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TANGENT 2-FIELDS ON EVEN-DIMENSIONAL NONORIENTABLE MANIFOLDS

BY

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ABSTRACT. This paper uses the Postnikov decomposition of a nonsimple fibration to describe the obstructions to a tangent 2-field on an even-dimensional nonorientable manifold.

Our purpose in this article is to prove the following theorem.

MAIN THEOREM. *Let M be a closed, nonorientable manifold of dimension m, where m is even and $m > 2$. Let $\mathcal{X}(M)$ in $H^m(M;\, Z_{w_1(M)})$ and $\mathcal{W}_{m-1}(M)$ in $H^{m-1}(M;\, Z_{w_1(M)})$ denote the twisted Euler class of M and the $(m-1)$ Stiefel-Whitney class of M, respectively. Then M has a 2-field if, and only if, $\mathcal{X}(M) = 0$ and $\mathcal{W}_{m-1}(M) = 0$.*

The vector field problem has a long history, and in the special case of 2-fields a great deal is already known. In order to put this result in perspective it will be helpful to give a brief statement of the known results. For details and an extensive bibliography the reader is referred to the excellent survey paper by Emery Thomas [**10**].

Thomas [**10**], Frank [**3**], and Atiyah [**1**] have described completely the necessary and sufficient conditions for a closed orientable manifold to have a 2-field. Suppose M is such a manifold of dimension m. Let $w_{m-1}(M)$ in $H^{m-1}(M;\, Z_2)$ be the $(m-1)$ Stiefel-Whitney class, $X(M)$ the Euler class, $k(M)$ the real Kervaire semicharacteristic, and $\sigma(M)$ the signature of M. Their results may then be summarized as follows.

Dimension of M mod 4	Necessary and sufficient conditions for a 2-field on M
3	M always has a 2-field
2	$X(M) = 0$
1	$w_{m-1}(M) = 0$ and $k(M) = 0$
0	$w_{m-1}(M) = 0$, $X(M) = 0$, and $\sigma(M) \equiv 0 \bmod 4$

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In addition, Atiyah and Dupont [2] have given necessary and sufficient conditions for a 2-field on a nonorientable $4k+1$ manifold which satisfies $w_1^2(M)=0$. Their result is somewhat complicated, and we will not attempt to state it here.

A summary of nonsimple Postnikov theory. Postnikov theory in the nonsimple case has been developed extensively by McClendon [5], Siegel [7], and C. A. Robinson [6]. Before proceeding to the proof of the Main Theorem we summarize their results in a form which will be convenient for our application.

Let $F \to E \xrightarrow{p} B$ be a fibration which is not necessarily simple, that is, $\pi_1(B)$ may act nontrivially on $\pi_*(F)$. We will assume throughout that F, E, B are all path connected and that F is simply connected. In this way we get a natural action of $\pi_1(B)$ on the homotopy of F. Let G be an abelian group and $h\colon \pi_1(B) \to \mathrm{Aut}(G)$ a homomorphism determining up to isomorphism a local coefficient system G_h on B. Define a map $\tilde{h}\colon \pi_1(B) \to \mathrm{Aut}(H^q(F; G))$ in the following way. Given an element b in $\pi_1(B)$, b determines an automorphism $\bar{b}\colon F \to F$ in the usual way. Let

$$\bar{b}^*\colon H^q(F; G) \to H^q(F; G)$$

be the induced isomorphism in cohomology. The element b also determines an isomorphism $h(b)\colon G \to G$ which induces a coefficient isomorphism

$$h(b)_*\colon H^q(F;\ G) \to H^q(F;\ G).$$

Now define $\tilde{h}(b)$ to be the composite $h(b)_* \circ \bar{b}^*$. The map $\tilde{h}$ determines a new local system on B denoted $H^q(F; G)_{\tilde{h}}$. We now have the following "twisted Serre spectral sequence".

