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GENERATION OF FINITE GROUPS OF LIE TYPE

BY

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ABSTRACT. Let p be an odd prime and G a finite group of Lie type in characteristic other than p. Fix an elementary abelian p-subgroup of $\mathrm{Aut}(G)$. It is shown that in most cases G is generated by the centralizers of the maximal subgroups of E. Results are established concerning the notions of layer generation and balance, and the strongly p-embedded subgroups of $\mathrm{Aut}(G)$ are determined.

1. Introduction. Let p be an odd prime, G a finite group, and $E \leqslant \mathrm{Aut}(G)$. Suppose that E is an elementary abelian p-group and set $C_G^0(E) = \langle C_G(F)\colon F$ maximal in $E\rangle$. A very useful tool in the study of finite groups is the basic generation result: $C_G^0(E) = G$, provided the order of G is relatively prime to p. The order restriction on G is essential. For example, if $G = S_{2p}$ is the symmetric group on $2p$ letters and E a Sylow p-subgroup of G, then $C_G^0(E) < G$. In this paper we consider the case of G a finite group of Lie type and we show that generation usually holds. In addition, we prove results concerning the notions of layer generation, balance, and we determine the strongly p-embedded subgroups of $\mathrm{Aut}(G)$.

Let $G = G(q)$ be a finite simple group of Lie type defined over a field of characteristic r, and let $E \leqslant \mathrm{Aut}(G) = Y$ be an elementary abelian p-group with $r \neq p$, a prime. In order to state more useful results we consider multipliers and therefore introduce the following notation. Let $\tilde{G}$ denote an image of the universal covering group of G (see **[25]** and (13.1), (13.3) of **[23]**) and let $\tilde{G} \trianglelefteq \tilde{Y}$ be such that $\tilde{G}/Z(\tilde{G}) = G$ and $\tilde{Y}/Z(\tilde{G}) = Y$. Now let $\tilde{E}$ be a p-subgroup of $\tilde{Y}$ such that $\tilde{E}Z(\tilde{G})/Z(\tilde{G}) = E$. We will prove the following results.

THEOREM 1. *Assume that E is contained in the subgroup of* $\mathrm{Aut}(G)$ *generated by inner and diagonal automorphisms. Then either $C_{\tilde{G}}^0(\tilde{E}) = \tilde{G}$ or one of the following holds*:

(i) $p = 3$ *and* $G \cong PSp(n,2)'$, $PSU(n,2)$, *or* $O^{\varepsilon}(n,2)'$, $\varepsilon = (-1)^{n/2}$.

(ii) $p = 3$ *and* $G \cong G_2(2)'$, $F_4(2)$, $E_6(2)$, ${}^2E_6(2)$, $E_7(2)$, *or* $E_8(2)$.

(iii) $p = 3$ *and* $G \cong PSL(3^k, 4)$, ${}^2F_4(2)'$, $F_4(4)$, $E_6(4)$ *or* $E_8(4)$.

(iv) $p = 5$ *and* $G \cong {}^2F_4(2)'$.

Moreover, in each of the cases (i)–(iv) *there exists a triple* $(\tilde{G}, \tilde{E}, p)$ *for which* $C_{\tilde{G}}^0(\tilde{E}) < \tilde{G}$.

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THEOREM 2. *Let* G_1 *be the subgroup of* $\mathrm{Aut}(G)$ *generated by inner and diagonal automorphisms of* G. *Assume that* $E \nleq G_1$. *Then either* $C_{\tilde{G}}^0(\tilde{E}) = \tilde{G}$, *or one of the following holds*:

(i) *If* $e \in E - G_1$, *then e induces a field automorphism on G and the pair* $(O^{r'}(C_G(e)), E \cap G_1)$ *satisfies one of the conditions* (i), (ii), *or* (iv) *of Theorem* 1.

(ii) $p = 5$ *and* $GE \cong \mathrm{Aut}(Sz(2^5))$.

(iii) $p = 3$ *and* $GE \cong \mathrm{Aut}(L_2(8))$.

(iv) $p = 3$ *and* $GE \cong D_4(2)\langle\sigma\rangle$, $D_4(4)\langle\sigma\rangle$, $O^{3'}(\mathrm{Aut}(D_4(8)))$, ${}^3D_4(2)\langle\sigma\rangle$, *where in each case* σ *is a graph automorphism of order* 3.

In each of the cases (i)–(iv), *there exists a triple* $(G, \tilde{E}, p)$ *for which* $C_{\tilde{G}}^0(\tilde{E}) < \tilde{G}$.

An immediate corollary to Theorems 1 and 2 is the following.

THEOREM 3. *If* $p \geqslant 7$ *or if* $r > 2$, *then* $C_G^0(E) = G$.

The above results can be used to obtain results on "layer generation" and "balance". With the above notation let $C_{\tilde{G}}^{\xi}(\tilde{E}) = \langle O^{r'}(C_{\tilde{G}}(F))\colon F \text{ maximal in } \tilde{E}\rangle$. As we will see in §2, $O^{r'}(C_{\tilde{G}}(F)) = E(C_{\tilde{G}}(F))$, if $q \geqslant 4$.

THEOREM 4 (LAYER GENERATION). *If* $E \cong Z_p \times Z_p$, *then one of the following holds*:

(i) $C_{\tilde{G}}^0(\tilde{E}) < \tilde{G}$.

(ii) $C_{\tilde{G}}^{\xi}(\tilde{E}) = \tilde{G}$.

(iii) $p \mid q - 1$, $G \cong L_p(q)$, $E \leqslant G_1$ (*as in Theorem* 2), *and* $\tilde{E}$ *is nonabelian*.

(iv) $p \mid q + 1$, $G \cong U_p(q)$, $E \leqslant G_1$, *and* $\tilde{E}$ *is nonabelian*.

THEOREM 5 (BALANCE). *If* $E \cong Z_p \times Z_p$, *then* $\bigcap_{e \in E^\#} O_{p'}(C_G(e)) = 1$ *and one of the following holds*:

(i) $\bigcap_{e \in E^\#} O_{p'}(C_X(e)) = 1$, *whenever* $G \leqslant X \leqslant \mathrm{Aut}(G)$.

(ii) $p \mid q - 1$, $G \cong L_p(q)$, $E \leqslant G_1$ (*as in Theorem* 2), *and* $\tilde{E}$ *is nonabelian*.

(iii) $p \mid q + 1$, $G \cong U_p(q)$, $E \leqslant G_1$, *and* $\tilde{E}$ *is nonabelian*.

THEOREM 6. *Assume* $p \geqslant 7$ *and* $(p, |\overline{W}|) = 1$, *where* $\overline{W}$ *is the Weyl group of the associated algebraic group of G. Then* $C_{\tilde{G}}^{\xi}(\tilde{E}) = \tilde{G}$.

THEOREM 7. *Let p be an arbitrary prime and assume that* $X \leqslant \mathrm{Aut}(G)$ *is a proper strongly p-embedded subgroup of GX. If* $m_p(X) \geqslant 2$, *then X normalizes a Sylow p-subgroup of G and one of the following holds*:

(i) $p = 3$ *and* $G \cong L_3(4)$;

(ii) $p = 5$ *and* $G \cong Sz(2^5)$ *or* ${}^2F_4(2)'$; *or*

(iii) $G \cong L_2(q)$, $U_3(q)$, $Sz(q)$, *or* ${}^2G_2(q)'$, *and q is a power of p*.

Much of the paper is concerned with the proof of Theorem 1, which is carried out in §§4 and 5. Theorem 2 is proved in §6 and the remaining results are proved in §7. A considerable amount of work is involved in describing examples where generation fails to hold. This contributes substantially to the length of the paper.

In many cases it is easy to prove that generation holds. For example, if $\tilde{E}$ normalizes a proper r-subgroup of $\tilde{G}$, then $\tilde{G} = C_{\tilde{G}}^{\xi}(\tilde{E}) = C_{\tilde{G}}^0(\tilde{E})$ (see (2.3)). Also when $m_p(E) \geqslant 3$ and p does not divide the order of the Weyl group of the overlying algebraic group, then, inductively, the situation is well behaved (see (2.3), (2.5), (2.6), and (2.7)(iv)). The more difficult cases are when p is small or when p is large with $m_p(E) = 2$. When generation fails to hold the group E typically has large p-rank, though not necessarily equal to the p-rank of G.

We make the following additional remarks concerning the proof of Theorem 1. The classical groups are handled using the standard module together with elementary facts on generation. For the exceptional groups we use more machinery. We introduce the algebraic group, $\bar{G}$, giving rise to G in order to get information on centralizers. In fact, we pass to a universal group where centralizers are better behaved. In §3 we use $\bar{G}$ to prove the existence of certain subgroups of G that contain E. The tool for this is Lang's theorem, and as a result we obtain many of the embeddings that Stensholt gets in **[26]**. However we need a bit more than his results give. The generation is then proved by using induction and classification theorems of Aschbacher **[2]** and Timmesfeld **[27]**.

For the most part our notation is standard. E_{p^n} denotes the elementary abelian group of order p^n. For X a group, let $E(X)$ denote the join of all quasisimple subnormal subgroups of X. We also make use of the following abuse of notation. Let G be a group with $X \subseteq G$ and $Y, Z \leqslant G$. By $X^Y \cap Z$ we mean the set of Y-conjugates of X contained in Z. The term Chevalley group refers to a group of Lie type generated by its root subgroups.

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2. Notation and preliminary lemmas. We will use the following notation. Let $G = G(q)$ be a simple Chevalley group (normal or twisted) defined over a field $\mathbf{F}_q$ of characteristic r, and let $\bar{G}$ be the overlying algebraic group defined over the algebraic closure, K, of $\mathbf{F}_r$. Then $G = O^{r'}(\bar{G}_\sigma)$ for σ an endomorphism of $\bar{G}$ satisfying $|\bar{G}_\sigma| < \infty$. Fix a prime $p \neq r$ and a subgroup $E \cong E_{p^n}$, for $n > 1$, of $\bar{G}_\sigma$. So E is in the subgroup of $\mathrm{Aut}(G)$ generated by inner and diagonal automorphisms.

As centralizers are better behaved in universal covering groups we introduce the following groups. $\hat{G}$ denotes the universal covering group of $\bar{G}$ (see **[25]**). Then σ induces an endomorphism of $\hat{G}$, again called σ, and $G_0 = \hat{G}_\sigma$ is a central extension of G. Let E_0 denote the Sylow p-subgroup of the preimage in $\hat{G}$ of E. Notice that E_0 may not be abelian and E_0 need not be in $\hat{G}_\sigma = G_0$.

If Y acts on a group X let $C_X^0(Y) = \langle C_X(Y_1)\colon Y_1 \text{ maximal in } Y\rangle$ and $C_X^r(Y) = \langle O^{r'}(C_X(Y_1))\colon Y_1 \text{ maximal in } Y\rangle$. Set $D = C_G^0(E)$ and $D_0 = C_{G_0}^0(E_0)$. The first observation is

(2.1) (i) $D_0 = G_0$ implies $D = G$.

(ii) $C_{G_0}^r(E_0) = G_0$ implies $C_G^r(E) = G$.

PROOF. This follows easily by taking homomorphic images.

In view of (2.1) we work primarily with G_0. The (B, N)-pair notation for G_0 will be standard. Let B_0 be a fixed Borel subgroup of G_0 with $B_0 = UH_0$, where H_0 is a Cartan subgroup and $U = O_r(B_0)$. The image of B_0 in G is B. Notice that $U \cong UZ(G_0)/Z(G_0)$. Σ denotes a root system for G_0, so $U = \prod_{\alpha \in \Sigma^+} U_\alpha$, a product of root subgroups. Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be a fundamental set of roots for Σ, let

$N_0/H_0 = W$, the Weyl group of G_0, and choose fundamental reflections $s_i \in \langle U_{\alpha_i}, U_{-\alpha_i}\rangle$ for $i = 1,\dots,n$ in the usual way.

In the group $\hat{G}$ we use similar notation, getting subgroups $\hat{B}$, $\hat{H}$, $\hat{U}$, $\hat{N}$, $\hat{U}_\alpha$, etc. The root system $\hat{\Sigma}$ may differ from that of Σ and we will not always assume that $B_0 \leqslant \hat{B}$ or that $\hat{B}$ is σ-invariant. Similarly, for $\overline{G}$.

If P is a parabolic subgroup of G_0 with $B_0 \leqslant P < G_0$ we define the opposite parabolic subgroup, P_1, as follows. Write $P = O_r(P)L$, where L is the Levi factor of P and $O_r(P)$ is a product of root subgroups for positive roots. Then $P_1 = O_r(P_1)L$, where $O_r(P_1)$ is the product of those root subgroups U_α, such that $U_{-\alpha} \leqslant O_r(P)$.

(2.2) Let $B_0 \leqslant P < G_0$ be a proper parabolic subgroup of G_0 and let P_1 be the opposite parabolic subgroup. Then $G_0 = \langle O_r(P), O_r(P_1)\rangle$.

PROOF. Say $P = \langle B_0, s_{i_1},\dots,s_{i_k}\rangle$. Then $P = O_r(P)L$ where

$$L = H_0\langle U_{\pm\alpha_i}\colon i = i_1,\dots,i_k\rangle.$$

Moreover $O_r(P)$ is the product of those root subgroups $U_\alpha \leqslant U$ such that the coefficient of α_i in α is positive for some $i \in \{1,\dots,n\} - \{i_1,\dots,i_k\}$. Also $P_1 = O_r(P_1)L$ with $O_r(P_1)$ the product of the root subgroups $U_{-\alpha}$ for roots $\alpha \in \Sigma$ with $U_\alpha \leqslant O_r(P)$.

Let $X = \langle O_r(P), O_r(P_1)\rangle$. Then $L \leqslant N(X)$ and, for $i \in \{1,\dots,n\} - \{i_1,\dots,i_k\}$, $\langle U_{\pm\alpha_i}\rangle \leqslant X \leqslant N(X)$. As $G_0 = \langle U_{\pm\alpha_i}\colon i = 1,\dots,n\rangle$, we have $X \trianglelefteq G_0$ and so $X = G_0$.

(2.3) Let Y be a noncyclic elementary abelian p-subgroup in $\mathrm{Aut}(G_0)$ and suppose Y normalizes a proper parabolic subgroup of G_0. Then $C^r_{G_0}(Y) = G_0$. In particular $C^0_{G_0}(Y) = G_0$.

PROOF. Suppose Y normalizes $P_0 \geqslant B_0$. Then $C^r_{G_0}(Y) \geqslant O_r(P_0)$. By (2.2) it will suffice to show that, for some $g \in P_0$, Y^g normalizes P_1, where P_1 is opposite to P_0. For then $C^r_{G_0}(Y^g) = G_0$ and we have the result. Consider $GY \leqslant \mathrm{Aut}(G_0)$. Let $P = P_0/Z(G_0)$, so $Y \leqslant N(P)$. Now $N(P) \cap GY = O_r(P)\overline{L}$, where $\overline{L} \cap G$ is the Levi factor, L, of P and $\overline{L}$ is the extension of L by certain combinations of diagonal, field, and graph automorphisms of G. It is easily verified that $\overline{L}$ stabilizes $P_1/Z(G_0)$. Since $p \neq r$ we have Y conjugate in P to a subgroup of $\overline{L}$, as needed.

Next we give several general results on centralizers of semisimple elements. What is involved is the statement of known results about centralizers in the algebraic group and the restrictions to the finite groups.

(2.4) (i) Let x be a semisimple element in $\hat{G}$. Then $C_{\hat{G}}(x) = C_{\hat{G}}(x)^0$.

(ii) Let $x \in \overline{G}$ be semisimple. Then $C_{\overline{G}}(x)/C_{\overline{G}}(x)^0$ is isomorphic to a subgroup of $Z(\hat{G})$.

(iii) Let $x \in \overline{G}$ be semisimple and suppose x has finite order with $(|x|, |Z(\hat{G})|) = 1$. Then $C_{\overline{G}}(x) = C_{\overline{G}}(x)^0$.

PROOF. (i) and (ii) follow from (4.4) on p. 204 of [**22**], while (iii) follows from (4.5) (again p. 204 of [**22**]).

(2.5) Let $x \in G_0$ (or, $x \in G$ with $(|x|, |Z(\hat{G})|) = 1$) be semisimple. Then $C_{\hat{G}}(x)$ (respectively, $C_{\overline{G}}(x)$) is the central product of $E(C_{\hat{G}}(x))$ (respectively, $E(C_{\overline{G}}(x))$) and a torus. Also, $C_{G_0}(x)$ (respectively, $C_G(x)$) contains a normal subgroup, X, where X is a central product of Chevalley groups (normal or twisted) with an abelian

r'-subgroup. $C_{G_0}(x)$ (respectively, $C_G(x)$) normalizes each of the factors of X and induces inner and diagonal automorphisms on each factor. In particular, $O^{r'}(X) = O^{r'}(C_{G_0}(x))$.

PROOF. Consider $C_{\hat{G}}(x)_\sigma$ (respectively $C_{\bar{G}}(x)_\sigma$) and apply (2.4).

(2.6) Let x and X be as in (2.5).

(i) $O_{r'}(C_{G_0}(x)) \leqslant Z(C_{G_0}(x))$ (respectively $O_{r'}(C_G(x)) \leqslant Z(C_G(x))$).

(ii) $O^{r'}(X)$ is a central product of Chevalley groups each defined over some extension field of $\mathbf{F}_q$.

PROOF. (i) is clear from (2.5). For (ii) assume that L is a component of $C_{\hat{G}}(x)$ (respectively $C_{\bar{G}}(x)$) and let $\{L = L_1, \ldots, L_k\}$ be the orbit of L under $\langle\sigma\rangle$. Then $\sigma^k = \delta$ normalizes each of $L_1, \ldots, L_k$. It will suffice if we can show that $O^{r'}(L_\delta)$ is a Chevalley group (normal or twisted) defined over $\mathbf{F}_{q^k}$. Suppose $Y = Y^\delta$ for a root subgroup, Y, of $\hat{G}$ with $Y \leqslant L$. It will suffice, in this case, to show $|Y_\delta| = q^k$. Say $Y = \hat{U}_\alpha^g$, where $\alpha \in \hat{\Sigma}$ and $g \in \hat{G}$. Then $g\delta g^{-1} \in N(\hat{U}_\alpha) \cap \hat{G}\delta$. As $\hat{G}_\sigma$ is defined over $\mathbf{F}_q$, $\hat{G}_\delta$ is defined over $\mathbf{F}_{q^k}$ and there is a conjugate τ of δ with $\tau \in N(\hat{U}_\alpha)$ and $|(\hat{U}_\alpha)_\tau| = q^k$. We have $g\delta g^{-1} \in \hat{G}\tau$, say $g\delta g^{-1} = y\tau$ for $y \in \hat{G}$. Then $y \in N_{\hat{G}}(\hat{U}_\alpha)$, so y acts on $\hat{U}_\alpha$ as multiplication by a scalar. It now follows from (11.2) of **[23]** that $|\hat{U}_\alpha \cap C(y\tau)| = q^k$, and so $|Y_\delta| = q^k$, as needed. If no such Y exists, then $\hat{G} = F_4(K)$, $Sp(4, K)$, or $G_2(K)$ and δ^2 fixes a root subgroup of L. Here σ^2 is a field automorphism of $\hat{G}$, with $\hat{G}_{\sigma^2}$ defined over $\mathbf{F}_q$ and σ a graph automorphism of $\hat{G}_{\sigma^2}$. Argue as above with δ^2 and σ^2, and the result follows.

(2.7) (i) E_0 and E normalize maximal tori of $\hat{G}$ and $\bar{G}$, respectively.

(ii) If $p \nmid |\hat{W}|$, then E_0 is contained in a maximal torus of $\hat{G}$. Similarly for $E \leqslant \bar{G}$.

(iii) If $(p, |Z(\hat{G})|) = 1$ and if $E \geqslant F \cong Z_p \times Z_p$, then F is contained in a maximal torus of $\bar{G}$.

(iv) If E contains a subgroup $F \cong E_{p^m}$ for $m > 1$ with F in a maximal torus of $\bar{G}$, then some maximal subgroup of F is contained in a proper parabolic subgroup of $\bar{G}_\sigma$.

(v) If $\alpha \in \hat{\Sigma}$ is a long root and $g \in \hat{G}$ such that $\hat{U}_\alpha^{g\sigma} = \hat{U}_\alpha^g$, then $(\hat{U}_\alpha^g)_\sigma$ is a long root subgroup of G_0.

PROOF. Consider $E \leqslant \bar{G}$. The reference for (i), (ii) and (iii) is Springer-Steinberg **[22]**. Namely, (i) is a consequence of (5.17), (ii) follows from (i), and (iii) follows from (5.1). To get (iv) assume that F is as given with $F \leqslant \bar{H}$. Then F normalizes each root subgroup $\bar{U}_\alpha$ for $\alpha \in \bar{\Sigma}$, inducing a cyclic group on $\bar{U}_\alpha$ (as $\bar{H}$ induces a 1-dimensional torus on $\bar{U}_\alpha$). So there is a root $\alpha \in \bar{\Sigma}^+$ and hyperplane $F_0 < F$ with $F_0 \leqslant C_{\bar{G}}(\bar{U}_\alpha)$. Then $F_0 \leqslant C(\langle \bar{U}_{\pm\alpha}\rangle)$ and $C_{\bar{G}}(F_0)$ contains a nontrivial component. Since σ normalizes $C_{\bar{G}}(F_0)$, $C_{\bar{G}}(F_0)_\sigma$ contains a nontrivial r-subgroup, say I. Then $F_0 \leqslant C_G(I)$ which is contained in a proper parabolic subgroup of $\bar{G}_\sigma$ (Borel-Tits **[6]**). (v) follows from Lang's theorem ((10.1) of **[23]**) and the fact that $N_{\hat{G}}(\hat{U}_\alpha)$ is connected.

