$, we are done. Otherwise, for $1 \leq i < j$ we have $q_{i-1*} \leq b$ while $q_{i-1*} \not\leq q_i$ by 2.1(i), wherefore $b \neq q_{i*}$. Thus b is join-reducible.

Conditions (ii) and (iii), $a \leq a_1 < a_0$ and $b \leq b_0$, are immediate.

(iv) If $a \leq b_0$, then using $q_k A_2 p$ we have $q_k \kappa(p) = p_* = a_* \leq b_0$, so we may apply (W) to the inclusion

$$q_k \kappa(p) \leq p_{**} + \sum_{i < j-1} q_{i*} + q_{j-1*}.$$

Now $q_k \not\leq b_0$ (the right-hand side) since by 2.1(ii), $b_0 \leq \kappa(q_k)$. Also $\kappa(p) \not\leq b_0$, for since $q_k A p$ we have $q_k \leq \kappa(p)^* = p + \kappa(p)$, while $p + b_0 \leq \kappa(q_k)$ as before. Of course $q\kappa(p) = p_* \not\leq p_{**}$, and $p_* \not\leq q_{i*}$ for $0 \leq i \leq j-1$ by the choice of j . This leaves only the possibility $p_* = q_{j-1*}$, which however would imply $q_{j-1*} < p_* < q_j$, contrary to 2.1(i). Therefore $a \not\leq b_0$.

(v) If $b \leq a_0$, then $q_{j-1*} \leq b \leq a_0 \leq q_k$. If $j = k$, this contradicts 2.1(i), so we may assume $j < k$. Now $q_{j-1*} \not\leq \kappa(p)$, or else we would have $q_{j-1*} \leq q_k \kappa(p) = p_* \leq q_j$, contrary to 2.1(i). Therefore $\kappa(p)^* = p + \kappa(p) = q_{j-1*} + \kappa(p)$, since $p \leq p_* + q_{j-1*} \leq q_{j-1*} + \kappa(p)$ and $q_{j-1*} \leq q_k \leq \kappa(p)^*$. Applying $(SD_{\vee})$ yields $\kappa(p)^* = p q_{j-1*} + \kappa(p)$. Since $p_* \leq \kappa(p)$, however, we cannot have $p q_{j-1*} < p$; therefore $p \leq q_{j-1*}$. If $j = 1$, this contradicts one of our original assumptions; otherwise we continue. Now $q_{j-1*} < q_k$, because $p_* \leq q_k$ but $p_* \not\leq q_{j-1*}$. Also, since $j < k$, we have $k - (j - 1) \geq 2$, whence by 2.1(iv), $q_{j-1} \leq q_{j-1*} + q_{k*} = q_{k*}$. Thus $p < q_{j-1} < q_k \leq \kappa(p)^*$, and $q_{j-1} A p$. This, however, contradicts the minimality of our original cycle. We conclude that $b \not\leq a_0$.

(vi) If $a_0 \leq a_1 + b_0$, then $j \neq k$ (i.e., $a_0 \neq q_k$), for by 2.1(ii), $b_0 \leq \kappa(q_k)$. Thus $j < k$, so that $a_0 \not\leq \kappa(q_{j-1})$ by the choice of a_0 , whence $q_{j-1} \leq q_{j-1*} + a_0$. Since also $a_0 \leq a_1 + b_0$ and $p_{**} < p_* = a$, we compute

$$a_0 + b_0 = a_0 + \sum_{i < j} q_{i*} = a_1 + \sum_{i < j-1} q_{i*} + q_{j-1*}.$$

Applying $(SD_{\vee})$ in its more general form, we obtain $a_0 + b_0 = a_1 + \sum_{i < j} q_{i*} + a_0 q_{j-1}$. However, in (v) we showed that $q_{j-1*} \not\leq q_k$, so $q_{j-1} \not\leq a_0$. Hence $a_0 q_{j-1} \leq q_{j-1*}$, and

$$a_0 + b_0 = a_1 + \sum_{i < j} q_{i*}.$$

But $a_1 + \sum_{i < j} q_{i*} \leq \kappa(q_{j-1})$ by the choice of a_0 and 2.1(ii), while $q_{j-1} \leq b_0 \leq a_0 + b_0$, so this is a contradiction. Thus $a_0 \not\leq a_1 + b_0$.

(vii) Finally $b_0 \not\leq a_1 + b$, else $q_{j-1} \leq b_0 \leq a_1 + b = a_1 + \sum_{i < j} q_{i*} \leq \kappa(q_{j-1})$ as above, a contradiction.

By Lemma 2.3 then, this configuration cannot occur, and we conclude that $\sum_{i < j} q_i \leq \kappa(p)$.

The second type of configuration we wish to consider which does not occur in a finite lattice satisfying (W) is also found in [15, Lemma 7.2]. It was inspired by P. Whitman's result [24] that a subset X of a lattice satisfying (W) generates a free lattice iff $a \not\leq \sum F$ and $a \not\geq \prod F$ whenever $a \in X$ and F is a finite subset of X with $a \notin F$ (cf. Jónsson [13]).

LEMMA 2.5. *A finite lattice L satisfying (W) cannot contain elements a, b, c such that the following conditions hold.*

- (i) $b(c + ab) \not\leq a$.
- (ii) $a(c + ab) \not\leq b$.
- (iii) $ab \not\leq c$.
- (iv) $a \not\leq b + c$.
- (v) $b \not\leq a + c$.

SKETCH OF PROOF. Supposing that (i)–(v) hold, let $a_1 = a(b + c)$ and $b_1 = b(a + c)$. Then $a_1 < a$ and $b_1 < b$, and it is fairly simple to show that (i)–(v) hold with a and b replaced by a_1 and b_1 . Thus we obtain two infinite descending chains, contrary to the finiteness of L .

We will most often use, instead of Lemma 2.5, the following simplified, dualized, and disguised version of this configuration (cf. H. Rolf [22]).

LEMMA 2.6. *A finite lattice L satisfying (W) cannot contain elements a, a_0, b, b_0 such that the following conditions hold.*

- (i) $a \leq a_0$.
- (ii) $b \leq b_0$.
- (iii) $a \not\leq b_0$.
- (iv) $b \not\leq a_0$.
- (v) $a_0 \not\leq a + b$.
- (vi) $b_0 \not\leq a + b$.
- (vii) $a_0b \not\leq a$.
- (viii) $ab_0 \not\leq b$.

PROOF. Supposing that (i)–(viii) hold, let $c = a_0b_0$. Then $a \not\leq b + c$ since $b + c \leq b_0$, and similarly $b \not\leq a + c$. By (W), $c \not\leq a + b$; for (v) and (vi) say that $a_0, b_0 \not\leq a + b$, while $a_0b \leq c$ and $ab_0 \leq c$ imply $c \not\leq a, b$ using (vii) and (viii). On the other hand, conditions (vii) and (viii) say directly that $bc \not\leq a$ and $ac \not\leq b$. The five noninclusions we have just shown are stronger than the duals of the conditions of Lemma 2.5, and we conclude that the configuration cannot occur in L .

(Conversely, if a, b, c satisfy $a \not\leq b + c$, $b \not\leq a + c$, $c \not\leq a + b$, $bc \not\leq a$ and $ac \not\leq b$, we may let $a_0 = a + c$ and $b_0 = b + c$. It is straightforward to check that 2.6(i)–(viii) hold.)

Now it turns out that the configuration of Lemma 2.6 also tends to arise when we consider sequences of the form $q_0 B \dots B q_k A p$. All of our applications of Lemma 2.6 are included in the following lemma.

LEMMA 2.7. Let $\langle q_0, \dots, q_k, p \rangle$ with $k \geq 1$ be a subsequence of a minimal cycle in an S -lattice L , and let j be fixed with $1 \leq j \leq k$. Assume that for some $t \in L$ the following conditions are satisfied.

(i) $\langle q_0, \dots, q_k \rangle$ is a B -type sequence.

(ii) $p_* \not\leq q_{0*}$.

(iii) $p_* \leq q_{j*}$.

(iv) $p < t \leq \kappa(p)^*$.

(v) $t \leq \kappa(q_{j-1})$.

(vi) $t \not\leq p + \sum_{i < j-1} q_{i*} + q_{j-1}$.

(vii) p B q_0 does not hold.

Then $\sum_{i < j} q_{i*} \not\leq \kappa(p)$.

PROOF. Suppose that $\sum_{i < j} q_{i*} \leq \kappa(p)$, whence by the free star principle and (vii) we also have $\sum_{i < j} q_i \leq \kappa(p)$. Let us apply Lemma 2.6 with

$$a = p + \sum_{i < j} q_{i*}, \quad a_0 = t + \sum_{i < j} q_{i*},$$

$$b = p_* + \sum_{i < j-1} q_{i*} + q_{j-1}, \quad b_0 = \kappa(p).$$

We need to check conditions (i)–(viii) of Lemma 2.6.

Conditions (i)–(v) are easy. That $a \leq a_0$ and $b \leq b_0$ are consequences of our assumptions, and $a \not\leq b_0$ is clear. By hypothesis (v) and 2.1(ii), we have $a_0 \leq \kappa(q_{j-1})$, whereas $q_{j-1} \leq b$, from which $a_0 \not\leq b$ follows. By hypothesis (vi), $t \not\leq a + b = p + \sum_{i < j-1} q_{i*} + q_{j-1}$, which yields $a_0 \not\leq a + b$.

