

## TAUBERIAN CONDITIONS FOR $L^1$ -CONVERGENCE OF FOURIER SERIES

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*Dedicated to Jovan Karamata*

**ABSTRACT.** It is proved that Fourier series with asymptotically even coefficients and satisfying  $\lim_{\lambda \rightarrow 1} \limsup_{n \rightarrow \infty} \sum_{j=n}^{[\lambda n]} j^{p-1} |\Delta \hat{f}(j)|^p = 0$ , for some  $1 < p \leq 2$ , converge in  $L^1$ -norm if and only if  $\|\hat{f}(n)E_n + \hat{f}(-n)E_{-n}\| = o(1)$ , where  $E_n(t) = \sum_{k=0}^n e^{ikt}$ . Recent results of Stanojević [1], Bojanic and Stanojević [2], and Goldberg and Stanojević [3] are special cases of some corollaries to the main theorem.

**1. Introduction.** The space  $L^1(T)$  of complex functions integrable on  $T = \mathbf{R}/2\pi\mathbf{Z}$  does not admit convergence in norm. Consequently, convergence in norm of the partial sums  $S_n(f) = S_n(f, t) = \sum_{|j| \leq n} \hat{f}(j)e^{ijt}$  to  $f \in L^1(T)$  cannot be characterized in terms of Fourier coefficients without additional assumptions about the sequence  $\{\hat{f}(n)\}$ .

In the case of even coefficients ( $\hat{f}(n) = \hat{f}(-n)$  for all integers  $n$ ) satisfying certain regularity and/or speed conditions, it is well known that

$$(1.1) \quad \|S_n(f) - f\| = o(1), \quad n \rightarrow \infty,$$

is equivalent with

$$(1.2) \quad \hat{f}(n) \lg n = o(1), \quad n \rightarrow \infty.$$

(A survey of classical and recent results of this kind can be found in [1, 2 and 3].)

Most recent results concerning the equivalence between (1.1) and (1.2) are due to Stanojević [1], Bojanic and Stanojević [2] and Goldberg and Stanojević [3].

In [1] it is proved that if  $\{\hat{f}(n)\}$  is even and satisfies

$$(1.3) \quad \frac{1}{n} \sum_{k=1}^n k |\Delta \hat{f}(k)| = o(1), \quad n \rightarrow \infty,$$

and

$$(1.4) \quad n \Delta \hat{f}(n) = O(1), \quad n \rightarrow \infty,$$

then (1.1) is equivalent with (1.2).

Goldberg and Stanojević [3] proved that if

$$(1.5) \quad \{(\hat{f}(n) - \hat{f}(-n)) \lg n\} \text{ is a null-sequence of bounded variation,}$$

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and if for some  $1 < p \leq 2$

$$(1.6) \quad \frac{1}{n} \sum_{j=n}^{2n} j^p |\Delta \hat{f}(j)|^p = o(1), \quad n \rightarrow \infty,$$

then (1.1) if and only if (1.2). An earlier result of Bojanic and Stanojević [2] is a corollary to the Goldberg-Stanojević theorem.

In this paper I shall extend and generalize the Goldberg-Stanojević theorem in two ways. Instead of (1.5), a weaker condition will be assumed, i.e.,

$$(AE) \quad \frac{1}{n} \sum_{j=1}^n |\hat{f}(j) - \hat{f}(-j)| \lg j = o(1), \quad n \rightarrow \infty,$$

$$\lim_{\lambda \rightarrow 1} \limsup_{n \rightarrow \infty} \sum_{j=n}^{[\lambda n]} |\Delta(\hat{f}(j) - \hat{f}(-j))| \lg j = 0,$$

and (1.6) will be relaxed as follows:

$$(HK) \quad \lim_{\lambda \rightarrow 1} \limsup_{n \rightarrow \infty} \sum_{j=n}^{[\lambda n]} j^{p-1} |\Delta \hat{f}(j)|^p = 0,$$

for some  $1 < p \leq 2$ .

A sequence of complex numbers satisfying (AE) is called *asymptotically even*. Clearly, every even sequence satisfies (AE).

The condition (HK) is a Tauberian condition of Hardy-Karamata [4] kind. Plainly (1.6) implies (HK).

As a consequence of the main theorem it will follow that the condition (1.3) is superfluous, and that (1.4) can be weakened if a certain speed of  $\|\sigma_n(f) - f\|$  is assumed, where  $\sigma_n(f)$  is the Fefér sum of  $S_n(f)$ .

