

PERMUTATION-PARTITION PAIRS II: BOUNDS ON THE GENUS OF THE AMALGAMATION OF GRAPHS¹

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ABSTRACT. Bounds are derived on the extent to which the parameter $\mu(P, \Pi)$ can fail to be additive over disjoint permutations. This is done by associating an Eulerian digraph to each such pair and relating the maximum orbiticity $\mu(P, \Pi)$ to the decompositions of this digraph's arc set into arc disjoint cycles. These bounds are then applied to obtain information about the genus of the amalgamation of graphs.

This note extends the theory of permutation-partition pairs to obtain bounds on the genus of the amalgamation of graphs. While familiarity with [S] would be helpful to the reader, this paper is essentially self-contained, except for Theorem 7.

We note here that Lemma 3 of [S] makes a false assertion about the winding number $\omega(P, \Pi)$. A counterexample to the lemma is obtained by choosing $P_1 = (123)$, $P_2 = (654)$, and $\Pi = \{\{1, 4\}, \{2, 5\}, \{3, 6\}\}$. Here $\omega(P_1, \Pi) = \omega(P_2, \Pi) = 1$ whereas $\omega(P_1 P_2, \Pi) = 0$. This also invalidates the lower bound of Theorem 22 of [S] on the genus of the amalgamation of graphs over three points.

This article has the following layout. First a digraph $T(P, \Pi)$ is associated to every pair (P, Π) . Then the maximum orbiticity $\mu(P, \Pi)$ is related to certain "admissible" cycles of $T(P, \Pi)$. A new parameter $\mu^*(P, \Pi)$ is defined and is shown to approximate $\mu(P, \Pi)$. The nonadditivity of $\mu^*(P, \Pi)$ is quantified and this information is used to bound the nonadditivity of $\mu(P, \Pi)$. These bounds are then used to obtain bounds on the genus of the amalgamation of graphs.

Throughout this paper (P, Π) will denote a fixed $PP(n, k)$ pair. In other words, we are given a set S of n bits, P is a permutation of this set, and Π is a partition of S into k nonempty subsets $\{\Pi_i\}_{i=1}^k$.

In order to make this note as self-contained as possible, we repeat here the definition of the *maximum orbiticity*. If P is any permutation then $\|P\|$ denotes the number of orbits of P . If Π is any partition of a set S , then $S(\Pi)$ denotes the set of permutations whose orbits equal the members of Π . Finally, $\mu(P, \Pi) = \max\{\|PQ\| \mid Q \in S(\Pi)\}$.

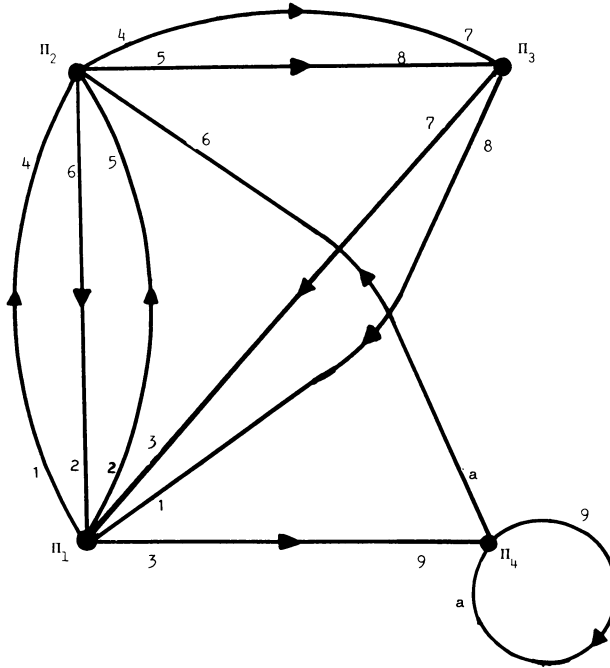
In studying these (permutation-partition) pairs it was found helpful to visualize the relationship between the components P and Π of the pair (P, Π) by means of a *transition digraph* $T(P, \Pi)$. The vertices of this digraph are the members of Π and each bit b contributes an arc labelled $[b, bP]$. Figure 1 contains an example.