For (vi), suppose $b_0 \leq a + b$. Then we would have $t \leq \kappa(p)^* = p + \kappa(p) \leq a + b$, contrary to hypothesis (vi) again. Thus $b_0 \not\leq a + b$.

(vii) If $a_0 b \leq a$, then we can apply (W) to the inclusion

$$\left[t + \sum_{i < j} q_{i*} \right] \left[p_* + \sum_{i < j-1} q_{i*} + q_{j-1} \right] \leq p + \sum_{i < j} q_{i*}.$$

Now $t \not\leq a$ (the right-hand side) as $t \not\leq a + b$. Since $p < t \leq \kappa(q_{j-1})$ and $\sum_{i < j} q_{i*} \leq \kappa(q_{j-1})$ by 2.1(ii), we have $a \leq \kappa(q_{j-1})$. Hence the second term is not below a . On the other hand, we cannot have $a_0 b \leq p$, for that would imply $a_0 b \leq p \kappa(p) = p_*$, and thence $q_{j-1*} \leq a_0 b \leq p_* \leq q_j$, contrary to 2.1(i). Since $p_* \leq a_0 b$ and $p_* \not\leq q_{0*}$, we cannot have $a_0 b \leq q_{0*}$. For $1 \leq i < j$, though, $q_{i-1*} \leq a_0 b$ implies $a_0 b \not\leq q_{i*}$ by 2.1(i). Therefore $a_0 b \not\leq a$.

(viii) If $ab_0 \leq b$, then we may apply (W) to the inclusion

$$\left[p + \sum_{i < j} q_{i*} \right] \kappa(p) \leq p_* + \sum_{i < j-1} q_{i*} + q_{j-1}.$$

Now p is not below the right-hand side, b , since $b \leq \kappa(p)$. If $\kappa(p) \leq b$, then $b_0 = \kappa(p) \leq b \leq a + b$, contrary to condition (vi) which was proved above. On the other hand, $q_{j-1*} \leq ab_0$ implies $ab_0 \not\leq p_*$ since $p_* \leq q_j$. Because $p_* \leq ab_0$ and $p_* \not\leq q_{0*}$, we have $ab_0 \not\leq q_{0*}$. If $j = 1$, we must also note that $ab_0 \neq q_0$ since in this case $ab_0 \leq a \leq \kappa(q_{j-1}) = \kappa(q_0)$. Otherwise, for $1 \leq i < j - 1$, $q_{i-1*} \leq ab_0$ implies $ab_0 \not\leq q_i$ by 2.1(i); a fortiori $ab_0 \not\leq q_{i*}$ for $1 \leq i < j - 1$. Therefore $ab_0 \not\leq b$.

By Lemma 2.6 then, this configuration cannot exist in L , and we conclude that $\sum_{i < j} q_{i*} \not\leq \kappa(p)$.

3. Major lemmas. In the preceding section we developed two types of configuration lemmas, and showed how each can be applied to a situation where $\langle q_0, \dots, q_k \rangle$ is a B -type sequence and $q_k A p$. What is nice about Lemmas 2.4 and 2.7 is that, with rather similar hypotheses, they yield opposite conclusions. (Note that the argument of the former gives rise to infinite descending chains, while that of the latter yields infinite ascending chains.) Thus, in situations where neither one of our configurations gives the desired conclusion, we can try to play the two against one another to reach a contradiction.

In this section we will use the configurations to show that, in every situation where $\langle q_0, \dots, q_k \rangle$ is a B -type sequence and $q_k A p$, either $q_{0*} > p_{**}$ or $\langle q_0, \dots, q_k \rangle$ can be replaced by a longer B -type sequence (starting at q_0 and ending beyond q_k in our minimal cycle).

LEMMA 3.1. *If $\langle q_0, \dots, q_k \rangle$ with $k \geq 0$ is a B -type sequence and $q_k A_1 p$, then $q_{0*} > p_*$.*

PROOF. We will proceed by induction on k . The case $k = 0$ is trivial, for then $q_0 A p$ implies $q_0 > p$, whence $q_{0*} > p_*$. (The case $k = 1$ may also be found as Lemma 7.5 of [15].)

If $k > 0$, then $\langle q_1, \dots, q_k \rangle$ is also a B -type sequence. Therefore, by the inductive hypothesis, we may assume that $q_{1*} > p_*$. Hence it follows that $q_{0*} \not\leq p_*$, for else we would have $q_{0*} \leq p_* < q_1$, contrary to 2.1(i).

Suppose $q_{0*} \not\geq p_*$. By the preceding remark, we then have in fact $q_{0*} \not\geq p_*$. We will apply Lemma 2.2, mimicking the argument of Lemma 2.4.

We must choose a, a_0, a_1, b, b_0 . Now $q_k \kappa(p) \leq q_{k*}$ as $p < q_k$ implies $q_k \not\leq \kappa(p)$, and $q_k \kappa(p) \not\leq q_{0*}$ since $p_* \leq q_k \kappa(p)$. Therefore we may find $j \geq 1$ minimal with respect to the property $q_k \kappa(p) \leq q_{j*}$. Let $a = p + q_k \kappa(p)$, $b = p_* + \sum_{i < j} q_{i*}$, and $b_0 = p_* + \sum_{i < j-1} q_{i*} + q_{j-1}$. If $j = k$, let $a_0 = q_k$ and $a_1 = q_{k*}$. Otherwise we choose a_0 and a_1 as follows. Since $k - (j - 1) \geq 2$ if $j < k$, by 2.1(iv) we have $q_{k*} \not\leq \kappa(q_{j-1})$. On the other hand, $p_* \leq q_k \kappa(p) \leq q_{j*} \leq \kappa(q_{j-1})$ by 2.1(iii), whence by the free star principle $p \leq \kappa(q_{j-1})$. Thus also $p + q_k \kappa(p) \leq \kappa(q_{j-1})$. Pick a_0 minimal in the interval $[p + q_k \kappa(p), q_{k*}]$ with respect to the property $a_0 \not\leq \kappa(q_{j-1})$. Clearly a_0 is join-irreducible in the interval $[p + q_k \kappa(p), q_{k*}]$, so a_0 covers a unique element a_1 in the interval. Note that in either case, whether $j = k$ or $j < k$, we have $a_1 \leq \kappa(q_{j-1})$.

Now we must check that the seven conditions of Lemma 2.2 hold. Except for condition (i), this is done exactly as in the proof of Lemma 2.4, but using $q_k \kappa(p)$ in place of p_* , and p_* instead of p_{**} . (Where before we had $p_{**} < q_k \kappa(p) = p_*$ and $p_{**} \not\leq q_{0*}$, we now have $p_* < q_k \kappa(p)$ and $p_* \not\leq q_{0*}$.) This task will be left to the reader.

For condition (i), we have immediately that $a = p + q_k \kappa(p)$ is join-reducible because $q_k A_1 p$, i.e., $q_k \kappa(p) \not\leq p$ (see Figure 1). The argument that b is join-reducible is adapted as above from that given in the proof of Lemma 2.4.

By Lemma 2.2, we conclude that this configuration cannot exist in L , and hence $q_{0*} > p_{*}$.

Next, we must start considering what happens when $q_k A_2 p$. First, let us recall Lemma 7.1 from [15].

LEMMA 3.2. *If $p_0 A_2 p_1 A p_2$ in a minimal cycle, then $p_{1*} = p_2$ and $p_1 A_2 p_2$.*

SKETCH OF PROOF. Apply (W) to the inclusion $p_0 \kappa(p_1) = p_{1*} \leq p_2 + \kappa(p_2)$.

It is also true that a minimal cycle cannot contain more than two A_2 's consecutively (see the proof of Lemma 8.5 of [15]), but we will not use this fact. Indeed, the following lemma shows that, for our purposes, two A_2 's behave essentially like an A_1 .

LEMMA 3.3. *If $\langle q_0, \dots, q_k \rangle$ with $k \geq 0$ is a B-type sequence and $q_k A_2 p_1 A_2 p_2$, then $q_{0*} > p_{2*}$.*

PROOF. Again we proceed by induction, with the case $k = 0$ following trivially from $q_0 A p_1 A p_2$. Thus we may assume $k > 0$ and $q_{1*} > p_{2*}$.

Suppose $q_{0*} \not> p_{2*}$. Now $q_{0*} \not\leq p_{2*}$ since $p_{2*} < q_1$; hence we have $q_{0*} \not> p_{2*}$. (Note that this implies $p_2 \neq q_0$.) Observe that by Lemma 3.2, $p_{1**} = p_{2*}$. So choosing j minimal with respect to the property $p_2 (= p_{1*}) \leq q_{j*}$, let us apply Lemma 2.4 with $p = p_1$. It is easy to verify that the hypotheses of Lemma 2.4 hold, and we conclude that $\sum_{i < j} q_i \leq \kappa(p_1)$.

Now, however, we are in a position to apply Lemma 2.7 with $p = p_2$ and $t = p_1$. Condition 2.7(i)–(iv) are easy to check.

For (v), note that $p_{1*} = p_2 \leq q_{j*} \leq \kappa(q_{j-1})$ by 2.1(iii). Thus by the free star principle $p_1 = t \leq \kappa(q_{j-1})$.

