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TREES, GLEASON SPACES, AND COABSOLUTES OF $\beta N \sim N$

BY

SCOTT W. WILLIAMS

ABSTRACT. For a regular Hausdorff space X, let $\mathfrak{E}(X)$ denote its absolute, and call two spaces X and Y coabsolute ($\mathcal{G}$-absolute) when $\mathfrak{E}(X)$ and $\mathfrak{E}(Y)$ ($\beta\mathfrak{E}(X)$ and $\beta\mathfrak{E}(Y)$) are homeomorphic. We prove X is $\mathcal{G}$-absolute with a linearly ordered space iff the POSET of proper regular-open sets of X has a cofinal tree; a Moore space is $\mathcal{G}$-absolute with a linearly ordered space iff it has a dense metrizable subspace; a dyadic space is $\mathcal{G}$-absolute with a linearly ordered space iff it is separable and metrizable; if X is a locally compact noncompact metric space, then $\beta X \sim X$ is coabsolute with a compact linearly ordered space having a dense set of P-points; CH implies but is not implied by "if X is a locally compact noncompact space of π-weight at most 2^ω and with a compatible complete uniformity, then $\beta X \sim X$ and $\beta N \sim N$ are coabsolute."

A *tree* T is a POSET (partially ordered set) in which $]\leftarrow, t[$, the set of predecessors of t, is well ordered for each $t \in T$. The trees most familiar to topologists are the Cantor tree, the Souslin trees, and the Aronszajn trees **[Ku]**, **[Ru]**. In §1 we study conditions under which a given POSET contains a cofinal tree.

Recall **[Po]**, **[P.S.]** that if X is a space,[1] then the *absolute* $\mathfrak{E}(X)$ of X is the unique (up to a homeomorphism) extremally disconnected space that can be mapped irreducibly onto X by a perfect map. Following **[C.N.2]** call $\beta\mathfrak{E}(X)$ the *Gleason space* of X and denote it by $\mathcal{G}(X)$. Two spaces X and Y are *coabsolute* ($\mathcal{G}$-*absolute*) whenever $\mathfrak{E}(X)$ and $\mathfrak{E}(Y)$ (respectively, $\mathcal{G}(X)$ and $\mathcal{G}(Y)$) are homeomorphic. Designate $\mathfrak{R}(X)$ for the Boolean algebra of regular-open sets of X–then it is known that $\mathcal{G}(X) \cong \mathcal{G}(Y)$ iff $\mathfrak{R}(X) \equiv \mathfrak{R}(Y)$.

In §2, we begin an application of §1 to topology with several theorems. We prove:

(2.1) X is $\mathcal{G}$-absolute with a linearly ordered space if, and only if, $(\mathfrak{R}(X) \sim \{X\}, \supseteq)$ contains a cofinal tree.

(2.3) ((2.8)) A first countable (Moore) space is $\mathcal{G}$-absolute with a linearly ordered space iff it has a dense linearly ordered (metrizable) subspace.

(2.10) A dyadic space is $\mathcal{G}$-absolute with a linearly ordered space iff it is separable and metrizable.

We also give (2.6 and 2.7) sufficient conditions (dependent on certain cardinal functions) for a space X to have a dense linearly ordered subspace.

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[1]All spaces are infinite Hausdorff and regular.

In §3 we consider coabsolutes of Stone-Čech remainders. For a noncompact completely regular space X, let $X^* = \beta X \sim X$. We prove

(3.5) A locally compact noncompact metric space X has X^* coabsolute with a linearly ordered space having a dense set of P-points.

(3.9) If X is a locally compact noncompact metric space of density at most 2^ω, then X^* is coabsolute with one of $\mathbf{N}^*$, $\mathbf{R}^*$, or $\mathbf{N}^* + \mathbf{R}^*$.

(3.13) The following is implied by **CH**, and consistent with and independent of $\neg$**CH**: If X is locally compact noncompact, has $\pi w(X) \leq 2^\omega$, and if X admits a complete uniformity, then X^* is coabsolute with $\mathbf{N}^*$.

A result of importance in §§2 and 3 is the Stone duality theorem [**C.N.2**]. The Stone space of a Boolean algebra B is denoted by $\mathrm{St}(B)$. "$\equiv$" between POSETS or Boolean algebras means "is isomorphic to".

We assume **ZFC**. **CH** is the continuum hypothesis, **SH** is Souslin's hypothesis, and **MA** is Martin's axiom. All cardinals and ordinals are von Neumann ordinals, so $\beta < \alpha$ means $\beta \in \alpha$. ω_0 is denoted by ω and if κ is a cardinal, 2^κ is the cardinal number of the set of subsets of κ. $|X|$ means the cardinality of X. If A and B are sets ${}^AB$ denotes the set of functions from A to B. The standard binary tree [**Ku**] of height λ, an ordinal, is $\{f \in {}^\alpha 2\colon \alpha \in \lambda\}$ ordered by $f \leq g$ when $f = g \restriction \mathrm{dom}(f)$ and is denoted by $\mathrm{TREE}(\lambda)$.

$\mathbf{N}$ is the space of natural numbers, **Irr** is the space of irrationals, and $\mathbf{R}$ is the space of reals. "int" and "cl" are the interior and closure operators. "$\cong$" between spaces means "is homeomorphic" and $A \sim B$ means the complement of B in A. "Σ" and "$+$" denote free union.

1. Trees in POSETS. Suppose P is a POSET, $p \in P$, and $Q \subseteq P$. Q is *cofinal* if $r \in P \Rightarrow \exists q \in Q$ with $r \leq q$. Q is a *filter* if $p > q \in Q \Rightarrow p \in Q$. $p \in P$ is *compatible*[2] with Q if, for each $q \in Q$, $\exists r$ with $p \leq r$ and $q \leq r$. P is *separative* if, for each pair $p, q \in P$, $p \nleq q \Rightarrow \exists r \geq q$ with r and p incompatible. If F is a cofinal filter in a separative POSET P, then each maximal incompatible family of F is a maximal incompatible family of P. On the other hand, if I is a maximal incompatible family of P, then $\{p \in P\colon \exists i \in I, i \leq p\}$ is a cofinal filter of P.

Suppose T is a tree and α is an ordinal, then

$$\mathrm{lv}(T, \alpha) = \{t \in T\colon]\leftarrow, t[\text{ has order type } \alpha\}$$

is the *αth level* of T also denoted by $\mathrm{lv}(\alpha)$ when there is no confusion. $T(\alpha) = \bigcup \{\mathrm{lv}(\beta)\colon \beta \in \alpha\}$ is the *αth subtree*. The *height* of T is

$$h(T) = \inf\{\alpha\colon \mathrm{lv}(\alpha) = \varnothing\}.$$

A *branch* b of T is a maximal linearly ordered subset of T and $\mathrm{ord}(b)$ denotes the order type of b.

If every linearly ordered subset of a POSET P is bounded above, then Zorn's lemma provides P with a cofinal tree of height 1. However, within any POSET P we may build a tree T, recursively, by the subtrees $T(\alpha)$, having special properties:

[2] This is the first of several traditional [**Bu**], [**Je**] definitions (also separative, κ-distributive, κ-closed) for which we have reversed the usual order relation to maintain the orientation upwards for trees.

1.1. LEMMA. *If P is a POSET, then there is a tree $T \subseteq P$ satisfying*:

(1) $\mathrm{lv}(0)$ *is a maximal incompatible family of P.*

(2) *If, for $\alpha \in h(T)$, b is a branch of $T(\alpha)$ bounded above in P, then* $\mathrm{lv}(\alpha)$ *contains a maximal incompatible family of $\bigcap\{]t, \to[: t \in b\}$, where each $]t, \to[= \{p \in P: t < p\}$.*

(3) $\bigcap\{]t, \to[: t \in b\} \subseteq T$ *for each branch b of T.*

Whenever a tree T satisfies 1.1(1)–(3) in a POSET P, T will be called an *unbounded tree* of P. Since a cofinal tree in a POSET will be unbounded it is useful to define the ordinal invariants (under isomorphism)

$$\#(P) = \inf\{h(T): T \text{ is an unbounded tree of } P\}$$

and $w\#(P) = \inf\{\#([p, \to[): p \in P\}$.

1.2. THEOREM. *For a tree T in a POSET P the following hold*:

(1) *T is unbounded if $p \in P$, $(p \nleqslant t \, \forall t \in T) \Rightarrow p$ is compatible with at least two incompatible elements of T.*

(2) *If P is separative and T is unbounded, then $(p \nleqslant t \, \forall t \in T) \Rightarrow p$ is compatible with two elements of some level of T.*

(3) *If P is separative, then $\#([p, \to[) \leqslant \#(P) \, \forall p \in P$.*

(4) [Ny] *If every compatible pair of elements of P is linearly ordered and if T is unbounded, then T is cofinal in P.*

PROOF. (1) Certainly T satisfies 1.1(1), for otherwise some element of P would be compatible with no elements of T. If $\lambda \leqslant h(T)$ and if p is a successor of each member of a branch b of $T(\lambda)$, then $b = \{t \in T(\lambda): p \text{ and } t \text{ are compatible}\}$. So 1.1(3) is immediate while 1.1(2) uses, as well, the observations for 1.1(1).