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$$P = (1\ 4\ 7\ 3\ 9\ a\ 6\ 2\ 5\ 8),$$

$$\Pi = \{\Pi_i\}_{i=1}^4, \quad \Pi_1 = \{1, 2, 3\}, \quad \Pi_2 = \{4, 5, 6\},$$

$$\Pi_3 = \{7, 8\}, \quad \Pi_4 = \{9, a\}.$$

FIGURE 1

Note that in general $T(P, \Pi)$ is Eulerian in the sense that the indegree of each vertex equals its outdegree. If $c = bP$ then we refer to b and c as the initial and terminal labels, respectively, of the arc $[b, c]$. A sequence of arcs $[b_1, c_1], [b_2, c_2], \dots, [b_s, c_s]$ is said to form a *circuit* of $T(P, \Pi)$ if for all $i = 1, 2, \dots, s$, $c_i \equiv b_{i+1} \pmod{\Pi}$ (addition mod s). The circuit is a *cycle* if whenever $i \neq j$ we also have $b_i \not\equiv b_j \pmod{\Pi}$. A circuit is *admissible* if for each $i = 1, 2, \dots, s$, $c_i \neq b_{i+1}$ (addition modulo s). Thus in Figure 1 the circuit $[3, 9], [a, 6], [4, 7], [8, 1]$ is admissible, whereas the circuit $[3, 9], [a, 6], [5, 8], [7, 3]$ fails to be admissible at Π_1 .

The following two theorems show that each admissible cycle makes a positive contribution to $\mu(P, \Pi)$. Unfortunately, in making this positive contribution a given admissible cycle may destroy the admissibility of other cycles; for this reason it does not suffice to merely count the admissible cycles of $T(P, \Pi)$.

THEOREM 1. Let (P, Π) be a pair and suppose $b \equiv c \pmod{\Pi}$. Set

$$\bar{P} = \begin{cases} P(b\ c)(b\ bP)/(b) & \text{if } b \neq bP \text{ and } c \neq b, bP, \\ P(b\ c)/(b) & \text{if } c = bP \text{ and } c \neq b, \\ P/(b) & \text{if } b = bP, \end{cases}$$

$$\bar{\Pi} = \Pi|_{S - \{b\}}.$$

Then

- (1) $\mu(P, \Pi) \geq \mu(\bar{P}, \bar{\Pi})$,
- (2) $\mu(P, \Pi) = 1 + \mu(\bar{P}, \bar{\Pi})$, if $c = bP$ and $c \neq b$,
- (3) $\mu(P, \Pi) = 1 + \mu(\bar{P}, \bar{\Pi})$ if $b = bP$ and $\{b\} \in \Pi$,
- (4) $\mu(P, \Pi) = \mu(\bar{P}, \bar{\Pi})$ if $b = bP$ and $\{b\} \notin \Pi$.

PROOF. First suppose $b = bP$ and $\{b\} \in \Pi$. In other words, (b) is a singleton orbit of P and of every $Q \in S(\Pi)$. Hence, if $Q \in S(\Pi)$ is such that $\|PQ\| = \mu(P, \Pi)$, then if $\bar{Q} = Q/(b)$, we have $\bar{Q} \in S(\bar{\Pi})$, $\|PQ\| = \|\bar{P}(b)\bar{Q}\| = 1 + \|\bar{P}\bar{Q}\|$, and so $\mu(P, \Pi) \leq 1 + \mu(\bar{P}, \bar{\Pi})$. Conversely, if $\bar{Q} \in S(\bar{\Pi})$ is such that $\|\bar{P}\bar{Q}\| = \mu(\bar{P}, \bar{\Pi})$, set $Q = (b)\bar{Q}$. Clearly $Q \in S(\Pi)$ and $\|PQ\| = 1 + \|\bar{P}\bar{Q}\|$. Consequently $\mu(P, \Pi) \geq 1 + \mu(\bar{P}, \bar{\Pi})$. This proves (3).

Next suppose $b = bP$, but $\{b\} \notin \Pi$. Let $\bar{Q} \in S(\bar{\Pi})$ be such that $\|\bar{P}\bar{Q}\| = \mu(\bar{P}, \bar{\Pi})$. Set $Q = (ab)\bar{Q}$ where a is an arbitrary bit such that $a \equiv b \pmod{\Pi}$. Then

$$(5) \quad \|PQ\| = \|\bar{P}(b)(ab)\bar{Q}\| = \|\bar{P}(ab)\bar{Q}\| = \|\bar{P}\bar{Q}\|.$$

Hence $\mu(P, \Pi) \geq \mu(\bar{P}, \bar{\Pi})$. Conversely, if $Q \in S(\Pi)$ is such that $\|PQ\| = \mu(P, \Pi)$, set $\bar{Q} = (ab)Q/(b)$ where $a = bQ^{-1} \neq b$. Again (5) holds, this time implying that $\mu(P, \Pi) \leq \mu(\bar{P}, \bar{\Pi})$. This concludes the proof of (4). We may now assume that $b \neq bP$.