From our application of Lemma 2.4 above, we have $p_1 \not\leq p_2 + \sum_{i < j} q_i$, from which (vi) follows immediately.

If (vii) failed, i.e., $p_2 B q_0$, then we could apply (W) to the inclusion

$$q_k \kappa(p_1) = p_2 \leq p_{2*} + q_0.$$

Of course, $q_k \not\leq p_{2*} + q_0$ since by 2.1(ii), $p_{2*} + q_0 \leq \kappa(q_k)$. Likewise $\kappa(p_1) \not\leq p_{2*} + q_0$, for otherwise $q_k \leq \kappa(p_1)^* = p_1 + \kappa(p_1) \leq p_1 + q_0 \leq \kappa(q_k)$, a contradiction. Surely $p_2 \not\leq p_{2*}$, and $p_2 \not\leq q_0$ since $p_{2*} \not\leq q_{0*}$. Thus (vii) holds.

Lemma 2.7 then yields $\sum_{i < j} q_{i*} \not\leq \kappa(p_2)$. That being the case, we may apply (W) to the inclusion

$$q_k \kappa(p_1) = p_2 \leq p_{2*} + \sum_{i < j} q_{i*}.$$

However, q_k is not below the right-hand side since $p_{2*} + \sum_{i < j} q_{i*} \leq \kappa(q_k)$ by 2.1(ii). Similarly $\kappa(p_1)$ is not below the right-hand side, for otherwise $q_k \leq \kappa(p_1)^* = p_1 + \kappa(p_1) \leq p_1 + \sum_{i < j} q_{i*} \leq \kappa(q_k)$, a contradiction. Of course $p_2 \not\leq p_{2*}$, while $p_2 \not\leq q_{i*}$ for $i < j$ by the choice of j .

Since this (W)-failure cannot occur, it must be that our original assumption $q_{0*} \not> p_{2*}$ was wrong, as desired.

So it remains for us to consider what happens when $q_k A_2 p B r$. One case of this situation is reasonably easy, so let us do it first.

LEMMA 3.4. *Let $\langle q_0, \dots, q_k \rangle$ with $k \geq 0$ be a B -type sequence, and let $q_k A_2 p$. If $p B q_0$ does not hold and $q_0 \not\leq \kappa(p)$, then $q_{0*} > p_{**}$.*

PROOF. The case $k = 0$ is trivial, so we may assume $k \geq 1$. Suppose $q_{0*} \not\geq p_{**}$, and let us apply Lemma 2.4.

To verify the hypotheses of that lemma, we need only show that $q_{0*} \neq p_{**}$, the rest being immediate. Since $q_0 \not\leq \kappa(p)$ and $p B q_0$ does not hold, we have $q_{0*} \not\leq \kappa(p)$, whence $q_{0*} \neq p_{**}$.

We conclude from Lemma 2.4 that $\sum_{i < j} q_i \leq \kappa(p)$, which contradicts our hypothesis that $q_0 \not\leq \kappa(p)$. Therefore $q_{0*} > p_{**}$.

Now we must deal with the case $q_k A_2 p B r$ with $q_0 \leq \kappa(p)$, where it will not be possible to conclude that $q_{0*} > p_{**}$ (see Figure 5). Most of our effort will be involved in showing that we can obtain a longer B -type sequence when this fails to occur. Our next three lemmas provide the preliminaries for Lemma 3.8.

FIGURE 5

LEMMA 3.5. Let $\langle q_0, \dots, q_k \rangle$ with $k \geq 1$ be a B-type sequence, and let $q_k A_2 p$. If $q_0 \leq \kappa(p)$ and $q_{0*} \nless p_*$, then $\sum_{i < k} q_i \leq \kappa(p)$.

PROOF. First of all, we note that in this situation $q_{0*} \neq p_*$, so that in fact we have $q_{0*} \nless p_*$. If $k = 1$, this follows from 2.1(i), for $p_* < q_k = q_1$ while $q_{0*} \nless q_1$. If $k > 1$, then $q_k \nless \kappa(q_0)$ by 2.1(iv), whence $q_{0*} = p_*$ would imply $q_0 \leq q_{0*} + q_{k*} = q_{k*}$. But then we would have $q_0 \leq q_k \kappa(p) = p_* = q_{0*}$, a contradiction.

Thus we may choose $j \geq 1$ minimal with respect to the property $q_{j*} \geq p_*$. We want next to apply a variation of Lemma 2.4 to show that $\sum_{i < j} q_i \leq \kappa(p)$.

So assume $\sum_{i < j} q_i \nless \kappa(p)$, in which case of course $j > 1$. We may then follow almost exactly the proof of Lemma 2.4, except that in applying Lemma 2.3, choose $b = \sum_{i < j} q_{i*}$ and $b_0 = \sum_{i < j-1} q_{i*} + q_{j-1}$. All subsequent references to p_{**} in the proof must then be omitted or have p_{**} replaced by p_* , as appropriate. The lone exception to this rule is that in order to show $b \neq q_{0*}$ in the proof of (i)', observe that $b = \sum_{i < j} q_{i*} \nless \kappa(p)$ by the free star principle, while $q_{0*} \leq \kappa(p)$ by assumption. The details of verifying that this all works will be left to the reader.

We conclude from this argument that $\sum_{i < j} q_i \leq \kappa(p)$. If $j = k$, this is the desired result. So assume $j < k$ and $\sum_{i < k} q_i \nless \kappa(p)$, and let us apply Lemma 2.7 with $t = \kappa(p)^*[p + \sum_{i < j} q_{i*}]$. Hypotheses (i)–(iii) of Lemma 2.7 hold immediately, while (vii) is a consequence of $q_0 \leq \kappa(p)$. This leaves hypotheses (iv)–(vi) for us to verify.

(iv) Clearly $p \leq t \leq \kappa(p)^*$. Note $q_{j-1*} \leq t\kappa(p)$. Therefore we cannot have $p = t$, for that would imply $q_{j-1*} \leq p\kappa(p) = p_* \leq q_j$, contrary to 2.1(i). Hence $p < t$.

(v) Since $p_* \leq q_{j*} \leq \kappa(q_{j-1})$, the free star principle yields $p \leq \kappa(q_{j-1})$. Combining this with 2.1(ii) and (iii), we have $t \leq p + \sum_{i < j} q_{i*} \leq \kappa(q_{j-1})$, as desired.

(vi) If $t \leq p + \sum_{i < j-1} q_{i*} + q_{j-1}$, then we may apply (W) to the inclusion

$$\kappa(p)^* \left[p + \sum_{i < j} q_{i*} \right] \leq p + \sum_{i < j-1} q_{i*} + q_{j-1}.$$

If $\kappa(p)^*$ is below the right-hand side, then $q_k \leq \kappa(p)^* \leq p + \sum_{i < j-1} q_i \leq \kappa(q_k)$ using 2.1(ii), a contradiction. Skipping over the second possible inclusion for a moment, note that the left-hand side $t \nless p$ since $p < t$ was shown above in (iv). Likewise, $p_* < t$ implies $t \nless q_{i*}$ for $i \leq j-1$ by the choice of j , while $t \neq q_{j-1}$ as $t \leq \kappa(q_{j-1})$ by (v) above.

So suppose that the second term, and hence in particular q_{j*} , is below the right-hand side. Recall that $j < k$, whence 2.1(iii) implies $q_{j*} = q_j(q_{j*} + q_{j+1*})$. Then we may apply (W) to the inclusion

$$(\dagger) \quad q_j(q_{j*} + q_{j+1*}) \leq p + \sum_{i < j-1} q_{i*} + q_{j-1}.$$

Since $p_* \leq q_{j*} \leq \kappa(q_j)$, by the free star principle $p \leq \kappa(q_j)$. As also $\sum_{i < j-1} q_{i*} + q_{j-1} \leq \kappa(q_j)$ by 2.1(ii), we have $q_j \nless p + \sum_{i < j-1} q_{i*} + q_{j-1}$.

Suppose $q_{j+1*} \leq p + \sum_{i < j-1} q_{i*} + q_{j-1}$. Then, using 2.1(iv), we obtain

$$u \equiv p + \sum_{i < j-1} q_{i*} + q_{j-1} = p + \sum_{i < j-1} q_{i*} + q_{j+1*}$$

whence, by $(SD_{\vee})$, $u = p + \sum_{i < j-1} q_{i*} + q_{j-1} q_{j+1*}$. Since $q_{j-1} < u$, we have $u \not\leq \kappa(q_{j-1})$. However, $p + \sum_{i < j-1} q_{i*} \leq \kappa(q_{j-1})$ as in (v) above, so that $q_{j-1} q_{j+1*} \not\leq \kappa(q_{j-1})$, which can only happen if $q_{j-1} \leq q_{j+1*}$. It follows from this that $j+1 < k$, for otherwise (i.e., if $j+1 = k$) we would have $q_{j-1*} < q_{j-1} < q_{k*} \kappa(p) = p_* < q_j$, contrary to 2.1(i). But then $q_{j+1*} = q_{j+1}(q_{j+1*} + q_{j+2*})$, and

