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ANALOGUES OF THE DENJOY-YOUNG-SAKS THEOREM

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ABSTRACT. In this paper, an analogue of the Denjoy-Young-Saks theorem concerning the almost everywhere classification of the Dini derivates of an arbitrary real function is established in both the case where the exceptional set is of first category and the case where it is σ-porous. Examples are given to indicate the sharpness of these results.

1. Introduction. Throughout this paper, S denotes an arbitrary subset of the real line R; and f denotes an arbitrary mapping of S into R.

For each x in S, the collection $\mathfrak{D}_f(x)$ of *bilateral derivates* of f at x consists of all extended real numbers β for which there exists a sequence $\{[s_n, t_n]\}$ of nondegenerate intervals with $x \in [s_n, t_n]$, $s_n, t_n \in S$, $[s_n, t_n] \to x$, and $[f(t_n) - f(s_n)]/[t_n - s_n] \to \beta$; by requiring further that $x = t_n$ [resp., $x = s_n$], we obtain the collection $\mathfrak{D}_f^-(x)$ [resp., $\mathfrak{D}_f^+(x)$] of *left* [resp., *right*] *derivates* of f at x; and, by removing the requirement that $x \in [s_n, t_n]$, we obtain the collection $\mathfrak{D}_f^*(x)$ of *strong derivates* of f at x.

The symbols $D_-f(x)$, $D^-f(x)$, $D_+f(x)$, $D^+f(x)$ denote the *Dini derivates* of f at x, where the first two [the last two] are the inf and sup of $\mathfrak{D}_f^-(x)$ [of $\mathfrak{D}_f^+(x)$]; the *extreme bilateral derivates* $\underline{D}f(x)$ and $\overline{D}f(x)$ of f at x are the inf and sup of $\mathfrak{D}_f(x)$; and, the *extreme strong derivates* $D_*f(x)$ and $\mathfrak{D}^*f(x)$ of f at x are the inf and sup of $\mathfrak{D}_f^*(x)$. (We use the conventions $\inf \varnothing = +\infty$ and $\sup \varnothing = -\infty$.) It is a simple exercise to show that

$$(1)\quad \underline{D}f(x) = \min\{D_-f(x), D_+f(x)\} \quad \text{and} \quad \overline{D}f(x) = \max\{D^-f(x), D^+f(x)\}.$$

If the four Dini derivates at x are equal, their common value is called the *derivative* of f at x and is denoted $f'(x)$; if the extreme strong derivates at x are equal, their common value is called the *strong derivative* of f at x and is denoted $f^*(x)$.

The following remarkable theorem was established in the early part of this century by A. Denjoy, G. C. Young and S. Saks. (See [**1**] for a discussion of this theorem and its consequences.)

DENJOY-YOUNG-SAKS THEOREM. $S = A \cup B \cup C \cup D \cup E$, *where*

$$A = \{f' \text{ exists and is finite}\},$$

$$B = \{-\infty = D_-f < D^-f = D_+f < D^+f = +\infty\},$$

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$C = \{-\infty = D_+f < D^+f = D_-f < D^-f = +\infty\}$,
$D = \{D_-f = D_+f = -\infty$ *and* $D^-f = D^+f = +\infty\}$, *and*
$E =$ *a set of* (*Lebesgue*) *measure* 0.

We shall prove the following analogues, give examples to indicate their sharpness, and point out some of their immediate consequences.

THEOREM 1. $S = A \cup B \cup C \cup E$, *where*

$A = \{D_-f = D_+f = D_*f$ *and* $D^-f = D^+f = D^*f\}$,
$B = \{-\infty = D_-f \leqslant D_+f \leqslant D^-f \leqslant D^+f = +\infty\}$,
$C = \{-\infty = D_+f \leqslant D_-f \leqslant D^+f \leqslant D^-f = +\infty\}$, *and*
$E =$ *a set of first* (*Baire*) *category*.

THEOREM 2. $S = A \cup B \cup C \cup E$, *where*

$A = \{D_-f = D_+f$ *and* $D^-f = D^+f\}$,
$B = \{-\infty = D_-f \leqslant D_+f \leqslant D^-f \leqslant D^+f = +\infty\}$,
$C = \{-\infty = D_+f \leqslant D_-f \leqslant D^+f \leqslant D^-f = +\infty\}$, *and*
$E =$ *a σ-porous set*.