Let $\bar{Q} \in S(\bar{\Pi})$ be such that $\|\bar{P}\bar{Q}\| = \mu(\bar{P}, \bar{\Pi})$. Set $Q = (bc)\bar{Q}$. Then, if $c \neq bP$, b ,

$$\|PQ\| = \|\bar{P}(bbP)(bc)(bc)\bar{Q}\| = \|\bar{P}(bbP)\bar{Q}\|,$$

and since b is not in the domain of either \bar{P} or \bar{Q} it follows that $\|PQ\| = \|\bar{P}\bar{Q}\|$. On the other hand, if $c = bP$ and $c \neq b$, then

$$(6) \quad \|PQ\| = \|\bar{P}(bc)(bc)\bar{Q}\| = \|\bar{P}(b)\bar{Q}\| = 1 + \|\bar{P}\bar{Q}\|.$$

Hence, in either case,

$$\mu(P, \Pi) \geq \|PQ\| \geq \|\bar{P}\bar{Q}\| = \mu(\bar{P}, \bar{\Pi}).$$

Finally, to prove (2) note that it follows from (6) that if $c = bP \neq b$ then $\mu(P, \Pi) \geq 1 + \mu(\bar{P}, \bar{\Pi})$. Hence it suffices to prove the reverse inequality. Suppose $Q \in S(\Pi)$ is such that $\mu(P, \Pi) = \|PQ\|$. If $cQ = b$, set $\bar{Q} = (bc)Q/(b)$. Then $\bar{Q} \in S(\bar{\Pi})$ and

$$\|PQ\| = \|\bar{P}(bc)(bc)\bar{Q}\| = \|\bar{P}(b)\bar{Q}\| = 1 + \|\bar{P}\bar{Q}\|,$$

and so $1 + \mu(\bar{P}, \bar{\Pi}) \geq \mu(P, \Pi)$.

On the other hand, if $cQ = a \neq b$, set $Q' = Q(bc)(ab)$. Then, $Q' \in S(\Pi)$ and

$$\|PQ'\| = \|PQ(bc)(ab)\|.$$

But $bPQ(bc) = a$ because $cQ \neq b, c$; and hence

$$\|PQ'\| = \|PQ(bc)(ab)\| = \|PQ(bc)\| + 1 \geq \|PQ\|.$$

In other words, $\|PQ'\| = \mu(P, \Pi)$ and $cQ' = cQ(bc)(ab) = b$, so that the previous argument applies if Q is replaced by Q' . Hence $\mu(P, \Pi) = 1 + \mu(\bar{P}, \bar{\Pi})$. Q.E.D.

We note here that this theorem is both a refinement of Lemma 12 of [S] and a variation of a technique employed by Walkup in [W]. In order to clarify the content of the above theorem we offer the following example.

EXAMPLES 2. Let $\Pi = \{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8\}, \{9\}, \{a\}\}$, $c = 1$, $b = 2$. If $P = (1\ 4\ 7\ 3\ 9\ a\ 6\ 2\ 5\ 8)$, then $\bar{P} = (1\ 4\ 7\ 3\ 9\ a\ 6)(5\ 8)$; if $P = (1\ 4\ 7\ 3\ 9\ a\ 6)(2\ 5\ 8)$, then $\bar{P} = (1\ 4\ 7\ 3\ 9\ a\ 6\ 5\ 8)$, and in both cases $\bar{\Pi} = \{\{1, 3\}, \{4, 5, 6\}, \{7, 8\}, \{9\}, \{a\}\}$. If $P = (2\ 1\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ a)$, then $\bar{P} = (1\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ a)$ with $\bar{\Pi}$ as above. Finally, if $b = 9$ and $P = (9)(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ a)$, then $\bar{P} = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ a)$ and $\bar{\Pi} = \{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8\}, \{a\}\}$.