$$q_{j+1}(q_{j+1*} + q_{j+2*}) \leq p + \sum_{i < j-1} q_{i*} + q_{j-1}$$

is a W-failure. To see this, first note that $p_* < q_{j*} \leq \kappa(q_{j+1})$, whence by the free star principle $p \leq \kappa(q_{j+1})$. Thus the whole right-hand side is below $\kappa(q_{j+1})$, and not above q_{j+1} . Likewise the right-hand side is below $\kappa(q_j)$ while, by 2.1(iv), $q_{j+2*} \not\leq \kappa(q_j)$. Therefore $q_{j+2*} \not\leq p + \sum_{i < j-1} q_{i*} + q_{j-1}$. Now considering the part of the right-hand side obtained by replacing q_{j-1} with q_{j-1*} , note $p + \sum_{i < j-1} q_{i*} + q_{j-1*} \leq \kappa(q_{j-1})$, while by 2.1(iv) the left-hand side $q_{j+1*} \not\leq \kappa(q_{j-1})$. Thus q_{j+1*} is not below any of those terms, which leaves us to deal with the only remaining possibility, $q_{j+1*} = q_{j-1}$. Supposing that to hold, we have $q_{j+1*} \leq \kappa(p)$, whence by the free star principle $q_{j+1} \leq \kappa(p)$. Note also that $q_{j+1*} \not\leq p_*$, for $p_* \not\leq q_{j+1**} = q_{j-1*}$ by the choice of j , and $p_* \neq q_{j+1*} = q_{j-1}$ since $p_* \leq q_{j*}$ and $q_{j-1} \not\leq q_{j*}$ (as a consequence of 2.1(i)). Thus, using induction, we may apply our lemma to the shorter B -type sequence $\langle q_{j+1}, \dots, q_k \rangle$ to obtain $\sum_{j+1 \leq i < k} q_i \leq \kappa(p)$. Now also we are assuming $\sum_{i < j} q_i \leq \kappa(p)$ and $\sum_{i < k} q_i \not\leq \kappa(p)$. Therefore $q_j \not\leq \kappa(p)$, and since $p_* \leq q_j$ that means $p \leq q_j$. In fact, $p < q_j$ since elements of a minimal cycle are of course distinct. Note that $j+1 < k$ since $q_{j+1} \leq \kappa(p)$ and $q_k \not\leq \kappa(p)$. Hence, using 2.1(iv) and $(\dagger)$, we may calculate

$$q_j \leq q_{j*} + q_{k*} \leq \left[p + \sum_{i < j-1} q_{i*} + q_{j-1} \right] + \kappa(p)^* = \kappa(p)^*.$$

Thus we have shown that $q_j A p$, contrary to the minimality of our cycle. We conclude that $q_{j+1*} \not\leq p + \sum_{i < j-1} q_{i*} + q_{j-1}$.

We are not finished yet with the inclusion $(\dagger)$, as we must show that q_{j*} is not below one of the terms on the right-hand side. Now $q_{j*} \not\leq q_{i*}$ for $i < j-1$ since j is minimal such that $p_* \leq q_{j*}$, and of course $q_{j*} \neq q_{j-1}$ by 2.1(i). Suppose $q_{j*} \leq p$. Then since $p_* \leq q_{j*}$ we have either $q_{j*} = p$ or $q_{j*} = p_*$. Also this implies $j < k-1$, for by 2.1(i) we cannot have $q_{k-1*} \leq p < q_k$. Hence, using 2.1(iv), $q_j \leq q_{j*} + q_{k*} \leq p + q_{k*} = q_{k*}$. In case $q_{j*} = p$, this means $p < q_j < q_k \leq \kappa(p)^*$, so that $q_j A p$, contrary to the minimality of our cycle. But if $q_{j*} = p_*$, then $q_j \leq \kappa(p)$ since $p_* \leq q_j$ and $p \not\leq q_j$, so we have $q_j \leq q_k \kappa(p) = p_* = q_{j*}$, a contradiction. Therefore $q_{j*} \not\leq p$.

We conclude that $(\dagger)$ does not hold. Hence hypothesis (vi) of Lemma 2.7 is valid, and that lemma applies to yield $\sum_{i < j} q_{i*} \not\leq \kappa(p)$. This, however, contradicts the conclusion which we obtained earlier by applying a modification of Lemma 2.4. Therefore $\sum_{i < k} q_i \leq \kappa(p)$, as desired.

LEMMA 3.6. *Let $\langle q_0, \dots, q_k \rangle$ with $k \geq 1$ be a B -type sequence, and let $q_k A_2 p$. If $q_0 \leq \kappa(p)$ and $q_{0*} \not\leq p_*$, then $p = q_{k*}$.*

PROOF. Suppose $p < q_{k*}$, and let us apply Lemma 2.7 with $t = q_{k*}$ and $j = k$ (so there is no minimality condition assumed on j this time). As in the proof of Lemma 3.5, we may assume $q_{0*} \not\leq p_*$, which is hypothesis 2.7(ii). Of the remaining hypotheses, (i), (iii)–(v) and (vii) follow immediately from our assumptions, leaving (vi) for us to verify.

If (vi) fails, we have

$$q_{k*} = q_k \kappa(q_k) \leq p + \sum_{i < k-1} q_{i*} + q_{k-1},$$

in which case we may apply (W). Now the right-hand side is below $\kappa(q_k)$ by 2.1(ii); hence $q_k \not\leq p + \sum_{i < k-1} q_{i*} + q_{k-1}$. Skipping over the second possible inclusion for a moment, we note that $q_{k*} \not\leq p$ since $p < q_{k*}$ by assumption. For $i < k-1$, 2.1(iv) says that $q_{k*} \not\leq \kappa(q_i)$, whence $q_{k*} \not\leq q_{i*}$. Moreover $q_{k*} \not\leq q_{k-1}$, for 2.1(i) implies $q_{k*} \neq q_{k-1}$, whence we need only show $q_{k*} \not\leq q_{k-1*}$. If $k = 1$, $q_{1*} \not\leq q_{0*}$ follows from $p_* < q_{1*}$ and $p_* \not\leq q_{0*}$; while if $k > 1$, $q_{k*} \not\leq q_{k-1*}$ because $q_{k*} \not\leq \kappa(q_{k-2})$ by 2.1(iv) and $q_{k-1*} \leq \kappa(q_{k-2})$ by 2.1(iii).

So suppose $\kappa(q_k) \leq p + \sum_{i < k-1} q_{i*} + q_{k-1}$, whence by 2.1(ii), $\kappa(q_k) = p + \sum_{i < k-1} q_{i*} + q_{k-1}$. Then we claim that $\kappa(p)[\sum_{i < k} q_i] \not\leq \kappa(q_k)$, for otherwise we could apply (W) to the inclusion $\kappa(p)[\sum_{i < k} q_i] \leq p + \sum_{i < k-1} q_{i*} + q_{k-1}$ to obtain a contradiction. For $\kappa(p)$ is not below $\kappa(q_k)$, since $q_k A_2 p$ implies $\kappa(q_k) B^d \kappa(p)$, which makes these elements incomparable. Of course $q_k \not\leq \kappa(q_k)$, wherefore $\sum_{i < k} q_i \not\leq \kappa(q_k)$. Observe, using Lemma 3.5, that $p_* + \sum_{i < k} q_i \leq \kappa(p)[\sum_{i < k} q_i]$, whence it will suffice to show that $p_* + \sum_{i < k} q_i$ is not below any term on the right-hand side. Since $p < q_k$ and $q_{k-1} \not\leq q_k$ (by 2.1(i)), $p_* + \sum_{i < k} q_i \not\leq p$. Because $p_* \not\leq q_{0*}$, and $q_{i-1} \not\leq q_{i*}$ for $i > 0$ (by 2.1(i) again), we have $p_* + \sum_{j < k} q_j \not\leq q_{i*}$ for $0 \leq i < k-1$. It remains to show that $p_* + \sum_{i < k} q_i \not\leq q_{k-1}$. If $k = 1$, note $p_* \neq q_0$ since $p_* < q_k = q_1$ and $q_0 \not\leq q_1$; combined with $p_* \not\leq q_{0*}$ this yields $p_* \not\leq q_0 = q_{k-1}$. If $k > 1$, the statement follows from $q_{k-2} \not\leq q_{k-1}$.

Therefore, using the above claim, Lemma 1.1(ii) and Lemma 3.5, we calculate that

$$\sum_{i < k} q_i = q_{k*} + \kappa(p) \left[\sum_{i < k} q_i \right].$$

Hence by $(SD_{\vee})$, since $q_k \kappa(p) = p_* < q_{k*}$, $\sum_{i < k} q_i = \sum_{i < k} q_i + q_{k*} \leq \kappa(q_k)$, which is of course a contradiction.

Thus $\kappa(q_k) \not\leq p + \sum_{i < k-1} q_{i*} + q_{k-1}$, so hypothesis (vi) of Lemma 2.7 holds. We conclude by that lemma that $\sum_{i < k} q_{i*} \not\leq \kappa(p)$, which contradicts Lemma 3.5. Hence $p = q_{k*}$.

LEMMA 3.7. Let $\langle q_0, \dots, q_k \rangle$ with $k \geq 1$ be a B-type sequence, and let $q_k A_2 p_0 B p_1$. If $q_0 \leq \kappa(p_0)$ and $q_{0*} \not\leq p_{0*}$, then $\sum_{i < k} q_i \leq p_0 + p_1$.