Each of Theorems 1 and 2 will be shown to be a consequence of a respective theorem from cluster set theory concerning the angular boundary behavior of a real-valued function defined on a subset of a half-plane. We begin with a lemma that will be used in both proofs.

2. Key lemma. Set $H = \{(x, y): x \in R, y > 0\}$ and $T = \{(x, y) \in H: x + y, x - y \in S\}$. Then for $z = (x, y)$ define the function $F: T \to R$ by

$$F(z) = F(x, y) = \frac{f(x + y) - f(x - y)}{2y}.$$

If Ω is a subset of H and x is a point of R, the *cluster set* $C_\Omega(F, x)$ *of* F *at* x *relative to* Ω is the collection of all extended real numbers β for which there exists a sequence $z_1, z_2, \ldots$ of points in $\Omega \cap T$ with $z_n \to x$ and $F(z_n) \to \beta$. If l_θ is the ray that emanates from x and makes the angle θ ($0 < \theta < \pi$) with the positive real axis, then the symbol $C(F, x, \theta)$ is used in place of $C_{l_\theta}(F, x)$.

The region between two distinct rays in H that emanate from the point x in R is referred to as a Stolz angle at x. The symbol $\mathcal{K}(F)$ is used to denote the set of all points x in R at which the relations $C_{\Delta_1}(F, x) = C_{\Delta_2}(F, x) \neq \varnothing$ hold for each pair of Stolz angles Δ_1, Δ_2 at x.

If x is a point of S, then every point in T has a representation $(x + \alpha h, h)$ for some $\alpha \in R$ and some $h > 0$. Furthermore,

$$F(x + \alpha h, h) = \frac{f(x + [\alpha + 1]h) - f(x + [\alpha - 1]h)}{2h}. \tag{2}$$

By taking $\alpha = 1$ and -1 in (2), we see that

$$C(F, x, \pi/4) = \mathfrak{D}_f^+(x) \quad \text{and} \quad C(F, x, 3\pi/4) = \mathfrak{D}_f^-(x).$$

Also, for $\alpha \neq 1$ or -1, we will find it useful to write (2) in the form

$$F(x+\alpha h, h) = \frac{1+\alpha}{2}\cdot\frac{f(x+[\alpha+1]h)-f(x)}{[\alpha+1]h} + \frac{1-\alpha}{2}\cdot\frac{f(x+[\alpha-1]h)-f(x)}{[\alpha-1]h}. \tag{3}$$

LEMMA 1. *If $x \in \mathcal{K}(F)$, then the following statements are valid*:

(i) $\mathfrak{D}_f^-(x)$ *or* $\mathfrak{D}_f^+(x)$ *bounded implies* $D_-f(x) = D_+f(x)$ *and* $D_-f(x) = D^+f(x)$.

(ii) $\underline{D}f(x) \neq -\infty$ *or* $\overline{D}f(x) \neq +\infty$ *implies* $D_-f(x) = D_+f(x)$ *and* $D^-f(x) = D^+f(x)$.

PROOF OF (i). Suppose that $\mathfrak{D}_f^+(x)$ is bounded and $x \in \mathcal{K}(F)$. Let $\{z_n \equiv (x+\alpha_n h_n, h_n)\}$ be a sequence of points in the intersection of T and the Stolz angle

$$\Delta_\varepsilon \equiv \{(x+\alpha h, h): h > 0 \text{ and } 1 < \alpha < 1+\varepsilon\} \qquad (\varepsilon > 0)$$

with $h_n \to 0$, $\alpha_n \to \alpha \in [1, 1+\varepsilon]$, and $F(z_n) \to \beta$. Since $D_+f(x)$ and $D^+f(x)$ are finite and since $\alpha_n + 1 > 0$ and $\alpha_n - 1 > 0$ for each n, it follows from (3) that

$$\frac{[1+\alpha]}{2}D_+f(x) + \frac{[1-\alpha]}{2}D^+f(x) \leqslant \beta \leqslant \frac{[1+\alpha]}{2}D^+f(x) + \frac{[1-\alpha]}{2}D_+f(x).$$

Since $1 \leqslant \alpha \leqslant 1+\varepsilon$ and $D^+f(x) = D_+f(x) + \delta$ for some $\delta \geqslant 0$, it follows that

$$C_{\Delta_\varepsilon}(F, x) \subset [D_+f(x) - \delta\varepsilon/2, D^+f(x) + \delta\varepsilon/2].$$