(2) We note that if I is a maximal incompatible family of a separative POSET P and if $p \in P$ is compatible with precisely one $i \in I$, then $i \leqslant p$. Otherwise, we may find $q \in P$ so that $p < q$ with q and i (and hence $I \cup \{q\}$) incompatible.

So if $p \in P$ is compatible with at most one element of each level, then 1.1(1) and (2) force p to be an upper bound of a branch of T. So $p \in T$, by 1.1(3).

(3) Suppose U is an unbounded tree of P, $\#(P) = h(U)$, and $p \in P$. If $p \leqslant u$ for some $u \in U$, then $U \cap [p, \to[$ is an unbounded tree of $[p, \to[$. So assume $p \nleqslant u \, \forall u \in U$. From (2) $\exists$ a first $\alpha_0 \in h(U)$ such that p is compatible with two elements of $\mathrm{lv}(U, \alpha_0)$. We may find a maximal incompatible family I of $\{q \in P: \exists u \in \mathrm{lv}(U, \alpha_0),\ u < q,\ p < q\}$. Starting with

$$T(\alpha_0 + 1) = U(\alpha_0) \cup I \cup \{u \in \mathrm{lv}(U, \alpha_0): p \text{ and } u \text{ are incompatible}\}$$

we can build, using (2) to obtain each level, an unbounded tree T of P such that if $\alpha \in h(T)$ and $t \in \mathrm{lv}(T, \alpha)$, then there is a $u \in \mathrm{lv}(U, \alpha) \ni t \leqslant u$. So $h(U) = h(T)$. Since $I \subseteq [p, \to[$, 1.1 shows that $T \cap [p, \to[$ is an unbounded tree of $[p, \to[$. Therefore,

$$h([p, \to[) \leqslant h(T) = \#(P).$$

(4) This generalization of "every linearly ordered set has a well-ordered cofinal subset" is proved similarly to (2). $\square$

For a POSET P and a cardinal κ, P is called *κ-distributive* if each intersection of at most κ cofinal filters in P is cofinal; P is *κ-closed* whenever each increasing sequence of length at most κ is bounded above. In forcing arguments the above properties have seen considerable activity [**Bu**].

1.3. LEMMA [**Bu**, 3.11]. *Let P be a POSET and κ a cardinal.*

(1) *If P is κ-closed, it is κ-distributive.*

(2) *If λ is the first cardinal such that P is not λ-distributive (or λ-closed), then either $\lambda = \omega$ or λ is a successor cardinal.*

1.4. LEMMA. *Suppose P is a POSET with no maximal elements, T is an unbounded tree of P, and κ is a cardinal.*

(1) *If P is κ-distributive and has no maximal elements, then $\kappa^+ \leqslant h(T)$ and* $\mathrm{lv}(T, \alpha)$ *is a maximal incompatible family of P $\forall \alpha \in \kappa^+$.*

(2) *If P is κ-closed, $\kappa <$* $\mathrm{cf}(\mathrm{ord}(b))$ $\forall$ *branches b of T.*

PROOF. As (2) is obvious, we suppose P is κ-distributive. Let α be the first ordinal such that $\mathrm{lv}(\alpha)$ is not a maximal incompatible family of P. If $\alpha \in \kappa^+$

$$C = \bigcap \{ \bigcup \{[t, \rightarrow [: t \in \mathrm{lv}(\beta)\} : \beta \in \alpha\}$$

is a cofinal filter of P. Given $c \in C$ we may choose $t_\beta \in \mathrm{lv}(\beta) \forall \beta \in \alpha$ such that $t_\beta \leqslant c$. As T is a tree, c is an upper bound of the branch $b = \{t_\beta : \beta \in \alpha\}$ of $T(\alpha)$. From 1.1(2), c is compatible with some element of $\mathrm{lv}(\alpha)$ bounding b from above. Thus, $\mathrm{lv}(\alpha)$ is a maximal incompatible family of C, and hence, of P. As this is absurd, we must have $\alpha \geqslant \kappa^+$. $\square$

1.5. THEOREM. *If P is a separative POSET without maximal elements, then $w\#(P) = \sup\{\kappa^+ : P$ is κ-distributive$\}$.*

PROOF. We denote the right-hand side of the above equation by λ. Then 1.4(1) shows $\lambda \leqslant \#(P)$. We consider two cases.

Case 1. $w\#(P) = \#(P)$. Suppose $\{F_\alpha : \alpha \in \lambda\}$ is a family of cofinal filters in P. Let $p \in P$ be arbitrary and $J = \{q \in P$: p and q are incompatible$\}$. We build, recursively, a tree T in $]p, \rightarrow[$.

Let $T(0) = \varnothing$. Suppose we are given an ordinal α, $\alpha \leqslant \lambda$, such that for each $\beta < \lambda$, we have found $T(\beta)$ subject to the restriction

($*$) If $\gamma < \beta$, then $\mathrm{lv}(T, \gamma)$ is a maximal incompatible family of $F_\gamma \cap\,]p, \rightarrow[$.

If α is a limit ordinal, we must set $T(\alpha) = \bigcup\{T(\beta) : \beta < \alpha\}$. If α is a successor ordinal, then $|\alpha| < \lambda$ since λ is a cardinal. Set

$$F = F_\alpha \cap \left(\bigcap \{ \bigcup \{]t, \rightarrow [: t \in \mathrm{lv}(T, \beta)\} : \beta + 1 < \alpha\}\right).$$

So $J \cup F$ is a cofinal filter of P. Now suppose $r \in J \cup F$ and $r > p$.

Clearly $r \in F$ and $\exists r_\beta$ for each $\beta < \alpha$ such that $r_\beta < r$ and $r \in \mathrm{lv}(T, \beta)$. Since $T(\alpha - 1)$ is a tree, $\{r_\beta : \beta + 1 < \alpha\}$ is a branch of $T(\alpha - 1)$. Since F is now a cofinal filter of $]p, \rightarrow[$, there is a maximal incompatible family I of $]p, \rightarrow[$ contained in F. Set $T(\alpha) = F \cup T(\alpha - 1)$. As our hypothesis ($*$) is now met, the construction of T is complete when we let $T = T(\lambda)$.

Now let us suppose that $\lambda < \#(P)$. Then T is not an unbounded tree of $]p, \to[$. From $(*)$, each $\mathrm{lv}(T, \alpha)$ is a maximal incompatible family of $]p, \to[$, so 1.1 implies that T has a branch b bounded above. If $\mathrm{ord}(b) < \lambda$, then there is a $t \in \mathrm{lv}(T, \mathrm{ord}(b))$ compatible with every element of b. Since T is a tree, t is an upper bound for b. As the latter is impossible, $\mathrm{ord}(b) = \lambda$. Now if s is an upper bound for b, then $(*)$ implies $p < s$ and $s \in \bigcap\{F_\alpha\colon \alpha < \lambda\}$. As p was chosen arbitrarily, $\bigcap\{F_\alpha\colon \alpha < \lambda\}$ is a cofinal filter of P. So $\lambda \not< \lambda^+$. This is a contradiction. Therefore, $\lambda = \#(P)$.

Case 2. $w\#(P) < \#(P)$. Choose a maximal incompatible family $I \subseteq P$ with

$$w\#(]p, \to[) = \#(]p, \to[)\ \forall p \in I.$$

Since $\bigcup\{]p, \to[\colon p \in I\}$ is a cofinal filter of P, $\#(P) = \sup\{\#(]p, \to[)\colon p \in I\}$. On the other hand, we may add $\bigcup\{]p, \to[\colon p \in I\}$ to any family of cofinal filters. So, from case 1, $\#(]p, \to[) \leqslant \lambda\ \forall p \in I$. Therefore, $\lambda = \#(P)$. $\square$

We observe that case 2 of 1.5 also shows

1.6. COROLLARY. *$\#(P)$ is a cardinal whenever P is a separative POSET.*

1.7. THEOREM. *Suppose P is a separative POSET for which $\#(P) = w\#(P)$. If P has a cofinal family the union of $\#(P)$ incompatible families, then P has a cofinal tree of height $\#(P)$.*

PROOF. WLOG we assume $\bigcup\{I_\alpha\colon \alpha \in \#(P)\}$ is cofinal in P where each is a maximal incompatible family of P containing no maximal elements of P. We construct an unbounded tree T, modifying the successor ordinal steps in 1.1, as follows:

If $\alpha < \#(P)$, $|\alpha| < \#(P)$ from 1.5. By 1.4

$$C = \left(\bigcup\{[i, \to[\colon i \in I_\alpha\}\right) \cap \left(\bigcap\{\bigcup\{[t, \to[\colon t \in \mathrm{lv}(T(\alpha-1), \beta)\}\colon \beta < \alpha - 1\}\right)$$

is a cofinal filter of P. So there is a maximal incompatible family I of $P \ni I \subseteq C$. Set $T(\alpha) = T(\alpha - 1) \cup I$.