The next result contains the application of the work of Aschbacher **[2]** and Timmesfeld **[27]**.

(2.8) Assume $G \not\cong {}^2F_4(q)'$ and $(p, q) \neq (3, 2)$. Let V be the center of a long root subgroup of G_0 with $V \leqslant D_0$. Set $X = \langle V^{G_0} \cap D_0\rangle$. Then one of the following holds:

(i) $И(r, E_0) \neq \{1\}$ and $D_0 = G_0$.

(ii) $X/Z(X)$ is a direct product of groups of Lie type each defined over $\mathbf{F}_q$, and each component of X is generated by conjugates of V. If X_1 is a component of X, then $V^G \cap X_1$ consists of the centers of long root subgroups of X_1 (or possibly short root subgroups if $X_1/Z(X_1) \cong PSp(4, 2^b)$, $F_4(2^b)$, or $G_2(3^b)$).

(iii) $r = 3$, $|V| = 3$, and setting $\bar{X} = X/O_2(X)$, $\bar{X}/Z(\bar{X})$ is as in (ii).

(iv) $r = 2$, $|V| = 2$, and setting $\bar{X} = X/O_3(X)$, $E(\bar{X}/Z(\bar{X}))$ is a central product $Y_1 \cdots Y_k$, where, for $i = 1, \ldots, k$, $Y_i \trianglelefteq \bar{X}$ and Y_i is either a Chevalley group over $\mathbf{F}_2$ (as in (ii)) or $E(Y_i)$ is one of the following: $PSO^{\pm}(n,2)'$, $PSO^{\pm}(n,3)'$, A_n, F_{22}, F_{23}, F'_{24}. Also, $\bar{X}/E(\bar{X})$ is solvable.

PROOF. Suppose (i) is false. By (2.3) we have $И(r, E_0) = \{1\}$. In particular $O_r(X) = 1$. Let $O_\infty(X)$ be the solvable radical of X and set $Y = O_\infty(X)V_1$, where $V_1 \in V^G \cap X$. For $g \in G_0$ arbitrary, $\langle V_1, V_1^g \rangle$ is either an r-group or isomorphic to $SL(2, q)$ (see (12.1) of [3]). Suppose $|V_1| > 3$. Then $\langle V_1, V_1^y \rangle$ is an r-group for each $y \in Y$, and we conclude that $V_1 \leqslant O_r(Y)$ (see [1]). But $O_r(O_\infty(X)) \leqslant O_r(X) = 1$ and hence $[V_1, O_\infty(X)] = 1$. We then conclude that $O_\infty(X) = Z(X)$ for $q = |V_1| > 3$.

Next suppose that $q = 2$ or 3. Then V_1 centralizes any chief factor of Y having order prime to 6. If s is a prime, then $O_{s',s}(Y)$ is the intersection of the centralizers in Y of all s-chief factors of Y. Therefore $V_1 \leqslant \bigcap_s O_{s',s}(Y) = Y_0$, where the intersection is taken over all primes greater than 3. If $X_0 = O_{\{2,3\}}(Y_0 \cap O_\infty(X))$ and $R \in \mathrm{Syl}_r(X_0)$, then $X = X_0 N_X(R)$. Since $p \neq 2, 3$, this contradicts $И(r, E_0) = \{1\}$ unless $R = 1$. Therefore $R = 1$ and $X_0 = O_3(X)$ or $O_2(X)$ according to whether $q = 2$ or 3. In addition, V_1 centralizes $O_\infty(X)/X_0$ for each $V_1 \in V^G \cap X$, and so X centralizes $O_\infty(X)/X_0$.

Let $Y = X/Z(X)$ if $|V| > 3$. If $|V| = 2$ or 3, then set $Y = \bar{X}/Z(\bar{X})$, where $\bar{X} = X/O_3(X)$ or $X/O_2(X)$, respectively. Let $I = F^*(Y)$, so that $I = E(Y)$ (by the above). Write $I = I_1 \cdots I_k$, a central product of components. For $V_1 \in V^G \cap X$, V_1 permutes $\{I_1, \ldots, I_k\}$, and the above property of V_1 forces $V_1 \leqslant N(I_j)$ for $j = 1, \ldots, k$. Therefore, $I_j \trianglelefteq Y$ for $j = 1, \ldots, k$.

Fix $j = 1, \ldots, k$ and choose $V_1 \in V^G \cap X$ such that $[I_j, V_1] \neq 1$. Let $\tilde{V}_1$ denote the image of V_1 in Y and set $J = I_j\tilde{V}_1$. Then $J'' = J' = I_j = F^*(J)$. If q is even we can apply Theorem 1 of [27] to determine the structure of J. If q is odd apply (4.2) of [29] to see that for $j \in J$ with $K = \langle \tilde{V}_1, \tilde{V}_1^j \rangle \cong SL_2(q)$, the class K^J satisfies Hypothesis Ω in [2]. Therefore, Theorem 1 of [2] applies and yields the structure of I_j (we note that $D_4(2)$ should be included in Theorem 1 of [2], but this is not a possibility here due to the fact that $K \cong SL_2(q)$).

We conclude from the above that either I_j is a group of Lie type in characteristic r and $\tilde{V}_1^{\#}$ consists of elements of long root subgroups of J (possibly short root subgroups if I_j is isomorphic to $PSp(2n, 2^b)$, $F_4(2^b)$, or $G_2(3^b)$) or $q = 2$ and one of the following holds:

(i) $J \cong O^{\pm}(n, 2^b)$ and $v \in \tilde{V}_1^{\#}$ is a transvection;

(ii) $J \cong O^{\pm}(n, s)$, $s = 3$ or 5, and $v \in \tilde{V}_1^{\#}$ is a reflection;

(iii) $J \cong S_n$ and $v \in \tilde{V}_1^{\#}$ is a transposition; or

(iv) $J \cong A_6, J_2, F_{22}, F_{23}$, or F_{24}.

If $q = 2$, then for $v \in \tilde{V}_1^{\#}$, r^J is a class such that for $v_1, v_2 \in v^J$, $|v_1 v_2| = 1, 2, 3$, or 4. So by the main results of Fischer **[11]** and Timmesfeld **[27]** we see that the cases J_2 and $O^{\pm}(n, 5)$ do not occur. At this stage we may assume that J is a group of Lie type in characteristic r and the elements of $\tilde{V}_1^{\#}$ are long root elements of J (possibly short root elements in the three exceptional cases).

We claim that $\tilde{V}_1$ is contained in the center of a root subgroup of J. This is trivial if $|\tilde{V}_1| = r$ or if J has Lie rank 1. So suppose $|\tilde{V}_1| > r$ and J has Lie rank at least 2. Since $|\tilde{V}_1| > r$, $Y = X/Z(X)$ and $V_1 \leqslant J^{(\infty)} = J_0$, a component of X. If the claim is false, then there exist elements $a, b \in V_1^{\#}$ and centers of long root subgroups, $U_1 \neq U_2$ of J_0, such that $a \in U_1$ and $b \in U_2$. Fix $g \in J_0$ such that $\langle U_1, U_1^g \rangle \cong SL_2(q_1)$, where $q_1 = |U_1|$. As $\langle a, a^g \rangle$ is not an r-group, $\langle V_1, V_1^g \rangle \cong SL_2(q)$, and so $\langle b, b^g \rangle$ is not an r-group. Also, $O^{r'}(C_G(\langle a, a^g \rangle)) = O^{r'}(C_G(\langle V_1, V_1^g \rangle)) = O^{r'}(C_G(\langle b, b^g \rangle))$; in each case the centralizer has the form $O^{r'}(L)$, where L is a Levi factor of $C_G(V_1)$. Similarly, $E = O^{r'}(C_{J_0}(\langle U_1, U_1^g \rangle)) = O^{r'}(C_{J_0}(\langle b, b^g \rangle)) = O^{r'}(C_{J_0}(\langle U_2, U_2^g \rangle))$. But $\langle U_1, U_1^g \rangle = O^{r'}(C_{J_0}(E)) = \langle U_2, U_2^g \rangle$. Since $[a, b] = 1$, we must have $U_1 = U_2$, a contradiction.

Finally, we must prove that $\tilde{V}_1$ is the center of a long root subgroup of J. If not, then the previous paragraph implies the existence of a group $U > \tilde{V}_1$ with U the center of a long root subgroup of J. Choose $j \in J$ such that $\langle U, U^j \rangle \cong SL_2(q_1)$, where $q \mid q_1$ and $q_1 = |U|$. If $r = 2$, then we see that for $a \in \tilde{V}_1^{\#}$, there exists $j_1 \in J$ such that $|aa^{j_1}| = q_1 + 1$. But this contradicts the fact that $\langle V_1, V_1^g \rangle$ is an r-group or $SL_2(q)$ for each $g \in G$. Suppose r is odd. Then Dickson's theorem ((2.8.4) of **[30]**) implies that for suitable $j_1 \in J$ either $\langle \tilde{V}_1, \tilde{V}_1^{j_1} \rangle \geqslant SL(2, q_1)$ or $q_1 = 9$ and $\langle \tilde{V}_1, \tilde{V}_1^{j_1} \rangle \cong SL_2(5)$. This is a contradiction.

(2.9) Let $E \leqslant F \leqslant G$ with F an abelian p-group, and let F_0 be the Sylow p-subgroup of the preimage in G_0 of F.

(a) $C_G^0(F) \leqslant C_G^0(E)$ and $C_{G_0}^0(F_0) \leqslant C_{G_0}^0(E_0)$.

(b) $C_G^r(F) \leqslant C_G^r(E)$ and $C_{G_0}^r(F_0) \leqslant C_{G_0}^r(E_0)$.

Proof. Let F_1 be maximal in F. Then either $E \leqslant F_1$ or $E \cap F_1$ is maximal in E. In either case $C_G(F_1) \leqslant C_G(E \cap F_1) \leqslant C_G(E_1)$ for some maximal subgroup E_1 of E. The lemma follows.

(2.10) Let $X = X_1 \cdots X_k \leqslant G_0$ be a product of pairwise commuting groups of Lie type each defined over a field of characteristic r. Let $Y \leqslant X$ be an elementary abelian p-group and $(p, |Z(X)|) = 1$. Suppose that Y projects onto a cyclic subgroup, Y_i, of $X_i/Z(X_i)$, for $i = 1, \ldots, k$. Then

(a) $X \leqslant C_{G_0}^0(Y)$.

(b) If r divides $|C_{X_i}(Y_i)|$ for some $i \in \{1, \ldots, k\}$, then $C_{G_0}^r(Y) = G_0$.

Proof. (a) follows by looking at suitable hyperplanes of Y–namely, the kernels of the projection maps. (b) follows from (2.3) and (3.12) of **[6]**.

(2.11) Suppose $E \cong Z_p \times Z_p$, and E_0 is contained in the maximal torus $\hat{H}$ of $\hat{G}$, where $\hat{B} = \hat{U}\hat{H}$ is a Borel subgroup of $\hat{G}$ and $\hat{U} = \hat{B}_u$ (unipotent radical). Then there exist at least three maximal subgroups of E_0 with nontrivial centralizers in $\hat{U}$.

Proof. Suppose false. Then clearly E_0 centralizes no element of $\hat{U}^{\#}$ and we may then write $E_0 = E_1 E_2$ where, for $i = 1, 2$, E_i contains $E_0 \cap Z(G_0)$ as a subgroup of index p and no element of $E_0 - (E_1 \cup E_2)$ centralizes an element of $\hat{U}^{\#}$. Let $\Delta_i = \{\alpha \in \hat{\Sigma}: E_i \leqslant C(\hat{U}_\alpha)\}$. As E_0 induces Z_p on each root subgroup $\hat{U}_\alpha$ for $\alpha \in \hat{\Sigma}$ we have $\hat{\Sigma} = \Delta_1 \cup \Delta_2$, a disjoint union. Also $\Delta_1 \neq \varnothing \neq \Delta_2$ for otherwise $E_i \leqslant C(\hat{G})$ for $i = 1$ or 2. Finally we observe that $\Delta_i = -\Delta_i$ for $i = 1, 2$.

We claim that $\alpha \in \Delta_1$ and $\beta \in \Delta_2$ implies $[\hat{U}_\alpha, \hat{U}_\beta] = 1$. Otherwise $\alpha + \beta \in \hat{\Sigma}$ and $\hat{U}_{\alpha+\beta} \leqslant [\hat{U}_\alpha, \hat{U}_\beta]$ (this follows from the commutator relations). Now fix $e \in E_1 - Z(\hat{G})$. Then e is trivial on $\hat{U}_\alpha$, nontrivial on $\hat{U}_\beta$, and hence nontrivial on $\hat{U}_{\alpha+\beta}$. Consequently $\alpha + \beta \notin \Delta_1$. Similarly $\alpha + \beta \notin \Delta_2$, a contradiction. This proves the claim.

Let $\hat{G}_i = \langle \hat{U}_\alpha: \alpha \in \Delta_i \rangle$ for $i = 1, 2$. Then $[\hat{G}_1, \hat{G}_2] = 1$, $\hat{G}_1\hat{G}_2 = \hat{G}$, and $\hat{G}_1 \cap \hat{G}_2 \leqslant Z(\hat{G})$. But this gives a direct decomposition of the simple group $\hat{G}/Z(\hat{G})$, which is impossible.

(2.12) Let T be any maximal torus of $\bar{G}$ with $T = T^\sigma$. Then $\bar{G}_\sigma = O^{r'}(\bar{G}_\sigma)T_\sigma$.

PROOF. The following is based on a suggestion of G. Lustig. Let π: $\hat{G} \to \bar{G}$ be the natural surjection and regard σ as an endomorphism of $\hat{G}$. Let $\hat{T}$ be the preimage in $\hat{G}$ of T and let $Z = Z(\hat{G})$. Then $\hat{T}$ is connected (see (3.2) and (8.2) of [**25**]). Write $X = \hat{G} \times_Z \hat{T}$ for the central product of $\hat{G}$ and $\hat{T}$ (amalgamation with respect to Z). Then σ induces an endomorphism of X, via $(\hat{g}, \hat{t})^\sigma = (\hat{g}^\sigma, \hat{t}^\sigma)$. Also $X = X^0$ and there is a well-defined homomorphism (of algebraic groups) given by $(\hat{g}, \hat{t})\theta = (\hat{g})\pi$. Then $\ker(\theta) = \{(1, \hat{t}): \hat{t} \in \hat{T}\}$. Now $\ker(\theta) \cong \hat{T}$ is connected, so Lang's theorem ([**18**] or (10.1) of [**23**]) implies that $(X_\sigma)\theta = \bar{G}_\sigma$.

Next we note that $(\hat{G}_\sigma)\pi = O^{r'}(\bar{G}_\sigma)$ (see (3.2) of [**25**] and (12.4) of [**23**]). Hence $(\hat{T}_\sigma)\pi = T_\sigma \cap O^{r'}(\bar{G}_\sigma)$. Let $g \in \bar{G}_\sigma - O^{r'}(\bar{G}_\sigma)$ and suppose that $(\hat{g}_1, \hat{t})\theta = g$ with $(\hat{g}_1, \hat{t}) \in X_\sigma$. Then, for some $z \in Z$, $\hat{g}_1^\sigma = \hat{g}_1 z^{-1}$ and $\hat{t}^\sigma = \hat{t}z$. As the cosets $gO^{r'}(\bar{G}_\sigma)$ vary, so does the element $z \in Z$. It follows that $|T_\sigma/(T_\sigma \cap O^{r'}(\bar{G}_\sigma))| = |\bar{G}_\sigma/O^{r'}(\bar{G}_\sigma)|$, proving the lemma.

(2.13) Let $G_1 = G_1^0$ be a quasisimple algebraic group over K and σ an endomorphism of G_1 with $|(G_1)_\sigma|$ finite. Let G_2 be the universal covering group of G_1 with $Z \leqslant Z(G_2)$ and $G_2/Z \cong G_1$. View σ as an endomorphism of G_2 (see (9.16) of [**23**]). Let $X = \{g \in G_2: g^\sigma = gz$ for some $z \in Z\}$. Then X is a group, $X \geqslant (G_2)_\sigma Z$, and $X/(G_2)_\sigma \cong Z$.

PROOF. Let $X = \{g \in G_2: g^\sigma = gz$ for some $z \in Z\}$. Clearly X is a subgroup containing Z and $(G_2)_\sigma$. The map $x \to x^{-1}x^\sigma$ is a homomorphism from X into Z. By Lang's theorem the map is surjective, so $X/(G_2)_\sigma \cong Z$.

(2.14) Let notation be as above and assume that G_1 is a simple group. Let π be the natural surjection from G_2 to G_1. Then $(G_1)_\sigma = X^\pi$ and $|(G_1)_\sigma: O^{r'}((G_1)_\sigma)| = |(Z)_\sigma|$.

PROOF. Clearly $X^\pi = (G_1)_\sigma$. Now set $X_0 = (G_2)_\sigma Z$. Then $X_0/(G_2)_\sigma \cong Z/Z_\sigma$, so $|X/X_0| = |Z_\sigma|$. As X_0 is the preimage under π of $O^{r'}((G_1)_\sigma)$, the result follows.

(2.15) Let G_1 be as in (2.14). Then $(G_1)_\sigma$ is the group $O^{r'}((G_1)_\sigma)$ together with all diagonal automorphisms. Moreover $(G_1)_\sigma/O^{r'}((G_1)_\sigma)$ is isomorphic to an image of Z of order equal to $|Z_\sigma|$. $(G_2)_\sigma$ is a universal covering group of $O^{r'}((G_1)_\sigma)$.

PROOF. For the last statement see 12.8 of [**23**]. For the second statement use (2.14) and (12.6) of [**23**]. The first statement follows from (12.3) and (11.6) of [**23**].

The following is useful in inductive situations and for various generalizations of the main theorems.

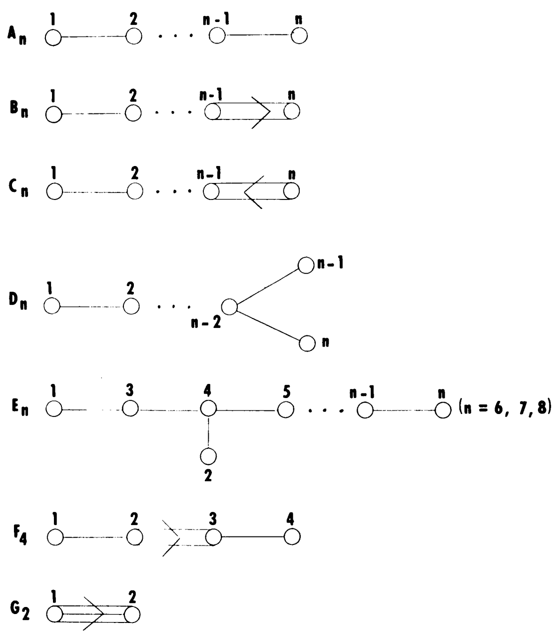
(2.16) Let $X = X_1 \cdots X_n$ be a central product of quasisimple groups and let E be an elementary abelian p-group acting on X. For each $i = 1,\ldots,n$ assume that $C^0_{X_i}(E_i) = X_i$, where E_i is the stabilizer in E of X_i. Then $C^0_X(E) = X$.

PROOF. We may assume that E is transitive on $\{X_1,\ldots,X_n\}$, so $E_i = E_j$ for each i, j and we set $F = E_1$. Clearly we may also assume $n > 1$. Let I be a maximal subgroup of E with $F \leqslant I$. Write $X = Y_1 \cdots Y_p$, where each Y_i is the product of the components of X in a fixed I-orbit. Inductively, $Y_i = C^0_{Y_i}(I)$ for $i = 1,\ldots,k$.

Fix $e \in E - I$ and let J be any maximal subgroup of I. Then for $y \in C_{Y_1}(J)$, we have $\prod_{i=0}^{p-1} y^{e^i}$ an element of $C_X(J\langle e\rangle)$. Do this for each maximal subgroup of I and conclude that $C^0_X(E)$ covers $C_{X/Z(X)}(e)$, so contains $C_X(e)'$. Now this happens for each $e \in E - I$. It follows that $C^0_X(E) = X$, as required.

Throughout this paper we will use the following labeling of Dynkin diagrams.

TABLE (2.17)



3. Subgroups of Chevalley groups. In this section we prove some results on generation of classical groups that will be used in §4. Also we establish the existence of certain subgroups of some exceptional groups of Lie type. These subgroups are similar to ones produced by Stensholt in [**26**]. However our methods are different, and yield a bit of additional information that we require in §5.