THEOREM 1 (SIEGEL [7]). *There exists a cohomology spectral sequence $E_r^{p,q}$ such that*

$$E_2^{p,q} = H^p(B; H^q(F; G)_{\tilde{h}}), \qquad E_r^{p,q} \Rightarrow H^*(E; p^{-1}G_h).$$

REMARKS. $p^{-1}G_h$ denotes the local coefficient system on E induced by the map $p\colon E \to B$. McClendon [5] has given a relative version of this theorem.

From this spectral sequence one obtains the following results.

COROLLARY 1. *Let $H^p(B;\ H^q(F;\ G)_{\tilde{h}}) = 0$ whenever $0 < q < n$ or $0 < p < m$. Then there is an exact sequence*

$$\cdots \to H^k(B; G_h) \xrightarrow{p^*} H^k(E; p^{-1}G_h) \xrightarrow{i^*} [H^k(F; G)]^{\tilde{h}}$$
$$\xrightarrow{\tau} H^{k+1}(B; G_h) \to \cdots \to H^{m+n-1}(E; p^{-1}G_h).$$

The group

$$[H^k(F; G)]^{\tilde{h}} = \{a \in H^k(F; G) \mid \tilde{h}(a) = a \text{ for all } a \text{ in } \pi_1(B)\},$$

that is, the subgroup of elements invariant under the action $\tilde{h}$.

COROLLARY 2. *Assume $H^p(B;\ H^q(F;\ G)_{\tilde{h}}) = 0$ for $0 < q < n$. Then*

$$p^*\colon H^i(B; G_h) \to H^i(E; p^{-1}G_h)$$

is an isomorphism for $i < n$, and the following sequence is exact.

$$0 \to H^n(B; G_h) \xrightarrow{p^*} H^n(E; p^{-1}G_h) \xrightarrow{i^*} [H^n(F; G)]^{\tilde{h}}$$
$$\xrightarrow{\tau} H^{n+1}(B; G_h) \xrightarrow{p^*} H^{n+1}(E; p^{-1}G_h).$$

These sequences will be important later on when we construct the Postnikov tower for the map p.

Next we consider classifying spaces for twisted cohomology. We will confine our attention to twisted integer cohomology, since this will suffice for our applications. A much more general treatment, however, is possible. Let h: $Z_2 \to \mathrm{Aut}(Z)$ be the nontrivial action of Z_2 on Z, and let $Z_2 \to \tilde{K} \to K(Z_2, 1)$ be the universal cover. A space of type $K(Z, n)$ can be constructed so that h determines an action $\hat{h}$ of Z_2 on $K(Z, n)$ by basepoint preserving homeomorphisms. Take the Cartesian product $\tilde{K} \times K(Z, n)$, and let Z_2 act by covering transformations on the first factor and by $\hat{h}$ on the second. Denote the quotient space of this action by $L(Z, n)$. Clearly, one has a fibration

$$K(Z, n) \xrightarrow{i} L(Z, n) \xrightarrow{p} K(Z_2, 1).$$

Furthermore, since the action of Z_2 on $K(Z, n)$ has a fixed point, the fibration has a canonical section s: $K(Z_2, 1) \to L(Z, n)$.

Let X be a complex and w: $X \to K(Z_2, 1)$ a fixed map. The map w determines a local system of integers on X, denoted Z_w. Let $[X, L(Z, n); w]$ denote the set of homotopy classes of lifts of w into $L(Z, n)$, where the homotopies are required to go through lifts of w. Then one has the following result.

THEOREM 2. *There is an element e_n in $H^n(L(Z, n); Z_p)$ with the property that the mapping*

$$\begin{array}{ccc} [X, L(Z, n); w] & \to & H^n(X; Z_w) \\ f & \to & f^*(e_n) \end{array}$$

is a bijection.