PROOF. Supposing that $\sum_{i < k} q_i \not\leq p_0 + p_1$, we will apply Lemma 2.3. Let

$$\begin{aligned} a &= q_k, & a_0 &= \left(\sum_{i < k} q_i \right) (p_0 + p_1), \\ b &= p_{0*} + p_{1*}, & b_0 &= \kappa(p_0) [p_0 + p_1]. \end{aligned}$$

Choose a_1 such that $a \leq a_1 < a_0$. (We will show in (ii) below that this is possible. The argument for (v) below shows that a_0 is meet-reducible, whence $a_0 \in J(L)$ and $a_1 = a_{0*}$.)

Now let us check conditions (i)' and (ii)–(vii) of Lemma 2.3.

(i) Certainly $a = q_k \in J(L)$. Note that by Lemma 3.6, $a_* = q_{k*} = p_0$. We will argue below that $p_{1*} \not\leq p_{0*}$ (i.e., $p_0 B_1 p_1$). On the one hand, this will show that b is join-reducible, since $p_{0*} \not\leq p_{1*}$ by 2.1(i). On the other hand, it also enables us to apply Lemma 7.3 of [15] to obtain $q_k \leq p_0 + p_1$. (To prove this claim directly, apply Lemma 2.5 with $a = p_0 + p_1 (= p_{0*} + p_1)$, $b = \kappa(p_0)$, $c = q_k$.) Since $p_0 = q_{k*} \leq \kappa(q_k)$, this means that $p_1 \not\leq \kappa(q_k)$, whence by the free star principle $p_{1*} \not\leq \kappa(q_k)$. Therefore $q_k \leq q_{k*} + p_{1*} = p_0 + p_{1*}$, i.e., $a \leq a_* + b$.

Assume $p_{1*} \leq p_{0*}$. Let us apply (W) to the inclusion

$$q_k(p_0 + q_{k-1}) = p_0 \leq p_{0*} + p_1.$$

(This is easily verified using 2.1(ii), Lemma 3.6 and $p_0 B p_1$.) If $q_k \leq p_{0*} + p_1$, then as above $q_k \leq p_0 + p_{1*} = p_0 < q_k$, a contradiction. If $q_{k-1} \leq p_{0*} + p_1$, then since $p_{0*} < q_{k*} \leq \kappa(q_{k-1})$ by 2.1(iii), we must have $p_1 \not\leq \kappa(q_{k-1})$. But then the free star principle yields $p_{1*} \not\leq \kappa(q_{k-1})$, contrary to $p_{1*} \leq p_{0*} \leq \kappa(q_{k-1})$. Of course $p_0 \not\leq p_{0*}$, and $p_0 \not\leq p_1$ because $p_0 B p_1$ (see Figure 1). Thus the assumption $p_{1*} \leq p_{0*}$ leads to a W-failure, wherefore we conclude $p_{1*} \not\leq p_{0*}$.

(ii) By what we have done so far, $a \leq a_0$. To see that $a \neq a_0$, suppose otherwise, in which case we may apply (W) to the inclusion

$$\left(\sum_{i < k} q_i \right) (p_0 + p_1) = q_k \leq p_0 + p_{1*}.$$

We are assuming that $\sum_{i < k} q_i \not\leq p_0 + p_1$, and $p_1 \not\leq p_0 + p_{1*}$ since $p_0 + p_{1*} \leq \kappa(p_1)$ by 2.1(ii) (or see Figure 1). On the other hand, $q_k \not\leq p_0 = q_{k*}$ and $q_k \not\leq p_{1*}$, for else $p_{0*} < q_k < p_1$, contrary to 2.1(i). Thus the assumption $a = a_0$ leads to a W-failure. We conclude that $a < a_0$, and a_1 may be chosen with $a \leq a_1 < a_0$.

(iii) Clearly $b \leq b_0$, for by definition $p_0 B p_1$ implies $p_{1*} \leq \kappa(p_0)$.

(iv) If $a \leq b_0$, we would have $p_0 < q_k = a \leq b_0 \leq \kappa(p_0)$, a contradiction. Thus $a \not\leq b_0$.

(v) If $b \leq a_0$, then $p_{1*} \leq \sum_{i < k} q_i$. Therefore, using $q_k \leq p_0 + p_{1*}$, we see that $\sum_{i < k} q_i = \sum_{i < k} q_i + p_0 + p_{1*}$, whence by $(SD_{\vee})$

$$\sum_{i < k} q_i = \sum_{i < k} q_i + p_0 + q_k p_{1*}.$$

However, $q_k \not\leq p_{1*}$ since $p_0 < q_k$ and $p_0 \not\leq p_{1*}$, whence $q_k p_{1*} \leq q_{k*} = p_0$. Thus, using 2.1(ii), $\sum_{i < k} q_i = \sum_{i < k} q_i + p_0 \leq \kappa(q_k)$, a contradiction. Hence $b \not\leq a_0$.

(vi) If $a_0 \leq a_1 + b_0$, then we may apply (W) to the inclusion

$$\left(\sum_{i < k} q_i \right) (p_0 + p_1) \leq a_1 + \kappa(p_0)[p_0 + p_1].$$

Now $\sum_{i \leq k} q_i$ is not below the right-hand side, since $a_1 + \kappa(p_0)[p_0 + p_1] \leq p_0 + p_1$, while by assumption $\sum_{i \leq k} q_i \not\leq p_0 + p_1$. If $p_1 \leq a_1 + b_0 \leq a_0 + b_0$, then we may easily see that $p_0 + p_1 = a_0 + b_0$, i.e.,

$$p_0 + p_1 = \left(\sum_{i \leq k} q_i \right) (p_0 + p_1) + \kappa(p_0)[p_0 + p_1].$$

Applying $(SD_{\vee})$ to this, we obtain

$$p_0 + p_1 = p_0 + p_1 \left(\sum_{i \leq k} q_i \right) + p_1 \kappa(p_0).$$

But $p_1 \not\leq \sum_{i \leq k} q_i$ by the argument of (v), and $p_1 \not\leq \kappa(p_0)$ by the definition of $p_0 B p_1$ (although $p_{1*} \leq \kappa(p_0)$). Therefore $p_1(\sum_{i \leq k} q_i) + p_1 \kappa(p_0) = p_{1*}$, and $p_0 + p_1 = p_0 + p_{1*} \leq \kappa(p_1)$, a contradiction. Hence $p_1 \not\leq a_1 + b_0$. Continuing with (W), of course the left-hand side $a_0 \not\leq a_1$, and since $a \leq a_0$, we have $a_0 \not\leq b_0$ by (iv). We conclude that $a_0 \not\leq a_1 + b_0$.

(vii) If $b_0 \leq a_1 + b$, then we may apply (W) to the inclusion

$$\kappa(p_0)[p_0 + p_1] \leq a_1 + p_{1*}$$

since $p_{0*} < a \leq a_1$. If $\kappa(p_0) \leq a_1 + p_{1*}$, then using Lemma 3.5 we have $\sum_{i \leq k} q_i \leq \kappa(p_0) \leq a_1 + p_{1*} \leq p_0 + p_1$. Since also $q_k \leq p_0 + p_1$, this implies $\sum_{i \leq k} q_i \leq p_0 + p_1$, contrary to assumption. If $p_1 \leq a_1 + p_{1*} = a_1 + b$, then again we would obtain $p_1 + p_0 = a_0 + b_0$, a possibility eliminated in the argument of (vi). On the other hand, since $b \leq b_0$ and $a_1 < a_0$, whereas $b \not\leq a_0$ by (v), we have $b_0 \not\leq a_1$. Finally, $p_{0*} \leq \kappa(p_0)[p_0 + p_1]$ while $p_{0*} \not\leq p_1$, whence $b_0 \not\leq p_{1*}$.

We conclude from Lemma 2.3 that this situation cannot occur in an S -lattice, whence $\sum_{i \leq k} q_i \leq p_0 + p_1$.

With these preliminaries out of the way, we are now in a position to take care of the case when $q_{0*} \not\leq p_{**}$. Roughly speaking, what we will show is this. If $\langle q_0, \dots, q_k \rangle$ is a B -type sequence which is either a sequence of B 's or arises from previous applications of this lemma, and if $q_k A_2 p_0 B p_1 B \dots B p_m$, and if furthermore $q_{0*} \not\leq p_{0**}$ (so that in particular, Lemma 3.4 does not apply), then by dropping q_k we can obtain a new, longer B -type sequence $\langle q_0, \dots, q_{k-1}, p_0, \dots, p_m \rangle$. (Of course, we still do not have a minimal cycle without q_k , for $q_{k-1} C p_0$ does not hold.) More precisely:

LEMMA 3.8. *Let*

$$\begin{array}{ccccccc} & r_{10} & B & \dots & B & r_{1k_1} \\ A_2 & r_{20} & B & \dots & B & r_{2k_2} \\ & & & \vdots & & \\ & & & \vdots & & \\ A_2 & r_{n0} & B & \dots & B & r_{nk_n}, \end{array}$$

with $n \geq 1$ and $k_i \geq 1$ for $1 \leq i \leq n$, be a subsequence of a minimal cycle in an S -lattice (where the above notation means that $r_{ik_i} A_2 r_{i+1,0}$ for $1 \leq i < n$). If $r_{10} \leq \kappa(r_{10})$ and $r_{10*} \not\prec r_{i0*}$ for $1 < i \leq n$, then the sequence obtained by omitting every r_{ik_i} with $1 \leq i < n$,

$$(\dagger\dagger) \quad \langle r_{10}, \dots, r_{1,k_1-1}, r_{20}, \dots, r_{i,k_i-1}, r_{i+1,0}, \dots, r_{nk_n} \rangle$$

is a B -type sequence.