Let W be an $\mathbf{F}_q$-space with a nondegenerate alternating or hermitian bilinear form, or quadratic form in case W is an orthogonal space. Say X is a subspace of W with $W = X \oplus X^\perp$. Define $\mathcal{G}(X)$ to be the group of isometries of W generated by those root subgroups trivial on $X^\perp$. For convenience there will be two cases where we vary from this definition, namely, the case when W is a unitary space with $\dim(X) = 1$

and W an orthogonal space with $\dim(X) = 2$. In the first case we let $\mathcal{I}(X)$ be the group of all isometries of W that are trivial on $X^\perp$. So $\mathcal{I}(X) \cong Z_{q+1}$. In the second case $\mathcal{I}(X)$ will be the subgroup of $SO(W)$ that is trivial on $X^\perp$. So here $\mathcal{I}(X) \cong Z_{q\pm 1}$, depending on whether X is anisotropic or not.

(3.1) Assume $W = W_1 \perp W_2 \perp W_3$ and that W_2 contains nonzero isotropic (singular if W is an orthogonal space) vectors. Then either

(i) $\mathcal{I}(W) = \langle \mathcal{I}(W_1 \perp W_2), \mathcal{I}(W_2 \perp W_3)\rangle$, or

(ii) $\mathcal{I}(W) \cong SO^+(6,2) \cong A_8$ and $\langle \mathcal{I}(W_1 \perp W_2), \mathcal{I}(W_2 \perp W_3)\rangle \cong A_7$.

Proof. Set $X = \langle \mathcal{I}(W_1 \perp W_2), \mathcal{I}(W_2 \perp W_3)\rangle$. It is easy to see that X is transitive on each $\mathcal{I}(W)$ orbit of V. Also we may assume $W_1 \neq 0 \neq W_3$. The idea is to show that X is flag-transitive in the sense of **[21]**. Once this is achieved we will be done by applying the main theorem of **[21]**.

Suppose W_3 contains an isotropic (singular) 1-space W_0, and write $W_3 = W_4 \perp \overline{W_0}$, where $\overline{W_0}$ is a nondegenerate 2-space containing W_0. Let P be the stabilizer in $\mathcal{I}(W)$ of W_0. Then P is a parabolic subgroup of $\mathcal{I}(W)$ and the transitivity of X implies that $G = PX$. Also $P = O_r(P)LH$, where $L = \mathcal{I}(W_1 \perp W_2 \perp W_4)$ and H is a Cartan subgroup of $\mathcal{I}(W)$ normalizing L. We may take $O_r(P)H \leqslant B \leqslant P$, where B is a Borel subgroup of $\mathcal{I}(W)$.

In view of the main theorem of **[21]** it will suffice to show $BX = \mathcal{I}(W)$, and for this we only need $P = B(P \cap X)$. However, inductively we have $L = \mathcal{I}(W_1 \perp W_2 \perp W_4) = \langle \mathcal{I}(W_1 \perp W_2), \mathcal{I}(W_2 \perp W_4)\rangle$ or $L \cong A_8$ and $\langle \mathcal{I}(W_1 \perp W_2), \mathcal{I}(W_2 \perp W_4)\rangle \cong A_7$. In either case we have $P = B(P \cap X)$. So we may now assume that W_3, and by symmetry W_1, contains no isotropic (singular) 1-spaces.

In particular W is not a symplectic space. If W is a unitary space, then $\dim(W_1) = \dim(W_3) = 1$. In this case let W_4 be an isotropic 1-space in $W_1 \perp W_3$ and P the stabilizer in $\mathcal{I}(W)$ of W_4. Then $P = O_r(P)\mathcal{I}(W_2)H$ and we have $P = B(X \cap P)$ for a Borel subgroup of $\mathcal{I}(W)$. As above this gives the result.

Now we may assume W is an orthogonal space. Write $W_2 = W_2' \perp W_2''$, where W_2' is a 2-space containing a singular 1-space. Let $W_3' = W_2'' \perp W_3$ and consider the decomposition $W = W_1 \perp W_2' \perp W_3'$. By the above arguments we may assume that W_3' contains no singular 1-space. So $\dim(W_1)$ and $\dim(W_3')$ are each at most 2.

If $\dim(W_1) = 1$, then $W_1 = \langle v\rangle$ for some nonisotropic vector, v, of W. Therefore X contains $\mathcal{I}(W_2' \perp W_3')$ which is of index 2 in $C_{\mathcal{I}(W)}(v)$ and X is transitive on $v^{\mathcal{I}(W)}$. But then X has index at most 2 in $\mathcal{I}(W)$, proving $X = \mathcal{I}(W)$. So we may assume $\dim(W_1) = 2$, and similarly $\dim(W_3') = 2$.

If q is odd we may write $W_1 = W_{11} \perp W_{12}$ with $\dim(W_{11}) = \dim(W_{12}) = 1$. Then argue as above, using the fact that $\mathcal{I}(W_{12} \perp W_2' \perp W_3') \leqslant X$ by induction. So assume q is even. Let W_0 be a singular 1-space in W_2' and P the stabilizer in $\mathcal{I}(W)$ of W_0. Then $P = O_r(P)LH$, where $O_r(P)H$ is contained in a Borel subgroup $B \leqslant P$ and $L = \mathcal{I}(W_1 \perp W_3')$. But $X \geqslant \mathcal{I}(W_1) \times \mathcal{I}(W_3') \cong Z_{q+1} \times Z_{q+1}$ and we have $L = (L \cap B)(\mathcal{I}(W_1) \times \mathcal{I}(W_3'))$. So $BX = \mathcal{I}(W)$ and we use **[21]** to complete the proof.

(3.2) Let $V = V_1 \perp V_2$ be an $\mathbf{F}_q$-space with a nondegenerate alternating or hermitian form, or a quadratic form. Assume $\dim(V_1) \geqslant \dim(V_2) \geqslant 2$, that $\dim(V_1) \geqslant 4$ if V is an orthogonal space, $\dim(V_2) \geqslant 3$ if V is orthogonal and $q = 3$, and $\dim(V_2) \geqslant 4$ if V is an orthogonal space and q is even. Let $X = Sp(V)$, $SU(V)$, or $O^{\pm}(V)$, whichever is appropriate, and let $I \leqslant X$ be a subgroup containing $C_X(V_1)$ and $C_X(V_2)$, but not stabilizing $\{V_1, V_2\}$. Then one of the following holds:

(i) $I = X$;

(ii) $X \cong SU(4,2)$; or

(iii) $X \cong O^+(8,2)$.

PROOF. Suppose $X \not\cong SU(4,2)$ or $O^+(8,2)$. Let $A = C_X(V_2)$. If $X \cong O^-(8,2)$, we may assume $A \cong O^-(4,2)$. There exists $g \in I$ such that $A^g \nleqslant N_X(A)$, and, for such a group A^g, choose an element $c \in A^g - N(A)$ with c a transvection or reflection (in case q is odd and $X = O^{\pm}(V)$). Then set $V_0 = [\langle A, c\rangle, V]$. We have $V_1 < V_0$ and $\dim(V_0) = \dim(V_1) + 1$.

Suppose $\operatorname{rad}(V_0) \neq 0$. Then $\operatorname{rad}(V_0)$, being A-invariant, is contained in V_2 and $V_0 = V_1 \perp \operatorname{rad}(V_0)$. As $\operatorname{rad}(V_2) = 0$ there is a 2-space V_3 of V_2 such that $\operatorname{rad}(V_3) = 0$ and $\operatorname{rad}(V_0) \leqslant V_3$. If V is an orthogonal space with q even and if $\operatorname{rad}(V_0)$ is nonsingular, then choose V_3 so that V_3 contains no singular 1-spaces. Then $I \geqslant A \times \mathcal{I}(V_3)$ and we will show that $\mathcal{I}(V_1 \perp V_3) \leqslant I$. To see this let P be the stabilizer in $\mathcal{I}(V_1 \perp V_3)$ of $\operatorname{rad}(V_0)$ and $P_0 = C_P(\operatorname{rad}(V_0))$. For the moment exclude the case where V is orthogonal and q is even. Then $P_0 = O_r(P)O^{r'}(A)$ and acts irreducibly on $O_r(P)/O_r(Z(P_0))$ (see §3 of **[10]**). Since $c \in N(P_0)$ but $c \notin N(A)$ we have $\langle A, c\rangle$ covering $O_r(P) \bmod O_r(Z(P_0))$. The results in §3 of **[10]** imply that $O_r(Z(P_0)) = \Phi(O_r(P))$, unless X is a symplectic group in characteristic 2. In the latter case $O_r(P)$ is indecomposable as an A-module, unless $X \cong Sp(4,2)$, where the result is easy to check. So we may now assume that $\langle A, c\rangle \geqslant P_0$, and so $P_0 \leqslant I$. Also $I \geqslant C_X(V_1) \geqslant C_X(V_1) \cap C_X(V_3^{\perp})$ and P_0 together with this last group generates a group containing $\mathcal{I}(V_1 \perp V_3)$. To complete this case consider the decomposition $V_1 \perp V_3 \perp (V_3^{\perp} \cap V_2)$ and apply (3.1) to conclude that $\mathcal{I}(V) \leqslant I$. So $X = I$.

Now suppose that V is orthogonal and q is even. If $\operatorname{rad}(V_0)$ is singular the same arguments work, once we note that I contains a transvection that stabilizes V_3 but not $\operatorname{rad}(V_0)$. So suppose $\operatorname{rad}(V_0)$ is nonsingular. Then $\langle A, c\rangle \leqslant P_0 = C(\operatorname{rad}(V_0)) \cap \mathcal{I}(V_1 \perp V_3) \cong Sp(k,q)$, where $k = \dim(V_1)$. We note that $\langle A, c\rangle = P_0$. To see this note that $V_1 \cap V_1^c$ is a hyperplane in V_1, so $|A \cap A^c|$ can be computed. Then count the number of elements in AA^c and conclude from **[17]** that $P_0 = \langle A, c\rangle$. Also I contains $\mathcal{I}(V_3) \cong Z_{q+1}$ and we have $I \geqslant \langle P_0, \mathcal{I}(V_3)\rangle = \mathcal{I}(V_1 \perp V_3)$. At this point we can choose a transvection $c' \in I$ such that $c' \notin N(A)$ and $\operatorname{rad}(V_0')$ is singular, where $V_0' = [\langle A, c'\rangle, V]$. Namely, first write $V = V_1 \perp V_3 \perp V_4$, where $V_4 = V_2 \cap V_3^{\perp}$. Let c_1 be a transvection in $C(V_1)$ with $[V, c_1] = v_3 + v_4$, where $v_3 \in V_3$, $v_4 \in V_4$, and $0 \neq v_4$ is singular. Let $g \in \mathcal{I}(V_1 \perp V_3)$ with $v_3^g \in V_1$. Then $c' = c_1^g$ is such a transvection. Now argue as above to get the result.

Next we suppose $\operatorname{rad}(V_0) = 0$. Then X is neither a symplectic space nor an orthogonal space with q even. If V is an orthogonal space with $q = 3$, let U be a nondegenerate 3-space of V_2 with $U \geqslant V_0 \cap V_2$. Otherwise let U be a nondegenerate 2-space of V_2 with $U \geqslant V_0 \cap V_2$. Unless V is orthogonal and $\dim(V_2) = 2$ we may assume that U contains an isotropic 1-space. Inductively we may assume $U = V_2$. Indeed, write $V = V_1 \perp U \perp (V_2 \cap U^{\perp})$ and apply (3.1).

Next we write $V_1 = V_1' \perp V_1''$, where V_1'' has dimension at least 2 if V is a unitary space and at least 4 if V is an orthogonal space. Further, choose V_1'' so that $V_1'' \perp V_2 \geqslant [V, c]$. Now consider the decomposition $V = V_1' \perp V_1'' \perp V_2$ and apply induction together with (3.1). We conclude that one of the following occurs: $I = X$, $V_1' = 0$, or $X = SU(n, 2)$ and V_1'' is a 2-space. In the latter case, we can replace V_1'' by a proper nondegenerate 3-space of V_1 containing V_1'', if $n \geqslant 6$. Here, (3.1) and induction yield $I = X$. So from now on we may assume that either $X \cong SU(5, 2)$ or $V_1' = 0$ and $X \cong SU(4, q)$, $q > 2$, $O^{\pm}(6, q)$, $q > 3$, or $O^{\pm}(7, 3)$.

Suppose $X \cong SU(5, 2)$. Then $A \cong SU(3, 2)$. The space $V_1 \cap V_1^c$ is c-invariant, while $[V, c] \nleqslant V_1$. It follows that $V_1 \cap V_1^c$ is a 2-space in $C_{V_0}(c)$, hence nondegenerate. As $A \cap A^c$ stabilizes $V_1 \cap V_1^c$, $A \cap A^c \cong S_3 \times Z_3$. Thus, $\langle A, A^c \rangle \geqslant |AA^c| \geqslant 12|A|$. We claim that $\langle A, c \rangle = SU(4, 2)$. As $\langle A, c \rangle \leqslant \mathcal{I}(V_0)$, $\langle A, c \rangle \leqslant SU(4, 2)$. Elementary order considerations show that either the claim holds or $\langle A, c \rangle$ has index 10 in $\mathcal{I}(V_0) \cong PSp(4, 3)$, so in the latter case $\langle A, c \rangle$ would be contained in a proper parabolic subgroup of $PSp(4, 3)$. Checking orders we have a contradiction. Therefore, $\mathcal{I}(V_0) \leqslant I$. Choose $g \in \mathcal{I}(V_2)$ such that $V = V_0 + V_0^g$. Then $\mathcal{I}(V_0^g) \leqslant I$ and by (3.1) $I \geqslant \langle \mathcal{I}(V_0), \mathcal{I}(V_0^g) \rangle = \mathcal{I}(V)$. From now on we assume $V_1' = 0$.

Suppose $X \cong SU(4, q)$ with $q > 2$. As above it will suffice to show that $\mathcal{I}(V_0) \leqslant I$. But this follows from a check of subgroups of $SU(3, q)$ (see **[5]** and **[14]**). Now suppose $X \cong O^{\pm}(6, q)$ with $q > 3$ or $X \cong O^{\pm}(7, 3)$. Here $\mathcal{I}(V_0) \cong PSp(4, q)$, $V_0 = \langle V_1, V_1^c \rangle$, and $V_1 \cap V_1^c$ is a 3-space stabilized by the involution c. Set $L = \langle A, c \rangle$, so that $|L| \geqslant |AA^c| = |A| \cdot |A\colon A \cap A^c|$. Computing the possible choices for $A \cap A^c$ (which stabilizes the hyperplane $V_1 \cap V_1^c$ of V_1), we have $|\mathcal{I}(V_0)\langle c \rangle\colon L| \leqslant \frac{1}{2}q^2$. Apply the theorem of **[17]** to conclude that $L = \mathcal{I}(V_0)\langle c \rangle$.

If $X = O^{\pm}(6, q)$ with $q > 3$, then choose $g \in \mathcal{I}(U)$ such that $V_0 \cap U \neq V_0^g \cap U$. Then $I \geqslant \langle \mathcal{I}(V_0), g, c \rangle$, and arguing as above we have $I = X$. This leaves the case $X \cong O^{\pm}(7, 3)$. Let $\langle v_0 \rangle = V_0 \cap U$ and choose $g \in \mathcal{I}(U)$ such that $\langle v_0, v_0^g \rangle$ is nondegenerate but contains isotropic vectors. Then (3.1) and induction yield $\mathcal{I}(\langle V_1, v_0, v_0^g \rangle) \leqslant I$. A further application of (3.1) shows that $X = I$.

Let $G_0 = G_0(q)$ be an exceptional Chevalley group (notation as in §2). We will produce certain subgroups L of G_0. These subgroups are described below in Table (3.3). Here $L_0 = O^{r'}(L)$, $C = C_L(L_0)$, and $F = L/L_0C$. In all cases each component, X, of L_0 is normal in L, and $L/C_L(X)$ induces inner and diagonal automorphisms of X.

To prove the existence of L we consider $G_0 = \hat{G}_0$ and argue as follows. We choose a certain σ-invariant subgroup $\hat{L}$ of $\hat{G}$, which usually is generated by root subgroups of $\hat{G}$. Then we produce an element $\hat{w}$ of the Weyl group of $\hat{G}$ such that $\hat{w}$ stabilizes $\hat{L}$. By Lang's theorem (**[18]** or (10.1) of **[23]**) $\hat{w}\sigma$ and σ are conjugate in $\hat{G}$. Consequently G_0 contains a $\hat{G}$-conjugate of $\hat{L}_{w\sigma}$, and this is the appropriate subgroup L. Recall that root diagrams are labeled as in Table (2.17).

Choose a σ-invariant Borel subgroup, $\hat{B}$, of $\hat{G}$ so that the (B, N)-structure of G_0 is obtained in the usual way from that of $\hat{G}$. So $H_0 = \hat{T}_{\sigma}$, for $\hat{T} = \hat{T}^{\sigma}$ a maximal torus in $\hat{B}$, and σ acts on the root subgroups of $\hat{B}_u = U$ in the usual way.

TABLE (3.3)

G_0	$L_0/Z(L_0)$	C	$Z(L_0)$	F
$E_6(q)$	$L_3(q) \times L_3(q) \times L_3(q)$	$Z_{(3,q-1)} \times Z_{(3,q-1)}$	$Z_{(3,q-1)} \times Z_{(3,q-1)}$	$Z_{(3,q-1)}$
	$L_3(q^2) \times U_3(q)$	$Z_{(3,q^2-1)}$	$Z_{(3,q^2-1)}$	$Z_{(3,q^2-1)}$
$E_7(q)$	${}^3D_4(q) \times L_2(q^3)$	$Z_{(2,q-1)}$	$Z_{(2,q-1)}$	$Z_{(2,q-1)}$
	${}^2E_6(q)$	Z_{q+1}	$Z_{(3,q+1)}$	$Z_{(3,q+1)}$
	$U_3(q) \times U_6(q)$	$Z_{(3,q+1)}$	$Z_{(3,q+1)}$	$Z_{(3,q+1)}$
$E_8(q)$	$D_4(q) \times D_4(q)$	$Z_{(2,q-1)} \times Z_{(2,q-1)}$	$Z_{(2,q-1)} \times Z_{(2,q-1)}$	$Z_{(2,q-1)} \times Z_{(2,q-1)}$
	${}^3D_4(q) \times {}^3D_4(q)$	1	1	1
	$L_5(q) \times L_5(q)$	$Z_{(5,q-1)}$	$Z_{(5,q-1)}$	$Z_{(5,q-1)}$
	$U_5(q) \times U_5(q)$	$Z_{(5,q+1)}$	$Z_{(5,q+1)}$	$Z_{(5,q+1)}$
	$L_3(q) \times E_6(q)$	$Z_{(3,q-1)}$	$Z_{(3,q-1)}$	$Z_{(3,q-1)}$
	$U_3(q) \times {}^2E_6(q)$	$Z_{(3,q+1)}$	$Z_{(3,q+1)}$	$Z_{(3,q+1)}$
	$PSO^-(12,q)' \times L_2(q^2)$	$Z_{(2,q-1)}$	$Z_{(2,q-1)}$	$Z_{(2,q-1)}$
	$L_9(q)$	$Z_{(3,q-1)}$	Z_n, $n = \dfrac{(9,q-1)}{(3,q-1)}$	Z_n, $n = \dfrac{(9,q-1)}{(3,q-1)}$
	$U_9(q)$	$Z_{(3,q+1)}$	Z_n, $n = \dfrac{(9,q+1)}{(3,q+1)}$	Z_n, $n = \dfrac{(9,q+1)}{(3,q+1)}$
${}^2E_6(q)$	$U_3(q) \times U_3(q) \times U_3(q)$	$Z_{(3,q+1)} \times Z_{(3,q+1)}$	$Z_{(3,q+1)} \times Z_{(3,q+1)}$	$Z_{(3,q+1)}$
$F_4(q)$	$L_3(q) \times L_3(q)$	$Z_{(3,q-1)}$	$Z_{(3,q-1)}$	$Z_{(3,q-1)}$
	$U_3(q) \times U_3(q)$	$Z_{(3,q+1)}$	$Z_{(3,q+1)}$	$Z_{(3,q+1)}$
$G_2(q)$	$L_3(q)$	$Z_{(3,q-1)}$	$Z_{(3,q-1)}$	1
	$U_3(q)$	$Z_{(3,q+1)}$	$Z_{(3,q+1)}$	1
$D_4(q)$	$L_3(q)$	$Z_{q-1} \times Z_{q-1}$	$Z_{(3,q-1)}$	$Z_{(3,q-1)}$
	$U_3(q)$	$Z_{q+1} \times Z_{q+1}$	$Z_{(3,q+1)}$	$Z_{(3,q+1)}$
${}^3D_4(q)$	$L_3(q)$	$Z_{(q^2+q+1)}$	$Z_{(3,q-1)}$	$Z_{(3,q-1)}$
	$U_3(q)$	$Z_{(q^2-q+1)}$	$Z_{(3,q+1)}$	$Z_{(3,q+1)}$

Let $\pi = \{\alpha_1, \ldots, \alpha_n\}$ be a fundamental system of roots for W. For the cases E_6, E_7, E_8 we label the Dynkin diagram as follows:

$$\begin{array}{c}\alpha_1\alpha_3\alpha_4\alpha_5\alpha_6\\ \alpha_2\end{array} \qquad \begin{array}{c}\alpha_1\alpha_3\alpha_4\alpha_5\alpha_6\alpha_7\\ \alpha_2\end{array} \qquad \begin{array}{c}\alpha_1\alpha_3\alpha_4\alpha_5\alpha_6\alpha_7\alpha_8\\ \alpha_2\end{array}$$

In all cases $-s$ will denote the positive root of highest height (with respect to π). Listed below are various other fundamental systems for E_6, E_7, E_8:

E_6 π_1 $\begin{array}{c}\alpha_5\alpha_6 t\alpha_1\alpha_3,\\ s\end{array}$ where $t = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$

E_7 π_2 $\begin{array}{c}\alpha_6\alpha_5\alpha_4\alpha_3\alpha_1 s\\ \alpha_2\end{array}$

π_3 $\begin{array}{c}st\alpha_6\alpha_5\alpha_4\alpha_2,\\ \alpha_7\end{array}$ where $t = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5$

E_8 π_4 $\begin{array}{c}\alpha_6\alpha_5\alpha_4\alpha_2 t_1 t_2\alpha_8,\\ \alpha_3\end{array}$ where $t_1 = -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6)$ and $t_2 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$

π_5 $\begin{array}{c}\alpha_2\alpha_4\alpha_3 ts\alpha_8\alpha_7,\\ \alpha_1\end{array}$ where $t = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$

π_6 $\begin{array}{c}\alpha_6\alpha_5\alpha_4\alpha_3\alpha_1 ts,\\ \alpha_2\end{array}$ where $t = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$

$$\begin{array}{lll} \pi_7 & \begin{array}{c} s\alpha_8\alpha_7\alpha_6\alpha_5\alpha_4\alpha_3, \\ t \end{array} & \text{where } t = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 \\ \pi_8 & \begin{array}{c} t_1\alpha_2\alpha_4\alpha_5\alpha_6\alpha_7 t_2, \\ \alpha_3 \end{array} & \begin{array}{l} \text{where } t_1 = \alpha_1 + \alpha_3 + \cdots + \alpha_8, \text{ and} \\ t_2 = -(\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8) \end{array} \end{array}$$

Since the Weyl group is transitive on the set of all fundamental systems of roots, we may choose $w_i \in W$ such that $\pi^{w_i} = \pi_i$, $i = 1, \ldots, 8$. For $i > 1$, w_i necessarily preserves the labeling of the Dynkin diagram. Setting $w_1 = (s_4 s_1 s_6)^{s_3 s_5 s_4 s_2}$, we also have this for $i = 1$.