The proof of this theorem can be found in Steenrod [**8**, 37.5]. It is not difficult to show that $s^*(e_n) = 0$ and $i^*(e_n) = \iota_n$, where ι_n in $H^n(K(Z, n); Z)$ is the fundamental class.

Let $k_0 \in K(Z, n)$ be the basepoint, and let $PK(Z, n)$ be the space of paths in $K(Z, n)$ starting at k_0. Since the action $\hat{h}$ of Z_2 on $K(Z, n)$ fixes the basepoint, Z_2 acts on $PK(Z, n)$ in the obvious way. As before, we can form the space $\tilde{K} \times PK(Z, n)$ and take the quotient of the Z_2 action. The resulting space will be denoted $\bar{P}L(Z, n)$ and is the total space of a fibration:

$$PK(Z, n) \to \bar{P}L(Z, n) \xrightarrow{\bar{p}} K(Z_2, 1).$$

Further, let $1 \times e$: $\tilde{K} \times PK(Z, n) \to \tilde{K} \times K(Z, n)$ be defined by $(1 \times e)(\tilde{k}, a) = (\tilde{k}, a(1))$. This map is a fibration with fibre $\Omega K(Z, n)$, and it is easily checked that $1 \times e$ is Z_2 equivariant. Thus $1 \times e$ induces a map $\bar{e}$: $\bar{P}L(Z, n) \to L(Z, n)$ which is also a fibration with fibre $\Omega K(Z, n)$. We can summarize the above constructions in the following commutative diagram.

$$\begin{array}{ccccc}
K(Z, n-1) & \to & PK(Z, n) & \xrightarrow{e} & K(Z, n) \\
\| & & \downarrow & & \downarrow \\
K(Z, n-1) & \to & \bar{P}L(Z, n) & \xrightarrow{\bar{e}} & L(Z, n) \\
& & \downarrow \bar{p} & & \downarrow p \\
& & K(Z_2, 1) & = & K(Z_2, 1)
\end{array}$$

Note that we have used the fact that $\Omega K(Z, n)$ has the same homotopy type as $K(Z, n-1)$.

There is one last space we will need. Let Z_2 act on $\Omega K(Z, n)$ in the manner analogous to the above and form the space $\bar{\Omega}L(Z, n)$. As before this space fibres over $K(Z_2, 1)$ with fibre $\Omega K(Z, n)$. It is easy to show that $\bar{\Omega}L(Z, n)$ has the same homotopy type as $L(Z, n-1)$.

Next we consider the analogue of a principal fibration in ordinary Postnikov theory. Let B be a space with a local system of integers Z_w determined by a map $w\colon B \to K(Z_2, 1)$. Let v be an element of $H^{n+1}(B; Z_w)$. Consider the diagram

$$\begin{array}{ccccc}
K(Z, n) & \xrightarrow{i} & E & \xrightarrow{\bar{v}} & \bar{P}L(Z, n+1) \\
& & \downarrow p_1 & & \downarrow \bar{e} \\
& & B & \xrightarrow{v} & L(Z, n+1).
\end{array}$$

The fibration p_1 is the pullback by v of $\bar{e}$. Thus

$$E = \{(b, u) \in B \times \bar{P}L(Z, n) \mid v(b) = \bar{e}(u)\}.$$

Observe that E maps to $K(Z_2, 1)$ by the composite wp_1. This defines an induced local system of integers on E. Define the space

$$E \oplus \bar{\Omega}L(Z, n+1) = \{((b, u), s) \in E \times \bar{\Omega}L(Z, n+1) \mid w(b) = p(s)\},$$

where p denotes the fibration $p\colon \bar{\Omega}L(Z, n+1) \to K(Z_2, 1)$. Define a map

$$(*) \qquad \mu\colon E \oplus \bar{\Omega}L(Z, n+1) \to E \quad \text{by } \mu((b, u), s) = (b, s+u).$$

