WEAKLY RAMSEY P POINTS

BY

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ABSTRACT. If the continuum hypothesis (CH) holds, then for any n Ramsey P point D and any $k > 1$ there exist many $n + k$ Ramsey P points which are immediate Rudin-Keisler successors of D . There exist (CH) many 5 Ramsey P points whose constellations are not linearly ordered.

A nonprincipal ultrafilter D on ω is n Ramsey ($n \in \omega$, $n \geq 1$) if n is minimal for D with the property that for every partition $F: [\omega]^2 \rightarrow n + 1$, there is a set $A \in D$ such that F omits a color on $[A]^2$. D is *weakly Ramsey* if D is n Ramsey for some $n \geq 1$. In [D], Dagenet showed (assuming CH) that, for every $n \geq 1$, there exist n Ramsey P points which are 2 square (an n Ramsey ultrafilter is 2 square [2 rangé in French] if it has n nonstandard constellations). In this paper, we show (CH) that there exist many weakly Ramsey P points which are not 2 square (this answers a question of Dagenet [D]). The underlying technique is a well-known sort of inductive construction; the specifics are in §2. §3 contains results from finite combinatorics which are used in the proofs of the main theorems in §§4 and 5. Throughout the paper, D and E are ultrafilters on ω , and f, g, h are functions from ω (or ω^2) to ω . $\mathbf{N}$ is the complete first order structure on ω , and an element of the universe of the ultrapower $D \text{ prod } \mathbf{N}$ is denoted $[f]_D$. We identify $\mathbf{N}$ with its canonical image in $D \text{ prod } \mathbf{N}$, and if $f(E) = D$, f^* denotes the canonical embedding of $D \text{ prod } \mathbf{N}$ into $E \text{ prod } \mathbf{N}$.

1. We assume basic results about P points, the Rudin-Keisler ordering (denoted $\leq$), and ultrapowers of $\mathbf{N}$ and their submodels (as contained in, for example, [P] and [B1]). E is a *strong immediate successor* of D if $D < E$ and $(\forall D' < E) D' \leq D$. An n coloring is a partition $G: [\omega]^2 \rightarrow n$; G is *irreducible on D* if $(\forall A \in D) |G''[A]^2| = n$ (" $''$ " means set image); otherwise G is *reducible on D* . D is $\leq n$ Ramsey if D is principal or D is k Ramsey for some $k \leq n$; D is $> n$ Ramsey if D is not $\leq n$ Ramsey, and similarly for $\geq n$ Ramsey and $< n$ Ramsey. Then D is $\leq n$ Ramsey iff $(\forall k \geq 1)$ every $(n + k)$ coloring is reducible on D . The following theorem consists of straightforward generalizations of results in [B1], in which the term "weakly Ramsey" denotes what herein is defined as "Ramsey or 2 Ramsey".

Received by the editors November 4, 1980.

1980 *Mathematics Subject Classification*. Primary 04A20; Secondary 03H15.

¹The results in this paper form part of the author's Ph.D. thesis submitted to the Horace H. Rackham School of Graduate Studies in the University of Michigan, and written under the supervision of Andreas Blass, to whom the author expresses his deepest gratitude.

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0002-9947/82/0000-0763/\$04.25

THEOREM 1.1. *Suppose $f(E) = D$ and E is n Ramsey. Then D is $\leq n$ Ramsey, and D is n Ramsey iff f is an E -isomorphism (i.e. one-to-one on a set in E). If D is $n - 1$ Ramsey, then E is a strong immediate successor of D .*

Since an ultrafilter is minimal iff it is Ramsey, it follows that if D is weakly Ramsey, then $(\exists E \leq D)(E \text{ is Ramsey})$. Since $\text{ZFC} \nVdash (\exists \text{ a Ramsey ultrafilter})$ (see [K]), then $\text{ZFC} \nVdash (\exists \text{ a weakly Ramsey ultrafilter})$.

Suppose that E is a P point, D is Ramsey and $f(E) = D$. A theorem of Puritz [P] shows that any two nonstandard submodels of $E \text{ prod } \mathbb{N}$ have intersection cofinal in $E \text{ Prod } \mathbb{N}$, and it follows that $f^* \text{ prod } \mathbb{N}$ is included in every nonstandard submodel of $E \text{ prod } \mathbb{N}$. Equivalently, $\forall g(\exists A \in E)(g \text{ is constant on } A \text{ or } f \text{ is } g \text{ fiberwise constant on } A)$. It also follows that if h is f fiberwise one-to-one on a set in E , then in fact h is one-to-one on a set in E .

THEOREM 1.2. *With D , E , and f as above, let $F: [\omega]^2 \rightarrow M$ ($M \in \omega$). Then $(\exists A \in E)|F''\{\{x, y\} \in [A]^2: f(x) \neq f(y)\}| = 1$.*

PROOF. Let $B_i = \{n \in \omega: \{m > n: F\{n, m\} = i\} \in E\}$ ($i = 0, 1, \dots, M - 1$). Without loss of generality, $B_0 \in E$, and we may assume f is finite-to-one on B_0 . For each $n \in B_0$, let $A_n \in E$, $A_n \subseteq \{m \in B_0: m > n \text{ and } F\{n, m\} = 0\}$, such that $A_{n+1} \subseteq A_n$. Define g by $g(n) = \text{greatest } k \text{ such that } n \in A_k \text{ if } n \in A_0$, and $g(n) = 0$ if $n \notin A_0$. Then if $n \in A_0$, $g(n) = k$ implies that $k < n$ and for all $j \in B_0$, $j < k \rightarrow F\{j, n\} = 0$. It is easy to check that g is not constant on any set in E , so there is a set $B \in E$ such that f is g fiberwise constant on B ; we can assume $B \subseteq B_0$. Define a partition G by

$$G\{s, t\} = \begin{cases} 0 & \text{if } (\forall z \in f^{-1}\{s\} \cap B)(\forall w \in f^{-1}\{t\} \cap B)F\{z, w\} = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Since D is Ramsey, there is a set $S \in D$ such that G assumes only one value η on $[S]^2$. We claim $\eta = 0$ (and then the proof is complete by setting $A = B \cap f^{-1}(S)$). Suppose, for contradiction, that $\eta = 1$, and let x_0 be the least element of X . For each $y \in S$, $y > x_0$, there exist $n_y \in B \cap f^{-1}\{x_0\}$, $m_y \in B \cap f^{-1}\{y\}$ such that $F\{n_y, m_y\} \neq 0$. Since $B \cap f^{-1}\{x_0\}$ is finite and S is infinite, $(\exists \bar{n} \in B \cap f^{-1}\{x_0\})(\bar{n} = n_y \text{ for infinitely many } y \in S)$. Since there is at most one m_y in each fiber of f , and since $g \upharpoonright B$ takes on different values on different fibers of f , there is a $\bar{y} \in S$ such that $\bar{n} = n_{\bar{y}}$, and $g(m_{\bar{y}}) > \bar{n} = n_{\bar{y}}$. But then $m_{\bar{y}} \in A_{g(m_{\bar{y}})} \subseteq A_{\bar{n}}$, so $F\{n_{\bar{y}}, m_{\bar{y}}\} = 0$, a contradiction. $\square$

COROLLARY 1.3. *With D , E , f as in the theorem, the range of f^* is coinital in the nonstandard part of $E \text{ prod } \mathbb{N}$.*

PROOF. Let $[h]_E \in E \text{ prod } \mathbb{N} - \mathbb{N}$. Define $G: [\omega]^2 \rightarrow 3$ by

$$G\{x, y\} = \begin{cases} 0 & \text{if } f(x) < f(y) \text{ and } h(x) < h(y), \\ 1 & \text{if } f(x) < f(y) \text{ and } h(x) > h(y), \\ 2 & \text{if } f(x) = f(y). \end{cases}$$