**2. Main theorem.** Fourier series considered throughout this section are the series with asymptotically even coefficients. That is

$$(2.1.1) \quad \frac{1}{n} \sum_{j=1}^n |\hat{f}(j) - \hat{f}(-j)| \lg j = o(1), \quad n \rightarrow \infty,$$

$$(2.1.2) \quad \lim_{\lambda \rightarrow 1} \limsup_{n \rightarrow \infty} \sum_{j=n}^{[\lambda n]} |\Delta(\hat{f}(j) - \hat{f}(-j))| \lg j = 0.$$

**MAIN THEOREM.** Let  $S[f] \sim \sum_{|n| < \infty} \hat{f}(n) e^{int}$  be the Fourier series of  $f \in L^1(T)$  with asymptotically even coefficients.

If for some  $1 < p \leq 2$

$$(HK) \quad \lim_{\lambda \rightarrow 1} \limsup_{n \rightarrow \infty} \sum_{j=n}^{[\lambda n]} j^{p-1} |\Delta \hat{f}(j)|^p = 0,$$

then  $\|S_n(f) - f\| = o(1)$ ,  $n \rightarrow \infty$ , if and only if

$$(2.2) \quad \|\hat{f}(n)E_n + \hat{f}(-n)E_{-n}\| = o(1), \quad n \rightarrow \infty,$$

where  $E_n(t) = \sum_{k=0}^n e^{ikt}$ .

PROOF. It suffices to show that

$$(2.3) \quad \limsup_{n \rightarrow \infty} \left| \|S_n(f) - f\| - \|\hat{f}(n)E_n + \hat{f}(-n)E_{-n}\| \right| = 0.$$

Let  $\lambda > 1$  and  $n > 1$ . Then the following identity can be established:

$$(2.4) \quad \begin{aligned} & S_n(f, t) - f(t) - (\hat{f}(n)E_n(t) + \hat{f}(-n)E_{-n}(t)) \\ &= \frac{[\lambda n] + 1}{[\lambda n] - n} [\sigma_{[\lambda n]}(f, t) - f(t)] - \frac{n + 1}{[\lambda n] - n} [\sigma_n(f, t) - f(t)] \\ &\quad - \frac{1}{[\lambda n] - n} \sum_{j=n-1}^{[\lambda n]} \hat{f}(j)E_j(t) - \frac{1}{[\lambda n] - n} \sum_{j=n-1}^{[\lambda n]} \hat{f}(-j)E_{-j}(t) \\ &\quad - \sum_{j=n}^{[\lambda n]-1} \frac{[\lambda n] + 1 - j}{[\lambda n] - n} [\Delta \hat{f}(j)] E_j(t) \\ &\quad - \sum_{j=n}^{[\lambda n]-1} \frac{[\lambda n] + 1 - j}{[\lambda n] - n} [\Delta \hat{f}(-j)] E_{-j}(t). \end{aligned}$$

The Dirichlet kernel can be written as

$$D_j(t) = E_j(t) + E_{-j}(t) - 1.$$

Thus the third and the fourth terms on the right-hand side of (2.4) can be grouped in the following way:

$$(2.5) \quad \begin{aligned} I_{1n} &= \frac{1}{[\lambda n] - n} \sum_{j=n}^{[\lambda n]} \hat{f}(j)D_j(t) \\ &\quad - \frac{1}{[\lambda n] - n} \sum_{j=n}^{[\lambda n]} (\hat{f}(j) - \hat{f}(-j))E_{-j}(t) + \frac{1}{[\lambda n] - n} \sum_{j=n}^{[\lambda n]} \hat{f}(j); \end{aligned}$$

and the fifth and the sixth terms as

$$(2.6) \quad \begin{aligned} I_{2n} &= \sum_{j=n}^{[\lambda n]-1} \frac{[\lambda n] + 1 - j}{[\lambda n] - n} [\Delta \hat{f}(j)] D_j(t) \\ &\quad - \sum_{j=n}^{[\lambda n]-1} \frac{[\lambda n] + 1 - j}{[\lambda n] - n} [\Delta(\hat{f}(j) - \hat{f}(-j))] E_{-j}(t) \\ &\quad + \sum_{j=n}^{[\lambda n]-1} \frac{[\lambda n] + 1 - j}{[\lambda n] - n} \Delta \hat{f}(j). \end{aligned}$$