It is clear that T is cofinal and $h(T) = \#(P)$. $\square$

1.8. COROLLARY. (1) (*Weiss*) *If P is a separative κ-closed POSET $\forall \kappa < \lambda$ and P has a cofinal family which is the union of λ incompatible families, then P has a cofinal tree.* (2) (*Davies*) *If P is a separative ω-distributive POSET and $|P| \leqslant \omega_1$, then P has a cofinal tree.*

PROOF. Observe that both results follow from 1.7 even if "separative" is removed from their hypothesis since we only used "P is κ-distributive $\forall \kappa < \#(P)$" in the proof of 1.7. $\square$

1.9. THEOREM. *If a separative POSET has a cofinal tree, then it has a cofinal tree of height $\#(P)$.*

PROOF. In proving 1.5, we observed that there is a maximal incompatible family I of P such that, $\forall i \in I$, $w\#([i, \to[) = \#([i, \to[)$ and

$$\#(P) = \sup\{\#([i, \to[)\colon i \in I\}.$$

So WLOG we may assume P has no maximal elements and $\#(P) = w\#(P)$. Suppose T is a cofinal tree in P, U is an unbounded tree in P, and $h(U) = \#(P)$. From 1.4 each $\mathrm{lv}(U, \alpha)$ is a maximal incompatible family of P. So we may choose a maximal incompatible family of P,

$$I_\alpha \subseteq \left(\bigcup \{[t, \rightarrow [: t \in \mathrm{lv}(U, \alpha)\}\right) \cap T.$$

If $\bigcup\{I_\alpha: \alpha \in \#(P)\}$ is cofinal, the theorem follows from 1.7. So suppose $t \in T$ such that $t \nleqslant i \,\forall i \in I_\alpha \forall \alpha \in \#(P)$. Since P is separative, we have, from 1.2(2), $\exists \alpha \in h(U)$ and u_0, $u_1 \in \mathrm{lv}(U, \alpha) \ni t$ is compatible with each u_n. Again using separativity $\exists$, for each $n \in \{0, 1\}$, a $t_n \in T$ such that $t < t_n$ and $u_n < t_n$. $\exists$ for each n, $i_n \in I_\alpha$ such that $u_n < i_n$ and t_n is compatible with i_n. Since T is a tree, either $i_n \geqslant t_n \geqslant t$ (a contradiction) or $i_n < t_n$. In the latter case, t and i_n are compatible so $i_0 < t$ and $i_1 < t$ (a contradiction). □

From 1.7, $2^\kappa = \kappa^+$ implies each κ-distributive POSET of cardinality at most 2^κ has a cofinal tree. Our next theorem represents an attempt at removing the set-theoretic hypothesis from this result.

1.10. LEMMA. *Suppose κ is an infinite cardinal, P is a separative κ-closed POSET, and $p \in P$ has no maximal successor; then p has 2^κ incompatible successors.*

PROOF. We can construct a tree in $[p, \rightarrow [$ isomorphic to $\mathrm{TREE}(\kappa + 1)$ by applying "separative" at successor ordinals to get two incompatible elements, and "κ-closed" at limit ordinals to get a single successor of a branch. The final level of $\mathrm{TREE}(\kappa + 1)$ contains 2^κ incompatible elements. □

1.11. LEMMA. *Suppose κ is an infinite cardinal and $P = \{p(\zeta): \zeta \in 2^\kappa\}$ is a κ-closed separative POSET. If $I = \{i(\alpha): \alpha \in 2^\kappa\}$ is an incompatible family in P, then there is a family $J \subseteq P$ subject to*

(1) $J = \{j(\alpha, \xi): (\alpha, \xi) \in 2^\kappa \times 2^\kappa\}$, *where* $i(\alpha) < j(\alpha, \xi)$ *for each* ξ.

(2) *If* $p(\zeta)$ *is compatible with an element of J but* $p(\zeta) \nleqslant j \,\forall j \in J$, *then*

$$|\{i \in I: p(\zeta) \text{ and } i \text{ are compatible}\}| \leqslant |\zeta|.$$

PROOF. J is constructed recursively via a diagonalization argument–we examine the βth step:

Suppose δ is the first element of 2^κ such that, $\forall \alpha \in 2^\kappa$, $p(\delta) \nleqslant i(\alpha)$ and $\forall \gamma \in \beta \,\forall \xi \in 2^\kappa$ $p(\delta) \nleqslant j(\gamma, \xi)$, but $p(\delta)$ and $i(\beta)$ are compatible. For some $q \in P$ with $p(\delta) < q$ and $i(\beta) < q$ we choose a family $\{j(\beta, \xi): \xi \in 2^\kappa\}$ of maximal incompatible successors of $i(\beta)$ to which q belongs. □

1.12. THEOREM. *Let κ be an infinite cardinal. If P is a separative κ-closed POSET with $|P| \leqslant 2^\kappa$, then P has a cofinal tree.*

PROOF. From 1.10, $|P| < 2^\kappa \Rightarrow P$ has a cofinal tree of height 1. So WLOG assume P has no maximal elements and we have a listing of P, $\{p(\zeta): \zeta \in 2^\kappa\}$.

We can construct, as in 1.1, using 1.10 and 1.11, an unbounded tree T of P subject to the additional conditions:

(4) $\mathrm{lv}(0)$ contains a successor of $p(0)$.

(5) If $\alpha \in h(T)$ and b is a branch of $T(\alpha)$ bounded above in P, then $|\operatorname{lv}(\alpha) \cap (\bigcap\{[t,\rightarrow[: t \in b\})| = 2^\kappa$.

(6) If $p(\zeta) \nleqslant t\, \forall t \in T(\alpha)$ for $\alpha \in h(T)$, then $\alpha \leqslant \zeta$ and $|\{t \in T(\alpha): p(\zeta)$ and t are incompatible$\}| \leqslant |\zeta|$.

Suppose $p = p(\zeta) \in P$ and $p \nleqslant t\, \forall t \in T$; then there is, by 1.2(2), a first α_0 such that p is compatible with two elements of $\operatorname{lv}(\alpha_0)$. Using 1.4(2) and following the proof of 1.10, we may build a tree $S \subseteq T$ consisting of elements compatible with p, each of whose levels is contained in a level of T, and which is isomorphic to TREE($\kappa + 1$). Since $\exists \lambda \in h(T)$ such that the last level of S is contained in $\operatorname{lv}(\lambda)$, p causes (6) to fail for $\alpha = \lambda$. $\square$

1.13. Lemma [**Je**, 29B]. *If P is a separative POSET, then there exists a unique (up to an isomorphism) complete Boolean algebra $\mathfrak{B}(P)$ for which P is cofinally embedded in $(\mathfrak{B}(P) - \{0\}, \geqslant)$.*

If B is a Boolean algebra such that $(B - \{1\}, \leqslant)$ or, equivalently, $(B - \{0\}, \geqslant)$, possesses (resp. a cofinal set P satisfying) the properties defined in this section for a POSET, we say for simplicity that B (resp. *cofinally*) possesses said property; therefore:

(i) Every Boolean algebra is separative.

(ii) No atomless σ-complete Boolean algebra is ω-closed.[3]

(iii) $\mathfrak{B}$(TREE(ω_1)) is cofinally ω-closed.

1.14. Corollary. *Suppose κ is an infinite cardinal. If $\kappa^+ = 2^\kappa$, then $\mathfrak{B}$(TREE(2^κ)) is the only complete atomless Boolean algebra which is cofinally κ-closed and has a cofinal set of cardinal 2^κ. If $2^{\kappa^+} = 2^\kappa$, then there are at least two complete atomless Boolean algebras cofinally κ-closed and having cofinal subsets of cardinal 2^κ.*

Proof. The Pressing Down Lemma [**Ku**] shows that, for each κ, $\mathfrak{B}(\mathrm{TREE}(\kappa^+)) \not\cong \mathfrak{B}(\mathrm{TREE}(\kappa^{++}))$. On the other hand (v) in the construction of T in 1.12 shows that if $\kappa^+ = 2^\kappa$ and P is a κ-closed separative POSET without maximal elements of cardinality 2^κ, then P has a cofinal tree isomorphic to $\bigcup\{\operatorname{lv}(\mathrm{TREE}(\kappa^+), \lambda): \lambda$ is a limit ordinal in $\kappa^+\}$. $\square$

1.15. *On products*. Suppose κ is a cardinal and $P(\alpha)$ is a POSET for each $\alpha \in \kappa$; there are two traditional definitions for partial orders on the Cartesian product $\Pi = \Pi\{P(\alpha): \alpha \in \kappa\}$:

(1) The *lexicographic product*, lex Π, is ordered by "$f < g$ whenever $\exists \alpha \in \kappa$ with $f(\alpha) < g(\alpha)$ and $f(\beta) = g(\beta)\, \forall \beta \in \alpha$". It is easy to see that lex Π *has a cofinal tree whenever $P(\alpha)$ has a cofinal tree $\forall \alpha \in \kappa$.*

(2) The *usual product* on Π, denoted by $\times\{P(\alpha): \alpha \in \kappa\}$, is ordered by "$f \leqslant g$ whenever $f(\alpha) \leqslant g(\alpha)\, \forall \alpha \in \kappa$". An easy application of the Pressing Down Lemma shows TREE(ω) $\times$ TREE(ω_1) has no cofinal tree. However, 1.7 shows that $P \times Q$ has a cofinal tree whenever each of P and Q have a cofinal tree and $w\#(P) = w\#(Q)$.