By the theorem, $\exists A \in E$ such that G assumes only one value η on pairs from A which lie in different fibers of f . Clearly $\eta \neq 2$, and it is easy to check $\eta \neq 1$, using that $[h]_E$ is nonstandard. So $\eta = 0$. Define g by

$$g(n) = \begin{cases} \min\{h(k): k \in A \cap f^{-1}\{n\}\} & \text{if } n \in f''A, \\ 0 & \text{if } n \notin f''A. \end{cases}$$

Then g is strictly increasing on $f''A \in D$, so $[g]_D$ is nonstandard, and $f^*([g]_D) = [g \circ f]_E \leq [h]_E$. $\square$

2. Let $p \in {}^\omega\omega$ (or $p \in {}^{\omega^2}\omega$). Let I be the ideal of subsets of ω (or ω^2) on which p is finite-to-one. A p -fiber measure is a map $\Gamma: I \rightarrow {}^\omega\omega$ satisfying:

- (i) $\Gamma(X)(n) \leq |X \cap p^{-1}\{n\}|$,
- (ii) $X \cap p^{-1}\{n\} \subseteq Y \cap p^{-1}\{n\} \rightarrow \Gamma(X)(n) \leq \Gamma(Y)(n)$, and
- (iii) if $(\forall n \in \omega) g(n) \leq \Gamma(Y)(n)$, then $(\exists X \subseteq Y) \Gamma(X) = g$.

An example is the *cardinality function* $C_{X,p}$ defined by $C_{X,p}(n) = |X \cap p^{-1}\{n\}|$.

A *cut* in $D \text{ prod } \mathbb{N}$ is a pair $\langle S, L \rangle$ where $S \cup L = D \text{ prod } \mathbb{N}$, $S \cap L = \emptyset$, and $(\forall a \in S)(\forall b \in L) a < b$. Suppose $p(E) = D$ and Γ is a p -fiber measure. The cut associated to p and E via Γ is defined by $L = \{a \in D \text{ prod } \mathbb{N}: \exists X \in E \cap I, [\Gamma(X)]_D \leq a\}$, $S = D \text{ prod } \mathbb{N} - L$. This notion, for the case $\Gamma(X) = C_{X,p}$ (the *cardinality cut associated to p and E*) was introduced by Blass in [B1]; results therein generalize trivially to give

THEOREM 2.1. *Let E be a P point, $p(E) = D$, p not an E isomorphism, D nonprincipal, Γ a p -fiber measure, and $\langle S, L \rangle$ the associated cut in $D \text{ prod } \mathbb{N}$. Then $\mathbb{N} \subseteq S$, $L \neq \emptyset$, and every countable subset of L has a lower bound in L .*

Any cut in $D \text{ prod } \mathbb{N}$ satisfying the conclusions of Theorem 2.1 is a *fair cut*. If $a_1, a_2, a_3, \dots$ is a strictly increasing ω -sequence from $D \text{ prod } \mathbb{N}$, the cut defined by $S = \{b \in D \text{ prod } \mathbb{N}: (\exists n \in \omega) b \leq a_n\}$ is always a fair cut, since $D \text{ prod } (\omega; <)$ is $\aleph_1$ -saturated (see [CK, p. 305]).

A *condition C* on a set X is simply a statement about X ; if the statement is true, we say X *satisfies C* . If $\langle S, L \rangle$ is a cut in $D \text{ prod } \mathbb{N}$, Γ a p -fiber measure, then a set X is *large* if $(\exists Y \subseteq X) Y \in I$ and $[\Gamma(Y)]_D \in L$.

THEOREM 2.2 (CH). *Let D be a P point, $\langle S, L \rangle$ a fair cut in $D \text{ prod } \mathbb{N}$, p the first projection from ω^2 to ω , Γ a p -fiber measure, $\{C_j: j \in J\}$ a set of $\leq \aleph_1$ conditions such that*

- (i) $(\exists X \subseteq \omega^2) [\Gamma(X)]_D \in L$,
- (ii) $(\forall X, A \subseteq \omega^2) [X \text{ large} \rightarrow (\exists Z \subseteq X) (Z \text{ large} \wedge (Z \subseteq A \vee Z \cap A = \emptyset))]$,
- (iii) $(\forall X \subseteq \omega^2) (X \text{ large} \rightarrow (\exists Y, Z \subseteq X) (Y, Z \text{ large} \wedge Y \cap Z = \emptyset))$,
- (iv) $(\forall X \subseteq \omega^2) (\forall j \in J) (X \text{ large} \rightarrow (\exists Y \subseteq X) (Y \text{ large} \wedge Y \text{ satisfies } C_j))$.

Then there exist $2^{\aleph_1}$ P points E on ω^2 such that $p(E) = D$, the associated cut in $D \text{ prod } \mathbb{N}$ (via Γ) is $\langle S, L \rangle$, and for each $j \in J$, E contains a set satisfying condition C_j .

PROOF. For each $Z \subseteq \omega^2$, let C_Z be the condition $(X \subseteq Z \vee X \subseteq \omega^2 - Z)$. For each $f \in {}^\omega\omega$ such that $[f]_D \in L$, let C_f be the condition $(X \in I \wedge [\Gamma(X)]_D < [f]_D)$. Let $\{D_\alpha: \alpha < \aleph_1\} = \{C_j: j \in J\} \cup \{C_Z: Z \subseteq \omega^2\} \cup \{C_f: [f]_D \in L\}$. Then the hypotheses of the theorem insure that $(\forall X \text{ large})(\forall \alpha < \aleph_1)(\exists Y \subseteq X)(Y \text{ large and } Y \text{ satisfies } D_\alpha)$.

For each $\rho: \aleph_1 \rightarrow 2$ and each $\alpha < \aleph_1$, we define large sets $A_\alpha^\rho \in I$ such that $A_{\alpha+1}^\rho$ satisfies D_α , the sequence $\{A_\alpha^\rho: \alpha < \aleph_1\}$ is almost decreasing for fixed ρ (i.e. $\alpha < \beta \rightarrow A_\beta^\rho - A_\alpha^\rho$ is finite), and if α_0 is minimal such that $\rho(\alpha_0) \neq \rho'(\alpha_0)$, then $A_\alpha^\rho = A_\alpha^{\rho'}$ for $\alpha < \alpha_0$ and $A_{\alpha_0}^\rho \cap A_{\alpha_0}^{\rho'} = \emptyset$. Then set $E_\rho = \{X \subseteq \omega^2: (\exists \alpha < \aleph_1) A_\alpha^\rho \subseteq X\}$.

The sets A_α^ρ are defined by induction on α . Let $B_0 \in I$ be large (by (i)), and let B_0^0, B_0^1 be disjoint large subsets of B_0 (by (iii)). Set $A_0^\rho = A_0^{\rho(0)}$. If $\alpha = \gamma + 1$, let $\eta = \rho \upharpoonright \alpha$, let $A_\beta^\eta = A_\beta^\rho$ for $\beta < \alpha$ (this is well defined by induction hypothesis), let B_η^0, B_η^1 be large disjoint subsets of A_γ^η , let Y_η^j be a large subset of B_η^j which satisfies D_γ , and let $A_\alpha^\rho = Y_{\rho \upharpoonright \alpha}^{\rho(\alpha)}$.