We say that the pair $(\bar{P}, \bar{\Pi})$ of Theorem 1 is obtained from (P, Π) by applying the *reduction* b/c (or b/b in the case where $b = bP$). The reader may find it helpful to visualize these reductions in terms of their effect on the transition digraph. In the case where b , c , and bP are all distinct, $T(\bar{P}, \bar{\Pi})$ is obtained from $T(P, \Pi)$ by deleting the arcs $[cP^{-1}, c]$, $[bP^{-1}, b]$, $[b, bP]$ and adding the arcs $[cP^{-1}, bP]$ and $[bP^{-1}, c]$. If $c = bP$ and $c \neq b$, the arcs $[bP^{-1}, b]$ and $[b, c]$ are deleted and the arc $[bP^{-1}, c]$ is gained. If labels are ignored, then in both cases two successive arcs of $T(P, \Pi)$ are replaced by an arc from the initial vertex of the first to the terminal vertex of the second. Finally, if $b = bP$ the reduction b/b has the effect of suppressing the loop $[b, bP]$ in the transition digraph.

THEOREM 3. If $T(P, \Pi)$ contains the admissible cycle $[b_1, c_1], [b_2, c_2], \dots, [b_s, c_s]$, and (P', Π') is the pair obtained from (P, Π) by successively applying the reductions $b_s/c_{s-1}, \dots, b_2/c_1, b_1/c_s$ to (P, Π) , then

$$\mu(P, \Pi) \geq 1 + \mu(P', \Pi').$$

PROOF. By induction on s . If $s = 1$, then the cycle is merely a loop $[b_1, c_1]$ with $c_1 = b_1P$, $c_1 \equiv b_1 \pmod{\Pi}$, and, because of admissibility, $c_1 \neq b_1$. Hence, by line (2) of Theorem 1, with $b_1 = b$ and $c_1 = c$, and $(P', \Pi') = (\bar{P}, \bar{\Pi})$, we have $\mu(P, \Pi) = 1 + \mu(P', \Pi')$.

Now assume that the theorem holds for all admissible cycles of length $s - 1$. Apply line (1) of Theorem 1 with $b_s = b$ and $c_{s-1} = c$. The derived transition digraph $T(\bar{P}, \bar{\Pi})$ has the still admissible cycle $[b_1, c_1], [b_2, c_2], \dots, [b_{s-1}, c_s]$ (because $b_{s-1} = c_{s-1}P^{-1}$ and $c_s = b_sP$). Applying the induction hypothesis to the above cycle, we get $\mu(P, \Pi) \geq 1 + \mu(P', \Pi')$. Q.E.D.

Lemma 12 of [S] is simply a restriction of the above theorem to the case $s = 2$.

We now define new parameters $\kappa(P, \Pi)$, $c(P, \Pi)$, and $\mu^*(P, \Pi)$. It will be shown that these parameters can be used to bound $\mu(P, \Pi)$. The advantage they offer is that they are theoretically more tractable than $\mu(P, \Pi)$.

The parameter $\kappa(P, \Pi)$ denotes the number of orbits of P that are completely contained in some member of Π . The parameter $c(P, \Pi)$ denotes the number of components of the unlabelled transition digraph $T(P, \Pi)$.

For the given partition Π let $S^*(\Pi)$ be the set of all those permutations σ of the bit set S such that each orbit of σ is completely contained in some member of Π . It is clear that $S^*(\Pi)$ is the direct sum of the symmetric groups on the members of Π . Define $\mu^*(P, \Pi) = \max\{\|PQ\| \mid Q \in S^*(\Pi)\}$.

PROPOSITION 4. $\mu^*(P, \Pi)$ is the largest integer m such that the arc set D of $T(P, \Pi)$ decomposes into m arc disjoint cycles.

PROOF. We show that each $Q \in S^*(\Pi)$ determines a decomposition of D into $\|PQ\|$ arc disjoint circuits and vice versa. Let $Q \in S^*(\Pi)$ be given. For each arc $[b, bP]$ of $T(P, \Pi)$ let the succeeding arc be $[bPQ, bPQP]$. In other words, given an arc of $T(P, \Pi)$ into some vertex Π_i , we apply Q to its terminal label in order to determine the initial label of the succeeding arc. For example, in Figure 1 the permutation $Q: (1\ 2\ 3)(4\ 5)(6)(7\ 8)(9)(a)$ determines the circuits $C_1 = [1, 4], [5, 8], [7, 3], C_2 = [2, 5], [4, 7], [8, 1]$, and $C_3 = [3, 9], [9, a], [a, 6], [6, 2]$. Since $Q|_{\Pi_i}$ is an arbitrary permutation of Π_i it follows that each such decomposition of the arc set D into circuits actually corresponds to some $Q \in S^*(\Pi)$. Finally since every circuit decomposes into cycles it follows that a maximum decomposition of D must consist of cycles. Q.E.D.