Let $G_0 = E_6(q)$. Then $\hat{G} = E_6(K)$ and σ is a field automorphism. Let $X_1 = \langle \hat{U}_{\pm\alpha_1}, \hat{U}_{\pm\alpha_3} \rangle$, $X_2 = \langle \hat{U}_{\pm\alpha_5}, \hat{U}_{\pm\alpha_6} \rangle$, and $X_3 = \langle \hat{U}_{\pm\alpha_2}, \hat{U}_{\pm s} \rangle$. Then $X_1 \cong X_2 \cong X_3 \cong SL(3, K)$ and $[X_1, X_2] = [X_2, X_3] = [X_1, X_3] = 1$. We note that $X_1 X_2 = X_1 \times X_2$. This can be seen by viewing $E_6(K) \leqslant E_7(K)$, then considering $X_1 X_2$ as a subgroup of $SL(7, K) \leqslant E_7(K)$. Conjugation within the Weyl group of G_0 yields $X_2 X_3 \sim X_1 X_3 \sim X_1 X_2 \cong X_1 \times X_2$. On the other hand, we claim that $Z(X_1 X_2 X_3) \cong Z_3 \times Z_3$ if $r \neq 3$. From the above we see that either this holds or $Z = Z(X_1 X_2 X_3) \cong Z_3 \times Z_3 \times Z_3$. If the latter holds consider $C_Z(\hat{U}_{\alpha_4}) = Z_0$. Then Z_0 also centralizes $\hat{U}_{-\alpha_4}$; hence $Z_0 \leqslant C(\langle U_{\pm\alpha_i} \mid i = 1, \ldots, 6 \rangle) = Z(\hat{G})$. However, $|Z_0| \geqslant 9$, whereas $|Z(\hat{G})| = 3$. This proves the claim and gives the group L_0 for the first case in $E_6(q)$. Namely, set $L_0 = (X_1)_\sigma (X_2)_\sigma (X_3)_\sigma$. Let $\langle z_i \rangle = Z(X_i)$ for $i = 1, 2, 3$. Notation may be chosen such that $z_1 z_2 z_3 = 1$. For each $i = 1, 2, 3$ we apply Lang's theorem ((10.1) of [**23**]) to obtain an element $a_i \in X_i$ such that $a_i^\sigma = a_i z_i$. Then $a^\sigma = a$, where $a = a_1 a_2 a_3$. Also $a \in N(L_0)$ (each $a_i \in N((X_i)_\sigma)$) and $a \notin L_0 Z(X_1 X_2 X_3)$ precisely if $3 \mid q - 1$. Let $L = L_0 \langle a \rangle$, L_0, or L_0 according to $q \equiv 1, -1$, or $0 \pmod 3$. Then L is the first of the groups for $E_6(q)$.

The element $w_1 \in W$ is an involution and it is clear that $X_1^{w_1} = X_2$, while $X_3^{w_1} = X_3$, and w_1 induces a graph automorphism on the Dynkin diagram of X_3. Let $g = w_1 \sigma$. Then $g \sim \sigma$ by Lang's theorem ((10.1) of [**23**]). We may take a coset representative of w_1 in $(G_0)_\sigma$, so $g^2 \in \sigma^2 H$ and another application of Lang's theorem yields $(X_i)_{g^2} \cong SL(3, q^2)$ for $i = 1, 2, 3$. Therefore, $(X_1 X_2)_g \cong SL(3, q^2)$. From the action of g on X_3 we have $(X_3)_g \cong SU(3, q)$ (see [**23**, (11.2) and (11.6)]). Let $L_0 = (X_1 X_2)_g (X_3)_g$. Since $X_1^g = X_2$ we have $Z(L_0) \cong Z_{(3, q^2 - 1)}$. If $(3, q^2 - 1) = 1$, set $L = L_0$, and if $3 \mid q^2 - 1$ set $L = L_0 \langle xy \rangle$, where $x \in X_1 X_2$, $y \in X_3$, $x^g = x z_1 z_2$, and $y^g = y z_3$. Then $L \leqslant \hat{G}_g \cong E_6(q)$ and we have the other subgroup for $E_6(q)$.

The above set-up can be used to describe the required subgroups for $F_4(q)$ and ${}^2E_6(q)$. Let τ be the graph automorphism of $\hat{G}$, defined with respect to $\hat{B}$, $\hat{H}$, $\hat{N}$, etc. Let $X = X_1 X_2 X_3$ and note that we may take $w \in \hat{G}_\tau \cong F_4(K)$. Consequently, $F_4(q) = (\hat{G}_\tau)_\sigma$ contains conjugates of $X_\tau \cap X_g$ and $X_\tau \cap X_\sigma$. Now τ centralizes X_3, so $z_3^\tau = z_3$. As $1 = (z_1 z_2 z_3)^\tau = z_1 z_2 z_3$ we must have $z_1^\tau = z_2$ and $z_2^\tau = z_1$. So $Z(X)_\tau = \langle z_3 \rangle$. Since $[X_3, \tau] = 1$, $(X_3)_g$ or $(X_3)_\sigma$ is contained in $X_\tau \cap X_g$ or $X_\tau \cap X_\sigma$, respectively. Also $((X_1 X_2)_\tau)_\sigma \cong SL(3, q)$ and $((X_1 X_2)_\tau)_g \cong SU(3, q)$. This gives the subgroups L_0 for $F_4(q)$ and the usual argument gives $L = L_0$ or $L_0 \langle a \rangle$, completing the case of $F_4(q)$. For $G_0 = {}^2E_6(q)$ note that $\tau\sigma \sim w_1 \tau\sigma$ so G_0 contains a conjugate of

$X_{w_1\tau\sigma}$. The element $w_1\tau\sigma$ stabilizes each X_i and $(X_i)_{w_1\tau\sigma} \cong SU(3, q)$. From here the usual argument shows that $L = X_{w\tau\sigma}$ is the required subgroup.

Next, let $G_0 = E_8(q)$ and $\hat{G} = E_8(K)$. We produce central products $\hat{X} = \hat{X}_1\hat{X}_2$ and $\hat{Y} = \hat{Y}_1\hat{Y}_2$ as follows. Let

$$\hat{X}_1 = \langle \hat{U}_{\pm\alpha_2}, \hat{U}_{\pm\alpha_3}, \hat{U}_{\pm\alpha_4}, \hat{U}_{\pm\alpha_5} \rangle$$

and

$$\hat{X}_2 = \langle \hat{U}_{\pm\alpha_7}, \hat{U}_{\pm\alpha_8}, \hat{U}_{\pm t_1}, \hat{U}_{\pm t_2} \rangle,$$

where $t_1 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$ and $t_2 = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$. Let $\hat{Y}_1 = \langle \hat{U}_{\pm\alpha_i} \mid 1 \leq i \leq 4 \rangle$ and $\hat{Y}_2 = \langle \hat{U}_{\pm\alpha_6}, \hat{U}_{\pm\alpha_7}, \hat{U}_{\pm\alpha_8}, U_{\pm s} \rangle$.

From the commutator relations we have $[\hat{X}_1, \hat{X}_2] = [\hat{Y}_1, \hat{Y}_2] = 1$. Each of $\hat{X}_1$ and $\hat{X}_2$ is a perfect central extension of $D_4(K)$ and each of $\hat{Y}_1$ and $\hat{Y}_2$ is a perfect central extension of $L_5(q)$. An easy computation shows that $Z(X_1) = Z(X_2) = 1$ or $Z_2 \times Z_2$, according to whether or not $r = 5$. Similarly, $Z(Y_1) = Z(Y_2) \cong 1$ or Z_5, according to whether or not $r = 2$. From the action of σ on $X = \hat{X}_1\hat{X}_2$ and on $Y = \hat{Y}_1\hat{Y}_2$ we obtain the required subgroup L $(L = X_\sigma$ or $Y_\sigma)$ with $L_0/Z(L_0) \cong D_4(q) \times D_4(q)$ or $L_5(q) \times L_5(q)$.

From the action of w_4 and w_5 on π we see that $X_i^{w_4} = X_i$ for $i = 1,2$ and $Y_i^{w_5} = Y_i$ for $i = 1,2$. Moreover, w_4 induces a triality graph automorphism on the Dynkin diagram of X_1 and X_2, while w_5 induces the involutory graph automorphism on the Dynkin diagram of Y_1 and Y_2. So considering $w_4\sigma$ and $w_5\sigma$ we obtain the groups L with $L_0/Z(L_0) \cong {}^3D_4(q) \times {}^3D_4(q)$ and $U_5(q) \times U_5(q)$. We can also restrict to $E_7(K) \leq E_8(K)$ and obtain the first assertion for $E_7(q)$. Namely, look at $X_1 \times X_3$, with $X_3 = \langle \hat{U}_{\pm\alpha_7} \rangle \times \langle \hat{U}_{\pm t_1} \rangle \times \langle \hat{U}_{\pm t_2} \rangle$, use the element $w_4\sigma$, and argue as usual.

Next, set $Z_1 = \langle \hat{U}_{\pm\alpha_1}, \ldots, \hat{U}_{\pm\alpha_6} \rangle$ and $Z_2 = \langle \hat{U}_{\pm\alpha_8}, \hat{U}_{\pm s} \rangle$. Then $[Z_1, Z_2] = 1$, Z_1 is a perfect central extension of $E_6(K)$ (in fact the universal group) and $Z_2 \cong SL(3, K)$. The argument used for the case $E_6(K)$ and the fact that $3 \nmid |Z(\hat{G})|$ imply that $Z(Z_1) = Z(Z_2)$. Letting $L_0 = (Z_1)_\sigma(Z_2)_\sigma$ we have $L_0/Z(L_0) \cong E_6(q) \times L_3(q)$ and $Z(L_0) \cong Z_{(3,q-1)}$. Fix $z \in Z(Z_1Z_2)^\#$ and choose $a \in Z_1$ and $b \in Z_2$ with $a^\sigma = az$ and $b^\sigma = bz^{-1}$. This is possible by Lang's theorem. Then σ centralizes ab and $L = L_0\langle ab \rangle$ satisfies the conditions for the appropriate subgroup of $E_8(q)$.

Let $g = w_6\sigma$. The usual arguments show that $L = (Z_1Z_2)_g$ is the twisted version of the previous example. Namely $O^{r'}(L) = L_0$ satisfies $L_0/Z(L_0) \cong {}^2E_6(q) \times U_3(q)$. A variation of this example leads to the second group for $E_7(q)$. Namely, use $g = w_2\sigma$ (acting on $E_7(K)$). As w_2 induces an involutory graph automorphism of Z_1, we see that $(Z_1)_g$ is the covering group of ${}^2E_6(q)$. Also, if we set $T = \{h(\chi) \mid \chi(\alpha_i) = 1$ for $i \neq 7\}$, then $[Z_1, T] = 1$ and T is a 1-dimensional torus. Now $Z(Z_1) \leq T$ and since $\alpha_7^{w_2} = -s$, g inverts T. Thus, $T_g \cong Z_{q+1}$ and $L = (Z_1T)_g$ is the desired group.

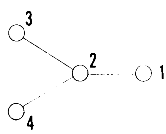
To complete $E_7(q)$, let $V_1 = \langle \hat{U}_{\pm\alpha_2}, \hat{U}_{\pm\alpha_4}, \ldots, \hat{U}_{\pm\alpha_7} \rangle$ and $V_2 = \langle \hat{U}_{\pm\alpha_1}, \hat{U}_{\pm s} \rangle$. Argue as usual to get $L = (V_1V_2)_{w_3\sigma}$ as indicated in Table (3.3).

We continue the analysis of $E_8(q)$. Set $D = \langle \hat{U}_{\pm\alpha_1}, \hat{U}_{\pm\alpha_3}, \ldots, \hat{U}_{\pm\alpha_8}, \hat{U}_{\pm s} \rangle$. Then D is a perfect central extension of $L_9(K)$ and checking the action of $\hat{H}$ we have

$Z(D) = 1$ or Z_3, according to $r = 3$ or $r \neq 3$. The covering group of D is $SL(9, K)$. Let $L = D_\sigma$ and $L_0 = O^{r'}(L)$. If $r \neq 3$, then (2.13) implies that $|L: L_0| = 3$. Also, considering the action of σ on $SL(9, K)$ we see that $3 \| q - 1$, then $L = L_0 \times Z_3$. So L satisfies the conditions in Table (3.3). To produce the twisted version of this example let $g = w_7\sigma$ and argue in the usual manner.

To complete the analysis of $E_8(q)$ we must produce a central extension of $PSO^-(12, q)' \times L_2(q^2)$. Let $J_1 = \langle \hat{U}_{\pm\alpha_2}, \ldots, \hat{U}_{\pm\alpha_7}\rangle$ and $J_2 = \langle \hat{U}_{\pm s}, U_{\pm t}\rangle$, where t is the root of highest height in the root system spanned by $\{\alpha_1, \ldots, \alpha_7\}$. Then $J_1 \cong D_6(K)$, $J_2 = \langle \hat{U}_{\pm s}\rangle \times \langle \hat{U}_{\pm t}\rangle$, w_8 induces a graph automorphism of J_1, and w_8 interchanges the components of J_2. Therefore, $(J_1 J_2)_{w_8\sigma}$ satisfies the necessary conditions.

For the remaining cases let $\hat{G} = D_4(K)$, with Dynkin diagram



Set $X_1 = \langle \hat{U}_{\pm\alpha_2}, \hat{U}_{\pm(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}\rangle$, so that $X_1 \cong SL(3, K)$. A direct check shows that $X_2 = C_{\hat{H}}(X_1)$ is a 2-dimensional torus. Indeed

$$C_{\hat{H}}(X_1) = \{h(\chi): \chi(\alpha_2) = 1, \chi(\alpha_1) = \zeta_1, \chi(\alpha_3) = \zeta_2, \chi(\alpha_4) = (\zeta_1\zeta_2)^{-1}\}.$$

As in previous cases we have $Z(X_1) \leqslant X_2$ and from here consideration of $L = (X_1 X_2)_\sigma$ gives the first assertion about $D_4(q)$. Now let $g = s_1 s_2 s_3$ and argue with $g\sigma$ to get the second assertion. The results for $G_2(q)$ are obtained by observing that the graph automorphism of order 3 of $\hat{G}$ centralizes X_1. Similarly for ${}^3D_4(q)$, but here one must first check the action of $g\sigma$ on X_2. But this is a straightforward computation. All entries in Table (3.3) have now been accounted for.

For later use we point out that in most of the situations of Table (3.3) the components are generaed by long root subgroups of G_0. In the notation of Table (3.3) let $L_0/Z(L_0) = C_1 \cdots C_k$, where the C_i are as indicated in column 2 of Table (3.3). Then each C_i is a component of $L_0/Z(L_0)$ except when $C_i \cong L_2(2)$, $L_2(3)$, or $U_3(2)$. Here we say C_i is a solvable component. In the case $U_{2n+1}(q)$, which has root system of type BC_n, we will call the long root subgroups those that are nonabelian of order q^3. We have the following:

(3.4) Let (G_0, L_0) be from Table (3.3). Except for the following cases, each C_i is generated by the images in $L_0/Z(L_0)$ of long root subgroups of G_0. In each case the indicated component (solvable component) is the only one so generated:

$$(E_6(q), L_3(q^2) \times U_3(q), U_3(q)), \quad (F_4(q), U_3(q) \times U_3(q), U_3(q)),$$
$$(F_4(q), L_3(q) \times L_3(q), L_3(q)), \quad (E_7, {}^3D_4(q) \times L_2(q^3), {}^3D_4(q)),$$
$$(E_8(q), PSO^-(12, q)' \times L_2(q^2), PSO^-(12, q)').$$

PROOF. In each case $L_0 = O^{r'}(L)$ where $L = \hat{L}_{w\sigma}$, $\hat{L}$ is a subgroup of $\hat{G}$ generated by certain root subgroups, and w (possibly $w = 1$) is an element of the Weyl group of G_0. In each case where w stabilizes a component $\hat{Y}$ of $\hat{L}$, w also stabilizes a long root subgroup of $\hat{L}$ and we easily have $\hat{Y}_{w\sigma}$ generated by long root subgroups. The exceptional cases listed are the ones where w has a nontrivial orbit on the components of $\hat{L}$.

(3.5) Let $\hat{G} = E_6(K)$, τ the field automorphism $k \to k^q$ and θ the graph automorphism of $\hat{G}$ (defined with respect to $\hat{B} = \hat{B}^\tau$, $\hat{H} = \hat{H}^\tau$, etc.).

(i) If $3 \mid q - 1$, then $\hat{N}_\tau$ contains a Sylow 3-subgroup of $\hat{G}_\tau$.

(ii) If $3 \mid q + 1$, then $\hat{G}_{\tau\theta}$ contains a subgroup Y with $Y/\mathrm{Fit}(Y) \cong O^2(W) \cong O^-(6,2)'$ and $\mathrm{Fit}(Y)$ is isomorphic to the direct product of 6 copies of Z_{q+1}. Y contains a Sylow 3-subgroup of $\hat{G}_{\tau\theta}$.

PROOF. The statements concerning Sylow 3-subgroups will follow from the order formulas for $\hat{G}_\tau$ and $\hat{G}_{\tau\theta}$, respectively (see (5.1) for example). Now τ acts on $\hat{N}$ and $\hat{N} = \hat{H}(\hat{N})_\tau$. Also $W \cong \hat{W} = \hat{N}/\hat{H} \cong O^-(6,2)$. Since $\hat{H}_\tau$ is isomorphic to the direct product of 6 copies of $Z_{q-1} \cong \mathbf{F}_q^\#$ we have (i).

For (ii) let τ, θ act on $\hat{N}$. Now τ centralizes $\hat{N}/\hat{H}$ and θ induces the graph automorphism on $\hat{N}/\hat{H}$. Let $g \in \hat{N}$ with $g\hat{H}$ the long word in the fundamental reflections generating $\hat{N}/\hat{H}$. Then for each $\alpha \in \hat{\Sigma}$, $\hat{U}_\alpha^{g\theta} = \hat{U}_{-\alpha}$. It follows that $g\theta$ centralizes $\hat{N}/\hat{H}$ and $g\theta$ inverts $\hat{H}$. By Lang's theorem $\tau\theta \sim g\tau\theta$ and by the above $\hat{H}_{g\tau\theta}$ is the direct product of 6 copies of Z_{q+1}. So $\hat{G}_{\tau\theta}$ contains a conjugate Y_1 of $\hat{N}_{g\tau\theta}$ and letting $Y/\mathrm{Fit}(Y_1) = O^2(Y_1/\mathrm{Fit}(Y_1))$ we have the desired group Y. (To check that $N_{g\tau\theta}$ covers $\hat{N}/\hat{H}$ use Lang's theorem once again.)

4. Classical groups. In this section we take G to be a classical group. Accordingly, let $G_0 = SL(V)$, $Sp(V)$, $SU(V)$, or $O(V)'$, where V is the appropriate module. This is in accordance with the notation in §2, except for the orthogonal group in odd characteristic. If $G = PSO^\pm(n,q)'$ with q odd, then the group G_0 of §2 may be an extension of $SO^\pm(n,q)$ by a group of order 2. But as $p > 2$ this does not affect generation. We omit the case of an orthogonal space of odd dimension and even characteristic, for we identify such groups with symplectic groups.