This map is the analogue of the principal action map for ordinary principal fibrations in the following sense. Suppose X is a complex, and $q\colon X \to B$ is a given map. Let $f, g\colon X \to E$ be two lifts of q, so that $p_1 f \simeq p_1 g \simeq q$. Define the space

$$X \oplus X = \{(x, x') \in X \times X \mid wq(x) = wq(x')\}.$$

PROPOSITION. *There exists a map $d\colon X \to \bar{\Omega}L(Z, n+1) \cong L(Z, n)$ such that the composite*

$$X \xrightarrow{\Delta} X \oplus X \xrightarrow{f \oplus d} E \oplus \bar{\Omega}L(Z, n+1) \xrightarrow{\mu} E$$

is homotopic to g. Moreover d satisfies the property that $pd = wq$, that is, the diagram

$$\begin{array}{ccc} X & \xrightarrow{d} & \bar{\Omega}L(Z, n+1) \\ & {\scriptstyle wq}\searrow \quad \swarrow{\scriptstyle p} & \\ & K(Z_2, 1) & \end{array}$$

commutes. Thus, by the identification

$$\bar{\Omega}L(Z, n+1) \cong L(Z, n),$$

we have $d \in H^n(X; q^{-1}Z_w)$.

COROLLARY. *If d represents the zero element of $H^n(X; q^{-1}Z_w)$, then f is homotopic to g.*

The map μ: $E \oplus \bar{\Omega}L(Z, n+1) \to E$ also has an inverse T defined by $T(b, s) = ((b, s), c)$ where $b \in B$, $s \in \bar{P}L(Z, n)$, and c is the constant path in $K(Z, n+1)$ over the point $w(b)$. It is not difficult to see that $\mu T \simeq 1_E$.

Having given a cursory description of the construction and properties of the spaces required for nonsimple Postnikov theory, we are in a position to describe the actual tower for the fibration $F \to E \xrightarrow{p} B$. To do this we need the following result.

THEOREM 3 (MCCLENDON [5]). *Assume F is $(n-1)$ connected for $n \geqslant 2$. Let ι in $H^n(F; \pi_n(F))$ be the fundamental class and h: $\pi_1(B) \to \mathrm{Aut}(\pi_n(F))$ the local system defined by the fibration. Then ι is in $[H^n(F; \pi_n(F))]^h$, and, hence, by Corollary 2 to Theorem 1, it is transgressive.*

The Postnikov tower for $F \to E \to B$ can now be constructed. Let ι in $H^n(F; \pi_n(F))$ be the fundamental class of F, and let h: $\pi_1(B) \to \mathrm{Aut}(\pi_n(F))$ be the local system on B defined by the fibration. By the previous theorem, $\tau(\iota) = k$, is defined and is an element of $H^{n+1}(B; \pi_n(F)_h)$. Construct the diagram

$$\begin{tikzcd} k(Z, n) \arrow[r] & E_1 \arrow[d, "p_1"] & \\ E \arrow[ur, "q_1"] \arrow[r, "p"'] & B \arrow[r, "k"'] & L(\pi_n(F), n+1) \end{tikzcd}$$

E_1 is the total space of the fibration which is the pullback by k of the fibration $\bar{P}L(\pi_n(F), n+1) \to L(\pi_n(F), n+1)$. Since $\tau(\iota) = k$, by exactness we must have $p^*(k) = 0$. Therefore, there exists a map q_1: $E \to E_1$ such that $p_1 q_1 = p$. By a standard argument, the fibre F_1 of q_1 is n-connected and $\pi_k(F_1) = \pi_k(F)$ for $k > n$. The space E_1 is the second stage of the Postnikov tower. To construct higher stages the above construction is iterated on the successive fibrations q_i.