1.16. Remarks. (1) Is it consistent that "every ω-distributive POSET of cardinality ω_1 is ω-closed?" Not in a model of **ZFC** + **SH**; however, Franklin Tall has

[3] In [**Wo2**], [**Wo3**] cofinally ω-closed Boolean algebras are called *Cantor-separable*.

communicated to the author Peter Davies' affirmative answer under the assumption of the consistency of certain large cardinal axioms.

(2) For the POSET P of nonempty clopen subsets of $\beta\mathbf{N} \sim \mathbf{N}$ (under "$\supseteq$") 1.5, 1.9, and 1.12 were proved, independently, in [**B.P.S.**].

(3) 1.8(2) is due to Peter Davies. 1.8(1) is an observation William Weiss made from one of our early results.

(4) Is it consistent with **ZFC** + **CH** that "there is precisely one complete atomless cofinally ω-closed Boolean algebra with a cofinal set of cardinality 2^{ω}?" See 3.13.

(5) For many POSETS P, $\#(P)$ is well defined by considering cofinal subsets of P. With proof similar to 1.2(3) and (4), this is true when P is either separative or when every compatible pair of elements of P are linearly ordered or when P is directed.

2. $\mathcal{G}$-absolutes of linearly ordered spaces. Recall [**Ju2**] if (X, τ) is a space, then a cofinal subset of $(\tau - \{\varnothing\}, \supseteq)$ is known as a *π-base* (*pseudobase* in [**C.N.2**]) and the *π-weight*, $\pi w(X)$, is the least cardinal possessed by a π-base for X. The *weight*, $w(X)$, is the least cardinal possessed by a base for τ.

2.1. THEOREM. *For a space X, the following are equivalent*:

(1) *X has a π-base with a cofinal tree.*

(2) *Every π-base of X has a cofinal tree.*

(3) *X is $\mathcal{G}$-absolute with a linearly ordered space.*

PROOF. If Y is the set of isolated points of X, then Y is a subset of every π-base for X, Y is the set of atoms of $\mathcal{R}(X)$, and

$$\mathcal{R}(X) \equiv \mathcal{R}\big(Y \dot{\cup} \operatorname{int}(X \sim Y)\big).$$

Since the free union of linearly ordered spaces is linearly ordered, we need only prove the theorem for $X \sim Y$. WLOG we assume X has no isolated points.

(1) $\Rightarrow$ (2). Let P be a π-base for X. Since a cofinal subset of a π-base is a π-base, we suppose that $(T, \supseteq)$ is a tree of nonempty open subsets of X such that T has the minimum possible height for a tree π-base for X. Since X has no isolated points, $h(T)$ and $\operatorname{ord}(b)$ are limit ordinals whenever b is a branch of T. We now construct, recursively, two trees $S_1 \subseteq T$ and $S_2 \subseteq P$.

For $i \in \{1,2\}$ let $S_i(0) = \varnothing$. Suppose we are given an ordinal $\alpha \leqslant h(T)$ such that $S_i(\beta)$ has been found, for each $\beta < \alpha$ and each $i \in \{1,2\}$, subject to the restriction $\gamma < \beta \Rightarrow$

(a) $\operatorname{lv}(S_i, \gamma)$ is a pairwise-disjoint family of nonempty open sets.

(b) If b is a branch of $S_1(\gamma)$ and if $\operatorname{int}(\bigcap b) \neq \varnothing$, then

$$\operatorname{int}\big(\bigcap b\big) \subseteq \operatorname{cl}\big(\{p \in \operatorname{lv}(S_2, \gamma) : p \subseteq \bigcap b\}\big).$$

(c) $\operatorname{lv}(S_1, \gamma) \subseteq T \sim T(\gamma)$.

(d) If $p \in \operatorname{lv}(S_2, \gamma)$, then $p \subseteq \operatorname{cl}(\{t \in \operatorname{lv}(S_1, \gamma) : t \subseteq p\})$.

If α is a limit ordinal, we set $S_i(\alpha) = \bigcup\{S_i(\beta) : \beta < \alpha\}\ \forall i$. If α is a successor ordinal, the choice of $\operatorname{lv}(S_i, \alpha - 1)$ is straightforward (given a π-base Q of a space Y

and a nonempty open set G of Y, G has a dense set which is the union of a family of pairwise-disjoint members of Q). Let $S_i(\alpha) = S_i(\alpha - 1) \cup \mathrm{lv}(S_i, \alpha - 1)$. Our recursion hypothesis is clearly met. So our construction of $S_i \forall i \in \{1, 2\}$ is complete when we set each $S_i = S_i(h(T))$.

Now (b) and (d) imply S_2 is a π-base for X iff S_1 is cofinal in T. (b) and (d) also imply that if $\alpha < h(T)$ and if b is a branch of $S_1(\alpha)$, then

$$\mathrm{int}\left(\bigcap b\right) \subseteq \mathrm{cl}\left(\{t \in \mathrm{lv}(S_1, \alpha) \colon t \subseteq \bigcap b\}\right).$$

From (c) and 1.1, S_1 is an unbounded tree of T. 1.2(4) shows S_1 is cofinal in T.

(2) $\Rightarrow$ (3). Since an infinite Hausdorff space contains an infinite family of nonempty pairwise-disjoint open sets, we suppose T is a cofinal tree in $\mathfrak{R}(X)$ satisfying

(iv) if $\alpha \in h(T)$ and b is a branch of $T(\alpha)$ bounded above in $\mathfrak{R}(X)$, then $\mathrm{lv}(T, \alpha)$ contains infinitely many elements each of whose closure is a subset of $\bigcap b$.

Following the standard [**Ku**] collapsing of Souslin and Aronszajn trees, order each $\mathrm{lv}(T, \alpha)$ so that $\mathrm{lv}(\alpha)$ and in the case of (iv), the successors of b in $\mathrm{lv}(\alpha)$, form a discrete linearly ordered set without endpoints.

Set $L = L(T) = \{b \colon b \text{ is a branch of } T\}$ and for $b_0, b_1 \in L$ define $b_0 < b_1$ if for some $\alpha \in h(T)$

$$\left(b_0 \cap \mathrm{lv}(T, \alpha)\right) < \left(b_1 \cap \mathrm{lv}(T, \alpha)\right) \quad \text{while } b_0 \cap T(\alpha) = b_1 \cap T(\alpha).$$

Thus, L is linearly ordered. Set $\psi(t) = \{b \in L \colon t \in b\}$ for each $t \in T$.

Since $\psi(t)$ has no endpoints, $\psi(t)$ is clopen; further, if in L, $b_0 < b_1 < b_2$, then by (iv), $\exists t \in b_1 \sim (b_0 \cup b_1) \ni \psi(t) \subseteq]b_0, b_1[$. So ψ embeds T cofinally within $\mathfrak{R}(L)$. Now apply 1.13 and the Stone duality.