THEOREM 5. For any $PP(n, k)$ pair (P, Π) ,

$$\begin{aligned} \mu^*(P, \Pi) - \kappa(P, \Pi) + c(P, \Pi) &\geq \mu(P, \Pi) \\ &\geq \mu^*(P, \Pi) - \kappa(P, \Pi) - 2 \sum_{i=2}^k (i-1)! \binom{k}{i}. \end{aligned}$$

PROOF. By induction on n . If $n = 1$ then all the parameters take the value 1 and we are done. Assume now that these inequalities hold for all pairs on less than n bits. We first deal with the case where $T(P, \Pi)$ has a loop $[b, bP]$.

If $c = bP \neq b$, apply the reduction b/c to obtain a pair (P', Π') such that $\mu^*(P, \Pi) = \mu^*(P', \Pi') + 1$, $\kappa(P, \Pi) = \kappa(P', \Pi')$, $c(P, \Pi) = c(P', \Pi')$, and $\mu(P, \Pi) = 1 + \mu(P', \Pi')$.

If $bP = b$ and $\{b\} \notin \Pi$ then the same reduction b/b now yields a pair (P', Π') such that $\mu^*(P, \Pi) = 1 + \mu^*(P', \Pi')$, $\kappa(P, \Pi) = 1 + \kappa(P', \Pi')$, $\mu(P, \Pi) = \mu(P', \Pi')$, and $c(P, \Pi) = c(P', \Pi')$.

If $bP = b$ and $\{b\} \in \Pi$ then the same reduction b/b now yields a pair (P', Π') such that $\mu^*(P, \Pi) = 1 + \mu^*(P', \Pi')$, $\kappa(P, \Pi) = 1 + \kappa(P', \Pi')$, $\mu(P, \Pi) = 1 + \mu(P', \Pi')$, and $c(P, \Pi) = 1 + c(P', \Pi')$.

In all of the above three cases the theorem follows by an application of the induction hypothesis to the pair (P', Π') .

Thus we may now assume that $T(P, \Pi)$ has no loops. It therefore follows that $\kappa(P, \Pi) = 0$. The left inequality is now clear and it remains to show that $\mu(P, \Pi) \geq \mu^*(P, \Pi) - 2 \sum_{i=2}^k (i-1)! \binom{k}{i}$.

Clearly we may assume that $\mu^*(P, \Pi) > 2 \sum_{i=2}^k (i-1)! \binom{k}{i}$, for otherwise the inequality holds by virtue of the fact that $\mu(P, \Pi)$ is positive. Let $\mathcal{C} = \{C_m \mid m = 1, 2, \dots, \mu^*(P, \Pi)\}$ be a decomposition of the arc set of $T(P, \Pi)$ into arc disjoint cycles. Since the complete symmetric digraph (with no multiple arcs) on k vertices possesses exactly $\sum_{i=2}^k (i-1)! \binom{k}{i}$ distinct nonloop cycles, it follows that \mathcal{C} contains a set of three cycles $\{C_1, C_2, C_3\}$ that traverse the same set of vertices of $T(P, \Pi)$ in the same order. Suppose their common length is s . We now construct an admissible