Proof of the Main Theorem. We begin by constructing a Postnikov tower for the fibration

$$V_{m,2} \to BO(m-2) \xrightarrow{p} BO(m)$$

where m is even, and $V_{m,2}$ is the Stiefel manifold of 2-frames in euclidean m space. The fibre $V_{m,2}$, hereafter denoted simply by V, is $(m-3)$ connected; $\pi_{m-2}(V) = Z$, and $\pi_{m-1}(V) = Z \oplus Z_2$. The action of $\pi_1 BO(n) = Z_2$ is nontrivial on $\pi_{m-2}(V)$ and on the Z-summand of $\pi_{m-1}(V)$.

LEMMA 1. *Let ι in $H^{m-2}(V; Z)$ be the fundamental class of V. Then $\tau(\iota) = \mathfrak{W}_{m-1}$ in $H^{m-1}(BO(m); Z_{w_1})$.*

LEMMA 2. *Let σ: $H^m(BO(m); Z_{w_1}) \to H^{m-1}(V; Z)$ denote the suspension map for the fibration p. Then $\sigma(\chi)$ contains a spherical class.*

REMARK. For the definition and properties of spherical classes the reader is referred to the article of McClendon [**5**].

These two lemmas imply that we may use $\mathfrak{W}_{m-1}$ and χ in the first stage of our Postnikov tower to kill off $\pi_{m-2}(V)$ and the Z-summand of $\pi_{m-1}(V)$. Thus we have the diagram

$$\begin{tikzcd}
 & E_1 \arrow[r, "k"] \arrow[d, "p_1"] & K(Z_2, m) \\
BO(m-2) \arrow[ur, dashed, "q_1"] \arrow[r, "p"] & BO(m) \arrow[r, "(\mathfrak{W}_{m-1}, \chi)"] & L(Z, m-1) \oplus L(Z, m)
\end{tikzcd}$$

$L(Z, m-1) \oplus L(Z, m)$ denotes the fibre product over $K(Z_2, 1)$ of $L(Z, m)$ and $L(Z, m-1)$. The map p lifts to a map q_1: $BO(m-2) \to E_1$, since $p^*(\chi)$ and $p^*(\mathfrak{W}_{m-1})$ are both zero. Convert the lift q_1 into a fibration with fibre F_1. Then F_1 is $(m-2)$ connected, and $\pi_{m-1}(F_1) = Z_2$. Let $u \in H^{m-1}(F_1; Z_2)$ be the fundamental class, and set $k = \tau(u)$ in $H^m(E_1; Z_2)$. The element k is the second Postnikov invariant for our fibration p. Let $K(Z_2, m-1) \to E_2 \xrightarrow{p_2} E_1$ be the principal fibration induced by k. We then have the following diagram.

$$\begin{tikzcd}
 & E_2 \arrow[d, "p_2"] & \\
 & E_1 \arrow[r, "k"] \arrow[d, "p_1"] & K(Z_2, m) \\
BO(m-2) \arrow[uur, "q_2"] \arrow[ur, "q_1"] \arrow[r, "p"] & BO(m) \arrow[r, "(\mathfrak{W}_{m-1}, \chi)"] & L(Z, m-1) \oplus L(Z, m)
\end{tikzcd}$$

The fibre F_2 of q_2 is $(m-1)$ connected, and therefore by a standard argument the map $q_{2\#}$: $[X, BO(m-2)] \to [X, E_2]$ is a bijection for a complex X of dimension less than or equal to m. Thus if we are given a map f: $X \to BO(m)$, f lifts to g: $X \to BO(m-2)$ if, and only if, f lifts to E_2. If $f^*(\chi) = f^*(\mathfrak{W}_{m-1}) = 0$, then f lifts to a map g: $X \to E_1$, and it is clear that f lifts to E_2 if, and only if, we can choose a lift g: $X \to E_1$ for which $g^*(k) = 0$.