We must make one additional change of notation regarding the group E_0. We want G_0E_0 to act on V. This certainly happens if $E_0 \leqslant G_0$ (as in the symplectic and orthogonal groups). But for $G_0 = SL(V)$ or $SU(V)$ this need not occur and we redefine the group E_0 as follows. Recall that we have $E \leqslant N_{\bar{G}}(G) = PGL(V)$ or $PGU(V)$. Let E_0 be the Sylow p-subgroup of the preimage in $GL(V)$ or $SU(V)$ of E. This change of notation does not affect generation.

Throughout this section we will assume $\dim(V) \geqslant 8$, if V is an orthogonal space.

To describe the cases where generation fails we make the following definition. Suppose $p = 3$, $q = 2$, and $G_0 \not\cong SL(V)$. Let $V = V_0 \perp V_1 \perp \cdots \perp V_k$ with $V_0 = C_V(E_0)$ and $V_i = V_i^{E_0}$ for $i = 1, \ldots, k$. We say this decomposition is *admissible* if $\dim(V_0) \leqslant 0, 1, 2$ according to whether V is symplectic, unitary, or orthogonal, and $\dim(V_1) = \cdots = \dim(V_k) = 1$ or 2 according to whether V is unitary or not. Note that $k \geqslant 2$.

(4.1) Assume E_0 is abelian. Then precisely one of the following holds:

(i) $D_0 = G_0$.

(ii) $p = 3$, $q = 2$, $G_0 \not\cong SL(V)$ and V contains an admissible decomposition. For any admissible decomposition $V = C_V(E_0) \perp V_1 \perp \cdots \perp V_k$, we have $D_0 \leqslant I$ and $E_0 \leqslant E_1$, where I is the stabilizer in G_0E_0 of $\{C_V(E_0), V_1, \ldots, V_k\}$ and $E_1 = O_3(I)$.

(iii) $p = 3$, $q = 2$, $G_0 \cong Sp(2n, 2)$ and $D_0 = O^\varepsilon(2n, 2)$, where $\varepsilon = (-1)^n$. More precisely, V has an admissible decomposition $V = V_1 \perp \cdots \perp V_k$, and for any such decomposition D_0 preserves the quadratic form which has value 1 on each vector in $V_i^{\#}$, for $i = 1, \ldots, k$.

In particular, either $D_0 = G_0$ or E_0 is elementary. Moreover, examples exist for each of (i), (ii) and (iii).

The proof of (4.1) will be carried out in a series of steps. We assume G_0 to be a counterexample of minimal order. By (2.3) we may assume that E_0 is contained in no proper parabolic subgroup of G_0E_0. A proper parabolic subgroup of G_0E_0 is the stabilizer of a particular proper subspace, isotropic subspace, or singular subspace of V, according to whether $G_0 \cong SL(V)$, $Sp(V)$ or $SU(V)$, or $O^{\pm}(V)'$. So E_0 stabilizes no such subspace. Since E_0 is not cyclic E_0 does not act irreducibly on V, and consequently $G_0 \not\cong SL(V)$. We will study E_0-invariant decompositions of V.

Write $V = C_V(E_0) \oplus V_1 \oplus \cdots \oplus V_k$, with each V_i irreducible E_0-invariant subspace of V. If $W \leqslant V$ with $W \cap W^\perp = 0$, let $\mathfrak{J}(W)$ be the group generated by all root subgroups of G_0 contained in $C(W^\perp)$. Except for certain small cases $\mathfrak{J}(W) = \mathfrak{I}(W)$ as defined in §3.

(4.2) We may assume that

(i) $V = C_V(E_0) \perp V_1 \perp \cdots \perp V_k$.

(ii) $C_V(E_0)$ has dimension at most 0, 1, 2 depending on whether G_0 is symplectic, unitary, or orthogonal.

(iii) $\dim(V_1) = \cdots = \dim(V_k)$.

(iv) If V is decomposed as in (i), then $\mathfrak{J}(C_V(E_0) \perp V_i) \leqslant D_0$ for $i = 1, \ldots, k$.

Proof. Clearly $V_1 \oplus \cdots \oplus V_k = [V, E_0]$ and so $V_1 \oplus \cdots \oplus V_k = C_V(E_0)^\perp$. As remarked above, V_1 is not isotropic (singular if V is an orthogonal space). If $\mathrm{rad}(V_1) \neq 0$ then $G_0 = SO^{\pm}(n, q)'$ for q a power of 2, and V_1 is a 1-space. But here E_0 must centralize V_1, whereas $V_1 \nleqslant C_V(E_0)$. So in all cases $\mathrm{rad}(V_1) = 0$ and consequently E_0 acts on $(C_V(E_0) \perp V_1)^\perp$. If necessary rechoose V_2 so V_2 is an irreducible E_0-invariant subgroup of $(C_V(E_0) \perp V_1)^\perp$. Continue in this way to get (i).

To get (ii) just notice that $C_V(E_0)$ contains no isotropic (singular) 1-space. For $i = 1, \ldots, k$, E_0 induces a cyclic group of order $\exp(E_0)$ on V_i. This determines $\dim(V_i)$ uniquely. So (iii) holds. To see (iv) note that, for $i = 1, \ldots, k$, E_0 induces a cyclic group of order $\exp(E_0)$ on $C_V(E_0) \perp V_i$ and that $\Phi(E_0) \leqslant C(G_0) \leqslant C(\mathfrak{J}(C_V(E_0) \perp V_i))$. So $\Phi(E_0)C_{E_0}(C_V(E_0) \perp V_i)$ is a maximal subgroup of E_0 centralizing $\mathfrak{J}(C_V(E_0) \perp V_i)$. The result follows.

(4.3) $k = 2$.

Proof. Suppose $k > 2$. First assume that either $q > 2$, $p > 3$, or E_0 is not elementary. In each of these cases we will show $D_0 = G_0$, by induction. Let $W_1 = C_V(E_0) \perp V_1 \perp \cdots \perp V_{k-1}$ and $W_2 = C_V(E_0) \perp V_2 \perp \cdots \perp V_k$. Inductively, $\mathfrak{J}(W_1), \mathfrak{J}(W_2) \leqslant D_0$ (here $\mathfrak{J}(W_i) = \mathfrak{I}(W_i)$ as defined in §3) and by (3.1) applied to the decomposition $V = V_1 \perp (W_1 \cap W_2) \perp V_k$ we conclude that either $G_0 \cong SO^+(6,2)'$ or $W_1 \cap W_2$ contains no isotropic (singular) 1-space. The former is out as $\dim(V) \geqslant 8$ in the orthogonal cases. Therefore $W_1 \cap W_2$ contains no isotropic (singular) 1-space. This is impossible for V a symplectic space. If V is unitary then $\dim(W_1 \cap W_2) = 1$, forcing $C_V(E_0) = 0$, $k = 3$, and $\dim(V_i) = 1$ for $i = 1,2,3$. If V is orthogonal then $\dim(W_1 \cap W_2) \leqslant 2$, forcing $\dim(V) < 8$, a contradiction.

Suppose then that $G_0 \cong SU(3,q)$. The proof of (4.2)(iv) shows that $\mathcal{J}(V_1 \perp V_2)$ and $\mathcal{J}(V_2 \perp V_3)$ are each in D_0. So by the results of **[5]** and **[14]**, $q = 2$ and $p = 3$, which is against our assumption.

To complete the proof of (4.3) we now assume $q = 2$, $p = 3$, and E_0 elementary abelian. As above we are done by (3.1) if $\mathcal{J}(W_1)$ and $\mathcal{J}(W_2)$ are each in D_0 (we are reduced to $G_0 \cong SU(3,2)$, where an easy check gives the result). Suppose then that $\mathcal{J}(W_1) \nleqslant D_0$. As E_0 is elementary each V_i is isomorphic to $Z_2 \times Z_2$. If G_0 is symplectic or orthogonal, then each V_i is a 2-space over $\mathbf{F}_2$, while if V is unitary each V_i is a 1-space over $\mathbf{F}_4$. So the decomposition is admissible. Let I be the stabilizer in G_0E_0 of $\{C_V(E_0), V_1, \ldots, V_k\}$. Then $E_0 \leqslant O_3(I) = E_1$, with E_1 elementary abelian.

We are assuming $D_0 < G_0$. Let F be a hyperplane of E_0. We will show that either $C_{G_0}(F) \leqslant I$ or that (4.1)(iii) holds. In the course of the proof it will become evident how to construct examples where (4.1)(ii) or (4.1)(iii) holds. For $i = 1, \ldots, k$ let $F_i = C_F(V_i)$ and $\hat{V}_i = C_V(F_i)$. So $\hat{V}_i \geqslant C_V(E_0) \perp V_i$ and $C_{G_0}(F)$ acts on $\hat{V}_i$ for $i = 1, \ldots, k$. In addition, $\hat{V}_i$ is the sum of $C_V(E_0)$ and certain of the subspaces in $\{V_1, \ldots, V_k\}$. Let I_i be the normalizer in I of $\hat{V}_i$. It will suffice to show that either (4.1)(iii) holds or, for each $i = 1, \ldots, k$, $C_{G_0}(F)|_{\hat{V}_i} \leqslant I_i|_{\hat{V}_i}$.

Fix $i \in \{1, \ldots, k\}$. Reorder, if necessary, so that $i = 1$ and $\hat{V}_1 = C_V(E_0) \perp V_1 \perp \cdots \perp V_m$, where $1 \leqslant m \leqslant k$. First suppose $m = 1$. If $F > F_1$, then $C_{G_0}(F)$ normalizes $[\hat{V}_1, F] = V_1$ and $C_{\hat{V}_1}(F) = C_V(E_0)$ so we easily have $C_{G_0}(F)|_{\hat{V}_1} \leqslant I_1|_{\hat{V}_1}$. This also holds if $F = F_1$ except if $G_0 = O(V)'$, $\dim(C_V(E_0)) = 2$ and $C_V(E_0)$ contains nonzero singular vectors. But in this case $C_{G_0}(F) = C_{G_0}(F_1) \geqslant \mathcal{J}(\hat{V}_1) = SO^-(4,2)' \cong A_5$. From here we obtain $D_0 = G_0$ using induction and (3.1) applied to the decomposition

$$V = V_2 \perp (C_V(E_0) \perp V_1) \perp (V_3 \perp \cdots \perp V_k).$$

So from now on assume $m \geqslant 2$.

Suppose $F = F_1$ so that $\mathcal{J}(\hat{V}_1) \leqslant C_{G_0}(F) \leqslant D_0$. If $k > m + 1$ set $W_1 = \hat{V}_1 \perp V_{m+1} \perp \cdots \perp V_{k-1}$ and $W_2 = \hat{V}_1 \perp V_{m+2} \perp \cdots \perp V_k$. Inductively, $C_{G_0}(F)|_{\hat{V}_1} \nleqslant I_1|_{\hat{V}_1}$ implies $\mathcal{J}(W_1), \mathcal{J}(W_2) \leqslant D_0$, so, by (3.1), $D_0 = G_0$. So now suppose $k = m + 1$. As F_1 acts on $V_k = V_{m+1}$ as a cyclic group we necessarily have $E_0 \cong Z_3 \times Z_3$. At this stage a check of the action of E_0 on V shows that for some proper subgroup, X, of E_0, $C_{G_0}(X)$ contains an element $g \in I$ with $V_k^g \neq V_k$. Then $D_0 \geqslant \langle \mathcal{J}(\hat{V}_1), \mathcal{J}(\hat{V}_1)^g \rangle = G_0$ by (3.1). This is a contradiction, so we now have $F > F_1$.

This implies that $C_{G_0}(F)$ acts on $[\hat{V}_1, F] = V_1 \perp \cdots \perp V_m$ and on $C_{\hat{V}_1}(F) = C_V(E_0)$. If $m = 2$, then an easy check of the group of isometries of $V_1 \perp V_2$ ($Sp(4,2)$, $GU(2,2)$, or $O^+(4,2)$) shows that $C_{G_0}(F)|_{\hat{V}_1} \leqslant I_1|_{\hat{V}_1}$. So we now suppose $m \geqslant 3$. We note that if $E_0 = E_1$, then order considerations show that $m \leqslant 2$, so here (4.1)(ii) holds.

Suppose G_0 is unitary. Then assuming $C_{G_0}(F)|_{\hat{V}_1} \nleqslant I_1|_{\hat{V}_1}$, F must have an eigenspace on $V_1 \perp \cdots \perp V_m$ of dimension at least 3, say $V_1 \perp \cdots \perp V_l$. Then consider decompositions of V of the form $V = U_1 \perp (V_1 \perp V_2 \perp V_3) \perp U_2$, where U_1 and U_2 are each sums of some of the subspaces $\{C_V(E_0), V_4, \ldots, V_k\}$. By minimality of G_0 and the fact that $\mathfrak{I}(V_1 \perp V_2 \perp V_3) \leqslant C(F)$ we conclude that, for $i = 1, 2$, D_0 contains $\mathfrak{I}(U_i \perp V_1 \perp V_2 \perp V_3)$, unless $V = U_i \perp V_1 \perp V_2 \perp V_3$. So by (3.1) we are done except in the case $k = 4$ and $C_V(E_0) = 1$. Here we can write $E_0 = F \times E_2 \cong Z_3 \times Z_3$, where E_2 is trivial on $V_1 \perp V_2 \perp V_3$ and fixed-point-free on V_4. It is then easy to produce a maximal subgroup E_3 of E_0 inducing scalars on $V_3 \perp V_4$. Therefore $C_{G_0}(E_3)$ contains an element g interchanging V_3 and V_4, while stabilizing $V_1 \perp V_2$. So

$$\begin{aligned} D_0 &\geqslant \langle \mathfrak{I}(V_1 \perp V_2 \perp V_3), \mathfrak{I}(V_1 \perp V_2 \perp V_3)^g \rangle \\ &= \langle \mathfrak{I}(V_1 \perp V_2 \perp V_3), \mathfrak{I}(V_1 \perp V_2 \perp V_4) \rangle \end{aligned}$$

and (3.1) yields a contradiction. So G_0 is not a unitary group.

In the other cases $\mathfrak{I}(V_1 \perp V_2 \perp V_3) \cong Sp(6,2)$ or $SO^-(6,2)'$. Now $Sp(6,2) \geqslant SO^-(6,2)' \cong PSU(4,2) \cong PSp(4,3)$ and from here we check that in either case $C(F) \cap \mathfrak{I}(V_1 \perp V_2 \perp V_3)$ contains $GU(3,2)$. We then argue as in the preceding paragraph with a decomposition $V = U_1 \perp (V_1 \perp V_2 \perp V_3) \perp U_2$. Set $W_i = U_i \perp V_1 \perp V_2 \perp V_3$ for $i = 1, 2$.

In the orthogonal case apply (3.1), induction, and use the fact that $C(F) \cap \mathfrak{I}(V_1 \perp V_2 \perp V_3) \leqslant C^0_{\mathfrak{I}(W_i)}(E_0)$. We reduce to the case $k = 3$ or 4. In the latter case $\mathfrak{I}(V_1 \perp V_2 \perp V_3) \leqslant D_0$ by induction and a direct calculation produces a hyperplane X of E_0 and element $g \in C(X) \cap I$ with $\{V_1, V_2, V_3, V_4\}^g = \{V_1, V_2, V_3, V_4\}$ and $V_4^g \neq V_4$. As before this leads to $D_0 = G_0$. If $k = 3$, then as $\dim(V) \geqslant 8$ we necessarily have $\dim(C_V(E_0)) = 2$ and $E_0 \cong Z_3 \times Z_3$. Inductively, $\mathfrak{I}(V_1 \perp V_2 \perp V_3) \leqslant D_0$. Also there is an element $e \in E_0^{\#}$ with $C(e)$ permuting $\{C_V(E_0), V_1, V_2, V_3\}$ and moving $C_V(E_0)$. Then (3.1) implies that $D_0 = G_0$.

Now suppose that $G \cong Sp(2n,2)$. Suppose $W_1, W_2 < V$. Then by induction $C^0_{\mathfrak{I}(W_i)}(E_0) = Sp(W_i)$ or $O^{\pm}(W_i)$, where the form is described as in (4.1)(iii). By (3.1) we then have $D_0 \geqslant O^\varepsilon(2n,2)$, where $\varepsilon = (-1)^n$. But $O^\varepsilon(2n,2)$ is maximal in G_0 (indeed G_0 is 2-transitive on the cosets of $O^\varepsilon(2n,2)$), so $D_0 = G_0$ or $D_0 = O^\varepsilon(2n,2)$. Thus we again reduce to the situation of $k = 3$ or 4. Also, $C_V(E_0) = 0$ in the symplectic case, so $G_0 \cong Sp(6,2)$ or $Sp(8,2)$. In the latter case we argue as before, so we are left with $G_0 \cong Sp(6,2)$, a group of order $2^9 \cdot 3^4 \cdot 5 \cdot 7$.

Here $C_{G_0}(F) \cong GU(3,2)$ and $F_1 = 1$. Viewing $E_0 \leqslant \mathfrak{I}(V_1) \times \mathfrak{I}(V_2) \times \mathfrak{I}(V_3)$ we easily have $\langle \mathfrak{I}(V_1) \times \mathfrak{I}(V_2) \times \mathfrak{I}(V_3), GU(3,2) \rangle = Z \leqslant D_0$. Checking a Sylow 2-subgroup of Z (which must contain Q_8 and an E_8) we have $|Z|$ divisible by $2^5 3^4$. A Sylow 3-subgroup of Z has the following orbits on $V^{\#}$:

$$\begin{aligned} O_1 &= \{v_i \colon v_i \in V_i^{\#}, i = 1,2,3\}, \\ O_2 &= \{v_i + v_j \colon i \neq j, v_i \in V_i^{\#}, v_j \in V_j^{\#}\} \end{aligned}$$

and

$$O_3 = \{v_1 + v_2 + v_3 : v_1 \in V_1^{\#}, v_2 \in V_2^{\#}, v_3 \in V_3^{\#}\}.$$

These have sizes 9, 27, 27 respectively. Suppose D_0 is transitive on $V^{\#}$. Then D_0 is transitive on the transvections of G_0 and as D_0 contains transvections, $G_0 = D_0$, a contradiction. So D_0 is not transitive on $V^{\#}$. Since elements of order 7 in G_0 are fixed-point-free on $V^{\#}$ this implies $7 \nmid |D_0|$. Suppose $5 \mid |D_0|$. An element of order 5 in G_0 fixes just 3 elements in $V^{\#}$, the nonzero vectors of a nondegenerate 2-space. Therefore either $O_1 \cup O_2$ or $O_1 \cup O_3$ is an orbit of D_0 on $V^{\#}$. Let $g \in D_0$, $|g| = 5$ and $v_1^g = v_1$. If $O_1 \cup O_2$ is an orbit of D_0, then g acts on $(O_1 \cup O_2) \cap v_1^{\perp} = \{v_1, V_2^{\#}, V_3^{\#}, v_1 + V_2^{\#}, v_1 + V_3^{\#}, V_2^{\#} + V_3^{\#}\}$. But this set contains 22 elements so g fixes an additional element of this set. This is a contradiction. Therefore if $5 \mid |D_0|$ the orbits of D_0 are $O_1 \cup O_3$ and O_2. But this just says that D_0 preserves the quadratic form taking the value 1 on $V_1^{\#} \cup V_2^{\#} \cup V_3^{\#}$. That is $D_0 \leqslant O^-(6,2)$ and has order divisible by $2^5 \cdot 3^4 \cdot 5$. Consequently $D_0 = O^-(6,2)$. So now suppose $5 \nmid |D_0|$. Then D_0 is a $\{2,3\}$-group. No 2-local subgroup of $Sp(6,2)$ has order divisible by 3^4, so D_0 is 3-local, but not 2-local. Easy arguments show that $F(D_0)$ must contain an elementary abelian subgroup of order 3^3, and this is not consistent with $D_0 \geqslant GU(3,2)$. Thus $D_0 \leqslant I$.

We have now completed the proof of (4.3) except for the construction of examples where (4.1)(iii) holds. Just argue as follows. Write $V = V_1 \perp \cdots \perp V_k$ with $k \geqslant 3$ and set $\langle x_i \rangle = O_3(\mathcal{J}(V_i))$. Let $E_0 = \langle x_1 x_2^{-1}, x_1 x_2 x_3, x_i \colon i \geqslant 4 \rangle$. Then $|E_0| = 3^{k-1}$ and maximal subgroups of E_0 have order 3^{k-2}. Given any such maximal subgroup, say F, there cannot exist a 4-set $\{i, j, l, m\}$ of $\{1, \ldots, k\}$ with $F_i = F_j = F_l = F_m$. For this contradicts $|F|$. The above arguments then imply that $C_{G_0}(F) \leqslant O^{\varepsilon}(2k, 2)$ with $\varepsilon = (-1)^k$. By choice of X, (4.1)(i) fails, so we must have $D_0 = O^{\varepsilon}(2k, 2)$. This completes the proof of (4.3).

(4.4) $k > 2$.

PROOF. Suppose $k = 2$ and notice that this implies E_0 has rank 2. The proof here will follow from (3.2) once certain small cases are handled. If $G_0 \cong SU(3,q)$, then $\dim(C_V(E_0)) = \dim(V_1) = \dim(V_2) = 1$ and the result follows from the results of **[5]** and **[14]** (recall, G simple implies $G_0 \not\cong SU(3,2)$). Also the case of $Sp(4,2)$ is easily checked. Suppose $C_V(E_0) \neq 0$. Then minimality of G_0 implies $\mathcal{J}(V_1 \perp V_2) \leqslant D_0$, so by (3.1) and (4.2)(iv) we are done. So assume $C_V(E_0) = 0$.