PROOF. We proceed by induction on n . If $n = 1$, then there are no A_2 's, and our lemma reduces to Lemma 2.1. Thus we may assume that $n > 1$ and the initial segment $\langle r_{10}, \dots, r_{i,k_i-1}, r_{i+1,0}, \dots, r_{n-1,k_{n-1}} \rangle$ of $(\dagger\dagger)$ is a B -type sequence. (This is the inductive hypothesis to which we will refer throughout the proof.) We must verify that conditions (i)–(iv) of Lemma 2.1 hold for our longer sequence.

Condition (i) requires that we show $p_* \not\leq q$ for each consecutive pair p, q from our sequence. This follows from the inductive hypotheses for consecutive pairs with p and q between r_{10} and $r_{n-1,k_{n-1}-1}$, and from Lemma 2.1(i) for pairs between r_{n0} and r_{nk_n} . For the remaining pair, we have $r_{n-1,k_{n-1}-1*} \not\leq r_{n0}$, because $r_{n-1,k_{n-1}-1} B r_{n-1,k_{n-1}} A_2 r_{n0}$ implies $r_{n0} < r_{n-1,k_{n-1}}$ but $r_{n-1,k_{n-1}-1*} \not\leq r_{n-1,k_{n-1}}$.

For (ii), we must show that for any q in our sequence

$$\sum \{p: p \text{ precedes } q\} \leq \kappa(q).$$

Again, for q between r_{10} and $r_{n-1,k_{n-1}-1}$ this follows from the inductive hypothesis. For $q = r_{n0}$, Lemma 3.5 applies to yield the desired result. But for $q = r_{ni}$ ($1 \leq i \leq k_n$), note that Lemma 1.3(ii) implies $\kappa(r_{n,i-1}) \leq \kappa(r_{ni})$, whence we may calculate (using induction and 2.1(ii))

$$\begin{aligned} \sum \{p: p \text{ precedes } r_{ni}\} &= \sum \{p: p \text{ precedes } r_{n,i-1}\} + r_{n,i-1} \\ &\leq \kappa(r_{n,i-1}) + r_{n,i-1} \leq \kappa(r_{ni}). \end{aligned}$$

Condition (iii) requires that for each consecutive pair p, q from our sequence $q_* \leq \kappa(p)$. As in (i), every pair except $r_{n-1,k_{n-1}-1}, r_{n0}$ is covered by either the inductive hypothesis or Lemma 2.1(iii). For this pair, we simply note that $r_{n0*} < r_{n-1,k_{n-1}-1*} \leq \kappa(r_{n-1,k_{n-1}-1})$, as desired.

For (iv), we must show that $q_* \not\leq \kappa(p)$ whenever p precedes q by at least two places in our sequence. If both p and q lie between r_{10} and $r_{n-1,k_{n-1}-1}$, or between r_{n0} and r_{nk_n} , then this is immediate. Therefore we may assume that $p = r_{im}$ for some $i < n$, and $q = r_{nj}$.

If $j = 0$, let r_{im} precede $r_{n-1,k_{n-1}-1}$, and note that Lemma 3.6 applies to yield $r_{n0} = r_{n-1,k_{n-1}-1*}$. Since the segment $\langle r_{10}, \dots, r_{s,n_s-1}, r_{s+1,0}, \dots, r_{n-1,k_{n-1}} \rangle$ is a B -type sequence, we have $r_{n0} = r_{n-1,k_{n-1}-1*} \not\leq \kappa(r_{im})$, whence by the free star principle $r_{n0*} \not\leq \kappa(r_{im})$.

So assume $j \geq 1$. Since $\kappa(r_{i0}) \leq \kappa(r_{i1}) \leq \dots \leq \kappa(r_{i,k_i-1})$ by Lemma 1.3(ii), while r_{ik_i} is not in our sequence, it suffices to show that $r_{nj*} \not\leq \kappa(r_{i,k_i-1})$ for each $i < n$. Moreover, by virtue of the free star principle, we need only show that $r_{nj} \not\leq \kappa(r_{i,k_i-1})$ for $i < n$.

If $j = 1$ and $i = n - 1$, we may apply Lemma 3.7 to obtain $r_{n-1, k_{n-1}-1} \leq r_{n0} + r_{n1}$. Since $r_{n0} = r_{n-1, k_{n-1}-1} \leq \kappa(r_{n-1, k_{n-1}-1})$, we must have $r_{n1} \not\leq \kappa(r_{n-1, k_{n-1}-1})$. Thus we may assume that either $j > 1$ or $i < n - 1$.

The final argument for (iv) is based on the following special case. If $r_{i+1,0} \leq r_{nj*}$, then $r_{nj} \not\leq \kappa(r_{i, k_i-1})$. To prove this claim, suppose that $r_{i+1,0} \leq r_{nj*}$ and $r_{nj} \leq \kappa(r_{i, k_i-1})$, and let us apply Lemma 2.7 with the following substitutions.

$$\langle q_1, \dots, q_k \rangle \leftrightarrow \langle r_{10}, \dots, r_{s, n_i-1}, r_{s+1,0}, \dots, r_{ik_i} \rangle, \quad p \leftrightarrow r_{i+1,0}, \\ t \leftrightarrow \kappa(r_{i+1,0})^* [r_{i, k_i-1*} + r_{nj}], \quad j \leftrightarrow (i, k_i).$$

Condition 2.7(i) follows from our inductive hypothesis. For 2.7(ii), note that $r_{i+1,0*} \not\leq r_{10*}$ by one of the hypotheses of Lemma 3.8, while equality is excluded by the remarks beginning the proof of Lemma 3.5. Condition 2.7(iii) is a consequence of $r_{ik_i} A r_{i+1,0}$.

For 2.7(iv), we clearly have $r_{i+1,0} \leq \kappa(r_{i+1,0})^* [r_{i, k_i-1*} + r_{nj}] \leq \kappa(r_{i+1,0})^*$ since $r_{i+1,0} < r_{nj}$. Moreover, the first inequality is strict because $r_{i, k_i-1*} \leq \kappa(r_{i+1,0})^* [r_{i, k_i-1*} + r_{nj}]$ by Lemma 3.5, while $r_{i, k_i-1*} \not\leq r_{i+1,0} = r_{i, k_i*}$ by 2.1(i).

Condition 2.7(v) follows from our assumption that $r_{nj} \leq \kappa(r_{i, k_i-1})$.

If 2.7(vi) fails, then we may apply (W) to the inclusion

$$\kappa(r_{i+1,0})^* [r_{i, k_i-1*} + r_{nj}] \leq r_{i+1,0} + \sum \{ r_{\alpha\beta*} : r_{\alpha\beta} \text{ precedes } r_{i, k_i-1} \} + r_{i, k_i-1}.$$

Now $\kappa(r_{i+1,0})^*$ is not below the right-hand side (RHS), since $r_{i, k_i} \leq \kappa(r_{i+1,0})^*$ while by the inductive hypothesis (using 2.1(ii)) $\text{RHS} \leq \kappa(r_{i, k_i})$. Similarly, since $\text{RHS} \leq \kappa(r_{nj})$ by condition 2.1(ii), which was proved above, we have $[r_{i, k_i-1*} + r_{nj}] \not\leq \text{RHS}$. On the other hand, recall from 2.7(iv) that $r_{i+1,0}$ is strictly below the left-hand side (LHS). Thus $\text{LHS} \not\leq r_{i+1,0}$. Moreover, since $r_{i+1,0} = r_{i, k_i*} \not\leq \kappa(r_{\alpha\beta})$ for $r_{\alpha\beta}$ preceding r_{i, k_i-1} by 2.1(iv) of the inductive hypothesis, we have $\text{LHS} \not\leq r_{\alpha\beta*}$. Finally, suppose $\text{LHS} \leq r_{i, k_i-1}$, whence $r_{i+1,0} = r_{i, k_i*} \leq r_{i, k_i-1}$. Then $r_{i, k_i-1} B_2 r_{i, k_i}$. It is easy to see that in a sequence of B 's, only the first B can be a B_2 . (For if $p B q B u$ and $u_* \leq q$, then $u_* \neq q$, so $u_* \leq q_* \leq \kappa(p)$, contrary to 2.1(iv). This is Lemma 7.6 of [15].) Therefore $k_i = 1$. Then $i \neq 1$, for if $r_{10} B_2 r_{11} A_2 r_{20}$ we would have $r_{20*} < r_{11*} < r_{10*}$, contrary to one of our original hypotheses. However, if $i > 1$ we have $r_{i0} B_1 r_{i1}$, because by the inductive hypothesis $r_{i0*} \leq \kappa(r_{i-1, k_{i-1}-1})$ and $r_{i1*} \not\leq \kappa(r_{i-1, k_{i-1}-1})$, whence $r_{i1*} \not\leq r_{i0*}$. Thus $\text{LHS} \not\leq r_{i, k_i-1}$, and 2.7(vi) holds.

For 2.7(vii), $r_{10} \leq \kappa(r_{i+1,0})$ implies $r_{i+1,0} B r_{10}$ does not hold.