Taking the norm of both sides of (2.5) we obtain

$$\begin{aligned}
 \|I_{1n}\| &\leq \frac{1}{[\lambda n] - n} \left\| \sum_{j=n}^{[\lambda n]} \hat{f}(j) D_j(t) \right\| \\
 &\quad + \frac{1}{[\lambda n] - n} \sum_{j=n}^{[\lambda n]} |\hat{f}(j) - \hat{f}(-j)| \lg j \\
 (2.7) \quad &\quad + \frac{1}{[\lambda n] - n} \sum_{j=n}^{[\lambda n]} |\hat{f}(j)| \\
 &= J_n + \frac{[\lambda n]}{[\lambda n] - n} \left( \frac{1}{[\lambda n]} \sum_{j=1}^{[\lambda n]} |\hat{f}(j) - \hat{f}(-j)| \lg j \right) \\
 &\quad + \frac{[\lambda n]}{[\lambda n] - n} \left( \frac{1}{[\lambda n]} \sum_{j=1}^{[\lambda n]} |\hat{f}(j)| \right).
 \end{aligned}$$

Applying first the Hölder inequality and then the Hausdorff-Young equality to  $J_n$ , we have

$$J_n \leq A_p \frac{[\lambda n]^{1/q}}{[\lambda n] - n} \left( \sum_{j=n}^{[\lambda n]} |\hat{f}(j)|^p \right)^{1/p} = A_p \frac{[\lambda n]}{[\lambda n] - n} \left( \frac{1}{[\lambda n]} \sum_{j=1}^{[\lambda n]} |\hat{f}(j)|^p \right)^{1/p},$$

where  $A_p$  is an absolute constant depending on  $p$ , and  $1/p + 1/q = 1$ . The last term in (2.7) is majorized by

$$\frac{[\lambda n]}{[\lambda n] - n} \left( \frac{1}{[\lambda n]} \sum_{j=1}^{[\lambda n]} |\hat{f}(j)|^p \right)^{1/p}.$$

Hence

$$\begin{aligned}
 \|I_{1n}\| &\leq C_1 \frac{[\lambda n]}{[\lambda n] - n} \left( \frac{1}{[\lambda n]} \sum_{j=1}^{[\lambda n]} |\hat{f}(j)|^p \right)^{1/p} \\
 &\quad + C_2 \frac{[\lambda n]}{[\lambda n] - n} \left( \frac{1}{[\lambda n]} \sum_{j=1}^{[\lambda n]} |\hat{f}(j) - \hat{f}(-j)| \lg j \right),
 \end{aligned}$$

where  $C_1$  and  $C_2$  are absolute constants.

In a similar manner we obtain

$$\|I_{2n}\| \leq C_3 \sum_{j=n}^{[\lambda n]} |\Delta(\hat{f}(j) - \hat{f}(-j))| \lg j + C_4 \left( \sum_{j=n}^{[\lambda n]} j^{p-1} |\Delta \hat{f}(j)|^p \right)^{1/p},$$

where  $C_3$  and  $C_4$  are absolute constants.

Combining estimates for both  $\|I_{1n}\|$  and  $\|I_{2n}\|$  we get

$$\begin{aligned}
 (2.8) \quad & \left| \|S_n(f) - f\| - \|\hat{f}(n)E_n + \hat{f}(-n)E_{-n}\| \right| \leq \frac{[\lambda n]}{[\lambda n] - n} \|\sigma_{[\lambda n]}(f) - f\| \\
 & + \frac{n+1}{[\lambda n] - n} \|\sigma_n(f) - f\| + C_1 \frac{[\lambda n]}{[\lambda n] - n} \left( \frac{1}{[\lambda n]} \sum_{j=1}^{[\lambda n]} |\hat{f}(j)|^p \right)^{1/p} \\
 & + C_2 \frac{[\lambda n]}{[\lambda n] - n} \left( \frac{1}{[\lambda n]} \sum_{j=1}^{[\lambda n]} |\hat{f}(j) - \hat{f}(-j)| \lg j \right) \\
 & + C_3 \sum_{j=1}^{[\lambda n]} |\Delta(\hat{f}(j) - \hat{f}(-j))| \lg j + C_4 \left( \sum_{j=n}^{[\lambda n]} j^{p-1} |\Delta \hat{f}(j)|^p \right)^{1/p}.
 \end{aligned}$$

Since for  $\lambda > 1$  we have  $\lambda n / ([\lambda n] - n) \sim \lambda / (\lambda - 1)$ ,  $n \rightarrow \infty$ , it follows that

$$\limsup_{n \rightarrow \infty} \frac{[\lambda n]}{[\lambda n] - n} C_n = 0,$$

for any null-sequence  $\{C_n\}$ .