For limit α , follow the limit ordinal case in Theorem 2 in [B1] (with $\Gamma(X)$ in place of C_X) to obtain, for each $\eta: \alpha \rightarrow 2$, a large Y_η such that $Y_\eta - A_\beta^\eta$ is finite for $\beta < \alpha$, $\rho \upharpoonright \alpha = \eta$. Let B_η^0, B_η^1 be large disjoint subsets of Y_η , and set $A_\alpha^\rho = B_{\rho \upharpoonright \alpha}^{\rho(\alpha)}$. $\square$

3. The following discussion, up to the statement of Theorem 3.1, is adapted from [AH]. A *system of colors* of length n is a sequence $\alpha = (\alpha_0, \dots, \alpha_n)$ of finite, nonempty sets. An α *pattern* Q consists of a finite, linearly ordered set X , and a function $f_Q: \bigcup_{j \leq n} [X]^j \rightarrow \bigcup_{j \leq n} \alpha_j$ such that $f_Q''[X]^j \subseteq \alpha_j$. Let $P = (X, f_P)$ and $Q = (Y, f_Q)$ be α patterns. P and Q are *isomorphic* if there exists an order preserving bijection $\psi: X \rightarrow Y$ such that $(\forall c \in [X]^{<n}) f_P(c) = f_Q(\psi''c)$. If $Z \subseteq X$ and $e, M \in \omega$, $F: [X]^e \rightarrow M$, then Z is *seminhomogeneous for F over f_P* iff $(\forall b, c \in [Z]^e)(\text{if } (b, f_P \upharpoonright [b]^{<n}) \cong (c, f_P \upharpoonright [c]^{<n}), \text{ then } F(b) = F(c))$. $Q \rightsquigarrow (P)_M^e$ means that for any partition $F: [Y]^e \rightarrow M$, there is a $Z \subseteq Y$ such that $(Z, f_Q \upharpoonright [Z]^{<n}) \cong P$ and Z is seminhomogeneous for F over f_Q . In [AH], Abramson and Harrington proved the following generalization of the finitary Ramsey theorem.

THEOREM 3.1. *Let $n, e, M \in \omega$, α a system of colors of length n , P an α pattern. Then there is an α pattern Q such that $Q \rightsquigarrow (P)_M^e$.*

COROLLARY 3.2. *Let $n, e_1, \dots, e_k, M_1, \dots, M_k \in \omega$, α and P as above. Then there is an α pattern Q such that $Q \rightsquigarrow (P)_{M_j}^{e_j}$ for all j , $1 \leq j \leq k$.*

PROOF. Let $Q_1 \rightsquigarrow (P)_{M_1}^{e_1}$, $Q_{j+1} \rightsquigarrow (Q_j)_{M_{j+1}}^{e_{j+1}}$. Then $Q_k \rightsquigarrow (P)_{M_j}^{e_j}$ for all j , $1 \leq j \leq k$. $\square$

For notational simplicity, we often fail to distinguish between an α pattern and its underlying set; furthermore, we write $f_P \upharpoonright Z$ rather than $f_P \upharpoonright [Z]^{<n}$. Temporarily fix, for the remainder of §3, $k \in \omega$, $k \geq 1$, and let α be the system of colors $(1, 1, k)$. If $P = (X, f_P)$ is an α pattern, we view f_P as a coloring on (only) the two element subsets of X . We say P is *scattered* if for every triple $(a, b, c) \in k \times k \times k$, there are $z_1, z_2, z_3 \in X$ with $f_P\{z_1, z_2\} = a$, $f_P\{z_2, z_3\} = b$, and $f_P\{z_3, z_1\} = c$.

LEMMA 3.3. Let $l \geq 3$, let Q and $P = (X, f_P)$ be scattered α patterns such that $P \rightsquigarrow (Q)_l^2$ and $|Q| \geq 4$.

(a) If g is a function with domain X , then $\exists Y \subseteq X$, $Y \cong Q$, such that g is one-to-one or constant on Y .

(b) There exist $Y_1, Y_2 \subseteq X$, $Y_1 \cap Y_2 = \emptyset$, $Y_1 \cong Y_2 \cong Q$.

(c) If $A \subseteq X$, then $(\exists Y \subseteq X) Y \cong Q$ such that $Y \subseteq A$ or $Y \cap A = \emptyset$.

(d) If $G: [X]^2 \rightarrow l$, then $(\exists Y \subseteq X) Y \cong Q$ such that $|G''[Y]^2| < k$.

PROOF. (a) Define $F: [X]^2 \rightarrow 2$ by

$$F\{x, y\} = \begin{cases} 0 & \text{if } g(x) = g(y), \\ 1 & \text{if } g(x) \neq g(y). \end{cases}$$

Then there is a $Y \subseteq X$, $Y \cong Q$, such that $(\forall x, y \in Y) F\{x, y\} = h(f_P\{x, y\})$ for a function $h: k \rightarrow 2$. It suffices to show that $|\text{range}(h)| = 1$, i.e. that F is homogeneous on Y . Suppose not, and let $\eta_0, \eta_1 < k$ with $h(\eta_i) = i$. Find $a, b, c \in Y$ with $f_Q\{a, b\} = f_Q\{b, c\} = \eta_0$ and $f_Q\{a, c\} = \eta_1$. Then $g(a) = g(b)$ and $g(b) = g(c)$, but $g(a) \neq g(c)$, a contradiction.

(b) Let $Y_1 \subseteq X$, $Y_1 \cong Q$, and write Y_1 as the disjoint union of two nonempty sets A and B . Let $C = X - Y_1$, and define $F: [X]^2 \rightarrow 3$ by

$$F\{x, y\} = \begin{cases} 0 & \text{if } \{x, y\} \subseteq A \text{ or } \{x, y\} \subseteq B, \\ 1 & \text{if } \{x, y\} \subseteq C, \\ 2 & \text{otherwise.} \end{cases}$$

Let $Y_2 \subseteq X$, $Y_2 \cong Q$, Y_2 semihomogeneous for F over f_P . We have $h: k \rightarrow 3$ with $F\{x, y\} = h(f_P\{x, y\})$ for $x, y \in Y_2$; it suffices to show $\text{range}(h) = \{1\}$.

For contradiction, suppose $0 \in \text{range}(h)$, say $h(\eta) = 0$. Then h must assume another value d , since otherwise $Y_2 \subseteq A$ or $Y_2 \subseteq B$, a contradiction since neither A nor B includes an isomorph of Q . Let $h(\rho) = d \neq 0$, and find $a, b, c \in Y_2$ with $f_P\{a, b\} = f_P\{b, c\} = \eta$ and $f_P\{a, c\} = \rho$. Thus, a and b , as well as b and c , are both in A or both in B , and so a and c are both in A or both in B , contradicting the choice of a, c , and ρ . So $0 \notin \text{range}(h)$.

Suppose now $h(\eta) = 2$ for some $\eta < k$. Then $(\exists \rho < k) h(\rho) = 1$, since any set on which F assumes only the value 2 has at most three elements. Let $a, b, c \in Y_2$ with $f_P\{a, b\} = f_P\{b, c\} = \rho$ and $f_P\{a, c\} = \eta$. Then both a and c are in C , contradicting $F\{a, c\} = 2$.

(c) Apply (a) to the characteristic function of A .

(d) Immediate by semihomogeneity. $\square$

4.

THEOREM 4.1 (CH) (BLASS [B1], FOR $n = k = 1$). Let D be an n Ramsey P point, $k \geq 1$. There exist $2^{2^0} n + k$ Ramsey P points E which are strong immediate successors of D .

PROOF. Fix R , a Ramsey ultrafilter, and $q \in {}^\omega \omega$ such that $q(D) = R$ (if $n = 1$, take $R = D$ and $q = \text{identity}$). Let $\langle S_R, L_R \rangle$ be the cardinality cut in R prod $\mathbb{N}$ associated to q and D . Fix $\bar{h} \in {}^\omega \omega$ such that $[\bar{h}]_R \in L_R$, and let H be an irreducible

n coloring on D . Let $s_0 = q^*([\bar{h}]_R) = [\bar{h} \circ q]_D$, $s_{i+1} = 2^s$, and define a cut $\langle S, L \rangle$ in $D \text{ prod } N$ by $S = \{t \in D \text{ prod } N : (\exists i \in \omega) t \leq s_i\}$. Then $\langle S, L \rangle$ is a fair cut, and S is closed under exponentiation, multiplication, and addition. Let p be the first projection from ω^2 to ω .