cycle of $T(P, \Pi)$ whose arcs all come from $C = C_1 \cup C_2 \cup C_3$. Let $[b_1, b_1P]$ be an arbitrary arc of C . Choose $[b_2, b_2P]$ from C so that $b_2 \neq b_1P$, but still $b_2 \equiv b_1P \pmod{\Pi}$. This is possible because there are three arcs in C whose initial vertex contains b_1P , but only one of them can have b_1P as its initial label (in fact the existence of two such arcs would have also been sufficient). This process is repeated until we have chosen $[b_1, b_1P], [b_2, b_2P], \dots, [b_{s-1}, b_{s-1}P]$ so that $b_iP \equiv b_{i+1} \pmod{\Pi}$ but $b_iP \neq b_{i+1}$ for $i = 1, 2, \dots, s-1$. Again, because there are three arcs in C from the vertex containing $b_{s-1}P$ to that containing b_1 , there is an arc $[b_s, b_sP]$ in C such that $b_s \equiv b_{s-1}P \pmod{\Pi}$, $b_1 \equiv b_sP \pmod{\Pi}$ but $b_s \neq b_{s-1}P$ and $b_1 \neq b_sP$. Thus we have constructed an admissible cycle $C_1^* \subseteq C$. Let C_2^* and C_3^* be two other cycles of length s whose union is $C - C_1^*$. Set $\mathcal{C}^* = [\mathcal{C} - \{C_1, C_2, C_3\}] \cup \{C_1^*, C_2^*, C_3^*\}$. Now apply Theorem 3 to C_1^* to obtain the pair (P', Π') . Note that $\mathcal{C}^* - \{C_1^*\}$ is a decomposition of the arc set of $T(P', \Pi')$ into arc disjoint cycles and hence $\mu^*(P', \Pi') \geq \mu^*(P, \Pi) - 1$. It follows from the remark preceding Theorem 3 that every cycle of the unlabelled $T(P', \Pi')$ is also a cycle of the unlabelled $T(P, \Pi)$. Consequently, since $T(P, \Pi)$ has no loops, $T(P', \Pi')$ has no loops either. Thus $\kappa(P', \Pi') = 0$. However, the pair (P', Π') has less than n bits and so the induction hypothesis may be applied to it. Hence,

$$\begin{aligned} \mu(P, \Pi) &\geq 1 + \mu(P', \Pi') \geq 1 + \mu^*(P', \Pi') - 2 \sum_{i=2}^k (i-1)! \binom{k}{i} \\ &\geq \mu^*(P, \Pi) - 2 \sum_{i=2}^k (i-1)! \binom{k}{i}. \quad \text{Q.E.D.} \end{aligned}$$

We next go on to study the extent to which $\mu^*(P, \Pi)$ can deviate from being additive over disjoint permutations.

LEMMA 6. *If P is the product of the disjoint permutations P_1 and P_2 , and $\Pi^{(i)}$ ($i = 1, 2$) is the partition obtained from Π by the suppression of all the bits not in P_i , then, if $k \geq 2$,*

$$\mu^*(P, \Pi) \leq \mu^*(P_1, \Pi^{(1)}) + \mu^*(P_2, \Pi^{(2)}) + (k-2)n/2k.$$

PROOF. Let $\lambda(P, \Pi)$ denote the number of loops in $T(P, \Pi)$. In other words, $\lambda(P, \Pi)$ denotes the number of bits b such that $b \equiv bP \pmod{\Pi}$. Since the permutations P_1 and P_2 are a disjoint factorization of P it follows that

$$\lambda(P, \Pi) = \lambda(P_1, \Pi^{(1)}) + \lambda(P_2, \Pi^{(2)}).$$

Every cycle of $T(P, \Pi)$ which is not a loop has length at least 2 and at most k . Hence, since $k \geq 2$,

$$n/k \leq \mu^*(P, \Pi) - \lambda(P, \Pi) \leq n/2.$$

If n_i is the number of bits in P_i , then $n = n_1 + n_2$ and

$$n_i/k \leq \mu^*(P_i, \Pi^{(i)}) - \lambda(P_i, \Pi^{(i)}) \leq n_i/2, \quad i = 1, 2.$$

$$\begin{aligned}
\therefore \mu^*(P, \Pi) &= \mu^*(P, \Pi) - \lambda(P, \Pi) + \lambda(P, \Pi) \leq n/2 + \lambda(P, \Pi) \\
&= (n_1 + n_2)/2 + \lambda(P_1, \Pi^{(1)}) + \lambda(P_2, \Pi^{(2)}) \\
&= n_1/k + n_2/k + (k-2)n/2k + \lambda(P_1, \Pi^{(1)}) + \lambda(P_2, \Pi^{(2)}) \\
&\leq \mu^*(P_1, \Pi^{(1)}) + \mu^*(P_2, \Pi^{(2)}) + (k-2)n/2k. \quad \text{Q.E.D.}
\end{aligned}$$

The above lemma is now applied to obtain bounds on the genus of the amalgamation of graphs.