Because ε is an arbitrary positive number and $x \in \mathcal{K}(F)$, for each Stolz angle Δ at x we have

$$C_\Delta(F, x) \subset [D_+f(x), D^+f(x)].$$

But $D_+f(x), D^+f(x) \in C(F, x, \pi/4)$, and hence

$$\inf C_\Delta(F, x) = D_+f(x) \quad \text{and} \quad \sup C_\Delta(F, x) = D^+f(x) \tag{4}$$

for each Stolz angle Δ at x.

Now, because of (4) and the equality $C(F, x, 3\pi/4) = \mathfrak{D}_f^-(x)$, the collection $\mathfrak{D}_f^-(x)$ is bounded, and an argument similar to the one given above with Δ_ε replaced by its reflection in the vertical at x yields

$$\inf C_\Delta(F, x) = D_-f(x) \quad \text{and} \quad \sup C_\Delta(F, x) = D^-f(x) \tag{5}$$

for each Stolz angle Δ at x. Then (4) and (5) imply

$$D_-f(x) = D_+f(x) \quad \text{and} \quad D^-f(x) = D^+f(x);$$

and, since it follows from our arguments that $\mathfrak{D}_f^-(x)$ is bounded if and only if $\mathfrak{D}_f^+(x)$ is bounded (for $x \in \mathcal{K}(F)$), the proof of (i) is complete.

PROOF OF (ii). Assume that x is in $\mathcal{K}(F)$ and that $\overline{D}f(x) \neq +\infty$. Since $\overline{D}f(x) = -\infty$ implies $f'(x) = -\infty$, we may assume that $\overline{D}f(x)$ is finite. Also, by (i) we need only consider the case when $D_-f(x) = D_+f(x) = -\infty$.

Suppose that

$$D^-f(x) < D^+f(x) - \delta \qquad (\delta > 0). \tag{6}$$

Let $\{z_n \equiv (x + \alpha_n h_n, h_n)\}$ be a sequence of points in the intersection of T and the Stolz angle

$$\Delta \equiv \{(x + \alpha h, h) \colon h > 0 \text{ and } -1/2 < \alpha < 1/2\}$$

with $h_n \to 0$, $\alpha_n \to \alpha \in [-1/2, 1/2]$, and $F(z_n) \to \beta$. Since $1 + \alpha_n > 1/2$ and $1 - \alpha_n > 1/2$ for each index, (3) and (6) yield the inequalities

$$\beta \leqslant ([1+\alpha]/2)D^+ f(x) + ([1-\alpha]/2)D^- f(x) < D^+ f(x) - \delta[1-\alpha]/2.$$

Hence, as $-1/2 \leqslant \alpha \leqslant 1/2$, we have $\beta < D^+ f(x) - \delta/4$; and consequently the supremum of $C_\Delta(F, x)$ is at most $D^+ f(x) - \delta/4$. But this contradicts $x \in \mathcal{K}(F)$, since $D^+ f(x) \in C(F, x, \pi/4)$. Thus we must have $D^- f(x) \geqslant D^+ f(x)$, and a similar argument will show that $D^+ f(x) \geqslant D^- f(x)$. Hence $D^- f(x) = D^+ f(x)$.

We have shown that for $x \in \mathcal{K}(F)$,

$$\overline{D}f(x) \neq +\infty \quad \text{implies } D_- f(x) = D_+ f(x) \quad \text{and} \quad D^- f(x) = D^+ f(x).$$

A similar proof yields the same implication for $\underline{D}f(x) \neq -\infty$, and (ii) is established.

From Lemma 1 we easily deduce the next lemma, from which Theorems 1 and 2 will readily follow.

LEMMA 2. *If $x \in \mathcal{K}(F)$, then either*
(i) $D_- f(x) = D_+ f(x)$ *and* $D^- f(x) = D^+ f(x)$,
(ii) $-\infty = D_- f(x) \leqslant D_+ f(x) \leqslant D^- f(x) \leqslant D^+ f(x) = +\infty$,
(iii) $-\infty = D_+ f(x) \leqslant D_- f(x) \leqslant D^+ f(x) \leqslant D^- f(x) = +\infty$, *or*
(iv) $[D_- f(x), D^- f(x)] \cap [D_+ f(x), D^+ f(x)] = \varnothing$.