LEMMA 3. *Define* $\bar{E}$ *to be the space* $E_1 \oplus L(Z, m-2) \oplus L(Z, m-1)$, *the fibre product over* $K(Z_2, 1)$. *Let* $\mu\colon \bar{E} \to E_1$ *be the map defined by* $(*)$, *and let* $e_{m-2} \in H^{m-2}(L(Z, m-2); Z_p)$ *and* $e_{m-1} \in H^{m-1}(L(Z, m-1); Z_p)$ *be the twisted fundamental classes of Theorem* 2. *Then we have the following formula.*

$$\mu^*(k) = k \otimes 1 \otimes 1 + 1 \otimes \mathrm{Sq}^1 e_{m-1} \otimes 1 + 1 \otimes 1 \otimes \mathrm{Sq}^2 e_{m-2} + w_2 \otimes 1 \otimes e_{m-2}.$$

The proofs of Lemmas 1–3 will be given later.

COROLLARY. *If* g_1 *and* g_2 *are two lifts of* f *to* E_1, *then there exist elements* u_1 *in* $H^{m-1}(X; Z_{f^*(w_1)})$ *and* $u_2 \in H^{m-2}(X; Z_{f^*(w_1)})$ *such that*

$$g_2^*(k) = g_1^*(k) + \mathrm{Sq}^1 u_1 + \mathrm{Sq}^2 u_2 + f^*(w_2) u_2.$$

PROOF. Use Lemma 3 and the Proposition.

The corollary shows that the elements $g^*(k)$ as g runs over lifts of f to E_1 form a coset in $H^m(X; Z_2)$ of the subgroup

$$\mathrm{Sq}^1 H^{m-1}\big(X; Z_{f^*(w_1)}\big) + \big(\mathrm{Sq}^2 + f^*(w_2)\big) H^{m-2}\big(X; Z_{f^*(w_1)}\big).$$

This immediately gives us the following theorem.

THEOREM 4. *Let* X *be a complex of dimension* m, *where* m *is even. Suppose* $f\colon X \to BO(m)$ *satisfies* $f^*(\mathcal{W}_{m-1}) = 0$ *and* $f^*(\chi) = 0$. *Then* f *lifts to* $f\colon X \to BO(m-2)$ *if, and only if,* $g^*(k) = 0$ *in*

$$H^m(X; Z_2) \Big/ \mathrm{Sq}^1 H^{m-1}\big(X; Z_{f^*(w_1)}\big) + \big(\mathrm{Sq}^2 + f^*(w_2)\big)\Big(H^{m-2}\big(X; Z_{f^*(w_1)}\big)\Big)$$

where g *is any lift of* f *to* E_1.

The Main Theorem is now an easy consequence of the following lemma.

LEMMA 4. *Let* M *be a nonorientable manifold of dimension* m. *The map* $\mathrm{Sq}^1\colon H^{m-1}(M; Z_{w_1(M)}) \to H^m(M; Z_2)$ *is onto.*

PROOF. We have the following commutative triangle:

$$\begin{tikzcd} H^{m-1}\big(M; Z_{w_1(M)}\big) \arrow[rr, "\mathrm{Sq}^1"] \arrow[dr, "\rho_2"'] & & H^m(M; Z_2) \\ & H^{m-1}(M; Z_2) \arrow[ur, "\mathrm{Sq}^1"'] & \end{tikzcd}$$

where ρ_2 is reduction mod 2. It is well known that $\mathrm{Sq}^1\colon H^{m-1}(M; Z_2) \to H^m(M; Z_2)$ is onto. But ρ_2 is also onto as can be seen from the following exact sequence:

$$\cdots \to H^{m-1}\big(M; Z_{w_1(M)}\big) \xrightarrow{\rho_2} H^{m-1}(M; Z_2) \xrightarrow{\beta} H^m\big(M; Z_{w_1(M)}\big) \to \cdots,$$

where β is the Bockstein homomorphism associated to the coefficient sequence $0 \to Z_{w_1} \to Z_{w_1} \to Z_2 \to 0$. Since $H^m(M; Z_{w_1(M)})$ is infinite cyclic and $H^{m-1}(M; Z_2)$

has order 2, we immediately see that β is zero, and hence ρ_2 is onto. Thus Sq^1: $H^{m-1}(M; Z_{w_1(M)}) \to H^m(M; Z_2)$ is the composition of onto maps and must itself be onto.