(3) $\Rightarrow$ (1). Suppose L is a linearly ordered space without isolated points; then $P = \{G \in \mathfrak{R}(L) \colon G \text{ is an open interval of } L\}$ is cofinal in $\mathfrak{R}(L)$. Suppose T is any unbounded tree of P and $]x, y[\in P$ such that $t \nsubseteq]x, y[\forall t \in T$. According to 1.2(2) $\exists \beta \in h(T)$, $]x_i, y_i[\in \mathrm{lv}(T, \beta)$ for $i \in \{0, 1\}$, such that $]x, y[\cap]x_i, y_i[\neq \varnothing$,

$$(*) \qquad x_0 < x < y_0 < y, \qquad x < x_1 < y < y_1,$$

and there are no points between y_0 and x_1. Since $\mathfrak{R}(L)$ is closed under intersection, $]x, y_0[\in P$. Again applying 1.2(2) $\exists \alpha \in h(T)$, $\alpha > \beta$, $\exists]x_j, y_j[\in \mathrm{lv}(T, \alpha)$ for $j \in \{2, 3\}$, such that $]x_0, y_0[\cap]x_j, y_j[\neq \varnothing$, $x_2 < x < y_2 < y_0$, $x < x_3 < y_0 < y_3$, and there are no points between y_2 and x_3. In particular, $y_0 \in]x_3, y_3[\sim]x_0, y_0[$. Since T is a tree, $\beta < \alpha$, and L has no isolated points, $]x_3, y_3[\cap]x_0, y_0[= \varnothing$ which contradicts $(*)$. So T is cofinal in P. Now apply the Stone duality. $\square$

2.2. Corollary. *For a space X, the following are equivalent*:

(1) *X has a σ-disjoint π-base.*

(2) *$\mathfrak{R}(X)$ has a cofinal tree and $\#(\mathfrak{R}(X)) \leqslant \omega$.*

(3) *$\exists$ a metric space M $\mathcal{G}$-absolute with X.*

Proof. (1) $\Rightarrow$ (2). This is a corollary of 1.7 since $\mathfrak{R}(X)$ is n-closed $\forall$ finite cardinals n.

(2) $\Rightarrow$ (3). In 2.1 (2) $\Rightarrow$ (3) the tree T may be assumed to have height ω (from 1.9). Thus, the space L is metrizable via $\rho(b_0, b_1) = 2^{-n}$ when $b_0 \cap \mathrm{lv}(n) \neq b_1 \cap \mathrm{lv}(n)$ and $b_0 \cap T(n) = b_1 \cap T(n)$.

(3) $\Rightarrow$ (1). Every metric space has a σ-disjoint base. $\square$

2.3. THEOREM. *Suppose X is a space whose every point has a well-ordered local base; then X is $\mathcal{G}$-absolute with a linearly ordered space iff X has a dense linearly ordered subspace.*

PROOF. We need only show "$\Rightarrow$". WLOG assume X has no isolated points. For each $x \in X$ let $\mathfrak{N}(x) \subseteq \mathfrak{R}(X)$ be a well-ordered, by "$\supseteq$", local base at x. From 2.1, $\mathfrak{R}(X)$ has a cofinal tree U. We construct a tree T and a function ψ: $T \to X$ as follows:

Let $T(1) = U(1)$ and $\psi(t) \in t$ be arbitrarily chosen for each $t \in T(1)$. Suppose we are given an ordinal λ such that $T(\alpha)$ and $\psi(t) \in t$ have been constructed $\forall \alpha \in \lambda\, \forall t \in T(\alpha)$ subject to the restrictions:

(i) $T(\alpha)$ satisfies 1.1(1) and (2).

(ii) $T(\alpha)$ satisfies (iv) of 2.1(2) $\Rightarrow$ (3).

(iii) If $\beta < \alpha$ and b is a branch of $T(\beta)$, then

$$|\{t \in \mathrm{lv}(T(\alpha), \beta) : t \subseteq \cap b \text{ and } t \notin U\}| \leqslant 1,$$

where equality holds only if ψ is constant on a tail of b.

(iv) If $\beta < \alpha$ and $t \in T(\beta)$, then $T(\alpha)$ has a branch b with $t \in b$ and $\psi(b) = \{\psi(t)\}$.

(v) If $s, t \in T(\alpha)$, $t \subsetneqq s$, and if $\psi(t) = \psi(s)$, then $t \in \mathfrak{N}(\psi(s))$.

If λ is a limit ordinal, set $T(\lambda) = \cup\{T(\alpha): \alpha \in \lambda\}$. If λ is a successor ordinal, then we will assign to each branch b of $T(\lambda - 1)$ a family $I(b)$ and we will set

$$T(\lambda) = T(\lambda - 1) \cup \big(\cup\, \{I(b) : b \text{ is a branch of } T(\lambda - 1)\}\big).$$

For $t \in T(\lambda) \sim T(\lambda - 1)$ and $t \subseteq \cap b$, we assign $\psi(t) = \psi(s)$ if ψ is constantly $\psi(s)$ on a tail of b and if $t \in \mathfrak{N}(\psi(s))$; otherwise, $\psi(t) \in t$ may be arbitrarily chosen.

Let $I(b) = \varnothing$ whenever $\mathrm{int}(\cap b) = \varnothing$. If $\mathrm{int}(\cap b) \neq \varnothing$ and ψ is not constant on a tail of b, we may choose, since X has no isolated points, a pairwise disjoint subfamily $I(b)$ of U with $\cup\{\mathrm{cl}(u): u \in I(b)\}$ a dense subset of $\mathrm{int}(\cap b)$. If $\mathrm{int}(\cap b) \neq \varnothing$ and ψ is constantly $\psi(s)$ on a tail of b with $s \in b$, then (v) implies $\mathrm{int}(\cap b)$ is a nbhd of $\psi(s)$. Choose $t \in \mathfrak{N}(\psi(s))$ with $\mathrm{cl}(t) \supsetneqq \mathrm{int}(\cap b)$. Then $I(b)$ will be the union of $\{t\}$ and an infinite pairwise disjoint subfamily of U such that $\cup\{\mathrm{cl}(u): u \in I(b) \sim \{t\}\}$ is a dense subset of $\mathrm{int}(\cap b) \sim \mathrm{cl}(t)$.

As the induction hypothesis is clearly satisfied, we can continue it until we have an unbounded tree T. Since $\psi(t) \in t\, \forall t \in T$, $\psi(T)$ is dense if T is cofinal in $\mathfrak{R}(X)$. So we suppose $u \in U$ such that $t \not\subseteq u\, \forall t \in T$. By 1.2(2) there is a first ordinal $\beta \in h(T)$ such that $u \cap t_0$, $u \cap t_1 \in \mathfrak{R}(X) \sim \{\varnothing\}$ for two distinct elements t_0, $t_1 \in \mathrm{lv}(T, \beta)$. So $u \subseteq \cap b$ for a branch b of $T(\beta)$, and $t_0 \cup t_1 \subseteq \cap b$. Since U is a tree, neither t_0 nor t_1 is in U. This contradicts (iii) for $\alpha = \beta + 1$. So T is cofinal in $\mathfrak{R}(X)$.

From (iv) and (v) there is for each $t \in T$ precisely one branch $b(t)$ of T such that $t \in b(t)$ and $b(t)$ is a local base at $\psi(t)$. So when $\{b(t): t \in T\}$ inherits the order given in 2.1(2) $\Rightarrow$ (3), the map $b(t) \to \psi(t)$ gives $\psi(T)$ a linear order generating the subspace topology inherited from X. $\square$

2.4. COROLLARY [**Wh**]. *A first countable space has a σ-disjoint π-base iff it has a dense metrizable subspace.*

PROOF. In 2.3 each $\mathfrak{N}(x)$ can be assumed to have order type ω, and, according to 2.2, $h(U) = \omega$. So $h(T) = \omega$. Following the proof of 2.2(2) $\Rightarrow$ (3) we see that $\psi(T)$ is metrizable. $\square$

As a Souslin line has no dense metrizable subspace, it is consistent with **ZFC** that "σ-disjoint π-base" cannot be replaced by "$\mathcal{G}$-absolute with a linearly ordered space" in 2.4. However, it is replaceable for the class of first countable spaces X which have $\#(\mathfrak{R}(X)) \leqslant \omega$. Therefore, we consider some cardinal functions which affect $\#(\mathfrak{R}(X))$.

Henceforth, *we shall use* $\#[w\#]$ *to denote* $\#(\mathfrak{R}(X))$ $[w\#(\mathfrak{R}(X))]$ when there is no confusion. A topological translation of 1.5 and 1.9 yields

2.5. LEMMA. *Let X be a space and $\{I_\alpha\colon \alpha \in \kappa\}$ be a collection of families of pairwise-disjoint nonempty open sets such that* $\mathrm{cl}(\bigcup I_\alpha) = X\,\forall \alpha \in \kappa$. *If $\kappa \leqslant w\#$, then there is an unbounded tree T of $\mathfrak{R}(X)$ satisfying*:

(1) $h(T) = \#$.

(2) $s, t \in T, s \subsetneqq t \Rightarrow \mathrm{cl}(s) \subseteq t$.

(3) $\mathrm{cl}(\bigcup \mathrm{lv}(\alpha)) = X\,\forall \alpha \in w\#$.

(4) *T satisfies* (iv) *of* 2.1(2) $\Rightarrow$ (3).

(5) $i \in I_\alpha \Rightarrow \exists\, t \in \mathrm{lv}(\alpha) \ni t \subseteq i$.

Further, if X is $\mathcal{G}$-absolute with a linearly ordered space, then T may be assumed to be cofinal in $\mathfrak{R}(X)$.