In the proofs of Theorems 1 and 2 we shall use the fact that condition (iv) of Lemma 2 can be satisfied only at a countable set of points in S; this is a simple exercise, and we omit the proof.

3. Proof of Theorem 1. The following theorem was discovered independently by E. F. Collingwood [**2**], E. P. Dolženko [**3**], and P. Erdös and G. Piranian [**5**]. (Their result for a half-plane was stated for functions defined throughout the half-plane, but their proofs work as well as for functions defined only on a subset of the half-plane.)

THEOREM A. *If $\Omega \subset H$ and $G\colon \Omega \to R$ is arbitrary, then for all x, except for a set of first category, $C_\Omega(G, x) \neq \varnothing$ implies $C_\Delta(G, x) = C_\Omega(G, x)$ for each Stolz angle Δ at x.*

For $\Omega = T$ and $G = F$, Theorem A says that there exists a subset E of S of first category such that for each $x \in S - E$, we have $C_\Delta(F, x) = C_T(F, x)$ for each Stolz angle Δ at x. If $x \in S$ and $\overline{\Delta}_0 = \{(x + \alpha h, h) \colon h > 0 \text{ and } -1 \leqslant \alpha \leqslant 1\}$, then $C_{\overline{\Delta}_0}(F, x) = \mathfrak{D}_f(x)$; therefore, we deduce that $C_T(F, x) = \mathfrak{D}_f(x)$ for each $x \in S - E$. Since $C_T(F, x) = \mathfrak{D}_f^*(x)$ for each $x \in S$, it follows that $\underline{D}f(x) = D_* f(x)$ and $\overline{D}f(x) = D^* f(x)$ for each $x \in S - E$. Theorem 1 now follows from (1) and Lemma 2.

4. Proof of Theorem 2. The *porosity* of a set $E \subset R$ at the point $x \in R$ is the value

$$\limsup_{r \downarrow 0} \frac{l(x, r, E)}{r},$$

where $l(x, r, E)$ denotes the length of the largest open interval contained in the set $(x - r, x + r) \cap (R - E)$. The set E is *porous* if it has positive porosity at each of its points, and it is *σ-porous* if it is a countable union of porous sets. The notion of porosity is due to E. P. Dolženko [**4**], and L. Zajíček [**10**] has shown that the σ-porous sets form a proper subclass of the class of all sets that are both of first category and of measure 0.

Dolženko [**4**] proved the following result. (Again, his result was stated for functions defined throughout the half-plane, but his proof can be adapted to this setting.)

THEOREM B. *If $\Omega \subset H$ and G: $\Omega \to R$ is arbitrary, then for all x, except for a σ-porous set, $C_\Delta(G, x) \neq \varnothing$ for some Stolz angle Δ at x implies $x \in \mathcal{K}(G)$.*

Since $C(F, x, \pi/4) \neq \varnothing$ for all but countably many points x in S, an application of Theorem B with $\Omega = T$ and $G = F$ yields a subset E of S such that E is σ-porous and $S - E \subset \mathcal{K}(F)$. Theorem 2 now follows from Lemma 2.

5. Examples. We shall now give examples to indicate the sharpness of Theorems 1 and 2. The first example shows that it is not possible to make Theorem 1 or 2 better resemble the Denjoy-Young-Saks theorem by replacing the set A by the union of the two sets

$$A_1 = \{f' \text{ exists and is finite}\}, \quad \text{and}$$
$$A_2 = \{D_-f = D_+f = -\infty \text{ and } D^-f = D^+f = +\infty\}.$$

EXAMPLE 1. *There exists a strictly increasing continuous function f on* $[0, 1]$ *such that $f'(x) = +\infty$ for a residual set of points x in* $(0, 1)$.

PROOF. See for example [**8**, pp. 214–215].

It is also not possible to replace the set A in Theorem 1 or 2 by the union of the three sets

$$A_1 = \{f' \text{ exists}\},$$
$$A_2 = \{D_-f = D_+f = -\infty \text{ and } D^-f = D^+f\,(\text{finite})\}, \quad \text{and}$$
$$A_3 = \{D_-f = D_+f\,(\text{finite}) \text{ and } D^-f = D^+f = +\infty\}.$$

EXAMPLE 2. *There exists a strictly increasing continuous function g on* $[0, 1]$ *such that*

$$(7)\quad D_-g(x) = D_+g(x) = D_*g(x) = 0 \quad \textit{and} \quad D^-g(x) = D^+g(x) = D^*g(x) = 1$$

for a residual set of points x in $(0, 1)$.