To prove the Main Theorem let f: $M \to BO(m)$ be a map classifying $T(M)$, the tangent bundle of M. By hypothesis, $f^*(\chi)$ and $f^*(\mathfrak{W}_{m-1})$ are zero. Choose any lift of f to the space E_1; call it g. Then $g^*(k)$ is an element of the quotient group

$$H^m(M; Z_2)\Big/\mathrm{Sq}^1H^{m-1}\big(M; Z_{w_1(M)}\big) + \big(\mathrm{Sq}^2 + w_2(M)\big)H^{m-2}\big(M; Z_{w_1(M)}\big).$$

But this group is zero by Lemma 4, and so by Theorem 4 f lifts to $BO(m-2)$.

It only remains for us to prove Lemmas 1 through 3.

Proof of Lemma 1. Corollary 2 to Theorem 1 gives us the following exact sequence.

$$\to\big[H^{m-2}(V; Z)\big]^{\tilde{h}} \xrightarrow{\tau} H^{m-1}\big(BO(m), Z_{w_1}\big) \xrightarrow{p^*} H^{m-1}\big(BO(m-2); Z_{w_1}\big) \to .$$

The group on the left is Z, generated by ι according to Theorem 3. The kernel of p^* is generated by $\mathfrak{W}_{m-1}$, so by exactness $\tau(\iota) = \mathfrak{W}_{m-1}$.

Proof of Lemma 2. Consider the diagram

$$\begin{array}{ccccc} V_{m,2} & \to & BO(m-2) & \xrightarrow{p} & BO(m) \\ g\downarrow & & \downarrow & & \| \\ S^{m-1} & \to & BO(m-1) & \xrightarrow{p'} & BO(m) \end{array}$$

Let σ be the suspension for p and σ' the suspension for p'. By the Serre and Gysin sequences for p' it is easy to see that $\sigma'(\chi) = v$, where v is a generator of $H^{m-1}(S^{m-1}; Z)$. So $\sigma'(\chi)$ contains a spherical class, namely v. The map g is a fibration with fibre S^{m-2}. Consider the homotopy exact sequence for g:

$$\to \pi_{m-1}(V_{m,2}) \xrightarrow{g^*} \pi_{m-1}(S^{m-1}) \xrightarrow{\Delta} \pi_{m-2}(S^{m-2}) \xrightarrow{i^*} \pi_{m-2}(V_{m,2}) \xrightarrow{g^*} \pi_{m-2}(S^{m-1}).$$

Since $\pi_{m-2}(S^{m-1})$ is zero, we must have i_* onto. But $\pi_{m-2}(S^{m-2})$ and $\pi_{m-2}(V_{m,2})$ are both infinite cyclic, and consequently i_* is also 1-1. By exactness, this makes g_*: $\pi_{m-1}(V_{m-2}) \to \pi_{m-1}(S^{m-1})$ an epimorphism. This implies that $g^*(v)$ in $\sigma(\chi)$ is spherical.

Proof of Lemma 3. Consider the following diagram:

$$\begin{array}{ccc} BO(m-2) \oplus L(Z, m-2) \oplus L(Z, m-1) & \xrightarrow{q_1 \oplus 1 \oplus 1} & \bar{E} \xrightarrow{\mu} E_1 \\ \bar{s}\uparrow\downarrow\pi & & \downarrow p_1 \\ BO(m-2) & \xrightarrow{p} & BO(m) \end{array}$$