Thus Lemma 2.7 applies to yield $\sum \{ r_{\alpha\beta*} : r_{\alpha\beta} \text{ precedes } r_{i, k_i} \} \not\leq \kappa(r_{i+1,0})$. This, however, contradicts property 2.1(ii), which we have already proved. Therefore $r_{nj} \not\leq \kappa(r_{i, k_i-1})$ whenever $r_{i+1,0} \leq r_{nj*}$, as claimed.

Now fix $j \geq 1$. If 2.1(iv) fails with $q = r_{nj}$, then we may choose $i < n$ maximal such that $r_{nj} \leq \kappa(r_{i, k_i-1})$. Since $\kappa(r_{i, k_i-1}) \leq \kappa(r_{i, k_i})$ by Lemma 1.3(ii), we have $r_{nj} \leq \kappa(r_{i, k_i})$, so that $r_{i, k_i} \not\leq r_{i, k_i*} + r_{nj} = r_{i+1,0} + r_{nj}$. Thus $r_{i, k_i}(r_{i+1,0} + r_{nj}) = r_{i+1,0}$. On the other hand, $r_{nj*} \not\leq \kappa(r_{i+1,0})$. If $i < n - 1$, this follows from the maximality of i , the fact that $\kappa(r_{i+1,0}) \leq \kappa(r_{i+1, k_{i+1}-1})$, and the free star principle. If $i = n - 1$, we have shown that $j > 1$, whence $r_{nj*} \not\leq \kappa(r_{n0})$ follows from 2.1(iv). Thus $r_{i+1,0} \leq r_{i+1,0*} + r_{nj*}$. Combining these relations, we obtain the inclusion

$$r_{i, k_i}(r_{i+1,0} + r_{nj}) = r_{i+1,0} \leq r_{i+1,0*} + r_{nj*},$$

to which we may apply (W). Recall $r_{ik_i} \not\leq r_{i+1,0} + r_{nj}$ from above, whence r_{ik_i} is not below the right-hand side. By property 2.1(ii), which was proved above, we have $r_{i+1,0*} \leq \kappa(r_{nj})$, so that $r_{nj} \not\leq r_{i+1,0*} + r_{nj*}$; hence the second term is not below the right-hand side. Of course $r_{i+1,0} \not\leq r_{i+1,0*}$, while $r_{i+1,0} \not\leq r_{nj*}$ by the special case done earlier. Thus if (iv) fails, then so does (W), whence we conclude that (iv) must hold. Therefore $(\dagger\dagger)$ is a B -type sequence, as claimed.

4. The main result. In this section we will prove that an S -lattice cannot contain a cycle, which combined with Theorem 1.2 shows that every S -lattice can be embedded in a free lattice. We will follow the plan outlined in §1.

So suppose that L is an S -lattice containing a cycle. Since $p \leq A \leq q$ implies $p > q$, it is clear that we cannot have a cycle containing only A 's. By Lemma 1.3(ii), then neither can we have a cycle containing all B 's, for then L^d would contain a cycle with all A 's. So let

$$\begin{aligned} q_{10} B^{k_1} q_{1k_1} &= p_{10} A^{m_1} p_{1m_1} \\ &= q_{20} B^{k_2} q_{2k_2} = p_{20} A^{m_2} p_{2m_2} \\ &\quad \vdots \\ &= q_{n0} B^{k_n} q_{nk_n} = p_{n0} A^{m_n} p_{nm_n} = q_{10} \end{aligned}$$

be a minimal cycle in L , where $q_{10} B^{k_1} q_{1k_1}$ means that $q_{10} B q_{11} B \dots B q_{1k_1}$, etc.

Claim 4.1. $n > 1$. For suppose $n = 1$. Then $k_1 > 1$, otherwise we would have $q_{10} = p_{1m_1} < p_{10} = q_{11}$, whereas $q_{10} B q_{11}$ implies that these elements are incomparable. Lemma 1.3 then implies that also $m_1 > 1$, since A 's and B 's are interchanged in the dual cycle. If $p_{10} A_1 p_{11}$, Lemma 3.1 applies, yielding $q_{10*} > p_{11*} > p_{1m_1*}$. If $p_{10} A_2 p_{11}$, then, by Lemma 3.2, $p_{11} A_2 p_{12}$, so that Lemma 3.3 applies to give $q_{10*} > p_{12*} \geq p_{1m_1*}$. Thus in either case $q_{10*} > p_{1m_1*} = q_{10*}$, a contradiction. Hence $n > 1$.

Claim 4.2. The cycle contains no A_1 . For suppose (wlog) that say $p_{nj} A_1 p_{n,j+1}$ for some j , whence by Lemma 3.2 we have $p_{n0} A_1 p_{n1}$. Let us begin by considering the subsequence

$$q_{10} B^{k_1} q_{1k_1} = p_{10} A^{m_1} p_{1m_1}.$$

One of Lemmas 3.1, 3.3, 3.4 or 3.8 applies to this subsequence, with the consequence that either

(a) $q_{10**} \geq p_{1m_1**} = q_{20**}$ or

(b) $\langle q_{10}, \dots, q_{1,k_1-1}, q_{20}, \dots, q_{2k_2} \rangle$ is a B -type sequence. (Indeed, (a) holds unless Lemma 3.8 applies.) Now if (a) holds, we proceed to apply one of the same four lemmas to $q_{20} B^{k_2} q_{2k_2} = p_{20} A^{m_2} p_{2m_2}$, obtaining either

(a)' $q_{20**} \geq p_{2m_2**} = q_{30**}$ or

(b)' $\langle q_{20}, \dots, q_{2,k_2-1}, q_{30}, \dots, q_{3k_3} \rangle$, is a B -type sequence.

Otherwise (b) holds, in which case one of the four lemmas applies to the sequence

$$\langle q_{10}, \dots, q_{1,k_1-1}, q_{20}, \dots, q_{2k_2} = p_{20}, \dots, p_{2m_2} \rangle,$$

yielding one of the conclusions

(a)'' $q_{10**} \geq p_{2m_2**} = q_{30**}$ or

(b)'' $\langle q_{10}, \dots, q_{1,k_1-1}, q_{20}, \dots, q_{2,k_2-1}, q_{30}, \dots, q_{3k_3} \rangle$, is a B -type sequence.

Continuing in this manner, we obtain a sequence of indices $i_1 = 1 < i_2 < \dots < i_t < n$ (where possibly $t = 1$) such that

(a)''' $q_{10**} \geq q_{i_2 0**} \geq \dots \geq q_{i_t 0**}$ and

(b)''' $\langle q_{i_1 0}, \dots, q_{j, k_j-1}, q_{j+1, 0}, \dots, q_{n k_n} \rangle$ is a B -type sequence.

At this point, since we have (b)''' and $q_{n k_n} = p_{n0} A_1 p_{n1}$, Lemma 3.1 applies to yield $q_{i_t 0**} > p_{n1**} > p_{n k_n **} = q_{10**}$, contradicting (a)'''. Therefore the cycle contains no A_1 .

If we assume instead that $p_{n0} A_2 p_{n1} A_2 p_{n2}$, the same arguments give (a)''' and (b)''', whence Lemma 3.3 yields $q_{i_t 0**} > p_{n2**} \geq p_{n k_n **} = q_{10**}$, with the same contradiction. Hence we have also shown

Claim 4.3. The cycle contains no consecutive pair of A_2 's.

We conclude then that $m_j = 1$ for all j , $1 \leq j \leq n$. Since these considerations also apply to the dual cycle, Lemma 1.3 gives us $k_j = 1$ for all j . Thus we may simplify notation by relabeling the cycle

$$q_1 B p_1 A_2 q_2 B \dots p_{n-1} A_2 q_n B p_n A_2 q_1.$$

Claim 4.4. Every B in the cycle is a B_1 . Suppose on the contrary that say $q_n B_2 p_n$, i.e., $p_{n*} < q_{n*}$. Then as before we can obtain a sequence of indices $i_1 = 1 < i_2 < \dots < i_t < n$ such that

(a)'''' $q_{1**} \geq q_{i_2 **} \geq \dots \geq q_{i_t **}$ and

(b)'''' $\langle q_{i_1}, \dots, p_{n-1} \rangle$ is a B -type sequence.

Now either Lemma 3.4 or Lemma 3.7 applies to $\langle q_{i_1}, \dots, p_{n-1} \rangle$ and $p_{n-1} A_2 q_n$. If Lemma 3.4 applies, we obtain immediately $q_{i_t **} \geq q_{n**}$. If Lemma 3.7 applies, we must observe that $p_{n-1} \not\leq q_n + p_n$. For $q_n \leq p_{n-1*} \leq \kappa(p_{n-1})$, whence also $p_{n*} < q_{n*} \leq \kappa(p_{n-1})$. By the free star principle, $p_n \leq \kappa(p_{n-1})$, whence $q_n + p_n \leq \kappa(p_{n-1})$. Therefore by Lemma 3.7, $q_{i_t*} > q_{n*}$. Thus in either case we have $q_{i_t **} \geq q_{n**}$. But then $q_{1**} \geq q_{i_t **} \geq q_{n**} > p_{n**} > q_{1**}$, a contradiction. Therefore $q_j B_1 p_j$ for all j , $1 \leq j \leq n$.