THEOREM 7. *Let G, G^1 , and G^2 be graphs such that $G = G^1 \vee_U G^2$ and suppose that $U = 1, 2, \dots, k$ and $n = \sum_{v \in U} \deg v$; then, for $k \geq 2$,*

$$\begin{aligned}
\gamma(G^1) + \gamma(G^2) + k - 1 &\geq \gamma(G) \\
&\geq \gamma(G^1) + \gamma(G^2) - 1 - 2 \sum_{i=2}^k (i-1)! \binom{k}{i} - \frac{(k-2)n}{4k}.
\end{aligned}$$

PROOF. The left inequality follows immediately by a tube-adding argument. To prove the other inequality note that it may be assumed that U is an independent set of vertices since any edge, both of whose vertices are in U , can be subdivided without affecting the genus of any graph that contains it.

Let R be a rotation system defining a genus embedding of G . Set $R_w^i = R_w$ whenever $w \in [V(G^i) - U]$, $i = 1, 2$, and $\text{Ext} = \text{Ext}(R, U)$, $\text{Ext}^i = \text{Ext}(R^i, U)$. Since U is independent it follows that $\text{Int}(R, U) = R_1 \circ R_2 \circ \dots \circ R_k$. Hence, by Theorem 17 of [S]

$$r(R, U) = \|\text{Ext} \circ \text{Int}(R, U)\| = \|\text{Ext} \circ R_1 \circ R_2 \circ \dots \circ R_k\|.$$

Since genus embeddings maximize the number of regions,

$$r(R, U) = \mu(\text{Ext}, \{D_j(G)\}_{j=1}^k).$$

Similarly we may define the $R_j^i, j = 1, 2, \dots, k$, so that

$$r(R^i, U) = \mu(\text{Ext}^i, \{D_j(G^i)\}_{j=1}^k), \quad i = 1, 2.$$

Since the set U separates $G^1 - U$ from $G^2 - U$, it follows that Ext is the disjoint product $\text{Ext}^1 \circ \text{Ext}^2$. Set $\mu = \mu(\text{Ext}, \{D_j(G)\}_{j=1}^k)$ and $\mu_i = \mu(\text{Ext}^i, \{D_j(G^i)\}_{j=1}^k)$, $i = 1, 2$, and define μ^*, κ, c and μ_i^*, κ_i analogously. Then, since Ext^1 and Ext^2 are disjoint it follows that $\kappa = \kappa_1 + \kappa_2$. Clearly $c \leq k$. Hence,

$$\begin{aligned}
r(R, U) &= \mu \leq \mu^* - \kappa + c \\
&\leq \mu_1^* + \mu_2^* + (k-2)n/2k - \kappa_1 - \kappa_2 + k \\
&= \mu_1^* - \kappa_1 + \mu_2^* - \kappa_2 + (k-2)n/2k + k \\
&\leq \mu_1 + \mu_2 + 4 \sum_{i=2}^k (i-1)! \binom{k}{i} + (k-2)n/2k + k \\
&= r(R^1, U) + r(R^2, U) + 4 \sum_{i=2}^k (i-1)! \binom{k}{i} \\
&\quad + (k-2)n/2k + k.
\end{aligned}$$

Since $|V(G)| = |V(G^1)| + |V(G^2)| - k$, and $|E(G)| = |E(G^1)| + |E(G^2)|$, an application of the Euler-Poincaré formula yields the desired result. Q.E.D.

Comments. Let σ and ω denote the switching number and the winding number, respectively, as defined in [S]. It is then easily shown that if (P, Π) is a $PP(n, k)$ pair, then

$$\begin{aligned}\mu^*(P, \Pi) &= n && \text{if } k = 1, \\ \mu^*(P, \Pi) &= n - \kappa(P, \Pi) - \tfrac{1}{2}\sigma(P, \Pi) && \text{if } k = 2, \\ \mu^*(P, \Pi) &= n - \kappa(P, \Pi) - \tfrac{1}{2}\sigma(P, \Pi) - \tfrac{1}{2}\omega(P, \Pi) && \text{if } k = 3.\end{aligned}$$

It would be of interest to generalize this to higher values of k . Also, Theorems 10 and 13 of [S] indicate that the bounds given in Theorem 5 of this paper can be greatly improved.

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