After taking the limit superior of both sides of (2.8) we get

$$\begin{aligned}
 (2.9) \quad & \limsup_{n \rightarrow \infty} \left| \|S_n(f) - f\| - \|\hat{f}(n)E_n + \hat{f}(-n)E_{-n}\| \right| \\
 & \leq C_3 \limsup_{n \rightarrow \infty} \sum_{j=n}^{[\lambda n]} |\Delta(\hat{f}(j) - \hat{f}(-j))| \lg j \\
 & + C_4 \limsup_{n \rightarrow \infty} \left( \sum_{j=n}^{[\lambda n]-1} j^{p-1} |\Delta \hat{f}(j)|^p \right)^{1/p}.
 \end{aligned}$$

For  $\|\sigma_n(f) - f\| = o(1)$ ,  $n \rightarrow \infty$ ,  $\hat{f}(n) = o(1)$ ,  $n \rightarrow \infty$ , and

$$\frac{1}{[\lambda n]} \sum_{j=1}^{[\lambda n]} |\hat{f}(j) - \hat{f}(-j)| \lg j = o(1), \quad n \rightarrow \infty,$$

because of (2.1.1).

Taking the limit as  $\lambda \rightarrow 1$  of both sides of (2.9) we obtain

$$\begin{aligned}
 & \lim_{\lambda \rightarrow 1} \limsup_{n \rightarrow \infty} \left| \|S_n(f) - f\| - \|\hat{f}(n)E_n + \hat{f}(-n)E_{-n}\| \right| \\
 & \leq C_3 \lim_{\lambda \rightarrow 1} \limsup_{n \rightarrow \infty} \sum_{j=n}^{[\lambda n]} |\Delta(\hat{f}(j) - \hat{f}(-j))| \lg j \\
 & + C_4 \lim_{\lambda \rightarrow 1} \limsup_{n \rightarrow \infty} \left( \sum_{j=n}^{[\lambda n]} j^{p-1} |\Delta \hat{f}(j)|^p \right)^{1/p}.
 \end{aligned}$$

Because of (2.1.2) and (HK) we finally have (2.3), i.e.

$$\limsup_{n \rightarrow \infty} \left\| S_n(f) - f \right\| - \left\| \hat{f}(n)E_n + \hat{f}(-n)E_{-n} \right\| = 0.$$

This completes the proof of the main theorem.

**3. Corollaries and additional results.** By strengthening either (2.1.2) or (HK), or both, one can obtain a number of corollaries that, as a special case, contain the results of Stanojević [1], Bojanic and Stanojević [2], and Goldberg and Stanojević [3].

The class of complex null-sequences  $\{c_n\}$  satisfying

$$(3.1) \quad \frac{1}{n} \sum_{k=1}^n k |\Delta c_k| = o(1), \quad n \rightarrow \infty,$$

includes as a proper subclass null-sequences of bounded variation.

**COROLLARY 3.1.** *Let  $S[f] \sim \sum_{|n| < \infty} \hat{f}(n)e^{int}$  be the Fourier series of  $f \in L^1(T)$ , and let  $\{(\hat{f}(n) - \hat{f}(-n)) \lg n\}$  satisfy (3.1). If (HK) holds then*

$$\|S_n(f) - f\| = o(1), \quad n \rightarrow \infty,$$

*if and only if*

$$\hat{f}(n) \lg n = o(1), \quad n \rightarrow \infty.$$

**PROOF.** The condition (2.1.1) is satisfied. It remains to show that (2.1.2) holds. Since for  $\lambda > 1$

$$\begin{aligned} \sum_{j=-n}^{[\lambda n]} |\Delta(\hat{f}(j) - \hat{f}(-j))| \lg j &\leq \frac{1}{n} \sum_{j=n}^{[\lambda n]} j |\Delta(\hat{f}(j) - \hat{f}(-j))| \lg j \\ &\leq \frac{\lambda}{[\lambda n]} \sum_{j=1}^{[\lambda n]} j |\Delta(\hat{f}(j) - \hat{f}(-j))| \lg j \\ &\leq \frac{\lambda}{[\lambda n]} \sum_{j=1}^{[\lambda n]} j |\Delta[(\hat{f}(j) - \hat{f}(-j)) \lg j]| + \frac{\lambda}{[\lambda n]} \sum_{j=1}^{[\lambda n]} |\hat{f}(j) - \hat{f}(-j)| \lg(1 + 1/j)^{-j}, \end{aligned}$$