Let α be the system of colors $(1, 1, k)$. Define α patterns P'_i as follows: $P'_0 = \emptyset$, $P'_1 =$ any scattered α pattern with $|P'_1| \geq 4$, $P'_{n+1} \rightsquigarrow (P'_n)_{n+k+1}^2$. Define α patterns P_i by $P_0 = \emptyset$, $P_1 = P'_1$, $P_{m+1} = P'_l$, where l is minimal such that

$$|P'_l| \geq (n + k + 1)^{\sum_{j < m} |P_j|}.$$

Then, for $i \geq 1$, P_i is scattered and Lemma 3.3 holds with P_{i+1} and P_i in place of P and Q , and $l = n + k + 1$.

Fix $\bar{g} \in {}^\omega \omega$ with $[\bar{g}]_D \in L$. Let $X_0 \subseteq \omega^2$ such that $|X_0 \cap p^{-1}\{m\}| = |P_{\bar{g}(m)}|$, and order $X_0 \cap p^{-1}\{m\}$ by $\langle a, m \rangle < \langle b, m \rangle$ iff $a < b$. Define $G: [\omega^2]^2 \rightarrow n + k$ by

- (a) $(X_0 \cap p^{-1}\{m\}, G|(X_0 \cap p^{-1}\{m\})) \cong P_{\bar{g}(m)}$,
- (b) if $p(x) \neq p(y)$, then $G\{x, y\} = H\{p(x), p(y)\} + k$,
- (c) $G\{x, y\}$ is arbitrary if $p(x) = p(y)$ and $\{x, y\} \not\subseteq X_0$.

Define a p fiber measure Γ by $\Gamma(Y)(m) = M$, where M is maximal such that $(Y \cap X_0 \cap p^{-1}\{m\}, G \upharpoonright Y \cap X_0 \cap p^{-1}\{m\})$ includes an isomorph of P_M . Then $\Gamma(X_0) = \bar{g}$.

For each $f: \omega^2 \rightarrow \omega$, let C_f be the condition on X : f is p fiberwise constant on X or $(q \circ p)$ fiberwise one-to-one on X .

For each $F: [\omega^2]^2 \rightarrow n + k + 1$, C_F is the condition:

$$(\forall l \in p''X) |F''[X \cap p^{-1}\{l\}]^2| \leq k \text{ and}$$

$$(\forall x, y, v, w \in X) [q(p(x)) = q(p(y)) = q(p(v)) = q(p(w))$$

$$\wedge p(x) = p(y) \neq p(v) = p(w) \rightarrow F\{x, v\} = F\{y, w\}].$$

We postpone the verification of the hypotheses in Theorem 2.2 and show that the $2^{2^{\aleph_0}}$ P points E produced by that theorem satisfy the conclusion of the present theorem. Since E contains sets satisfying C_f for each $f: \omega^2 \rightarrow \omega$, it follows by the remarks preceding Theorem 2.1 that every $f: \omega^2 \rightarrow \omega$ is p fiberwise constant or (globally) one-to-one on a set in E . Hence E is a strong immediate successor of D .

By the definition of Γ , G assumes all $n + k$ values on any large set; since E consists of large sets, G is irreducible on E and so E is $\geq n + k$ Ramsey. Let $F: [\omega^2]^2 \rightarrow n + k + 1$, and let $X \in E$ satisfy C_F . Partition $p''X$ into $\binom{n+k+1}{k}$ pieces W_i such that if a and b are in W_i , then the same k colors are assumed on $[X \cap p^{-1}(a)]^2$ and on $[X \cap p^{-1}(b)]^2$. Since $p''X \in D$, one of these pieces $\bar{W}$ is in D ; let $\eta_0, \dots, \eta_{k-1}$ be the colors assumed on pairs in the same fiber of $Y = X \cap p^{-1}\bar{W} \in E$. By Theorem 1.2, there is a set $Y' \in E$ such that $F''\{[x, y] \in [Y']^2: q(p(x)) \neq q(p(y))\} = \{\rho\}$, for some $\rho < n + k + 1$. Let $Z = Y \cap Y' \in E$.

If $n = 1$, then $q = \text{identity}$, the second half of C_X is vacuous, and so $(\forall x, y \in Z)$,

$$F\{x, y\} = \begin{cases} \eta_i \text{ (some } i < k) & \text{if } p(x) = p(y), \\ \rho & \text{if } p(x) \neq p(y). \end{cases}$$

Thus F assumes at most $k + 1$ colors on $[Z]^2$, so E is $k + 1$ Ramsey.

If $n > 1$, define a partition $\bar{H}: [\omega]^2 \rightarrow n + k + 2$ by

$$\bar{H}\{l, m\} = \begin{cases} F\{x, y\} & \text{if } q(l) = q(m) \wedge x, y \in Z \wedge p(x) = l \wedge p(y) = m, \\ n + k + 1 & \text{if } q(l) \neq q(m) \vee l \notin p''Z \vee m \notin p''Z. \end{cases}$$

$\bar{H}$ is well defined since Z satisfies the second half of C_F . Since D is n Ramsey, $(\exists B \in D)[\bar{H}''[B]^2] \leq n$, and we can assume $B \subseteq p''Z$. $\bar{H}$ must assume the value $n + k + 1$ on $[B]^2$, so $\bar{H}$ assumes at most $n - 1$ values, say $\beta_0, \dots, \beta_{n-2}$, on $\{\{k, l\} \in [B]^2: q(k) = q(m)\}$. Let $V = Z \cap p^{-1}B \in V$. If $x, y \in V$, we have

$$F\{x, y\} = \begin{cases} \eta_i \text{ (some } i < k) & \text{if } p(x) = p(y), \\ \rho & \text{if } q(p(x)) \neq q(p(y)), \\ j \text{ (some } j < n - 1) & \text{if } q(p(x)) = q(p(y)) \wedge p(x) \neq p(y). \end{cases}$$

Thus, E is $n + k$ Ramsey.

It remains to verify the hypotheses of Theorem 2.2; (i) is satisfied by the set X_0 . For (ii), let X be large, and let $A \subseteq \omega^2$; without loss of generality $X \in I$ and $[\Gamma(X)]_D \in L$. For each $l \in \omega$ such that $\Gamma(X)(l) \geq 1$, apply Lemma 2.2(c) to $(X \cap p^{-1}\{l\}, G \upharpoonright X \cap p^{-1}\{l\})$ and $A \cap p^{-1}\{l\}$ to obtain $Y_l \subseteq X \cap p^{-1}\{l\}$ such that $(Y_l, G \upharpoonright Y_l) = P_{\Gamma(X)(l)-1}$ and $Y_l \subseteq A$ or $Y_l \cap A = \emptyset$. If $\Gamma(X)(l) = 0$, let $Y_l = \emptyset$. Let $B \in D$ such that $(\forall l \in B) Y_l \subseteq A$ or $(\forall l \in B) Y_l \cap A = \emptyset$; let $Z = X \cap p^{-1}B$. Then $[\Gamma(Z)]_D = [\Gamma(X)]_D - 1 \in L$ (since S is closed under the successor function), so Z is large and either $Z \cap X = \emptyset$ or $Z \subseteq X$.