At this point, we recall Lemma 7.3 of [15]: $p A_2 q B_1 r$ implies $p \leq q + r$ (cf. the proof of Lemma 3.7 above). Thus $p_{i-1} \leq q_i + p_i$ for all i , $1 \leq i \leq n$, where $p_0 \equiv p_n$. Also, to avoid a degenerate case in our final argument, we borrow our next claim from Lemma 8.5 of [15].

Claim 4.5. $n > 2$. For suppose $q_1 B_1 p_1 A_2 q_2 B_1 p_2 A_2 q_1$. Then using the above remark we see that

$$p_1 + p_2 = q_1 + p_1 = q_2 + p_2,$$

whence, by $(SD_\vee)$, $p_1 + p_2 = q_1 + q_2 + p_1 p_2$. Now $p_1 \not\leq p_2$ since $q_2 \leq p_1$ and $q_2 \not\leq p_2$, so $p_1 p_2 \leq p_{1*}$, wherefore

$$p_1 + p_2 \leq q_1 + p_{1*} \leq \kappa(p_1),$$

a contradiction.

Thus the following lemma will apply to our cycle.

LEMMA 4.6. Let $\langle r_0, \dots, r_5 \rangle$ be a subsequence of a minimal cycle such that $r_0 A_2 r_1 B_1 r_2 A_2 r_3 B_1 r_4 A_2 r_5$. Then

- (i) $r_0 \leq r_2 + r_4$ and
- (ii) $r_4 \not\leq r_0 + r_2$.

PROOF. As above, Lemma 7.3 of [15] gives us $r_0 \leq r_1 + r_2$ and $r_2 \leq r_3 + r_4$, whence in particular $r_4 \not\leq \kappa(r_2)$ since $r_3 \leq r_{2*} \leq \kappa(r_2)$. By the free star principle, $r_{4*} \not\leq \kappa(r_2)$. Now $r_2 \neq r_{4*}$ since $r_3 \leq r_2$ and $r_3 \not\leq r_{4*}$. Thus if $r_{4**} \leq \kappa(r_2)$, we would have $r_2 B r_{4*}$. If $r_{4*} = r_5$, this immediately contradicts the minimality of the cycle; otherwise $r_{4*} > r_5$, in which case $r_{4*} A r_5$ (since $r_5 < r_{4*} < r_4 \leq \kappa(r_5)^*$), again shortening the cycle. Therefore $r_{4**} \not\leq \kappa(r_2)$, i.e., $r_2 \leq r_{2*} + r_{4**}$. It follows in particular from this last statement that $r_{4**} \not\leq r_{3*}$, since $r_{3*} \leq r_{2*}$.

We can now prove a strong version of 4.6(ii). Note that $r_1 < r_0 \leq r_1 + r_2$ implies $r_0 + r_2 = r_1 + r_2$. Suppose $r_{4**} \leq r_0 + r_2$. Then we would have $r_1 + r_2 = r_1 + r_{2*} + r_{4**}$ whence by $(SD_{\vee})$

$$r_1 + r_2 = r_1 + r_{2*} + r_2 r_{4**}.$$

Since $r_3 \leq r_2$ and $r_3 \not\leq r_4$, we have $r_2 \not\leq r_{4**}$, so that $r_2 r_{4**} \leq r_{2*}$. Thus $r_1 + r_2 = r_1 + r_{2*} \leq \kappa(r_2)$, a contradiction. Therefore $r_{4**} \not\leq r_0 + r_2$.

Now suppose that 4.6(i) fails, i.e., $r_0 \not\leq r_2 + r_4$. Then $(r_0 + r_2)(r_2 + r_4)$ is meet-reducible, and hence join-irreducible. Let us apply Lemma 2.3 with

$$\begin{aligned} a &= r_2, & a_0 &= (r_0 + r_2)(r_2 + r_4), & a_1 &= a_{0*}, \\ b &= r_{3*} + r_{4**}, & b_0 &= r_{3*} + r_{4*}. \end{aligned}$$

We must check conditions (i)' and (ii)–(vii).

(i)' Certainly $a = r_2 \in J(L)$, and $a = r_2 \leq r_{2*} + r_{4**} = a_* + b$ by the above remarks. Also $r_{3*} \not\leq r_{4**}$ since $r_3 B r_4$, while $r_{4**} \not\leq r_3$ was shown above, so b is join-reducible.

(ii) Clearly $a \leq a_0$; we need to show that $a < a_0$, i.e., $(r_0 + r_2)(r_2 + r_4) \neq r_2$. Otherwise, we could apply (W) to the inclusion

$$(r_0 + r_2)(r_2 + r_4) = r_2 \leq r_{2*} + r_{4*}.$$

Now $r_0 \not\leq r_2 + r_4$ by assumption, so the first term is not below the right-hand side. On the other hand $r_2 \not\leq r_{2*}$ and $r_2 \not\leq r_{4*}$ (since $r_3 < r_2$ and $r_3 B r_4$), which means that the second term must be below the right-hand side, i.e., $r_4 \leq r_2 + r_{4*}$. But then, using $r_3 \leq r_{2*} < r_2 \leq r_3 + r_4$, we have $r_3 + r_4 = r_{2*} + r_{4*}$. Applying $(SD_{\vee})$ we obtain

$$r_3 + r_4 = r_3 + r_{4*} + r_{2*} r_4.$$

However, $r_4 \not\leq r_{2*}$ since $r_{4**} \not\leq \kappa(r_2)$, so $r_{2*} r_4 \leq r_{4*}$. Thus $r_3 + r_4 = r_3 + r_{4*} \leq \kappa(r_4)$, a contradiction. Therefore $(r_0 + r_2)(r_2 + r_4) \neq r_2$, so that $a < a_0$ whence $a \leq a_{0*} = a_1$, as desired.

(iii) $b \leq b_0$ is clear.

(iv) We have $r_3 < r_2 = a$, while $b_0 = r_{3*} + r_{4*} \leq \kappa(r_3)$ since $r_3 B r_4$. Hence $a \not\leq b_0$.

(v) $b \not\leq a_0$ follows from $r_{4**} \not\leq r_0 + r_2$.

(vi) If $a_0 \leq a_1 + b_0$, then we may apply (W) to the inclusion

$$(r_0 + r_2)(r_2 + r_4) = a_0 \leq a_1 + r_{4*},$$

where we have used $r_{3*} < r_2 = a \leq a_1$. Now $r_0 \not\leq a_1 + r_{4*}$ since $a_1 + r_{4*} \leq r_2 + r_4$. On the other hand $a_0 \not\leq a_1$, and $a_0 \not\leq r_{4*}$ since $r_2 \not\leq r_{4*}$. Therefore we must have $r_4 \leq a_1 + r_{4*}$, whence $r_4 \leq a_0 + r_{4*}$. But then, arguing as above, $r_3 + r_4 = a_0 + r_{4*}$, whence by $(SD_{\vee})$

$$r_3 + r_4 = r_3 + r_{4*} + r_4 a_0 = r_3 + r_{4*} \leq \kappa(r_4)$$

since $r_4 \not\leq a_0$ because $a_0 \leq r_0 + r_2$. This is a contradiction, whereupon we conclude $a_0 \not\leq a_1 + b_0$.

(vii) If $b_0 \leq a_1 + b$, then $r_{4*} \leq a_1 + r_{4**}$ (where again we have used $r_{3*} \leq a_1$). Now the last argument in (vi) shows that $r_4 \not\leq a_0 + r_{4*}$. Hence $r_4(a_0 + r_{4*}) = r_{4*}$, so we may apply (W) to the inclusion

$$r_4(a_0 + r_{4*}) = r_{4*} \leq a_1 + r_{4**}.$$

Of course $r_4 \not\leq a_1 + r_{4**}$ since $r_4 \not\leq a_0 + r_{4*}$, while $a_0 \not\leq a_1 + r_{4**}$ since $a_0 \not\leq a_1 + r_{4*}$ by (vi). On the other hand, $r_{4*} \not\leq a_1$ because $r_{4**} \not\leq r_0 + r_2$, and obviously $r_{4*} \not\leq r_{4**}$. Thus the assumption $b_0 \leq a_1 + b$ leads to a W-failure, whence $b_0 \not\leq a_1 + b$.

By Lemma 2.3, this configuration cannot exist in an S -lattice. Therefore 4.6(i) holds, as desired.

Now let us apply Lemma 4.6 to our current situation. By repeated application of 4.6(i),

$$p_1 \leq p_2 + p_3 \leq (p_3 + p_4) + p_3 = p_3 + p_4 \leq \cdots \leq p_{n-1} + p_n.$$

However, if we let $r_0 = p_{n-1}$, 4.6(ii) says that $p_1 \not\leq p_{n-1} + p_n$. This contradiction eliminates our last possibility for the existence of a cycle in an S -lattice.

Thus Jónsson's conjecture is true.

THEOREM 4.7. *A finite lattice is a sublattice of a free lattice iff it is semidistributive and satisfies (W).*

In closing, we remark that most of the problems discussed in [15] have now been solved. A notable exception to this is the problem of characterizing arbitrary sublattices of a free lattice. Distributive sublattices of a free lattice were described by Galvin and Jónsson [7], and the arguments of [15], based in large part on Kostinsky [16], show that a finitely generated lattice is embeddable in a free lattice iff it satisfies (W) and the generators are contained in $D(L) \cap D'(L)$. Beyond this little is known except a few necessary conditions (see [15, 6]). Perhaps the countable case would be a good place to start.

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