it follows that if  $\{(\hat{f}(n) - \hat{f}(-n)) \lg n\}$  satisfies (3.1) then (2.1.2) holds.

A special case of Corollary 3.1 is the Goldberg-Stanojević theorem. Indeed, let  $1 < \lambda \leq 2$ . Then

$$\sum_{j=n}^{[\lambda n]} j^{p-1} |\Delta \hat{f}(j)|^p \leq \frac{2}{n} \sum_{j=n}^{2n} j^p |\Delta \hat{f}(j)|^p.$$

**COROLLARY 3.2.** *Let  $S[f] \sim \sum_{|n| < \infty} \hat{f}(n)e^{int}$  be the Fourier series of  $f \in L^1(T)$ , and let (2.1.1) hold. If (HK) holds and if*

$$(3.2) \quad n[\Delta(\hat{f}(n) - \hat{f}(-n))] \lg n = O(1), \quad n \rightarrow \infty,$$

*then (1.1) is equivalent with (1.2).*

PROOF. Due to (3.2) we have  $\sum_{j=n}^{[\lambda n]} |\Delta(\hat{f}(j) - \hat{f}(-j))| \lg j \leq C \lg \lambda$ , where  $C$  is an absolute constant. Hence  $\{\hat{f}(n)\}$  is an asymptotically even sequence.

Since (3.2) is a summability condition in the sense of Hardy [6], from (2.1.1) it follows that

$$(3.3) \quad (f(n) - \hat{f}(-n)) \lg n = o(1), \quad n \rightarrow \infty.$$

But (3.3) implies that (2.2) is equivalent with (1.2), for  $\|D_n\| = (4/\pi^2) \lg n + O(1)$ ,  $n \rightarrow \infty$ .

COROLLARY 3.3. Let  $S[f] \sim \sum_{|n| < \infty} \hat{f}(n) e^{int}$  be the Fourier series of  $f \in L^1(T)$  with even coefficients. If (HK) holds then (1.1) is equivalent with (1.2).

A special case of this corollary is the main theorem of Bojanic and Stanojević [2].

PROOF. Every even sequence is asymptotically even.

COROLLARY 3.4. Let  $S[f] \sim \sum_{|n| < \infty} \hat{f}(n) e^{int}$  be the Fourier series of  $f \in L^1(T)$  with even coefficients. If  $n\Delta\hat{f}(n) = O(1)$ ,  $n \rightarrow \infty$ , then

$$\|S_n(f) - f\| = o(1), \quad n \rightarrow \infty$$

if and only if

$$\hat{f}(n) \lg n = o(1), \quad n \rightarrow \infty.$$

PROOF. The condition  $n\Delta\hat{f}(n) = O(1)$ ,  $n \rightarrow \infty$  implies (HK), for

$$\sum_{j=n}^{[\lambda n]} j^{p-1} |\Delta\hat{f}(j)|^p \leq C \lg \lambda,$$

where  $C$  is an absolute constant.

A special case of Corollary 3.4 is the Stanojević theorem.

In what follows it will be assumed that, for simplicity's sake,  $\{\hat{f}(n)\}$  are even sequences.

All classical conditions as well as (HK) imply that

$$(3.4) \quad n^\alpha \Delta\hat{f}(n) = o(1), \quad n \rightarrow \infty, \text{ for some } 0 < \alpha < 1.$$

It seems unlikely that (3.4) would imply that (1.1)  $\Leftrightarrow$  (1.2). But a slightly stronger form of (3.4) such as

$$(3.5) \quad n[\lg n]^{-1/p} \max_{n \leq j \leq n + [n/\lg n]} |\Delta\hat{f}(j)| = o(1), \quad n \rightarrow \infty,$$

for some  $1 < p \leq 2$  and  $1/p + 1/q = 1$ , and certain conditions on the speed with which  $\|\sigma_n(f) - f\|$  goes to zero as  $n \rightarrow \infty$  could imply that (1.1)  $\Leftrightarrow$  (1.2).