For (iii), let $X \in I$, $[\Gamma(X)]_D \in L$. Apply Lemma 3.3(b) in each fiber $X \in p^{-1}\{l\}$ such that $\Gamma(X)(l) \geq 1$ to obtain Y_l^0 and Y_l^1 with $Y_l^0 \cap Y_l^1 = \emptyset$, $Y_l^i \subseteq X \cap p^{-1}\{l\}$ and $(Y_l^i, G \upharpoonright Y_l^i) \cong P_{\Gamma(X)(l)-1}$. Let $Y = \bigcup Y_l^0$, $Z = \bigcup Y_l^1$.

For (iv), let $X \in I$, $[\Gamma(X)] \in L$, and let $f: \omega^2 \rightarrow \omega$. Apply Lemma 3.3(a) to each p -fiber of X and argue as above to obtain a large $Y' \subseteq X$ such that f is p fiberwise constant or p fiberwise one-to-one on Y' . If the former, then Y' satisfies C_f . If the latter, let $B \in D$ such that $(\forall a \in \omega) C_{B,q}(a) \leq \bar{h}(a)$ (there exists such a $B \in D$ since $[\bar{h}]_R \in L_R$). Let $Y = Y' \cap p^{-1}(B)$; Y is large. Temporarily fix $a \in (q \circ p)''Y$, and let $B \cap q^{-1}\{a\} = \{m_0, \dots, m_{l-1}\}$ and $\Gamma(Y)(m_i) = M_i$. Then $l < \bar{h}(a)$.

Reorder $\{m_i: 0 \leq i < l\}$ so that $M_i \geq M_{i+1}$. Choose $\Omega_i \subseteq Y \cap p^{-1}\{m_i\}$ with $\Omega_i \cong P_{M_i-2i}$ ($\Omega_i = \emptyset$ if $M_i - 2i \leq 0$). For each i , $0 \leq i < l - 2$, define $g_i: \Omega_i \rightarrow 2$ by $g_i(x) = 0$ iff $(\exists y \in \bigcup_{i < j < l} \Omega_j) f(x) = f(y)$. By Lemma 3.3(a), there is a set $\psi_i \subseteq \Omega_i$, $\psi_i \cong P_{M_i-2i-1}$, such that g_i is one-to-one or constant on ψ_i . For nonempty ψ_i , g_i is constant on ψ_i , since $|\psi_i| \geq 4$. For each i , we claim that $g_i''\psi_i = \{1\}$, since otherwise $|\psi_i| \leq \sum_{i < j < l} |\Omega_j|$; hence $|P_{M_i-2i-1}| \leq \sum_{i < j < l} |P_{M_i-2j}|$, contradicting the definition of $\{P_m\}$. Let $\psi_{l-1} = \Omega_{l-1}$, and let $\Delta_a = \bigcup_{0 \leq i < l} \psi_i$; then f is one-to-one on Δ_a .

Let $Z = \bigcup_{a \in (q \circ p)''Y} \Delta_a$. Then Z satisfies C_f , and for all $m \in \omega$, $\Gamma(Z)(m) \geq \Gamma(Y)(m) - 2\bar{h}(q(m)) - 1$, so $[\Gamma(Z)]_D \geq [\Gamma(Y)] - 2[\bar{h} \circ q] - 1$. Since $[\bar{h} \circ q]_D \in S$ and S is closed under addition and multiplication, it follows that $[\Gamma(Z)]_D \in L$, so Z is large.

Now suppose $X \in I$, $[\Gamma(X)]_D \in L$, and $F: [\omega^2]^2 \rightarrow n + k + 1$. Apply Lemma 3.3(d) to each fiber $X \cap p^{-1}\{l\}$ to obtain $A_l \subseteq X \cap p^{-1}\{l\}$, $A_l \cong P_{\Gamma(X)(l)-1}$ such

that $|F''[A_l]^2| \leq k$. Let $X' = \bigcup_{l \in \omega} A_l$; then X' is large and satisfies the first half of condition C_F . Let $B \in D$ such that $(\forall a \in \omega) C_{B,q}(a) \leq \bar{h}(a)$. Let $Y = X' \cap p^{-1}B$; Y is large. Fix $a \in (q \circ p)''Y$, let $B \cap q^{-1}\{a\} = \{m_0, \dots, m_{l-1}\}$, ordered so that if $M_i = \Gamma(Y)(m_i)$, then $M_i \geq M_{i+1}$. Choose $\Omega_i \subseteq Y \cap p^{-1}\{m_i\}$ such that $\Omega_i \cong P_{M_i-2i}$. For each i , $0 \leq i \leq l-2$, define a function g_i on Ω_i by $g_i(x) = \langle z, \eta \rangle$: $z \in \bigcup_{i < j < l} \Omega_j \wedge F\{x, z\} = \eta$. By Lemma 3.3(a), find $\psi_i \subseteq \Omega_i$, $\psi_i \cong P_{M_i-2i-1}$ such that g_i is one-to-one or constant on ψ_i . We claim that g_i is constant on ψ_i .

The number of possible functions $g_i(x)$, for $x \in \Omega_i$, is

$$\begin{aligned} (n+k+1)^{\sum_{i < j < l} |\Omega_j|} &= (n+k+1)^{\sum_{i < j < l} |P_{M_j-2j}|} \\ &\leq (n+k+1)^{\sum_{m < M_i-2i-1} |P_m|} < |P_{M_i-2i-1}| = |\psi_i|. \end{aligned}$$

Thus g_i is not one-to-one on ψ_i . Let $\psi_{l-1} = \Omega_{l-1}$; then if $i < j \leq l-1$, and $x, y \in \psi_i$ and $z \in \psi_j$, then $F\{x, z\} = F\{y, z\}$.

Now suppose $1 \leq j \leq l-1$ and $z \in \psi_j$. The number of possible ways to assign $n+k+1$ colors to pairs of the form $\{x, z\}$ for $x \in \bigcup_{0 \leq i < j} \psi_i$ is $(n+k+1)^j$ (by the last sentence of the last paragraph). For each j , $1 \leq j \leq l-1$, partition ψ_j into $(n+k+1)^j$ pieces so that if z and w lie in the same piece, then $(\forall x \in \bigcup_{i < j} \psi_i) F\{x, z\} = F\{x, w\}$. Apply Lemma 3.3(c) $(n+k+1)^j - 1$ times to ψ_j and this partition to obtain

$$\chi_j \subseteq \psi_j, \quad \chi_j \cong P_{M_j-2j-(n+k+1)^j}$$

such that χ_j is included in one of the pieces. Let $\chi_0 = \psi_0$.

Let $\Delta_a = \bigcup_{0 \leq j \leq l-1} \chi_j$. Then $(\forall x, y, z, w \in \Delta_a)[p(x) = p(y) \neq p(z) = p(w) \rightarrow F\{x, z\} = F\{y, w\}]$. Let $Z = \bigcup_{a \in (q \circ p)''Y} \Delta_a$. Then Z satisfies the second half, and hence all of, C_F . For all $m \in P''Z$,

$$\Gamma(Z)(m) \geq \Gamma(Y)(m) - 2\bar{h}(q(m)) - (n+k+1)^{\bar{h}(q(m))}.$$

Since $[h \circ q]_D \in S$ and S is closed under exponentiation, $[\Gamma(Z)]_D \in L$, so Z is large. $\square$

In [D], Dagenet showed (CH) that for any n , there exist n Ramsey P points E whose (nonstandard) constellations are linearly ordered in the form

$$\begin{array}{c} n \\ n-1 \\ \vdots \\ \vdots \\ 2 \\ 1 \end{array}$$

where the number j indicates that if $[f]_E$ is in that constellation, then $f(E)$ is j Ramsey. Such an E is said to be *strictly n Ramsey* (clearly any strictly n Ramsey ultrafilter is 2 square). The main result in [R] is that any 2 square ultrafilter is strictly n Ramsey for some $n \in \omega$. Our theorem shows (CH) that any linearly ordered, strictly increasing finite sequence which begins with 1 occurs as the constellation structure of a weakly Ramsey P point. Thus, for any $k \geq 3$, there exist (CH) many k Ramsey P points which are not 2 square.