Recall [**C.N.2**] that for a cardinal κ, a space X is *κ-Baire* if the intersection of at most κ many open dense subsets of X is dense [so Baire $=$ ω-Baire]; and $x \in X$ is a *P_κ-point* if it has a local base λ-closed $\forall \lambda < \kappa$ [so P-point $= P_{\omega_1}$-point]. X is an *almost P_κ-space* [**Le**] if the intersection of less than κ many nonempty open subsets of X has nonempty interior (equivalently, if $\mathfrak{R}(X)$ is cofinally λ-closed $\forall \lambda < \kappa$). A base (or π-base) for a space will be called *κ-disjoint* when it is the union of a collection of κ many families of pairwise disjoint sets (WLOG each family may be assumed to have union dense in X).

The following should be compared to [**C.N.1**, 3.1] and [**C.N.2**, 6.15].

2.6. THEOREM. *For a space X with a κ-disjoint π-base B,*

(1) $\kappa \geqslant \#$ (*so* $\pi w(X) \geqslant \#$),

(2) *if* $\kappa = w\#$, *X is κ-Baire, and if B is a base, then X has a dense subset of P_κ-points which is linearly orderable.*

PROOF. (1) From 1.6 there is a family J of pairwise-disjoint nonempty regular-open subsets of X such that $\bigcup J$ is dense in X, $w\#(\mathfrak{R}(G)) = \#(\mathfrak{R}(G))\,\forall G \in J$, and $\# = \sup\{\#(\mathfrak{R}(G))\colon G \in J\}$. From 2.5(5) $\kappa \geqslant \#(\mathfrak{R}(G))$ for each G.

(2) Let $B = \bigcup\{I_\alpha\colon \alpha \in \kappa\}$, where each I_α is pairwise-disjoint and has dense union. Let T be the tree guaranteed in 2.5 and set $D = \{\bigcap b\colon b$ is a branch of T, $\mathrm{ord}(b) = \kappa$, and $\bigcap b \neq \varnothing\}$. Since X is κ-Baire, 2.5(3) implies $\bigcup D$ is dense. Since B is a base and T is a tree, 2.5(5) implies b is a well-ordered local base at $\bigcap b$ and

$|\bigcap b| = 1$ whenever $\bigcap b \in D$. To see that $\bigcup D$ is linearly orderable, follow the last paragraph of 2.3. □

2.7. THEOREM. *Suppose X is a space whose diagonal is the intersection of κ many open subsets of $X \times X$; then $\kappa \geq \#$. Further, if $\kappa = w\#$, X is κ-Baire, and if X is the intersection of at most κ many open subsets of βX, then X has a dense subset of P_κ-points which is linearly orderable.*

PROOF. Let $\{\mathcal{O}(\alpha)\colon \alpha \in \kappa\}$ be a collection of open sets of $X \times X$ whose intersection is $\Delta = \{(x, x)\colon x \in X\}$. Given $x \in X$ and $\alpha \in \kappa$, there is a nbhd G of x such that $G \times G \subseteq \mathcal{O}(\alpha)$. Hence, for each $\alpha \in \kappa$ there is a pairwise-disjoint collection $I_\alpha \subseteq \mathfrak{R}(X)$ such that $\mathrm{cl}(\bigcup I_\alpha) = X$ and $G \times G \subseteq \mathcal{O}(\alpha) \forall G \in I_\alpha$. Let T be the tree of 2.5 constructed from $\{I_\alpha\colon \alpha \in \kappa\}$. If $\lambda < w\#$, then 2.5(5) implies $\Delta \neq \bigcap\{\mathcal{O}(\alpha)\colon \alpha \in \lambda\}$. Following 2.5(1) shows $\# \leq \kappa$.

For the further, let $\{H(\alpha)\colon \alpha \in \lambda\}$ be a family of open subsets of βX whose intersection is X, and suppose $\lambda \leq \kappa$. For $\alpha \in \kappa \sim \lambda$ set $H(\alpha) = \beta X$. For each $\alpha \in \kappa$, let

$$J_\alpha = \left\{H(\alpha) \cap \mathrm{int}_{\beta X}\left(\mathrm{cl}_{\beta X}(G)\right)\colon G \in I_\alpha\right\}.$$

Let $S \subseteq \mathfrak{R}(\beta X)$ be the tree of 2.5 constructed for βX from $\{J_\alpha\colon \alpha \in \kappa\}$. Using 2.5(2) we see $\bigcap b$ is a nonempty compact subset of βX whenever b is a branch of $S(\eta)$ for $0 < \eta \leq \kappa$. So b is always a local base for $\bigcap b$ when η is a limit ordinal. Since $\bigcap\{H(\alpha)\colon \alpha \in \lambda\} = X$, 2.5(5) implies $\bigcap b \subseteq X$ whenever b is a branch of $S(\eta)$ for $\lambda \leq \eta \leq \kappa$. Since $\Delta = \bigcap\{\mathcal{O}(\alpha)\colon \alpha \in \kappa\}$, 2.5(5) implies $|\bigcap b| = 1$ whenever b is a branch of S and $\mathrm{ord}(b) = \kappa$. As X is κ-Baire, 2.5(3) implies $\{x \in \bigcap b\colon b$ is a branch of S, $\mathrm{ord}(b) = \kappa\}$ is dense in X. Now follow the last paragraph of 2.3. □

2.8. COROLLARY. *A Moore space is $\mathcal{G}$-absolute with a linearly ordered space iff it has a dense metrizable subspace.*

PROOF. A Moore space is 1st countable so 2.3 applies. As it also has a G_δ-diagonal 2.7, and hence 2.2, applies. □

2.9. COROLLARY. *For a Čech-complete space X, the following hold:*

(1) *If $w\#(\mathfrak{R}(X)) > \omega$, then X has a dense open locally compact subspace.*

(2) *If $\#(\mathfrak{R}(X)) > \omega$ and X is perfectly normal, then X is not ω_1-Baire.*

(3) *If X has a G_δ-diagonal, then X has a dense metrizable linearly orderable G_δ-set.*

PROOF. (1) Consider $\mathrm{lv}(T, \omega)$ in the "further" of 2.7. (2) The Pressing Down Lemma implies $\mathrm{lv}(T, \omega_1) = \varnothing$ in the "further". (3) This follows immediately from the "further" and 2.2. □

Recall [**Ju1**] the cardinal κ is a *caliber* for a space X if each collection of κ many nonempty open subsets of X contains a centered subfamily of cardinality κ. [**Ju1**, A2.2] shows that if $X = {}^\kappa 2$ is given the Tychonov product topology, then ω_1 is a caliber for X whenever $\kappa \geq \omega_1$.

Suppose X is a space for which κ is a caliber; if Y is the image of X under a continuous surjection and T is an unbounded tree of $\mathfrak{R}(Y)$ satisfying 2.5(2) for Y, then $|T| < \kappa$ since the inverse image of T satisfies 2.5(2).

The next theorem is essentially the theme of **[Gv]**. The preceding paragraph yields for us a shorter proof.

2.10. THEOREM. *A dyadic space is $\mathcal{G}$-absolute with a linearly ordered space iff it is separable and metrizable.*

PROOF. ($\Rightarrow$) D is a dyadic space, if, by definition, D is a dense subset of a continuous image Y of ${}^{\kappa}2$, where, by **[Ju1**, 4.9] $\kappa = \pi w(Y) = w(Y)$. From 2.1 and the above $\kappa = \omega$ and so Y is a compact metric. □

2.11. COROLLARY **[Po]**. *If a dyadic space is coabsolute with a metric space, it is metrizable.*

2.12. REMARKS. (1) A space is non-Archimedean (see **[Ny]** for a survey) provided it has a base in which every pair of members are either related by $\subseteq$ or disjoint. By virtue of 1.2(4), these are spaces having inverted trees as bases. Observe that the space L of 2.1(2) $\Rightarrow$ (3) and $\psi(T)$ of 2.3 are non-Archimedean, while the dense subspaces in 2.6 and 2.7 are actually κ-metrizable. A technical modification of 2.1(2) $\Rightarrow$ (3) shows that "linearly ordered" in 2.1(3) can be replaced by "subspace of linearly ordered".

(2) Is there (in **ZFC**) a compact first countable space not the compactification of a linearly ordered space? We observe that the Pressing Down Lemma shows that if X is a Souslin line, then $X \times [0, 1]$ has no dense linearly ordered subspace.

(3) In **[Ta]** methods for recognizing Souslin trees in topologies are investigated, and equivalences and implications of **SH** are given. We observe (from 1.6 and 2.9) that SH is equivalent to the statement "if X is an ω_1-Baire Čech-complete perfectly normal space, then $\#(\mathfrak{R}(X)) \leq \omega$".

(4) Does every compact almost P-space of weight 2^{ω} contain a dense linearly ordered subspace? From **[C.N.1]** the answer is yes if **CH** is assumed. From 1.12 and 2.3, a compact almost P-space of π-weight 2^{ω} contains a dense non-Archimedean linearly ordered subspace whenever it contains a dense set of points with well-ordered local bases. The last condition is necessary as every linearly ordered subspace of $\mathcal{E}(\beta\mathbf{N} \sim \mathbf{N})$ contains no isolated points.