Let $\langle e_i \rangle = C_{E_0}(V_i)$ for $i = 1,2$. Recall that from (4.2)(iv) we have $\mathcal{J}(V_i) \leqslant D_0$ for $i = 1,2$. We first note that E can be embedded in a maximal torus of $\bar{G}$. To see this choose $e \in E_0$ with $E_0 = \langle e_1 \rangle \langle e \rangle$. Then $e_1 \in C_{\hat{G}}(e)$, so e_1 can be embedded in a σ-invariant maximal torus of $C_{\hat{G}}(e) = C_{\hat{G}}(e)^0$, which is a maximal torus of $\hat{G}$. Now pass to $\bar{G}$ to get the assertion. From (2.11) we have the existence of an element $f \in E - (\langle \bar{e}_1 \rangle \cup \langle \bar{e}_2 \rangle)$ such that $O^{r'}(C_G(f)) \neq 1$. If $f_0 \in E_0$ with $\bar{f}_0 = f$, then $Q = O^{r'}(C_{G_0}(f_0)) \neq 1$, although $O^{r'}(C(f_0) \cap (\mathcal{J}(V_1) \times \mathcal{J}(V_2))) = 1$.

Assume $(p,q) \neq (3,2)$. Then $Q \not\cong S_3$, so Q does not stabilize $\{V_1, V_2\}$ and we can apply (3.2) to conclude $D_0 = G_0$, except for a minor adjustment needed in case G_0 is an orthogonal group. Namely, (3.2) is stated for $X = O^{\pm}(V)$, rather than for $G_0 = \mathfrak{J}(V)$. This was done to ease the proof of that result. In our situation consider $O^+(V)' = G_0 \trianglelefteq O^+(V) = X$. $E_0 = \Omega_1(P)$ for $P \in \mathrm{Syl}_p(G_0)$, so the Frattini argument gives $X = G_0 N_X(E_0)$. Now $N_X(E_0) \leqslant N_X(D_0)$, so $D_0 N_X(E_0)$ is a subgroup of X properly containing $C_X(V_1)$ and $C_X(V_2)$. So (3.2) implies $X = D_0 N_X(E_0) \leqslant N_X(D_0)$, and hence $D_0 = G_0$.

Finally, assume $(p, q) = (3, 2)$. Here $G_0 \cong SU(6, 2)$, $E_0 \cong Z_3 \times Z_9$, and $\hat{G} \cong SL(6, K)$. If $f \in E_0$ and $|f| = 9$, then f has three distinct eigenvalues on $K \otimes V$ and so $C_{\hat{G}}(f) \cong SL(2, K) \times SL(2, K) \times SL(2, K)$. So $Q = O^{r'}(C_{G_0}(f)) \ncong S_3$ and the earlier argument shows that $D_0 = G_0$. This completes the proof of (4.4).

At this point the proof of (4.1) is complete and to complete the proof of Theorem 1 for G a classical group we must consider the case where E_0 is nonabelian. To this end we assume that Theorem 1 holds for groups of order less than $|G|$.

We assume E_0 nonabelian. Then $E_0' \cong Z_p$ and we may write $E_0 = E_1 E_2 E_3$, where E_1 is 1 or extraspecial of exponent p, $E_2 = Z(E_0)$, and $E_3 = 1$ or E_3 is nonabelian with a maximal cyclic subgroup. Since p is odd, $G_0 = SL(n, q)$ or $SU(n, q)$. The main result is the following.

(4.5) One of the following holds:

(i) $D_0 = G_0$.

(ii) $G_0 \cong SL(3^k, 4)$ and $p = 3$.

(iii) $G_0 \cong SU(n, 2)$ and $p = 3$.

In cases (ii) and (iii) there is a group $F_0 \leqslant G_0$ such that $F_0 \geqslant Z(G_0)$, $F_0/Z(G_0)$ is an elementary abelian 3-group and $C^0_{G_0}(F_0) \leqslant N_{G_0}(F_0)$.

We will prove (4.5) in several steps. Assume the result false and let G be a minimal counterexample.

(4.6) E_0 acts irreducibly on V.

PROOF. Suppose false and let V_1 be a proper E_0-invariant subspace of V with V_1 irreducible. By (2.3), $G_0 \ncong SL(n, q)$ and $\mathrm{rad}(V_1) = 0$. Then $V = V_1 \perp V_1^{\perp}$ is E_0-invariant. Continue in this way, obtaining $V = V_1 \perp \cdots \perp V_k$, with each V_i E_0-invariant and irreducible. As each V_i is a faithful module for $E_1 E_3$ we have $\dim(V_i) = k_i p^a$, where $|E_0 : E_2| = p^{2a}$. We may assume $q > 2$, for otherwise $Z(G_0) \cong Z_3$, $p = 3$, and (iii) holds. Suppose $k > 2$. By minimality $\mathfrak{J}(V_1 \perp \cdots \perp V_{k-1})$ and $\mathfrak{J}(V_2 \perp \cdots \perp V_k)$ (notation as before) are each in D_0, so we have $D_0 = G_0$ by (3.1). Therefore we assume $k = 2$.

Let $\langle z \rangle = \Omega_1(Z(E_0))$ and let g be the element of $GU(V)$ that is trivial on V_1 and induces z on V_2. Then $[E_0, g] = 1$, $|g| = p$, and we may assume $g \in E_0$ (see (2.9)). Let e be an element of order p in $E_1 E_3 - Z(E_0)$ and set $F = \langle e \rangle \times \langle g \rangle$. Now consider $C^r_{G_0}(F)$. Notice that $q \neq 2$, $q \neq 3$ (as $p \mid q + 1$), and $q = 4$ implies $p = 5$. Since E_0 centralizes F modulo $Z(G_0)$ we can apply induction, (2.6), and (2.16) to conclude that $C^r_{G_0}(F) \leqslant D_0$. Suppose that $C^r_{G_0}(F)$ stabilizes $\{V_1, V_2\}$. As $q \geqslant 4$, (2.6) implies that $C^r_{G_0}(F)$ has no subgroup of index 2 and so $C^r_{G_0}(F)$ acts on V_1 and on V_2. But checking the eigenspaces of e on V we see that $O^{r'}(C_{G_0}(e))$ does not fix V_1 and V_2. This is a contradiction. Therefore $C^r_{G_0}(F)$ does not stabilize $\{V_1, V_2\}$, neither does D_0, and by (3.2) we have $D_0 = G_0$. This proves (4.6).

In view of (4.6) we now have E_0 irreducible on V, and consequently E_2 is cyclic. From the representation theory of E_0 we have $\dim(V) = p^a$ or p^{a+1}, where $|E_1E_3: Z(E_1E_3)| = p^{2a}$ (use the fact that $E_2^p \leq Z(G_0)$). In fact, $\dim(V) = p^a$ if and only if E_2 is diagonalizable on V. In view of these facts we now assume $q > 4$ if $G_0 \cong SL(n, q)$ and $q \geq 4$ if $G_0 \cong SU(n, q)$.

(4.7) (i) $\dim(V) = p$;

(ii) $[E_2, G_0] = 1$; and

(iii) $E_1E_3 = E_1$ or E_3, with $|E_1E_3: Z(E_1E_3)| = p^2$.

PROOF. Suppose $\dim(V) \geq p^2$. Choose $e \in E_1E_3 - Z(E_1E_3)$ with $|e| = p$. From the representation theory of E_0 we see that $\langle e \rangle$ induces a multiple of the regular representation on V. Write $V = V_1 \oplus \cdots \oplus V_p$, where $\{V_1, \ldots, V_p\}$ is the collection of eigenspaces of e. Choose f of order p such that $\langle e, f \rangle$ is extraspecial of order p^3. Redefining E_1, if necessary, we may assume $f \in E_1$ and write $E_1E_3 = \langle e, f \rangle E_4$ with $[\langle e, f \rangle, E_4] = 1$.

If $G_0 = SL(V)$, then $\langle e, f \rangle$ reducible on V implies $\langle e, f \rangle$ normalizes a parabolic subgroup of G_0. So (2.3) implies that $C^r_{G_0}(\langle e, f \rangle) = G_0$. Then induction, (2.6), and (2.16) imply that $G_0 = C^r_{G_0}(\langle e, f \rangle) \leq D_0$. So we now assume $G_0 \cong SU(V)$. Now $\langle f \rangle$ transitively permutes $\{V_1, \ldots, V_p\}$ and an easy computation shows that $V = V_1 \perp \cdots \perp V_p$.

Choose $g \in E_4 - Z(E_0)$. Then $\langle e, g \rangle$ is abelian of rank 2. The inductive argument of the last paragraph will work once we show $C^r_{G_0}(\langle e, g \rangle) = G_0$. For this first consider $\langle e, g \rangle$ acting on $V_1 \perp V_2$. We have $\mathfrak{J}(V_1) \times \mathfrak{J}(V_2) \leq C^r_{G_0}(\langle e, g \rangle)$. The element g has p eigenvalues on V and $\dim(V_1 \perp V_2) \geq 2p$. It follows that $O^{r'}(C(g)) \cap \mathfrak{J}(V_1 \perp V_2)$ does not stabilize $\{V_1, V_2\}$. Consequently $\mathfrak{J}(V_1 \perp V_2) \leq C^r_{G_0}(\langle e, g \rangle)$ by (3.2). Similarly, $\mathfrak{J}(V_i \perp V_j) \leq C^r_{G_0}(\langle e, g \rangle)$ for each $i \neq j$. Now, repeated use of (3.1) gives $G_0 = C^r_{G_0}(\langle e, g \rangle)$. We have now proved that $\dim(V) = p$, so (i) holds. Since E_1E_3 is absolutely irreducible on V, (ii) follows. Also, (iii) follows from (i).

In view of (4.7)(iii) we alter our notation, if necessary, so that $E_0 = E_1E_2$, where $[E_2, G_0] = 1$, $|E_1: Z(E_1)| = p^2$, and E_1 acts irreducibly on V.

(4.8) $E_1E_3 = E_3$.

PROOF. Assume $G_0 \cong SL(p, q)$ or $SU(p, q)$ and $E_1E_3 \neq E_3$. Then $E_1E_3 = E_1 = \langle e, f \rangle$, where $|e| = |f| = p$. Recall that $q \geq 4$ and $q > 4$ if $G_0 \cong SL(n, q)$. As $p \mid q - 1$ or $p \mid q + 1$ according to $G_0 \cong SL(V)$ or $SU(V)$ we can diagonalize e and f. Write $V = W_1 \oplus \cdots \oplus W_p$, where $W_i = \langle v_i \rangle$ is the eigenspace for eigenvalue α^{i-1} of e. Here α is an element of order p in $\mathbf{F}_q^{\#}$ ($\mathbf{F}_{q^2}^{\#}$ in the unitary case), and f may be chosen so that $v_i^f = v_{i+1}$ for $i = 1, \ldots, p - 1$. We get a basis of eigenvectors for f by setting $w_1 = v_1 + \cdots + v_p$ and $w_i = w_{i-1}^{e^{-1}}$ for $i = 2, \ldots, p$. Then $w_i^f = \alpha^{i-1} w_i$.

Now $C_{G_0}(e)$ and $C_{G_0}(f)$ stabilize the eigenspaces of e and f, respectively. So $C_{G_0}(e)$, $C_{G_0}(f)$ consists of all diagonal matrices in G_0 with respect to the ordered bases $\{v_1, \ldots, v_p\}$, $\{w_1, \ldots, w_p\}$. Choose $h \in C_{G_0}(e)$ having eigenvalues $\beta, \gamma, \delta, \ldots, \delta$ in the basis $\{v_1, \ldots, v_p\}$ and $k \in C_{G_0}(f)$ with eigenvalues $\varepsilon, \eta, \ldots, \eta$ in the basis $\{w_1, \ldots, w_p\}$. These elements must be chosen so that $\beta\gamma\delta^{p-2} = \varepsilon\eta^{p-1} = 1$.

Consider the basis $\{v_1, \ldots, v_{p-1}, w_1\}$ of V and check that in this ordered basis h, k are given by the following matrices:

$$h = \begin{pmatrix} \beta & & & & & & \\ & \gamma & & & & & \\ & & \delta & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ \beta-\delta & \gamma-\delta & 0 & \cdot & \cdot & 0 & \delta \end{pmatrix},$$

$$k = \begin{pmatrix} \eta & & & & \frac{1}{p}(\varepsilon-\eta) \\ & \cdot & & & \cdot \\ & & \cdot & & \vdots \\ & & & \eta & \frac{1}{p}(\varepsilon-\eta) \\ & & & & \varepsilon \end{pmatrix}.$$

Notice that h and k each stabilize $V_0 = \langle v_1, v_2, w_1 \rangle$ and $V_1 = \langle v_3 - v_4, \ldots, v_3 - v_p \rangle$, and $V = V_0 \oplus V_1$ unless $p \equiv 2 \pmod r$ and $V_0 \cap V_1 = \langle v_3 + \cdots + v_p \rangle$. In the unitary case $[W_i, W_j] = 0$ if $i \neq j$ and $[V_0, V_1] = 0$.

Let X be the subgroup of G_0 generated by all such elements h, k above, and let X_0 and X_1 denote the restrictions of X to V_0 and V_1, respectively. Then X_0 contains all matrices of the form

$$\begin{pmatrix} \beta & 0 & 0 \\ 0 & \gamma & 0 \\ \beta-\delta & \gamma-\delta & \delta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \eta & 0 & \frac{1}{p}(\varepsilon-\eta) \\ 0 & \eta & \frac{1}{p}(\varepsilon-\eta) \\ 0 & 0 & \varepsilon \end{pmatrix},$$

where $\beta\gamma\delta^{p-2} = \varepsilon\eta^{p-1} = 1$. Considering the action of the elements h above we see that an X_0-invariant subspace is necessarily a sum of the spaces $\langle v_1 \rangle, \langle v_2 \rangle, \langle v_3 + \cdots + v_p \rangle$. Now consider the action of all possible elements k, with k as above. We conclude that either X_0 is irreducible on V_0 or $\langle v_3 + \cdots + v_p \rangle$ is invariant. In the latter case either $p \equiv 2 \pmod r$ or $\varepsilon = \eta$ for all elements k above, this forcing $p = q - 1$ or $p = q + 1$, depending on whether $G_0 \cong SL(V)$ or $SU(V)$.

Suppose X_0 is irreducible on V_0, and consider the image $\bar{X}_0$ of X_0 in $PGL(3, q)$ (respectively $PGU(3, q)$). Using the results of Bloom **[5]** and Hartley **[14]** we show that $\bar{X}_0 \geqslant PSL(3, q)$ (respectively $PSU(3, q)$). This involves checking the group $\bar{X}_0$ against the lists of groups presented in **[5]** and **[14]**. The following facts are useful in such a check. Let I be the group of all elements h above, restricted to V_0. Then $I \cong Z_{q\mp 1} \times Z_{q\mp 1}$ and has image in $PGL(3, q)$ ($PGU(3, q)$) isomorphic to $(Z_{q\mp 1} \times Z_{q\mp 1})/Z_p$. If $p = 3$, $\bar{I}$ consists of matrices of determinant 1, while if $p > 3$, $\bar{I} \cap PSL(3, q)$ ($\bar{I} \cap PSU(3, q)$) has index at most 3. Choosing $\bar{h} \in \bar{I}$ with $|h| = p$ and h not scalar on V_0 we have $\bar{I} \leqslant C_{\bar{X}_0}(\bar{h})$. Comparing this with centralizers in the various proper subgroups of $PSL(3, q)$ ($PSU(3, q)$), most cases are eliminated. Also one should recall that $\bar{X}_0$ is irreducible on V_0, that $\bar{k} \in \bar{I}$ only when $\bar{k}$ is scalar on V_0, and that when $p = 3$ we may take $\bar{h} = \bar{e}$ and conclude that $C_{\bar{X}_0}(\bar{e})$ contains a subgroup isomorphic to $\bar{I}$, for any $e \in E_1 - Z(E_1)$.

Once we have $\bar{X}_0 \geqslant PSL(3, q)$ $(PSU(3, q))$ we conclude that X' contains $(C(V_1) \cap N(V_0))'$. This is because X induces a group of scalar matrices on V_1. Now the same argument can be carried out with $\langle v_i, v_j, w_1 \rangle$ replacing V_0, where $i \neq j$ and $i, j \leqslant p - 1$. It is now fairly easy to see that $D_0 = G_0$. If $G_0 = SU(p, q)$ this follows from repeated use of (3.1). For $SL(p, q)$ one can use (B, N)-pairs.

Next suppose $p \equiv 2 \pmod r$. Here r is odd and $V_0 \cap V_1 = \langle v_3 + \cdots + v_p \rangle$. On $V_0/\langle v_3 + \cdots + v_p \rangle$ X_0 contains all matrices of the form

$$\begin{pmatrix} \beta & 0 \\ 0 & \gamma \end{pmatrix} \quad \text{and} \quad \frac{1}{2}\begin{pmatrix} \eta + \varepsilon & \eta - \varepsilon \\ \eta - \varepsilon & \eta + \varepsilon \end{pmatrix},$$

where $\gamma = \beta^{-1}$ and $\eta^{p-1}\varepsilon = 1$. Considering subgroups of $PSL(2, q)$ we conclude that either $q \pm 1 = 2p$ or X_0 induces on $V_0/\langle v_3 + \cdots + v_p \rangle$ a subgroup of $GL(2, q)$ (respectively $GU(2, q)$), containing $SL(2, q)$. Excluding the case $q \pm 1 = 2p$ and using the fact that q is odd, we conclude $X_0^{(\infty)}$ contains a subgroup $Y \cong SL(2, q)$ and that there exists $V_2 < V_0$ such that V_2 is Y-invariant and $V_0 = V_2 \oplus (V_0 \cap V_1)$ (an orthogonal decomposition in the unitary case). As $X^{(\infty)}$ centralizes V/V_0, $C_V(Z(Y))$ is a complement to V_2 in V, and so Y is generated by groups of transvections in G_0 (centers of long root subgroups). So by McLaughlin **[19]** or Wagner **[28]** we have $D_0 = G_0$.

We are left with the case $p \equiv 2 \pmod r$ and $q + 1 = 2p$, and the case $p = q \pm 1$. In the first case G_0 is unitary, $r = 3$, and $q = 3^{2^b}$ for some integer b, while in the second case we have $r = 2$. In either case, let $\hat{V}_0 = \langle v_1, v_2, w_1, w_2 \rangle$ and let Y be the subgroup of D_0 generated by all elements h_1, h, k_1, k, where h_1, h are diagonal $(1, -1, \ldots, -1)$ and $(\gamma^{-1}, \gamma, 1, \ldots, 1)$ in the ordered basis $\{v_1, \ldots, v_p\}$, and k_1, k are the same, but in the ordered basis $\{w_1, \ldots, w_p\}$.

For $\gamma \neq 1$, each of h, h have fixed space in $\hat{V}_0$ of dimension 2. Indeed, h centralizes $\langle v_3 + \cdots + v_p,\ \alpha^{-2}v_3 + \cdots + \alpha^{-(p-1)}v_p \rangle$ and k centralizes $\langle w_3 + \cdots + w_p,\ \alpha^{-2}w_3 + \cdots + \alpha^{-(p-1)}w_p \rangle$. Using these facts one checks that Y is irreducible on $\hat{V}_0$. In fact, if D is the subgroup of Y generated by those h, k for which $|\gamma| = p$, then D is absolutely irreducible on $\hat{V}_0$ (consider eigenspaces of h, k). Now, Y centralizes a $(p - 4)$-subspace of V, so $V = \hat{V}_0 \oplus \hat{V}_1$, with Y trivial on $\hat{V}_1$. The sum is orthogonal in case G_0 is a unitary group. We identify Y with its restriction to $\hat{V}_0$ and note that by McLaughlin **[19]** and Wagner **[28]** we may assume that Y does not contain a full group of transvections.

First suppose that $p = q \pm 1$. Here we apply the result of Mwene **[20]** to get a contradiction, although a couple of remarks are in order. Firstly, for the $SU(4, q)$ case, we cannot simply apply Mwene's main theorem on subgroups of $SL(4, q^2)$, because that result only gives the maximal subgroups of $SL(4, q^2)$. However, the proof in **[20]** actually describes all subgroups of $SL(4, q^2)$. Secondly, in checking Y against the proper subgroups of $SL(4, q)$ (resp. $SL(r, q^2)$), our previous remarks are sufficient to rule out all but one possibility—the case where Y preserves a nondegenerate symplectic form on $\hat{V}_0$. So, suppose Y fixes such a form, [,]. Then $pv_1 = w_1 + \cdots + w_p$ implies that $0 = [w_3 + \cdots + w_p, pv_1 - w_1 - w_2] = [w_3 + \cdots + w_p, v_1] + [w_3 + \cdots + w_p, w_1] + [w_3 + \cdots + w_p, w_2]$. Checking eigenspaces of the elements k above, we see that $w_3 + \cdots + w_p$ must be orthogonal (under [,]) to $\langle w_1, w_2 \rangle$, and so $[w_3 + \cdots + w_p, v_1] = 0$. This implies $w_3 + \cdots + w_p \in \langle v_1, v_3 + \cdots + v_p, \alpha^{-2}v_3 + \cdots + \alpha^{-(p-1)}v_p \rangle$, which is false. So Y fixes no such form, and the case $p = q \pm 1$ is out.