PROOF. Let Z be any dense G_δ-subset of $(0, 1)$ that has measure zero. According to C. Goffman [**7**, Proof of Theorem 1], there exists a measurable set $M \subset (0, 1)$ such that the upper and lower densities of M at each $x \in Z$ are 1 and 0. (These densities are the largest and smallest of the numbers d for which there is a sequence $\{[a_n, b_n]\}$ of nondegenerate intervals with

$$x \in [a_n, b_n], [a_n, b_n] \to x, \quad \text{and} \quad \text{meas}(M \cap [a_n, b_n])/(b_n - a_n) \to d.)$$

Define $g(x)$ to be the measure of $M \cap (0, x)$. Clearly, $\underline{D}g(x) = 0$ and $\overline{D}g(x) = 1$ for each $x \in Z$. Because of (1) and Theorem 1, there exists a set E of first category such that (7) is satisfied for each $x \in Z - E$.

It is also not possible to replace the set A in Theorem 1 or 2 by the union of the two sets

$$A_1 = \{f' \text{ exists}\}, \quad \text{and}$$
$$A_2 = \{D_-f = D_+f \text{ (finite) and } D^-f = D^+f \text{ (finite)}\}.$$

EXAMPLE 3. *There exists a strictly increasing continuous function h on* $[0, 1]$ *and a number* $\tau \in (0, 1)$ *such that*

$$D_-h(x) = D_+h(x) = \tau \quad \text{and} \quad D^-h(x) = D^+h(x) = +\infty \tag{8}$$

for a residual set of points x in R.

PROOF. Let Z and g be the set and function described in the proof of Example 2, and define h_0 to be the inverse of g. Then h_0 maps $[0, \tau]$ onto $[0, 1]$, where $\tau = g(1) < 1$; furthermore, $\underline{D}h_0(x) = 1$ and $\overline{D}h_0(x) = +\infty$ for each x in $g(Z)$, which is a dense G_δ-subset of $(0, \tau)$. Therefore, if we set $h(x) = h_0(\tau x)$, then h is an automorphism of $[0, 1]$ with $\underline{D}h(x) = \tau$ and $\overline{D}h(x) = +\infty$ for each x in $W \equiv \{x: \tau x \in g(Z)\}$, which is a dense G_δ-subset of $(0, 1)$. By (1) and Theorem 1 it follows that there exists a set E of first category such that (8) is satisfied for each $x \in W - E$.

We shall now show that it is not possible to change some or all of the inequalities in the definition of the sets B and C in Theorem 1 or 2 to equalities.

EXAMPLE 4. *There exists a bounded Baire one function f on* $[0, 1]$ *such that*

$$-\infty = D_-f(x) < D_+f(x) < D^-f(x) < D^+f(x) = +\infty \tag{9}$$

for a residual set of points x in $(0, 1)$.

PROOF. Let Z, E and g be as described in the proof of Example 2.

For each positive integer n, choose $\delta_n \in (0, 1/n2^n)$ such that

$$|g(a) - g(b)| < 2^{-n-1} \quad \text{for } a, b \in [0, 1] \text{ and } |a - b| < \delta_n.$$

Then choose a partition $x_0^n, x_1^n, \ldots, x_{m_n}^n$ of $[0, 1]$ with $0 = x_0^n < x_1^n < \cdots < x_{m_n}^n = 1$ and $x_{j+1}^n - x_j^n < \delta_n$ for each $j = 0, 1, \ldots, m_n - 1$. Define the function g_n by $g_n(x_j^n) = 2^{-n}$ for $j = 0, 1, \ldots, m_n$ and $g(x) = 0$ otherwise. Then set $f = g + \sum_{n=1}^{\infty} g_n$, and set $[0, 1]^* = [0, 1] - \{x_j^n\}_{n \geq 1; 0 \leq j \leq m_n}$.