5. In this section we show (CH) that there exist 5 Ramsey P points whose constellation structure is

In a sense, this is the best possible result, for a result of the author [R] shows that for $k \leq 4$, any k Ramsey ultrafilter has a linearly ordered constellation lattice.

We will start with any Ramsey ultrafilter R and then simultaneously construct 2 Ramsey ultrafilters D_0 and D_1 , with $g(D_0) = g(D_1) = R$ (for some $g \in {}^\omega\omega$) such that if E is any ultrafilter on ω^2 which extends the filter $D_0 \times D_1$ and includes the set $\{(x, y): g(x) = g(y)\}$, then E will be the desired 5 Ramsey P point. The construction of D_0 and D_1 follows Theorem 2.2, except that hypothesis (ii) of that theorem, which insures that the resulting filter is an ultrafilter, is not satisfied. Thus we construct a filter F which then extends to the ultrafilters D_0 and D_1 (by conditions C_Z below). Conditions C_f force the ultrafilter E to be a P point, which in conjunction with condition C_H will show that E is 5 Ramsey.

Let α be the system of colors $(1, 2)$. Thus an α pattern $P = (X, f_P)$ is simply a partition of X into 2 pieces. We denote the ordering on X by $<$, and let $P^i = X^i = \{x \in X: f_P\{x\} = i\}$ ($i = 0, 1$). Let $P_0 = \emptyset$, $P_1 = \{x_1, \dots, x_8\}$ where $x_i < x_{i+1}$ and $P_1^0 = \{x_1, x_2, x_3, x_4\}$. Suppose we have defined P_m so that

$$(*) \quad (\forall x \in P_m^0)(\forall y \in P_m^1)x < y.$$

Let $P'_{m+1} \rightsquigarrow (P_m)_6^l$ for $l = 2, 3, 4$. Let P_{m+1} have the same underlying set as P'_{m+1} , let $f_{P_{m+1}} = f_{P'_{m+1}}$, and define the ordering $<$ on P_{m+1} from the ordering $<'$ on P'_{m+1} by $z < \omega$ iff $(f_{P_{m+1}}(z) = f_{P_{m+1}}(w) \wedge z <' w) \vee (z \in P_{m+1}^0 \wedge w \in P_{m+1}^1)$. Then P_{m+1} satisfies $(*)$, and it is easy to check that $P_{m+1} \rightsquigarrow (P_m)_6^l$ for $l = 2, 3, 4$.

Assume CH, let R be a Ramsey ultrafilter, and let $\langle S, L \rangle = \langle \mathbb{N}, R \text{ prod } \mathbb{N} - \mathbb{N} \rangle$. Then $\langle S, L \rangle$ is a fair cut. Let p be the first projection from ω^2 to ω , and let $\bar{X} \subseteq \omega^2$ such that $|\bar{X} \cap p^{-1}\{n\}| = |P_n|$. Define an α pattern G^n in each fiber $p^{-1}\{n\} \cap \bar{X}$ such that $G^n \cong P_n$ (ordered by $(a, n) < (b, n)$ iff $a < b$). Define Γ by $\Gamma(X)(n) = \text{maximum } M \text{ such that } (X \cap \bar{X} \cap p^{-1}\{n\}, G^n \upharpoonright X \cap \bar{X} \cap p^{-1}\{n\}) \text{ includes an isomorph of } P_M$. For any $Y \subseteq \omega^2$, let

$$Y^i = \bigcup_{n \in \omega} \{x \in Y \cap \bar{X} \cap p^{-1}\{n\}: G^n\{x\} = i\} \quad (i = 0, 1),$$

and let $\Delta_n Y = (Y^0 \cap p^{-1}\{n\}) \times (Y^1 \cap p^{-1}\{n\})$.

For each $H: [\omega^2]^l \rightarrow 6$, for $l = 2, 3, 4$, let C_H be the condition: $X \subseteq \bar{X}$ and $(\forall n \in p''X) X \cap p^{-1}\{n\}$ is semihomogeneous for H over G^n .

For each $Z \subseteq \omega^2$, C_Z is the condition $(X^0 \subseteq Z \vee X^0 \cap Z = \emptyset) \wedge (X^1 \subseteq Z \vee X^1 \cap Z = \emptyset)$.

For each $f: \bigcup_{n \in \omega} \Delta_n X \rightarrow \omega$, C_f is the condition: $X \subseteq \bar{X}$ and at least one of the following holds:

$$(1) (\forall x \in X^0)(\forall y, z \in X^1)[p(x) = p(y) = p(z) \rightarrow f(x, y) = f(x, z)],$$

(2) $(\forall x, y \in X^0)(\forall z \in X^1)[p(x) = p(y) = p(z) \rightarrow f(x, y) = f(y, z)]$, or

(3) $(\forall m, n \in p''X)(m \neq n \rightarrow f''\Delta_m X \cap f''\Delta_n X = \emptyset)$.

Hypothesis (i) of Theorem 2.2 is satisfied by $\bar{X}$. (iii) can be verified easily using the following

LEMMA 5.1. *Let $m \geq 1$. Then $(\exists Q_0, Q_1 \subseteq P_m)(Q_0 \cong Q_1 \cong P_{m-1} \wedge Q_0 \cap Q_1 = \emptyset)$.*

PROOF. Similar to Lemma 3.3(b) (let $Q_0 = A \cup B$, $A \cap B = \emptyset$, $A \cap Q_0^i \neq \emptyset$, and $B \cap Q_0^i \neq \emptyset$ for $i = 0, 1$; define F as in Lemma 3.3(b)). $\square$

The parts of (iv) which demand large sets satisfying C_H can be verified by arguments like those in §4; for C_Z , use the following

LEMMA 5.2. *If $m \geq 2$, $A \subseteq P_m$, then $(\exists Q \subseteq P_m)[Q \cong P_{m-1} \wedge (\forall i < 1)(Q^i \subseteq A \vee Q^i \cap A = \emptyset)]$.*

PROOF. Let $F: [P_m]^2 \rightarrow 2$ be defined by $F\{x, y\} = 1$ iff $|\{x, y\} \cap A| = 1$. Let $Q \subseteq P_m$, $Q \cong P_{m-1}$, Q semihomogeneous for F over f_p . Since Q^i is homogeneous for F and $|Q^i| > 3$, it follows that $F''[Q^i]^2 = \{0\}$. $\square$

We now verify hypothesis (iv) for C_f .

LEMMA 5.3. *If f is any function defined on $P_m^0 \times P_m^1$, then $\exists Q \subseteq P_m$, $Q \cong P_{m-1}$, such that at least one of the following holds:*

- (1) $(\forall x \in Q^0) f$ is constant on $\{x\} \times Q^1$,
- (2) $(\forall x \in Q^1) f$ is constant on $Q^0 \times \{x\}$, or
- (3) for each $(x, y) \in Q^0 \times Q^1$, f is one-to-one on both $\{x\} \times Q^1$ and $Q^0 \times \{y\}$.