Recall that $\bar{E} = E_1 \oplus L(Z, m-2) \oplus L(Z, m-1)$, the fibre product over $K(Z_2, 1)$. It is easy to see that the map π has a section $\bar{s}$. Furthermore, if we let v be the composite $\mu \circ (q_1 \oplus 1 \oplus 1)$, then $v\bar{s} \simeq q_1$. By the work of Emery Thomas **[9]**, there exist a map τ_1 and a short exact sequence in Z_2 cohomology

$$0 \to H^m(E_1) \xrightarrow{v^*} H^m(BO(m-2) \oplus L(Z, m-2) \oplus L(Z, m-1))$$
$$\xrightarrow{\tau_1} H^{m+1}(BO(m)).$$

Moreover, τ_1 commutes with Steenrod squares and is an $H^*(BO(n); Z_2)$ module map. As described earlier, there is a map $T\colon E_1 \to \bar{E}$ such that $\mu T \simeq 1_{\bar{E}}$. Therefore,

$$\mu^*(k) = k \otimes 1 \otimes 1 + \sum_j a_j \otimes b_j \otimes c_j$$

where a_j is in $H^*(E_1; Z_2)$, b_j is in $H^*(L(Z, m-2); Z_2)$, and c_j is in

$$H^*(L(Z, m-1); Z_2),$$

with $\deg a_j + \deg b_j + \deg c_j = m$ and $\deg b_j + \deg c_j > 0$. This last condition implies that $\Sigma a_j \otimes b_j \otimes c_j$ is in $\ker \bar{s}^*$. The second Postnikov invariant k is in $\ker q_1^*$, so $v^*(k) = \Sigma a_j \otimes b_j \otimes c_j$ is in $\ker \bar{s}^* \cap \ker \tau_1$ by the short exact sequence above.

LEMMA 5. *The intersection of* $\ker \bar{s}^*$ *with* $H^m(BO(m-2) \oplus (L(Z, m-1); Z_2))$ *consists of the elements* $1 \otimes \mathrm{Sq}^1 e_{m-1} \otimes 1$, $1 \otimes 1 \otimes \mathrm{Sq}^2 e_{m-2}$, $w_1 \otimes 1 \otimes \mathrm{Sq}^1 e_{m-2}$, *and* $w_2 \otimes 1 \otimes e_{m-2}$.

PROOF. Consider the mod 2 Serre spectral sequence for the fibration π (see previous diagram). Since π has a section, and since we are taking mod 2 cohomology, an easy argument shows that the spectral sequence collapses, that is, $E_2 = E_\infty$. The conclusion of the lemma now follows easily.

We now compute the action of the map τ_1 on these elements. Recall that by construction we have $\tau_1(1 \otimes e_{m-1} \otimes 1) = w_m$, where w_m is the mod 2 reduction of χ, and $\tau_1(1 \otimes 1 \otimes e_{m-2}) = w_{m-1}$. Using these formulas and those of Wu **[11]** we obtain

$$\begin{aligned}
\tau_1(1 \otimes \mathrm{Sq}^1 e_{m-1} \otimes 1) &= \mathrm{Sq}^1 w_m = w_1 w_m, \\
\tau_1(1 \otimes 1 \otimes \mathrm{Sq}^2 e_{m-2}) &= \mathrm{Sq}^2 w_{m-1} = w_2 w_{m-1} + w_1 w_m, \\
\tau_1(w_1 \otimes 1 \otimes \mathrm{Sq}^1 e_{m-2}) &= w_1^2 w_{m-1}, \\
\tau_1(w_2 \otimes 1 \otimes e_{m-2}) &= w_2 w_{m-1}.
\end{aligned}$$

Therefore, $\ker \tau_1 \cap \ker \bar{s}^*$ must consist of the single element

$$1 \otimes \mathrm{Sq}^1 e_{m-1} \otimes 1 + 1 \otimes 1 \otimes \mathrm{Sq}^2 e_{m-2} + w_2 \otimes 1 \otimes e_{m-2}.$$

By the short exact sequence given earlier, this gives us the required formula for $\mu^*(k)$.

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