Suppose $q + 1 = 2p$. Choose h as above with $|h| = q + 1$ and write $h = h_2 h_p$, where $|h_2| = 2$, $|h_p| = p$, and $[h_2, h_p] = 1$. Similarly, for $k = k_2 k_p$. Then $D = \langle h_p, k_p \rangle$ and previous remarks show that $C_I(D) \leqslant Z(I) \cong Z_2$, where $I = SU(4, q)$. Since the Sylow p-subgroups of I are abelian, this immediately implies $O_p(D) = 1$.

Let $s \neq p$ be prime. We claim $O_s(Y) \leqslant Z(I) \cong Z_2$. Suppose otherwise, and let J be minimal normal in Y with $J \leqslant O_s(Y)$. As D is absolutely irreducible, $s \neq r$ and one of $[J, h_p]$ or $[J, k_p]$ is nontrivial. By symmetry, we assume the former. If $s \neq 2$, then $[J, h_p]\langle h_p \rangle$ is a Frobenius group and standard arguments from representation theory imply $\dim(\hat{V}_0) \geqslant p$, a contradiction. The same argument can be used when $s = 2$, unless J is extraspecial. In the latter case, set $J_0 = [J, h_p] \cap C(h_2)$. Then $J_0 \nleqslant Z(I)$ and $\langle h_p \rangle$ is fixed-point-free on $J_0/Z(I)$. But h_p normalizes no such 2-group in $C_I(h_2)$. This proves the claim, from which it follows that $F^*(Y) = E(Y)$ or $E(Y)Z(I)$.

The group $C_I(h_2)$ satisfies $C_I(h_2)^{(\infty)} \cong SL(2, q) \circ SL(2, q)$, with h in one of the factors and h_1 inducing an outer diagonal automorphism on this factor. From earlier assumptions, neither factor is contained in Y. Therefore, $\langle h_p \rangle \leqslant O_p(C_Y(h_2))$.

The restriction $q + 1 = 2p$ forces $PSU(4, q)$ to have sectional 2-rank 4. In particular, $E(Y)$ is the product of at most 2 components and if there are two, then they are each of sectional 2-rank 2. Each of h_p and k_p must stabilize each component of Y, inducing a nontrivial automorphism on at least one component. From the above paragraph we conclude the components of Y are normalized by $\langle h, k \rangle$.

By Gorenstein-Harada [**31**], the components of Y have known structures. Also, we have $\langle h_p \rangle \leqslant O_p(C_Y(h_2))$ and similarly for k_2. It follows (use [**3**] and Table 1 of [**29**]) that each component of D is of Lie type in odd characteristic. Since $\dim(\hat{V}_0) = 4$, the Sylow s-subgroups of Y are abelian for all primes $s \neq 3$. Assume that J is a component of Y and of Lie type in characteristic $s > 3$. By the above, $J \cong SL(2, s^c)$ or $PSL(2, s^c)$. As $PSL(2, s^c)$ contains a Frobenius group of order $\frac{1}{2}s^c(s^c - 1)$, and since $\dim(\hat{V}_0) = 4$, we must have $s^c \leqslant 9$. However, $p \mid |J|$ and $s \neq 3$. This forces $s^c = 5 = p$, and contradicts the facts $\langle h_p \rangle \leqslant O_p(C_Y(h_2))$ and $\langle k_p \rangle \leqslant O_p(C_Y(k_2))$.

At this point we have established that the components of Y are of Lie type in characteristic 3. Each of h_p and k_p induce inner automorphisms on $E(Y)$. For otherwise, a field automorphism is induced and we obtain a contradiction, as before, by considering certain Frobenius groups. Therefore $h_p, k_p \in E(Y)$. If Y has two components, X_1 and X_2, then each has the form $SL(2, 3^e)$ or $PSL(2, 3^e)$ and each has order divisible by p. As X_1 centralizes a Sylow 3-subgroup, R, of X_2, $R \times X_2$ is in a proper parabolic subgroup of I. One checks that this forces R to consist of transvections and $X_2 = SL(2, 3^e) = SL(2, q)$ is generated by full groups of transvections. This is a contradiction. Thus, $E(Y)$ is quasisimple.

Suppose $E(Y)/Z(E(Y)) \cong PSL(2, 3^e)$. Since $h_p \in E(Y)$, $p \mid 3^e \pm 1$ and this forces $q \mid 3^e$ (use primitive divisors). As $C(h_p) \cap YZ(I) \geqslant Z(I) \times \langle h_2 \rangle \times \langle h_1 \rangle$, and since $YZ(I)/Z(I) \leqslant \mathrm{Aut}(PSL(2, 3^e))$, some element of $\langle h_1, h_2 \rangle$ induces an involutory field automorphism of $E(Y)$. As $\langle h_p \rangle \leqslant O_p(C_Y(h_2))$, this element cannot be h_2. Hence both h_1 and h_1h_2 induce field automorphisms. The containment $E(Y) \leqslant SU(4, q)$, forces $3^e = q^2$, so $E(Y) \cong SL(2, q^2)$ or $PSL(2, q^2)$. Now, $E(C_I(h_1)) = SU(3, q)$, with natural action on $\langle v_1 \rangle^{\perp} \cap V_0$. We may write $J = O^{p'}(E(Y) \cap C(h_1))$ and $J = \langle h_p, h_p^j \rangle$, for some $j \in J$. Then $J \cong SL(2, q)$ or $PSL(2, q)$. But h_p is trivial on a 2-space of $\langle v_1 \rangle^{\perp} \cap V_0$, so J is trivial on a 1-space. It then follows that $J \cong SL(2, q)$ and is generated by full groups of transvections. This is a contradiction.

The proper parabolic subgroups of I have at most one noncyclic composition factor and this is isomorphic to $PSL(2, q)$ or to $PSL(2, q^2)$. It follows that $E(Y)/Z(E(Y)) \cong PSL(3, 3^e)$, $PSU(3, 3^e)$, $PSL(4, 3^e)$, $PSU(4, 3^e)$, or $G_2(3^e)$ with e a power of 2. Since $h_p \in E(Y)$, an order argument shows that either $a \mid e$ or $E(Y)$ is a 4-dimensional symplectic or unitary group with $a \mid 2e$. As $E(Y) < I$, we use additional order arguments to conclude $E(Y)/Z(E(Y)) \cong PSU(3, q)$, $PSp(4, q)$, or $PSp(4, \sqrt{q})$. In the first two cases, let j be an involution in $E(Y) - Z(I)$ with $j(Z(E(Y)))$ 2 central. Then $C_{E(Y)}(j)$ contains an $SL(2, q)$ component with center $\langle j \rangle$. This component is necessarily one of $C_I(j)$ (as $j \sim h_2$ in I), hence generated by full groups of transvections. We are assuming this false, so these cases are out. For the last case one argues that in $\mathrm{Aut}(PSp(4, \sqrt{q}))$ the centralizer of an element of order p is cyclic of order $2p$. As $\langle h_1, h_2 \rangle \leqslant C(h_p)$, there is some involution $h \in \langle h_1, h_2 \rangle \cap C(E(Y))$. But then $C_I(j)$ has a section isomorphic to $PSp(4, \sqrt{q})$, whereas $E(C_I(j)) \cong SU(3, q)$ or $SL(2, q) \times SL(2, q)$. This is a contradiction, completing the proof of (4.8).

(4.9) $E_1E_3 \neq E_3$.

PROOF. Suppose $E_1E_3 = E_3$ and write $E_3 = \langle e, f \rangle$ with $|e| = p$ and $|f| = p^n$. Recall that $G_0 \ncong SL(3, 4)$. Let $1 \neq \alpha$ be an eigenvalue of e on V and let $V_i = \langle v_i \rangle$ be the eigenspace of e with corresponding eigenvalue α^{i-1}. First suppose that $p^n \mid q - 1 \mid$ ($p^n \mid q + 1$ in the unitary case). Then f is diagonalizable with eigenvalues $\beta, \beta\alpha, \ldots, \beta\alpha^{p-1}$, where β has order p^n in the multiplicative group of the underlying field ($\mathbf{F}_q$ or $\mathbf{F}_{q^2}$). So there is a scalar transformation z on V such that $|fz| = p$ and fz has distinct eigenvalues. Replacing E_3 by $\langle e, fz \rangle$ we reduce to (4.8). So from now on we may assume $p^n \nmid q - 1$ ($p^n \nmid q + 1$ in the unitary case).

This forces $\langle f \rangle$ to act irreducibly on V. In an appropriate extension field, f has the eigenvalues above, so $f \notin G_0$. Now $GL(p, q) = G_0Z\langle f \rangle$ ($GU(p, q) = G_0Z\langle f \rangle$), where $Z = Z(GL(p, q))$ ($Z(GU(p, q))$). So $H = C(f) \cap GL(p, q)$ (respectively, $H = C(f) \cap GU(p, q)$) is contained in $I = N(D_0)$. Indeed, $(H \cap G_0)Z\langle f \rangle \leqslant D_0Z\langle f \rangle \leqslant N(D_0)$. We may assume $I \ngeqslant SL(p, q)$ ($I \ngeqslant SU(p, q)$).

Now H is cyclic of order $q^p - 1$ $(q^p + 1)$, so I is transitive on 1-spaces of V if $G_0 \cong SL(p, q)$.

Let $W = C(e) \cap GL(p, q)$ (respectively $C(e) \cap GU(p, q)$) and $W_1 = O_p(W)$. We claim that $W \leqslant I$. To see this first note that $W_1 \cong (Z_{p^{n-1}})^p$ and has index p in a Sylow p-subgroup of $GL(p, q)$ $(GU(p, q))$. In fact $W_1\langle f \rangle \cong Z_{p^{n-1}} \operatorname{wr} Z_p$ is a Sylow p-subgroup of $G_0\langle f \rangle$. As $W \leqslant G_0 Z W_1$, it suffices to show $W_1 \leqslant I$. Let $W_0 = W_1 \cap G_0$, so $W_0 \leqslant D_0$. Suppose $W_0 \in \mathrm{Syl}_p(D_0)$. Since $O_p(Z(G_0)) < W_0$ we can apply easy transfer and fusion arguments to get $D_0 = D_1 \times O_p(Z(G_0))$ for some subgroup D_1. Since f normalizes W_0 but no proper subgroup of W_0 not containing $O_p(Z(G_0))$ we must have $W_0 \cap O^p(D_0) = 1$. That is $D_0 = O_{p'}(D_0)W_0$. Let s be a prime such that $s \mid q^p - 1$ (resp. $s \mid q^{2p} - 1$) but $s \nmid r^a - 1$ for $r^a < q$ (resp. $r^a < q^2$). Let $S \in \mathrm{Syl}_2(H)$. Then $S \leqslant G_0$, so $S \leqslant D_0$, and a Frattini argument shows that S is normalized by a conjugate of W_0. But $N(S)$ has p-rank 2, so $m_p(W_0) \leqslant 2$. Since $m_p(W_0) = p - 1$, this is a contradiction unless $p = 3$. But $p = 3$ is eliminated by considering the lists in Hartley [**14**] and Bloom [**5**]. So we now assume $p > 3$ and $W_0 \notin \mathrm{Syl}_p(D_0)$. If $W_0 < \hat{W}_0 \in \mathrm{Syl}_p(D_0)$, then there exists $g \in \hat{W}_0$ with $e^g = ez$ for some $1 \neq z \in Z(G_0)$. We may choose z such that $e^{gf} = e$. Now take $h \in W_1 - W_0$ such that $|fh| = p$. Then $\langle e, fh \rangle$ is extraspecial of exponent p, $fh \in G_0$, and $gfh \in C_{G_0}(e) \leqslant D_0$. So $gfh \in I$, and since $g, f \in I$, $h \in I$. But $W_1 = \langle W_0, h \rangle$ implies that $W_1 \leqslant I$ as claimed.

Suppose $G_0 \cong SL(p, q)$. Let K be the subgroup of W that is trivial on $\langle v_2, \ldots, v_p \rangle$. Then $[V, K] = \langle v_1 \rangle$. Let $A = \langle v_1, v_2 \rangle$. As I is transitive on 1-spaces, for each 1-space $\langle v \rangle \leqslant A$, there is a subgroup $K^g \leqslant I$ such that $[K^g, V] = \langle v \rangle$. Let X be the subgroup of I generated by all such K^g. Then $X|_A$ contains all diagonal matrices in the basis $\{v_1, v_2\}$. It follows that $X|_A$ contains all matrices of form $\left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right)$ for $x \in \mathbf{F}_q$, where the matrices are taken with respect to either the ordered basis $\{v_1, v_2\}$ or the ordered basis $\{v_2, v_1\}$. In either case one can then argue that X contains a full group of transvections. Therefore $D_0 = G_0$ by McLaughlin [**19**].

We are left with the case $G_0 = SU(p, q)$. The argument is similar to the above. As before let K be the subgroup of W centralizing $\langle v_2, \ldots, v_p \rangle$. Suppose that, for some $g \in I$, $\mathrm{rad}(\langle v_1, v_1^g \rangle) = 0$ and $v_1^g \notin v_1^\perp$. Set $A = \langle v_1, v_1^g \rangle$. Then $X = \langle K, K^g \rangle$ acts on A and on $A^\perp$. As X centralizes V/A and $V = A \perp A^\perp$ we conclude that X is trivial on $A^\perp$. Now K is transitive on the isotropic 1-spaces of A (since for each element $1 \neq k \in K$, k stabilizes only the spaces $\langle v_1 \rangle$ and $C_A(K)$, neither of which is isotropic), so it follows that $X|_A \geqslant SL(2, q)$. So I contains a group of transvections and $I \geqslant G_0$ by Wagner [**28**]. This is a contradiction. So now suppose that $v_1^g \in v_1^\perp = \langle v_2, \ldots, v_p \rangle$ whenever $\mathrm{rad}(\langle v_1, v_1^g \rangle) = 0$ and $g \in I$.

Recall that $C(f) = H \leqslant I$, and it is irreducible. It is then possible to choose $g \in I$ such that $v_1^g \notin \langle v_1 \rangle$ and $v_1^g \notin v_1^\perp$. Consequently $\mathrm{rad}(\langle v_1, v_1^g \rangle) \neq 0$. So $v_1^g = \beta v_1 + a$ for some $0 \neq a \in \langle v_2, \ldots, v_p \rangle$. Choose $i \geqslant 2$ such that $(a, v_i) \neq 0$, and set $A = \langle v_1, v_1^g, v_i \rangle$. Then $\mathrm{rad}(A) = 0$. Now $C(e)$ contains a subgroup I_0 isomorphic to $Z_{q+1} \times Z_{q+1} \times Z_{q+1}$ such that I_0 is faithful and diagonalizable on A and stabilizes $A^\perp$. Using the results of Hartley [**14**] and Bloom [**5**] we conclude that $X = \langle H_0, K^g \rangle$ contains $SL(2, q)$ (and generated by groups of transvections). As before the results of Wagner [**28**] imply $G_0 \leqslant I$. This is a contradiction, proving (4.9).

The proof of (4.5) will be complete once we show that for $G_0 \cong SL(3^k, 4)$ and $SU(n, 2)$ there is a 3-group $E_0 < G_0$ such that $C^0_{G_0}(E_0) \leqslant N_{G_0}(E_0)$. For $G_0 \cong SU(n, 2)$ this follows from (4.1). So suppose $G_0 \cong SL(3^k, 4)$. Let E_0 be an extraspecial 3-group of order 3^{2k+1} and of exponent 3. Then we can consider $E_0 < G_0$. If V is the module affording this representation, then each $e \in E_0 - Z(E_0)$ has 3 distinct eigenspaces on V of dimension 3^{k-1}. Using this and the fact that $C_{E_0}(e)$ is absolutely irreducible on each of these eigenspaces, we conclude that $C^0_{G_0}(E_0) \leqslant N_{G_0}(E_0)$. This completes the proof of (4.5).

5. Exceptional groups of Lie type. In this section G will denote either an exceptional group of Lie type or one of the groups $G_2(2)'$ or ${}^2F_4(2)'$. Notation will be as in §2. Namely, $\overline{G}$ is a simple algebraic group with $G = O^{r'}(\overline{G}_\sigma)'$. By (2.15), $\overline{G}_\sigma$ is G together with all diagonal automorphisms of G. We take $E \leqslant \overline{G}_\sigma$ and define $\hat{G}$, E_0, G_0, etc. as in §2. By (2.15) $|\overline{G}_\sigma : G|$ divides $|Z(\hat{G})|$. So except for the cases $p = 3$, $G \cong E_6(q)$ or ${}^2E_6(q)$, with $3 \mid q - 1$ or $3 \mid q + 1$, respectively, we necessarily have $E_0 \leqslant G_0$. We will prove

(5.1) With the above notation one of the following holds:

(i) $C^0_{G_0}(E_0) = G_0$.

(ii) $p = 3$ and $G \cong G_2(2)'$, $F_4(2)$, $F_4(4)$, $E_6(2)$, $E_6(4)$, ${}^2E_6(2)$, $E_7(2)$, $E_8(2)$, or $E_8(4)$. If $G \cong G_2(2)'$, then D_0 normalizes a Sylow 3-subgroup of G. In the other cases an example E_0 exists such that $C^0_{G_0}(E_0) \leqslant N_{G_0}(E_0)$.

(iii) $p = 5$, $G = {}^2F_4(2)'$, and $C^0_{G_0}(E_0) \leqslant N_{G_0}(E_0)$.

(iv) $p = 3$, $G = {}^2F_4(2)'$ and for $E_0 \cong Z_3 \times Z_3$ we have $C^0_{G_0}(E_0) \cong \mathrm{Aut}(L_3(3))$. On the other hand if $G_1 = {}^2F_4(2)$, then $C^0_{G_1}(E_0) = G_1$.

In this section we will prove (5.1). So assume (5.1) false and let G be a counterexample of minimal order. Let $\Phi_d(x)$ denote the cyclotomic polynomial with roots the primitive complex dth roots of 1. Then set $\varphi_d = \Phi_d(q)$. The group G and the order of G_0 is given in the following table.

TABLE (5.2)

G	$\lvert G_0 \rvert$
$G_2(q)$	$q^6\varphi_6\varphi_3\varphi_2^2\varphi_1^2$
${}^3D_4(q)$	$q^{12}\varphi_{12}\varphi_6^2\varphi_3^2\varphi_2^2\varphi_1^2$
${}^2F_4(q)$	$q^{12}\varphi_{12}\varphi_6\varphi_4^2\varphi_2^2\varphi_1^2$
$F_4(q)$	$q^{24}\varphi_{12}\varphi_8\varphi_6^2\varphi_4^2\varphi_3^2\varphi_2^4\varphi_1^4$
${}^2E_6(q)$	$q^{36}\varphi_{18}\varphi_{12}\varphi_{10}\varphi_8\varphi_6^3\varphi_4^2\varphi_3^2\varphi_2^6\varphi_1^4$
$E_6(q)$	$q^{36}\varphi_{12}\varphi_9\varphi_8\varphi_6^2\varphi_5\varphi_4^2\varphi_3^3\varphi_2^4\varphi_1^6$
$E_7(q)$	$q^{63}\varphi_{18}\varphi_{14}\varphi_{12}\varphi_{10}\varphi_9\varphi_8\varphi_7\varphi_6^3\varphi_5\varphi_4^2\varphi_3^3\varphi_2^7\varphi_1^7$
$E_8(q)$	$q^{120}\varphi_{30}\varphi_{24}\varphi_{20}\varphi_{18}\varphi_{15}\varphi_{14}\varphi_{12}^2\varphi_{10}^2\varphi_9\varphi_8^2\varphi_7\varphi_6^4\varphi_5^2\varphi_4^4\varphi_3^4\varphi_2^8\varphi_1^8$

We will say p is *associated* with φ_x if x is minimal subject to $p \mid \varphi_x$. Notice that if x is even, then φ_x divides $q^{x/2} + 1$. Also if p is associated with φ_x, then $p > x$. The next several lemmas deal with the easiest cases. These are the cases where the p-rank of G_0 is 1, where (2.3) applies, or where induction is easily applied. In effect, we reduce to the case where p is a small prime divisor of $q \pm 1$.

(5.3) Suppose p is associated with φ_x. None of the following can occur: $G = G_2(q)'$ and $x = 3, 6$; $G = {}^3D_4(q)$ and $x = 12$; $G = {}^2F_4(q)'$ and $x = 6, 12$; $G = F_4(q)$ and $x = 8, 12$; $G = {}^2E_6(q)$ and $x = 12, 18$; $G = E_6(q)$ and $x = 9, 12$; $G = E_7(q)$ and $x = 14, 18$; $G = E_8(q)$ and $x = 15, 20, 24$, or 30.

PROOF. This follows from (2.7)(ii) and (2.7)(iv). It is necessary to check that $p \nmid |\hat{W}|$, and then check that p does not divide the order of any maximal parabolic of G.

(5.4) Suppose p is associated with φ_x. None of the following can occur: $G = {}^2E_6(q)$ and $x = 8, 10$; $G = E_6(q)$ and $x = 4, 5, 8$; $G = E_7(q)$ and $x = 3, 4, 5, 7, 8, 9, 10, 12$; $G = E_8(q)$ and $x = 7, 9, 14, 18$.