PROOF. Define $H: [P_m]^3 \rightarrow 2$ by, for $x_1 < x_2 < x_3$, $H\{x_1, x_2, x_3\} = 0$ iff $[x_1, x_2 \in P_m^0 \wedge x_3 \in P_m^1 \wedge f(x_1, x_3) = f(x_2, x_3)] \vee [x_1 \in P_m^0 \wedge x_2, x_3 \in P_m^1 \wedge f(x_1, x_2) = f(x_1, x_3)]$. Let $Q \subseteq P_m$, $Q \cong P_{m-1}$, Q semihomogeneous for H over f_p . Then H assumes only one value on $s_0 = \{\{x_1, x_2, x_3\} \in [Q]^3: x_1, x_2 \in Q^0 \wedge x_3 \in Q^1\}$ and only one value on $s_1 = \{\{x_1, x_2, x_3\} \in [Q]^3: x_1 \in Q^0 \wedge x_2, x_3 \in Q^1\}$. If $H''s_0 = \{0\}$, then Q satisfies (2). If $H''s_1 = \{0\}$, then Q satisfies (1). If $H''s_0 = H''s_1 = \{1\}$, then Q satisfies (3). $\square$

Now suppose X is large and $f: \bigcup_{n \in \omega} \Delta_n X \rightarrow \omega$. Apply Lemma 5.3 in each fiber of $X \cap \bar{X}$ to obtain $Q_n \subseteq X \cap \bar{X} \cap p^{-1}\{n\}$, $Q_n \cong P_{\Gamma(X \cap \bar{X})-1}$, such that Q_n satisfies one of the conclusions of the lemma. Let $A \in R$ such that $(\forall n \in A) Q_n$ satisfies the same conclusion and let $Y = \bigcup_{n \in A} Q_n$. Then Y is large; if the conclusion satisfied in the fibers of Y is (1) or (2), then Y satisfies the same conclusion in C_f . Assume the conclusion satisfied is (3). Define $H: [A]^2 \rightarrow 2$ by, for $m < n$, $H\{m, n\} = 0$ iff $\Gamma(Y)(n) \leq \sum_{j \in A, j < m} |Q_j^0| \cdot |Q_j^1| + 1$.

Since R is Ramsey, there is a set $B \in R$, $B \subseteq A$, which is homogeneous for H . Then $H''[B]^2 = 1$, since otherwise $\{\Gamma(Y)(j): j \in B\}$ is bounded, a contradiction since $Y \cap p^{-1}B$ is large.

We now define, by induction on $n \in B$, sets $\bar{Q}_n \subseteq Q_n$, $\bar{Q}_n \cong \Gamma(Y)(n) - 1$, such that if $m, n \in B$, $m < n$, then $f''(\bar{Q}_m^0 \times \bar{Q}_n^1) \cap f''(\bar{Q}_n^0 \times \bar{Q}_m^1) = \emptyset$. If m_0 is the least element of B , let $\bar{Q}_{m_0} \subseteq Q_{m_0}$, $\bar{Q}_{m_0} \cong P_{\Gamma(Y \cap \bar{X})(m_0)-1}$. Let $n \in B$, and suppose we have

defined $\bar{Q}_m$ for $m < n$, $m \in B$. Define $\psi: [Q_n]^2 \rightarrow 2$ by $\psi\{x, y\} = 0$ iff $x \in Q_n^0 \wedge y \in Q_n^1 \wedge f(x, y) \notin \bigcup_{m < n; m \in B} f''\bar{Q}_m^0 \times \bar{Q}_m^1$. Let $\bar{Q}_n \subseteq Q_n$, $\bar{Q}_n \cong P_{\Gamma(Y)(n)-1}$, $\bar{Q}_n$ semihomogeneous for ψ over f_0 . Then ψ assumes only one value on $T = \{\{x, y\} \in [\bar{Q}_n]^2: x \in Q_n^0 \wedge y \in Q_n^1\}$. Since f is one-to-one on $\{x\} \times \bar{Q}_n^1$, f assumes at least $|\bar{Q}_n^1| \geq \Gamma(Y)(n) - 1 > \sum_{j < n; j \in B} |Q_j^0| \cdot |Q_j^1|$ values on $\bar{Q}_n^0 \times \bar{Q}_n^1$. Thus $\psi''T = \{0\}$; this completes the construction of $\{\bar{Q}_n: n \in B\}$.

Let $Z = \bigcup_{n \in B} \bar{Q}_n$. Then Z satisfies (3) in C_f , and Z is large, since $[\Gamma(Z)]_R + 2 \geq [\Gamma(X)]_R$; thus, hypothesis (iv) of Theorem 2.2 is satisfied.

Hypothesis (ii) is not satisfied by Γ, p, R , and $\langle S, L \rangle$. If we follow the proof of Theorem 2.2, omitting the steps where we satisfy (ii), then, for each $\rho: \aleph_1 \rightarrow 2$, we obtain a filter F_ρ on ω^2 (instead of an ultrafilter) such that for any condition C_H , C_f , or C_Z , F_ρ contains a set satisfying that condition. Fix $\rho: \aleph_1 \rightarrow 2$, and let $F = F_\rho$. Then $F \cup \{\bar{X}^i\}$ generates an ultrafilter $\bar{D}_i$ (for $i = 0, 1$), since F contains sets satisfying C_Z for each $Z \subseteq \omega^2$, and $\bar{D}_i$ is a P point. Since F contains sets satisfying C_H for all $H: [\omega^2]^2 \rightarrow 2$, it follows that $\bar{D}_i$ is 2 Ramsey (use Theorem 1.2).

For notational ease, we identify $\bar{X}$ with ω as follows. If $x, y \in \bar{X}$, then define $x \Delta y$ if $p(x) < p(y) \vee (p(x) = p(y) \wedge x < y)$, where $<$ is the ordering given by G^n . Then Δ well orders $\bar{X}$ in order type ω ; let $\phi: \bar{X} \rightarrow \omega$ be the order isomorphism. View $\bar{D}_i$ as an ultrafilter on $\bar{X}$, let $D_i = \phi(D_i)$ and $g = p \circ \phi^{-1}$. Then D_i is an ultrafilter on ω , and $g(D_i) = R$ ($i = 0, 1$).

Let p_0 and p_1 be the first and second projections, respectively, from ω^2 to ω . By Theorem 1 in [B2], there is an ultrafilter E on ω^2 such that $p_i(E) = D_i$ and $\{(x, y) \in \omega^2: g(x) = g(y)\} \in E$. We claim that any such E is a 5 Ramsey P point. First, the constellations of E are not linearly ordered ($\text{con}([P_0]_E)$ and $\text{con}([P_1]_E)$ are incomparable), so E is ≥ 5 Ramsey. Since E contains the set $\{(x, y): g(x) = g(y)\}$, it follows that the skies of $D_0 \text{ prod } \mathbb{N}$ and $D_1 \text{ prod } \mathbb{N}$ are embedded in the same sky $\bar{S}$ of $E \text{ prod } \mathbb{N}$ (by p_0^* and p_1^* , respectively), which must be the highest sky of $E \text{ prod } \mathbb{N}$ (for a discussion of skies, see [P] and [B2]). To show that E is a P point, we show that $\bar{S}$ is the only sky of $E \text{ prod } \mathbb{N}$.

Let $f \in {}^{(\omega^2)}\omega$. Define $\hat{f}: \bigcup_{n \in \omega} \Delta_n \bar{X} \rightarrow \omega$ by $\hat{f}(a, b) = f(\phi(a), \phi(b))$. Let $X \in E$ satisfy $C_{\hat{f}}$, and let $A^i = \phi''X^i$. Then $A_i \in D_i$, so $V = \{(x, y): x \in A_0, y \in A_1, g(x) = g(y)\} \in E$. Since X satisfies $C_{\hat{f}}$, one of the following holds:

- (1) f is p_0 fiberwise constant on V ,
- (2) f is p_1 fiberwise constant on V , or
- (3) $(\forall s, t \in V)(f(s) = f(t) \rightarrow g(p_0(s)) = g(p_0(t)))$.