PROPOSITION 3.1. Let  $S[f] \sim \sum_{|n| < \infty} \hat{f}(n) e^{int}$  be the Fourier series of  $f \in L^1(T)$  with even coefficients. If, for some  $1 < p \leq 2$ ,  $1/p + 1/q = 1$ , (3.5) holds and

$$(3.6) \quad \lg n \|\sigma_n(f) - f\| = o(1), \quad n \rightarrow \infty,$$

$$\lg n \left( \frac{1}{n} \sum_{j=n}^{n + [n/\lg n]} |\hat{f}(j)|^p \right)^{1/p} = o(1), \quad n \rightarrow \infty,$$

then (1.1) if and only if (1.2).

PROOF. Let  $m > n > 1$ . Then using the same technique as in the proof of the main theorem one can obtain the inequality

$$(3.7) \quad \begin{aligned} \left| \|S_n(f) - f\| - |\hat{f}(n)| \|D_n\| \right| &\leq \frac{m+1}{m-n} \|\sigma_m(f) - f\| + \frac{n+1}{m-n} \|\sigma_n(f) - f\| \\ &+ C_1 \left( \sum_{j=n}^m j^{p-1} |\Delta \hat{f}(j)|^p \right)^{1/p} \left( \frac{m}{n} \right)^{1/q} \\ &+ C_2 \left( \frac{1}{m-n} \sum_{j=n}^m |\hat{f}(j)|^p \right)^{1/p} \left( \frac{m}{m-n} \right)^{1/q}, \end{aligned}$$

where  $C_1$  and  $C_2$  are absolute constants.

Let  $m = n + [n/\lg n]$ . Then (3.7) becomes

$$\begin{aligned} \left| \|S_n(f) - f\| - |\hat{f}(n)| \|D_n\| \right| &\leq B_1 \left[ \lg \left( n + \left[ \frac{n}{\lg n} \right] \right) \right] \|\sigma_{n+[n/\lg n]}(f) - f\| \\ &+ B_2 [\lg n] \|\sigma_n(f) - f\| + B_3 \left( \sum_{j=n}^{n+[n/\lg n]} j^{p-1} |\Delta \hat{f}(j)|^p \right)^{1/p} \\ &+ B_4 \left( \frac{1}{n} \sum_{j=n}^{n+[n/\lg n]} |\hat{f}(j)|^p \right)^{1/p} \lg n \end{aligned}$$

where  $B_1, \dots, B_4$  are absolute constants.

Due to (3.5) and (3.6), for sufficiently large  $n$  we have

$$\left| \|S_n(f) - f\| - |\hat{f}(n)| \lg n \right| = O \left( n [\lg n]^{-1/p} \max_{n \leq j \leq n+[n/\lg n]} |\Delta \hat{f}(j)| \right) + o(1).$$

This completes the proof of Proposition 3.1.

If instead of  $\{[n/\lg n]\}$  we take a sequence of integers  $\{[n/L(n)]\}$  where  $L(n)$  is a slowly varying function in the sense of Karamata [5], such that  $L(n + [n/L(n)]) \geq L(n)$  for all  $n$  greater than some  $n_0$ , we can obtain a generalization of Proposition 3.1.

## REFERENCES

1. Časlav V. Stanojević, *Classes of  $L^1$ -convergence of Fourier and Fourier-Stieltjes series*, Proc. Amer. Math. Soc. **82** (1981), 209–215.
2. R. Bojanic and Č. V. Stanojević, *A class of  $L^1$ -convergence*, Trans. Amer. Math. Soc. (to appear).
3. R. R. Goldberg and Č. V. Stanojević,  *$L^1$ -convergence and Segal algebras of Fourier series*, preprint (1980).
4. J. Karamata, *Teorija i praksa Stieltjes-ova integrala*, Srpska Akademija Nauka, Beograd, 1949.
5. ———, *Sur un mode de croissance régulière des fonctions*, Mathematica (Cluj) **4** (1930), 38–53.
6. G. H. Hardy, *Theorems relating to the convergence and summability of slowly oscillating series*, Proc. London Math. Soc. Ser. 2 **8** (1909).

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