(5) 2.7 (for $\kappa = \omega$) and 2.8 were originally proved independently of White's result 2.4; however, 2.4 motivated 2.3. We observe that "Čech-complete" in 2.9 can be replaced by "p-space in the sense of Ar′hangelskii", and a similar generality works in 2.7.

3. Coabsolutes of Stone-Čech remainders. If Z is a zero set of a completely regular space X, then we let $Z^* = \{x \in \beta X \sim X\colon\ Z \in x\}$. So $X^* = \beta X \sim X$. It is well known (see **[En]**) that X^* can be nearly anything for a suitable pseudocompact space, and if X^* is dyadic, then X is pseudocompact. Thus, 2.10 especially encourages us to restrict our attention to a class of spaces in which every pseudocompact closed subspace is compact–in this case we consider the class of spaces with a compatible complete uniformity (which we will call *complete* spaces).[4] Further, if X is nowhere

[4]Shirota's theorem says that when we assume no measurable cardinals exist, a space is complete iff it is realcompact. So the reader may wish to replace "complete" with "realcompact" throughout this section.

locally compact, then X^* and X are dense in βX. Therefore, we restrict our attention to complete locally compact noncompact spaces. With minor changes in proof the following is [**Wo2**, 3.2]:

3.1. Lemma. *If X is complete, then X^* and $(\mathfrak{S}(X))^*$ are coabsolute.*

3.2. Lemma. *If X is a locally compact extremally disconnected noncompact space, then $\exists\kappa \geqslant 2^\omega$ and a family $\{D(\alpha)\colon \alpha \in \kappa\}$ satisfying*:

(1) *Each $D(\alpha)$ is the union of countably many pairwise-disjoint compact open subspaces of X.*

(2) *Either $D(\alpha) \cong N$ or $D(\alpha)$ has no isolated points.*

(3) *$\beta < \alpha \Rightarrow D(\beta)^* \cap D(\alpha)^* = \varnothing$.*

(4) *$X = \mathrm{cl}(\bigcup\{D(\alpha)^*\colon \alpha \in \kappa\})$ and each $D(\alpha)^*$ is open in X^*.*

Proof. Let $E(0) = \{\{x\}\colon \{x\}$ is open in $X\}$ and choose, by local compactness, a maximal collection $E(1)$ of pairwise disjoint nonempty open compact members of $\mathfrak{R}(\mathrm{int}(X \sim \bigcup E(0)))$. For each $n \in \{0,1\}$ let $\{e(\alpha, n)\colon \alpha \in \kappa(n)\}$ be a listing of $E(n)$ with a cardinal $\kappa(n)$.

If $\kappa(n)$ is finite, let $I(n) = \varnothing$. If $\kappa(n)$ is infinite, then choose $I(n)$ to be a (*maximal almost-disjoint*) family of countably infinite subsets of $\kappa(n)$ maximal w.r.t. "every intersection of distinct members is finite". We may choose $|I(n)| \geqslant 2^\omega$ since it is well known [**Ru**] that ω contains a maximal almost-disjoint family of cardinal 2^ω.

The desired family will be

$$\{D(i,n)\colon i \in I(n), n \in \{0,1\}\}, \quad \text{where } D(i,n) = \bigcup\{e(\alpha,n)\colon \alpha \in i\}.$$

Its cardinality is at least 2^ω since X is not compact. (1) and (2) are certainly satisfied. (3) follows since $I(n)$ is almost-disjoint or empty and $(\bigcup E(0)) \cap (\bigcup E(1)) = \varnothing$. If C is clopen in X and $\varnothing \neq C^*$, then $\exists n \in \{0,1\}$ and a countably infinite set $j \subseteq \kappa(n)$ such that $C \cap e(\alpha, n) \neq \varnothing$ for each $\alpha \in j$ (otherwise, $\bigcup(E(0) \cup E(1))$ would not be dense in X). As $I(n)$ is infinite and a maximal almost-disjoint family, $i \cap j$ is infinite for some $i \in I(n)$. (4) follows since

$$\varnothing \neq \left(\bigcup\{C \cap e(\alpha,n)\colon \alpha \in i \cap j\}\right)^* \subseteq C^* D(i,n)^*. \quad \square$$

3.3. Lemma [**F.G.**, 3.1]. *If X is realcompact and locally compact, then X^* is an almost P_{ω_1}-space.*

3.4. Theorem. *If X is a complete locally compact noncompact space each of whose nonempty open sets contains a nonempty open set of π-weight at most 2^ω, then X^* is coabsolute with a linearly ordered space having a dense set of P-points.*

Proof. From 3.1 we may assume X is extremally disconnected, locally compact, noncompact with the same π-weight conditions. Thus, in 3.2 we may assume $\pi w(e(\alpha, n)) \leqslant 2^\omega$, and hence, from extremal disconnectedness $2^\omega \leqslant \pi w(D(\alpha, i)^*) \leqslant (2^\omega)^\omega = 2^\omega$. Since each $D(\alpha, i)$ is σ-compact, each $D(\alpha, i)^*$ is an almost P_{ω_1}-space. From 1.3(2) and 1.12, each $\mathfrak{R}(D(\alpha, i)^*)$, and hence, $\mathfrak{R}(X^*)$ has a cofinal tree T whose branches fail to have countable cofinality. The desired space is the Dedekind completion (with endpoints) of the space L in 2.1(2) $\Rightarrow$ (3). $\square$

As each locally compact metric space is the free union of σ-compact spaces which must have π-weight at most ω, and as every locally compact metric space admits a complete uniformity, we have shown

3.5. COROLLARY. *A locally compact noncompact metric space X has X^* coabsolute with a linearly ordered space having a dense set of P-points.*

3.6. LEMMA. *If X is a locally compact, extremally disconnected, noncompact space with $\pi w(X) = \omega$, then X^* is homeomorphic to one of* $\mathbf{N}^*$, $(\mathfrak{S}(\mathbf{R}))^*$, *or* $\mathbf{N}^* + \mathfrak{S}(\mathbf{R})^*$.

PROOF. Following the proof of 3.2, $\pi w(X) = \omega$ yields $X = E(0) \cup E(1)$, where $E(0)$ is the closure of all of the at most ω isolated points of X and $E(1) = X \sim E(0)$. As X is extremally disconnected $X^* = E(0)^* \cup E(1)^*$.

When $E(0)$ is not compact, each clopen set free ultrafilter on $E(0)$ traces to an ultrafilter on $\operatorname{int}(E(0)) \cong \mathbf{N}$; therefore, $E(0)^* \cong \mathbf{N}^*$. When $E(1)$ is not compact, it is the disjoint union of ω many (since $\pi w(X) = \omega$) compact spaces of π-weight ω without isolated points. From [**Si**, 9c] each of the spaces is homeomorphic to $\mathfrak{S}$(Cantor set). Applying the same argument to $\mathbf{R}$, we have $E(1) \cong \mathfrak{S}(\mathbf{R})$. $\square$

3.7. LEMMA. *If K is a compact space and if, for each $n \in \{0, 1\}$, $X(n)$ is the free union of a cardinal $\kappa(n)$ many copies of K, where $\omega \leqslant \kappa(n) \leqslant 2^\omega$, then $X(0)^*$ and $X(1)^*$ contain homeomorphic dense open subspaces.*

PROOF. Write $X(n) = \Sigma\{K(\alpha)\colon \alpha \in \kappa(n)\}$, where $K(\alpha) \cong K$ for each α. Since $\kappa(n) \leqslant 2^\omega$, $\kappa(n)$ contains precisely 2^ω countably infinite subsets. Hence, following the argument in the proof of 3.2, we may choose a maximal almost-disjoint family $I(n)$ of countably infinite subsets of $\kappa(n)$ such that $|I(n)| = 2^\omega$. Then

$$\left\{\left\{x \in X(n)^*\colon \left(\left(\sum\{K(\alpha)\colon \alpha \in i\}\right) \sim K(\gamma)\right) \in x\, \forall \gamma \in i\right\}\colon i \in I(n)\right\}$$

is a pairwise-disjoint family of clopen subsets of $X(n)^*$ whose union is dense in $X(n)^*$ and whose members are homeomorphic to $(\Sigma\{K(\alpha)\colon \alpha \in \omega\})^*$. $\square$

3.8. THEOREM. *If X is a complete locally compact noncompact space of π-weight at most 2^ω, and if every nonempty open set of X contains a nonempty open set of countable π-weight, then X^* is coabsolute with one of* $\mathbf{N}^*$, $\mathbf{R}^*$, *or* $\mathbf{N}^* + \mathbf{R}^*$.