If (1) (resp. (2)) holds, then $[f]_E \in p_0^{**}D_0 \text{ prod } \mathbb{N}$ (resp. $[f]_E \in p_1^{**}D_1 \text{ prod } \mathbb{N}$). If (3) holds, then $g \circ p_0$ is f fiberwise constant on $V \in E$, so $\text{con}([g \circ p_0]_E)$ lies below $\text{con}([f]_E)$, and thus the sky of $[f]_E$ cannot be below the sky of $[g \circ p_0]_E$, which is $\bar{S}$. In all three cases, the sky of $[f]_E$ is $\bar{S}$ (or $[f]_E$ is standard). Thus E is a P point.

To show that E is 5 Ramsey, let $\Sigma: [\omega^2]^2 \rightarrow 6$. Since E is a P point and R is Ramsey, there is a set $V_1 \in E$ such that Σ assumes only one value β on $\{\{s, t\} \in [V_1]^2: g(p_0(s)) \neq g(p_0(t))\}$.

Define a partition $\Omega_3: [\bar{X}]^3 \rightarrow 6$ as follows. $\Omega_3\{x_1, x_2, x_3\}$ is arbitrary if

$$\neg(p(x_1) = p(x_2) = p(x_3)) \vee \{x_1, x_2, x_3\} \subseteq \bar{X}^i \quad \text{for } i = 0, 1.$$

Otherwise, for $x_1 < x_2 < x_3$, let

$$\Omega_3\{x_1, x_2, x_3\} = \begin{cases} \Sigma\{(\phi(x_1), \phi(x_2)), (\phi(x_1), \phi(x_3))\} & \text{if } x_1 \in \bar{X}^0 \wedge x_2, x_3 \in \bar{X}^1, \\ \Sigma\{(\phi(x_1), \phi(x_3)), (\phi(x_2), \phi(x_3))\} & \text{if } x_1, x_2 \in \bar{X}^0 \wedge x_3 \in \bar{X}^1. \end{cases}$$

Let $Y \in F$ satisfy C_{Ω_3} . By cutting down $p''Y$ appropriately, we can assume that Ω_3 takes only one value γ (resp. δ) on $\{\{x_1, x_2, x_3\} \in [Y]^3: p(x_1) = p(x_2) = p(x_3) \wedge x_1 \in Y^0 \wedge x_2, x_3 \in Y^1\}$ (resp. on $\{\{x_1, x_2, x_3\} \in [Y]^3: p(x_1) = p(x_2) = p(x_3) \wedge x_1, x_2 \in Y^0 \wedge x_3 \in Y^1\}$). Let $A_i = \phi''Y^i$ for $i = 0, 1$; then $A_i \in D_i$ and $V_2 = \{(x, y) \in \omega^2: x \in A_1, y \in A_2 \wedge g(x) = g(y)\} \in E$. We have $\Sigma''\{\{s, t\} \in [V_2]^2: p_0(s) = p_0(t)\} = \{\gamma\}$, and $\Sigma''\{\{s, t\} \in [V_2]^2: p_1(s) = p_1(t)\} = \{\delta\}$.

Define partitions Ω_4 and $\bar{\Omega}_4$ on $[X]^4$ as follows. Ω_4 and $\bar{\Omega}_4$ are arbitrary on $\{x_1, x_2, x_3, x_4\}$ if

$$\neg(p(x_1) = p(x_2) = p(x_3) = p(x_4)) \vee \neg|\{x_1, x_2, x_3, x_4\} \cap \bar{X}^0| = 2.$$

Otherwise, for $x_1 < x_2 < x_3 < x_4$, let

$$\Omega_4\{x_1, x_2, x_3, x_4\} = \Sigma\{(\phi(x_1), \phi(x_3)), (\phi(x_2), \phi(x_4))\},$$

and

$$\bar{\Omega}_4\{x_1, x_2, x_3, x_4\} = \Sigma\{(\phi(x_1), \phi(x_4)), (\phi(x_2), \phi(x_3))\}.$$

Let $Y \in F$ (resp. $\bar{Y} \in F$) satisfy C_{Ω_4} (resp. $C_{\bar{\Omega}_4}$). By cutting down $p''Y$ (resp. $p''\bar{Y}$) appropriately, we can assume that Ω_4 (resp. $\bar{\Omega}_4$) assumes only one value η (resp. $\bar{\eta}$) on $\{c \in [Y]^4: |p''c| = 1 \wedge |c \cap Y^0| = 2\}$ (resp. the same with $\bar{Y}$ for Y). Let $B_i = \phi''Y^i$, $\bar{B}_i = \phi''\bar{Y}^i$, $i = 0, 1$; then $B_i, \bar{B}_i \in D_i$. Let $V_3 = \{(x, y) \in \omega^2: x \in B_0, y \in B_1, g(x) = g(y)\}$, and $\bar{V}_3 = \{(x, y) \in \omega^2: x \in \bar{B}_0, y \in \bar{B}_1, g(x) = g(y)\}$; then $V_3, \bar{V}_3 \in E$.

Let $W = V_1 \cap V_2 \cap V_3 \cap \bar{V}_3$. Then $W \in E \wedge (\forall s \in W)g(p_0(s)) = g(p_1(s))$. Let us agree to write an element of $[W]^2$ as $\{(x, y), (z, w)\}$ if $x < z$ or $x = z$ and $y < w$. Then, if $\{(x, y), (z, w)\} \in [W]^2$, we have

$$\Sigma\{(x, y), (z, w)\} = \begin{cases} \beta & \text{if } g(x) \neq g(z), \\ \gamma & \text{if } x = z, \\ \delta & \text{if } y = w, \\ \eta & \text{if } x < z \wedge y < w \wedge g(x) = g(z), \\ \bar{\eta} & \text{if } x < z \wedge w < y \wedge g(x) = g(z). \end{cases}$$

By our notational convention and the fact that $W \subseteq \{(x, y): g(x) = g(y)\}$, we conclude that $|\Sigma''[W]^2| \leq 5$.

THEOREM 5.4 (CH). *There exist $2^{2^{\aleph_0}}$ 5 Ramsey P points E such that $E \text{ prod } \mathbb{N}$ has exactly 4 nonstandard constellations, two of which correspond to 2 Ramsey ultrafilters.*

Thus the constellation structure of these ultrafilters is

PROOF. It only remains to show that there are no other constellations. But if there were, then E would be 2 square, and then the constellations of E would be linearly ordered [R]. $\square$

6. Some natural questions follow. Does Theorem 4.1 hold if we replace “ P point” by “ultrafilter”? Does Theorem 5.4 hold with arbitrary $n \geq 6$ in the place of “5”? The ultimate problem here is to find a characterization of all those lattices which occur (CH) as the constellation structure of a weakly Ramsey ultrafilter; this problem, even restricted to P points, seems quite complicated.

REFERENCES

- [AH] F. G. Abramson and L. A. Harrington, *Models without indiscernibles*, J. Symbolic Logic **43** (1978), 572–600.
- [B1] A. Blass, *Ultrafilter mappings and their Dedekind cuts*, Trans. Amer. Math. Soc. **188** (1974), 327–340.
- [B2] ———, *Amalgamation of non-standard models of arithmetic*, J. Symbolic Logic **42** (1977), 372–386.
- [CK] C. C. Chang and H. J. Keisler, *Model theory*, North-Holland, Amsterdam, 1973.
- [D] M. Dagenet, *Ultrafiltres à la façon de Ramsey*, Trans. Amer. Math. Soc. **250** (1979), 91–120.
- [P] C. Puritz, *Ultrafilters and standard functions in non-standard arithmetic*, Proc. London Math. Soc. (3) **22** (1971), 705–733.
- [K] K. Kunen, *Some points in $\beta\mathbb{N}$* , Math. Proc. Cambridge Philos. Soc. **80** (1976), 289–312.
- [R] N. I. Rosen, *A characterization of 2 square ultrafilters*, J. Symbolic Logic (to appear).

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