If $\hat{g} = g \mid [0, 1]^*$, then it follows from the continuity of g that $\mathfrak{D}_{\hat{g}}^-(x) = \mathfrak{D}_g^-(x)$ and $\mathfrak{D}_{\hat{g}}^+(x) = \mathfrak{D}_g^+(x)$ for each $x \in [0, 1]^*$. Consequently, for each $x \in [0, 1]^*$, the set $\mathfrak{D}_f^-(x)$ [resp., $\mathfrak{D}_f^+(x)$] is equal to the union of $\mathfrak{D}_g^-(x)$ [resp., $\mathfrak{D}_g^+(x)$] and the collection of all left [resp., right] derivates of f at x that correspond to sequences that converge to x through $[0, 1] - [0, 1]^*$ from the left [resp., right]. It follows that

$$D^-f(x) = 1 \quad \text{and} \quad D_+f(x) = 0 \quad \text{for each } x \in (Z - E) \cap [0, 1]^*. \tag{10}$$

Now suppose that $x \in [0, 1]^*$ and N is a positive integer. Let j be such that $x_j^N < x < x_{j+1}^N$. Then

$$f(x_{j+1}^N) - f(x) > 2^{-N} \quad \text{and} \quad x_{j+1}^N - x < \delta_N < 1/N2^N;$$

hence

$$\left[f(x_{j+1}^N) - f(x)\right] / \left[x_{j+1}^N - x\right] > N,$$

and it follows that

$$D^+f(x) = +\infty. \tag{11}$$

Also, $f(x_j^N) - f(x) > 2^{-N-1}$ and $0 > x_j^N - x > -1/N2^N$. That is,

$$\left[f(x_j^N) - f(x)\right]/\left[x_j^N - x\right] < -N/2;$$

and hence

$$D_-f(x) = -\infty. \tag{12}$$

Finally, from statements (10) to (12), we have

$$D_-f(x) = -\infty, \qquad D_+f(x) = 0, \qquad D^-f(x) = 1, \qquad D^+f(x) = +\infty$$

for each x in $(Z - E) \cap [0, 1]^*$, which is a residual subset of $(0, 1)$.

We note that no continuous function can satisfy condition (9) at more than a first category set of points, as C. J. Neugebauer **[9]** has shown that a continuous function f has both $D_-f = D_+f$ and $D^-f = D^+f$ for all but a first category set of points.

Our final example shows that the inequalities in either set B or set C in Theorem 1 or 2 cannot all be made strict.

EXAMPLE 5. *There exists a bounded Baire one function f on* $[0, 1]$ *such that*

$$-\infty = D_-f(x) < D_+f(x) < D^-f(x) = D^+f(x) = +\infty$$

for a residual set of points x in $(0, 1)$.

PROOF. This proof is the same as that of Example 4, except that the function h of Example 3 is used instead of the function g of Example 2.

6. Some consequences of Theorems 1 and 2. We close by listing three results that follow immediately from Theorems 1 and 2. The first is a recent result of Evans and Humke **[6]**.

THEOREM 3. *If f: $S \to R$ is monotone, then the set $\{D_-f \neq D_+f$ or $D^-f \neq D^+f\}$ is σ-porous.*

THEOREM 4. *If f: $S \to R$ is arbitrary and P denotes the set of points where either the left or right derivative of f exists and is finite, then $\{x \in P$: $f'(x)$ does not exist$\}$ is σ-porous and $\{x \in P$: $f^*(x)$ does not exist$\}$ is of first category.*

THEOREM 5. *If f: $S \to R$ is locally Lipschitz, then the set $\{D_-f \neq D_+f$ or $D^-f \neq D^+f\}$ is σ-porous.*

This last theorem does not remain valid when the words "locally Lipschitz" are replaced by "absolutely continuous", as Evans and Humke **[6]** have proved that, given any set K of first category and measure 0, there exists an absolutely continuous function f: $R \to R$ with $D^-f(x) \neq D^+f(x)$ for all x in K. (With bounded variation being the property of concern in **[6]**, Evans and Humke showed only that their function is BV; however, it is easy to see that R can be expressed as a countable union of sets on each of which their function satisfies Lusin's condition (N), and consequently that f is AC.)

ADDED IN PROOF. After this paper was accepted for publication, the proof of Theorem 2 in the case when S equals the whole real line appeared in L. Zajíček's paper *On the symmetry of Dini derivates of arbitrary functions*, Comment. Math. Univ. Carolinae **22** (1981).

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