PROOF. Following the proof of 3.4, we observe that each $D(\alpha, i)$ may also be assumed here to have countable π-weight, and from their construction in 3.2 there are precisely 2^ω of the sets $D^*(\alpha, i)$ each of which is homeomorphic to one of the three spaces above by 3.6. Now apply 3.7 with $K = \mathbf{N}^*$ and $K = \mathbf{R}^*$. $\square$

3.9. COROLLARY. *Suppose X is a locally compact noncompact metric space of density at most 2^ω; then X^* is coabsolute with*

(1) $\mathbf{N}^*$, *if X has a dense discrete subspace,*

(2) $\mathbf{R}^*$, *if the set of isolated points of X has compact closure,*

(3) $\mathbf{N}^* + \mathbf{R}^*$, *otherwise.*

In [**Wo1, Wo2**] it is shown that if **CH** is assumed X^* is coabsolute with $\mathbf{N}^*$ whenever X is locally compact, noncompact, and either metric of density at most 2^ω

or with $|\mathfrak{R}(X)| = 2^\omega$; however, this follows from 1.14 which shows **CH** *implies* $\mathfrak{R}(\mathbf{N}^*) \equiv \mathfrak{R}(Y)$ *whenever* Y *is an almost P-space with* $\pi w(Y) = 2^\omega$ *and no isolated points*. We end this section with an example which shows 2^ω is essential in 3.4 and which allows us to remove **CH** from the hypothesis of Woods' results.

3.10. EXAMPLE. Suppose $\kappa > \omega$ is a cardinal and $\mathfrak{D}(\kappa) = \Sigma\{\mathfrak{D}(\kappa, n)\colon n \in \omega\}$, where each $\mathfrak{D}(\kappa, n) \cong {}^\kappa 2$ given the Tychonov product topology. If K denotes the linearly ordered space obtained from ordering ${}^{\omega_1}2$ lexicographically, then the following are equivalent:

(1) $\kappa \leqslant 2^\omega$.

(2) $\mathfrak{D}(\kappa)^*$ and K are coabsolute.

(3) $\mathfrak{D}(\kappa)^*$ is coabsolute with a linearly ordered space.

PROOF. For each ordinal $\alpha < \omega_1$, we set

$$A(\alpha) = \{Z(f)\colon \mathrm{dom}(f) = \alpha\}, \quad \text{where } Z(f) = \{g \in {}^\kappa 2\colon g \,|\, \alpha = f\},$$

and $L(\alpha) = \{\mathrm{int}((\bigcup\{Z(f, n)\colon n \in \omega\})^*)\colon \forall n \in \omega\, Z(f, n) \in A(\alpha) \text{ and } Z(f, n) \subseteq D(\kappa, n)\}$. Then $T = \{L(\alpha)\colon \alpha \in \omega_1\}$ is a tree in $\mathfrak{R}(\mathfrak{D}(\kappa)^*)$ whose αth level is $L(\alpha)$. We claim that T is an unbounded tree.

First we observe that $\mathfrak{D}(\kappa)^*$ has a π-base $P(\kappa)$ of sets of the form $\mathrm{int}(Z^*)$ such that for each $n \in \omega\, \exists h_n \in {}^\kappa 2$ and a countable set $C_n \subseteq \kappa$ such that $g \in Z \cap \mathfrak{D}(\kappa, n)$ iff $g \restriction C_n = h_n$. So if $\alpha < \omega_1$ is the first ordinal with $\omega_1 \cap (\bigcup\{C_n\colon n \in \omega\}) \subseteq \alpha$, then $\mathrm{int}(Z^*)$ intersects two elements of $L(\alpha + 1)$. From 1.2(1), T is unbounded.

(1) $\Rightarrow$ (2). From 1.9 and 1.12 $\mathfrak{R}(\mathfrak{D}(\kappa)^*)$ has a cofinal tree of height ω_1 since $\pi w(\mathfrak{D}(\kappa)^*) = 2^\omega$. From 3.3 it has a cofinal tree order isomorphic to the union of the limit ordinal levels of TREE(ω_1). If L is constructed as in 2.1(2) $\Rightarrow$ (3), then L is homeomorphic to a dense subspace of K.

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (1). As $\mathfrak{D}(\kappa)$ has caliber ω_1, it follows that every pairwise-disjoint family of $P(\kappa)$ has cardinality at most 2^ω. On the other hand, $\pi w(\mathfrak{D}(\kappa)^*)$ is at least $2^\omega + \kappa$. So if $\mathfrak{D}(\kappa)^*$ is coabsolute with a linearly ordered space, then $\mathfrak{R}(\mathfrak{D}(\kappa)^*)$ has a cofinal tree S of height ω_1. Since $|S| = 2^\omega \cdot \omega_1 = 2^\omega$ and S is a π-base of $\mathfrak{D}(\kappa)^*$, $\kappa \leqslant 2^\omega$. □

3.11. LEMMA. *If X is a locally compact complete noncompact space, then* $\#(\mathfrak{R}(X^*)) \leqslant \#(\mathfrak{R}(\mathbf{N}^*))$.

PROOF. From 3.1 and 3.2 $X = \Sigma\{X(n)\colon n \in \omega\}$, where each $X(n)$ is compact and extremally disconnected, can be assumed. One readily observes that the map $X(n) \to n$ induces a function from $\mathfrak{R}(\mathbf{N}^*)$ to $\mathfrak{R}(X^*)$ which takes unbounded trees to unbounded trees isomorphic to their pre-images. □

3.12. THEOREM. *The following are equivalent statements in* **ZFC**:

(1) $\mathbf{N}^*$ *and* $D(\omega_1)^*$ *are coabsolute.*

(2) $\#(\mathfrak{R}(\mathbf{N}^*)) = \omega_1$.

(3) *If X is a locally compact complete noncompact space and if $\pi w(X) \leqslant 2^\omega$, then X^* and $\mathbf{N}^*$ are coabsolute.*

PROOF. Since $\#(\mathfrak{R}(\mathfrak{D}(\omega_1)^*)) = \omega_1$, (1) $\Rightarrow$ (2). From 3.11 $\#(\mathfrak{R}(X^*)) = \omega_1$ so (2) $\Rightarrow$ (3) is similar to 3.10 (3) $\Rightarrow$ (1). (3) $\Rightarrow$ (1) is obvious. $\square$

As we have observed before **CH** implies $\#(\mathfrak{R}(\mathbf{N}^*)) = \omega_1$. However, as remarked in [**B.P.S.**] there are numerous models of **ZFC** + $\neg$**CH** in which $\#(\mathfrak{R}(\mathbf{N}^*)) = \omega_1$, for example in any model $\mathfrak{M}$ for which $\mathfrak{M} \models {}^\omega\omega$ to have an ω_1-scale [**He**] and $\mathfrak{M} \models \omega_1 < 2^\omega$. On the other hand, **MA** + $\neg$**CH** implies $\mathfrak{R}(\mathbf{N}^*)$ is cofinally κ-closed $\forall\kappa < 2^\omega$ [**Ru**], and hence, $\#(\mathfrak{R}(\mathbf{N}^*)) = 2^\omega > \omega_1$. Therefore, we have

3.13. COROLLARY. *The following statement is implied by* **CH**, *and is consistent with and independent of* $\neg$**CH**: *If X is a locally compact complete noncompact space with $\pi w(X) \leqslant 2^\omega$, then X^* and* $\mathbf{N}^*$ *are coabsolute.*

3.14. REMARKS. (1) 3.7 was communicated verbally to the author by S. Broverman for the case $|K| = 1$. The proof he gave is similar. R. G. Woods has informed us that 3.9(2) can also be proved using a recent result of E. K. van Douwen on remote points and a theorem in [**Wo3**].

(2) *Are* $\mathbf{N}^*$ *and* $\mathbf{R}^*$ *coabsolute*? As $\mathbf{N}^*$ is an almost P_κ-space iff $\mathbf{R}^*$ is an almost P_κ-space [**vD1**], we observe that a proof similar to that of 3.12 (2) $\Rightarrow$ (3) shows $\mathbf{N}^*$ and $\mathbf{R}^*$ are coabsolute whenever $\mathbf{N}^*$ is an almost P_κ-space $\forall\kappa < \#(\mathfrak{R}(N^*))$. The latter is true in a variety of situations (including **MA**). However, we do not know whether a negative answer is consistent.

(3) 3.10 was motivated by an example in [**vD.vM.**] called here $\mathfrak{D}(2^\omega)$.

(4) In the first draft of this paper we used a result in [**vD2**] to show 3.11. Originally we obtained 3.13 prior to 1.12 (and independently of [**B.P.S.**]); however, our proof was longer while a key lemma to our short proof in the first draft of this paper was false.

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ADDED IN PROOF. The answer to the questions in 2.12(2) is "yes" (Todorčévic); in 2.12(4) is "not $\mathfrak{D}(\omega_2)$" (myself) and it is consistent that X^* has no dense linearly ordered subspace for every non-pseudo-compact space X.

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DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK, BUFFALO, NEW YORK 14214