PROOF. The idea is as follows. Say, for example, that $G = {}^2E_6(q)$ and $x = 8$ or 10. From (5.2) we conclude that a Sylow p-subgroup of G_0 has order dividing $q^4 + 1$ or $q^5 + 1$, respectively. Also G_0 contains $SO^-(8, q)'$ and $SU(6, q)$ and order considerations show that a Sylow p-subgroup of G_0 is contained in one of these subgroups. As each of the subgroups is contained in a proper parabolic subgroup of G_0, we are done by (2.3).

The same argument works for the other cases. Let $E_0 \leqslant P \in \mathrm{Syl}_p(G_0)$. Below are the triples (G, I, φ_x) where $P \in \mathrm{Syl}_p(I)$ and I is involved in a proper parabolic subgroup of G_0.

TABLE

G	I	φ_x
$E_6(q)$	$L_6(q)$	φ_5
$E_6(q)$	$SO^+(10, q)'$	φ_4, φ_8
$E_7(q)$	$E_6(q)$	$\varphi_3, \varphi_4, \varphi_9, \varphi_{12}$
$E_7(q)$	$SO^+(12, q)'$	$\varphi_5, \varphi_8, \varphi_{10}$
$E_7(q)$	$L_7(q)$	φ_7
$E_8(q)$	$E_7(q)$	$\varphi_7, \varphi_9, \varphi_{14}, \varphi_{18}$

(5.5) Suppose p is associated with φ_1 or φ_2 and that G has Lie rank at least 4. In addition, assume $p \neq 3$ if $G = {}^2E_6(q)$ or $F_4(q)$, $p \neq 5$ if $G = {}^2E_6(q)$ and $5 \mid q + 1$, $p \neq 3, 5$ if $G = E_6(q)$, and $p \neq 3, 5, 7$ if $G = E_7(q)$ or $E_8(q)$. Then $C^0_{G_0}(E_0) = G_0$.

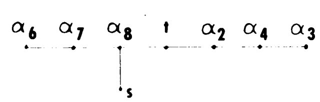
PROOF. Let s be the positive root of highest height in Σ and set $J_1 = \langle U_{\pm s} \rangle$. Choose conjugates $J_1, \ldots, J_k$ of $J_1 \cong SL(2, q)$ with k maximal such that $[J_i, J_j] = 1$ for $i \neq j$. This can be done so that each J_i is generated by a pair of opposite root subgroups of G_0 for roots in Σ. Then H_0 normalizes $J_1 \cdots J_k$. One checks that $k = 4$ if $G = F_4(q)$, ${}^2E_6(q)$, $E_6(q)$, $k = 7$ if $G = E_7(q)$, and $k = 8$ if $G = E_8(q)$. If $G = F_4(q)$, $E_7(q)$, or $E_8(q)$, then $O(H_0) \leqslant J_1 \cdots J_k$. In the other cases $J_1 \cdots J_k O(H_0) = J_1 \cdots J_k H_1$, where $H_1 = O(H_0) \cap C(J_1 \cdots J_k)$. So in all cases there is a subgroup $H_1 \leqslant H_0$ with $I = J_1 \cdots J_k O(H_0) = (J_1 \cdots J_k) \times H_1$.

From order considerations, keeping in mind the prime restrictions, we see that I contains a Sylow p-subgroup of G_0, so we may assume $E_0 \leqslant I$. Considering the projection of E_0 into $J_1 \cdots J_k$ we can apply (2.10) and get $I \leqslant C^0_{G_0}(E_0)$. In addition, for each $i = 1,\ldots,k$, E_0 acts on $G_i = E(C_{G_0}(J_i))$. Using §4 and induction we have $G_i \leqslant C^0_{G_0}(E_0)$ for $i = 1,\ldots,k$. An easy check gives $G_0 = \langle G_i\colon i = 1,\ldots,k\rangle$, proving the result.

(5.6) Suppose that p is associated with φ_x and one of the following holds: $G = E_6(q)$ and $x = 3$; $G = E_7(q)$ and $x = 6$; $G = E_8(q)$ and $x = 3,4$ or 6; $G = {}^2E_6(q)$ and $x = 6$. Then $D_0 = G_0$.

PROOF. First assume $x = 3$. If $G = E_6(q)$, then from Table (3.3) we see that G contains a central product $X_1X_2X_3$ of three copies of $SL(3,q)$. If $G = E_8(q)$, set $X_4 = \langle U_{\pm\alpha_8}, U_{\pm s}\rangle$, where $s \in \Sigma^+$ is the positive root of highest height. Then $X_4 \cong SL(3,q)$ and X_4 is centralized by $\langle U_{\pm\alpha_1},\ldots,U_{\pm\alpha_6}\rangle \cong E_6(q)$. So for $G = E_6(q)$ or $E_8(q)$, G_0 contains a central product $X = X_1\cdots X_k$ of copies of $SL(3,q)$, where $k = 3$ or 4 respectively. Now (5.2) implies that X contains a Sylow p-subgroup of G_0, so we may assume $E_0 \leqslant X$. A Sylow p-subgroup of X is abelian of rank k, and there is a subgroup $E_1 \cong Z_p \times Z_p$ such that $E_0 \leqslant C(E_1)$ and $E_1 \leqslant X_1X_2$. By (2.3) we have $G_0 = C^r_{G_0}(E_1)$ (since E_1 centralizes a unipotent element of $X_3^\#$). By (2.5), (2.6)(ii), and either minimality of G or the results of §4 we have $C^r_{G_0}(E_1) \leqslant C^0_{G_0}(E_0)$. This proves the result.

Essentially the same argument works for the other cases, but there are minor changes required. For $x = 4$, $G = E_8(q)$ and Table (3.3) shows that G_0 contains a central product of two copies of $D_4(q)$. Then G_0 contains a central product, X, of four copies of $O^-(4,q)' \cong L_2(q^2)$. From (5.2) one checks that a Sylow p-subgroup of X is also one for G_0 unless $p = 5$, in which case a Sylow 5-subgroup of X is of index 5 in one for G_0. Suppose this occurs. The arguments used in the verification of Table (3.3) suffice to show that $SU(5,q^2) \leqslant E_8(q)$. Namely, use the groups Y_1 and Y_2 (of §3) together with the fundamental system



where $t = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$. By orders we may take $E_0 \leqslant SU(5,q^2)$ and it follows that E_0 is conjugate to a subgroup of X. So in all cases we may take $E_0 \leqslant X$ and argue as before. The only difficulty is the possiblilty that $q = 2$, $p = 5$, and with E_1 as before, there is some $x \in E_1$ such that $C_{G_0}(x)$ contains a component of type $Sz(2^5)$ or ${}^2F_4(2)'$. The former case is out by (2.5). The latter case is also impossible, since in order to get an ${}^2F_4(2)'$ component of $C_{G_0}(x)$, $C_{\hat{G}}(x)$ would have to contain an $F_4(K)$ component. But this cannot happen since the components of $C_{\hat{G}}(x)$ all have root systems being a subset of the root system of type E_8.

For $x = 6$ again use Table (3.3) to conclude that ${}^2E_6(q)$ contains a central extension of $U_3(q) \times U_3(q) \times U_3(q)$, $E_7(q)$ a central extension of ${}^3D_4(q) \times L_2(q^3)$, and $E_8(q)$ a central extension of ${}^3D_4(q) \times {}^3D_4(q)$. We may assume that E_0 is contained in the appropriate central product (again by (5.2)). Now ${}^3D_4(q)$ contains a central extension of $L_2(q) \times L_2(q^3)$. So if $m_p({}^3D_4(q)) = 1$, this would force E_0 to centralize a nontrivial r-subgroup, whence (2.3) gives the result. So we may assume $m_p({}^3D_4(q)) = 2$ and choose a suitable E_1, as before.

In the next two results the contradiction is reached by combining the results of Table (3.3), (2.8), and the centralizer information in $\hat{G}$.

(5.7) Let p be associated with φ_x. In each of the following cases we have $C^0_{G_0}(E_0) = G_0$: $G = F_4(q)$ and $x = 4$; $G = {}^2E_6(q)$ and $x = 3, 4$; $G = E_6(q)$ and $x = 6$; $G = {}^3D_4(q)$ and $x = 1, 2$, but $p \neq 3$; $G = E_8(q)$ and $x = 8$.

PROOF. Suppose false. First we point out the existence of a large subgroup, X, of G. These are presented as triples $(G, \varphi_x, X/Z(X))$ and are as follows: (i) $(F_4(q), \varphi_4, PSO(9, q)')$; (ii) $({}^2E_6(q), \varphi_3, F_4(q))$; (iii) $({}^2E_6(q), \varphi_4, PSO^-(10, q)')$; (iv) $(E_6(q), \varphi_6, F_4(q))$; (v) $({}^3D_4(q), \varphi_1$ or $\varphi_2, G_2(q))$; (vi) $(E_8(q), \varphi_8, PSO^+(16, q)')$. In cases (i), (iii), and (vi) the existence of X follows from consideration of the extended Dynkin diagram of G. In the other cases X is obtained from the centralizer of a graph automorphism of G_0. In all cases X is generated by long root subgroups of G_0. Moreover from (5.2) we check that X contains a Sylow p-subgroup of G_0, so we take $E_0 \leqslant X$. We note that $m_p(E_0) = 2$. By minimality of G and by results of §4 we have $X \leqslant C^0_{G_0}(E_0)$. Let $D_0 = C^0_{G_0}(E_0)$.

Let V be a long root subgroup of G_0 with $V \leqslant X \leqslant D_0$. Set $I = \langle V^{G_0} \cap D_0 \rangle$. Then $D_0 \leqslant N_{G_0}(I) \geqslant X$ and the structure of I is given in (2.8). We may assume (2.8)(i) does not hold. Let $\bar{I} = I$ if $q > 3$, and $\bar{I} = I/O_3(I)$ or $I/O_2(I)$ if $q = 2$ or 3, respectively. Write $\bar{I} = \bar{I}_1 \cdots \bar{I}_k$, a central product as given in (2.8). As $X = \langle V^G \cap X \rangle$ we may take $X \leqslant I_1$. If $k > 1$, then some r-local subgroup of G_0 involves X, against $\text{И}(r, E_0) = 1$. So $i = 1$ and $I = I_1$. As $X \leqslant I$ it follows that $\bar{I}$ is a central extension of a group of Lie type defined over $\mathbf{F}_q$. Comparing orders we have $\bar{X} = \bar{I}$. In this comparison use the fact that $|\bar{I}|_r \leqslant |G_0|_r$ and the existence of primitive divisors. (A prime divisor s of $r^a - 1$ is a primitive divisor if $s \nmid r^b - 1$ for $b < a$. Such divisors exist except when $r^a = 2^6$ or $r^a = r^2$ with r a Mersenne prime.)

Now $D_0 \leqslant N(I)$, so D_0 acts on $\bar{X} = \bar{I}$, and as no parabolic subgroup of G_0 contains a section isomorphic to $\bar{X}/Z(\bar{X})$ we conclude that $\bar{X} = (O^{r'}(\bar{D}_0^{(\infty)}))$. We will show this to be impossible. Say $G \cong F_4(q)$ with q even. Then the Frattini argument shows that E_0 is normalized by an element of $\mathrm{Aut}(G_0)$ lying in the coset of a graph automorphism. But no such automorphism can normalize D_0. Say $G \cong F_4(q)$ with q odd. Here we use the embedding of $SO^-(4, q) \times SO^-(4, q) \leqslant PSO(9, q)'$ to conclude that E_0 centralizes a klein group $C \leqslant G_0$ with C contained in a parabolic subgroup of G_0. Then (2.3) gives $C^0_{G_0}(C) = G_0$. Now use induction and the results of §4 to conclude $D_0 = G_0$. For $G = {}^2E_6(q)$ we have $E_0 \leqslant J$ where $J \cong L_3(q) \times L_3(q^2)$ or $F_4(q)$, according to whether we are in case (ii) or (iii). We then have $J \leqslant I$, and order considerations contradict $\bar{X} = (O^{r'}(\bar{D}_0))^{(\infty)}$.

If $G \cong E_6(q)$, use Table (3.3) to get a subgroup J such that $J/Z(J) \cong L_3(q^2) \times U_3(q)$. By orders we may take $E_0 \leqslant J$ and thus $J \leqslant D_0$. But then $L_3(q^2) \times U_3(q) \leqslant \bar{X} = F_4(q)$, whereas a consideration of the parabolics of $F_4(q)$ shows this to be impossible. Similarly, if $G \cong {}^3D_4(q)$, then G_0 contains a subgroup isomorphic to $SL_2(q) \circ SL_2(q^3)$. But then E_0 is in such a subgroup and $\bar{D}_0$ contains $L_2(q) \times L_2(q^3)$. However $G_2(q)$ contains no such subgroup, so $(O^{r'}(\bar{D}_0^{(\infty)})) \neq \bar{X}$.

Finally, suppose $G = E_8(q)$, p is associated with $q^8 - 1$ and $\bar{X} \cong PSO^+(16, q)'$. Let M be the natural module for $Y = O^+(16, q)$, choose $e \in E_0^\#$, and regard $e \in Y$. Then e acts on M and we set $Y_0 = O^{r'}(C_Y(e))$. Write $M = M_1 \oplus M_2$, a sum of $\langle e \rangle$-invariant 8-spaces of M. One checks that $Y_0 = 1$, $O^-(8, q)'$, or $SL(2, q^4)$ depending on the action of e on M_1 and M_2. However, by (3.3), G_0 contains a central extension of $O^-(12, q)' \times U_3(q)$. So G_0 contains a central extension of $O^-(8, q)' \times O^+(4, q) \times U_3(q)$, and some element, e, of $E_0 - Z(G_0)$ centralizes a subgroup isomorphic to $SO^+(4, q) \times U_3(q)$. Moreover, each of the factors of the subgroup is generated by conjugates of V, so $C_{\bar{I}}(e)$ contains $SO^+(4, q) \times U_3(q)$, contradicting $\bar{X} = \bar{I}$. This completes the proof of (5.7).

(5.8) Let p be associated with φ_x. Then none of the following occur: $G = G_2(q)$, $p > 3$, and $x = 1, 2$; $G = E_8(q)$ and $x = 5, 10, 12$; $G = F_4(q)$, and $x = 3, 6$.

PROOF. As in the proof of (5.7) we produce a certain subgroup X of G. However this time X will be a central product of two quasisimple Chevalley groups. The triples $(G, \varphi_x, X/Z(X))$ are as follows: $(F_4(q), \varphi_3, L_3(q) \times L_3(q))$, $(F_4(q), \varphi_6, U_3(q) \times U_3(q))$, $(E_8(q), \varphi_5, L_5(q) \times L_5(q))$, $(E_8(q), \varphi_{10}, U_5(q) \times U_5(q))$, $(E_8(q), \varphi_{12}, {}^3D_4(q) \times {}^3D_4(q))$, $(G_2(q), \varphi_1$ or $\varphi_2, L_2(q) \times L_2(q))$. The existence of X follows from Table (3.3) in all but the first and last cases. In the first case let $X = X_1X_2$, where $X_1 = \langle U_{\pm\alpha_3}, U_{\pm\alpha_4} \rangle$ and $X_2 = \langle U_{\pm\alpha_1}, U_{\pm s} \rangle$. In the last case let $X = X_1X_2$, where $X_1 = \langle U_{\pm\alpha_2} \rangle$ and $X_2 = \langle U_{\pm s} \rangle$. (In each case s denotes the positive root of highest height in Σ.) Using Table (5.2) we check that X contains a Sylow p-subgroup of G_0, so we may take $E_0 \leqslant X$.

First we claim that each component of X has p-rank 1, so that $E_0 \cong Z_p \times Z_p$. This is clear from the theory of linear groups except for the case where $X = {}^3D_4(q) \times {}^3D_4(q)$. Suppose $m_p({}^3D_4(q)) > 1$. Here the argument of (5.6) applies. Namely, $E_0 \leqslant C(E_1)$ with $E_1 \cong Z_p \times Z_p$ and E_1 in one of the components of X. From here induction, the main results of §4, and (2.3) give a contradiction to $C^0_{G_0}(E_0) < G_0$. This proves the claim and we write $E = E_1 \times E_2$, where $E_i = E \cap X_i$ and X_1, X_2 are the components of X. Clearly $X \leqslant C^0_{G_0}(E_0)$.

Next we set $I = \langle V^{G_0} \cap D_0 \rangle$ for V a long root subgroup of G_0. We will determine the structure of I as in (5.7). We have $X \leqslant D_0 \leqslant N_{G_0}(I)$ and either by (3.4) or by construction, at least one of the components of X is generated by conjugates of V. We use (2.8) to obtain the structure of I. Set $\bar{I} = I$ if $q > 3$, $\bar{I} = I/O_2(I)$ if $q = 3$, and $\bar{I} = I/O_3(I)$ if $q = 2$. Now $X \leqslant D_0 \leqslant N_{G_0}(I)$ and at least one of the components of X is actually contained in I. By (2.3) and (2.8) the structure of $\bar{I}/Z(\bar{I})$ is known and $\bar{I} = \bar{I}_1 \cdots \bar{I}_k$, a central product, with $\bar{I}_i = \langle \bar{V}_i^{\bar{I}} \rangle$ for some $V_i \in V^G$ and $\bar{I}_i$ is either a Chevalley group over $\mathbf{F}_q$, or $q = 2$ and $E(\bar{I}_i)$ is described in (2.8)(iv). Since $m_p(E_0) = m_p(G_0) = 2$ and $И(r, E_0) = \{1\}$, we must have $X \leqslant N(\bar{I}_i)$ for $i = 1, \ldots, k$. As $X = X'$, X induces a group of inner automorphisms on each $\bar{I}_i$, and since $И(r, E_0) = \{1\}$ (by (2.3)) we conclude that $k \leqslant 2$. Indeed, if $k > 2$, then there exists $i \in \{1, \ldots, k\}$ such that $p \nmid |\bar{I}_i|$, and for this i, $[X, \bar{I}_i] = 1$. Now let $J = IX$ and $\bar{J} = IX$, $IX/O_2(I)$, or $IX/O_3(I)$, depending on whether $q > 3$, $q = 3$, or $q = 2$. Then either $\bar{J} \neq \bar{I}$, or $\bar{J} = \bar{I}\bar{L}$ where $[\bar{I}, \bar{L}] = 1$. In the second case we must have $k = m_p(\bar{I}) = m_p(\bar{L}) = 1$. This can be seen by using the facts that $m_p(G_0) = 2$ and $И(r, E_0) = \{1\}$. Consequently $\bar{E}_0 \cap \bar{L} \neq 1$, and we conclude that $\bar{L}$ is the image of one of the components of X. Write $\bar{J} = \bar{J}_1\bar{J}_2$ a central product, with $E(\bar{J}_1)$ quasisimple, and $E(\bar{J}_2)$ quasisimple or $\bar{J}_2 = 1$. Also we have $\bar{X} = (\bar{X} \cap \bar{J}_1)(\bar{X} \cap \bar{J}_2)$ and $X = X'$. Thus $\bar{I}'$ contains root elements. If $q = 2$, this rules out all exceptional cases of (2.8)(iv) other than the Fischer groups F_{22}, F_{23}, F_{24}. Each of $|F_{23}|$ and $|F_{24}|$ is divisible by 23, whereas $23 \nmid |G_0|$. So these cases are out. If $\bar{J}_i/Z(\bar{J}_i) \cong F_{22}$, then order arguments force $G_0 = E_8(2)$ and $\bar{J}\bar{X} \cong \bar{X}_1 \times F_{22}$. By [6] some parabolic of G_0 contains a Sylow r-subgroup of $\bar{X}_1$ in its unipotent radical and F_{22} in its Levi factor. Consideration of the nilpotence class of the Sylow r-subgroup of X_1 and order considerations lead to a contradiction. Therefore, $\bar{J} = \bar{J}_1\bar{J}_2$ with each of $\bar{J}_1$, $\bar{J}_2$ a central extension of a Chevalley group over $\mathbf{F}_q$.

We claim that $\bar{J} = \bar{X}$ and we indicate by example how this is proved. Say $G = E_8(q)$ and p is associated with φ_5. Then $X/Z(X) \cong L_5(q) \times L_5(q)$. Suppose that $\bar{X} \leqslant \bar{J}_1$. Checking orders of Chevalley groups defined over $\mathbf{F}_q$ we conclude that $\bar{J}_1$ must be an extension of $L_{10}(q)$. This implies that some p-local subgroup of G, and hence some parabolic subgroup of G, involves $L_9(q)$. The parabolic subgroup is necessarily the maximal parabolic of $E_8(q)$ that involves $E_7(q)$. Repeating this we see that some proper parabolic subgroup of $E_7(q)$ must involve $L_8(q)$. This is impossible (compare orders). So we may assume $\bar{X}_1 \leqslant \bar{J}_1$ and $\bar{X}_2 \leqslant \bar{J}_2$. Say $\bar{J}_i > \bar{X}_i$. Considering possible choices for $\bar{J}_i$ we can argue as above. For example if $\bar{J}_i$ involves $L_6(q)$, then some proper parabolic subgroup of G_0 involves $L_5(q) \times L_5(q)$ and we argue as above to get a contradiction. Thus $\bar{J} = \bar{X}$ as claimed. The other cases are similar, but there is one troublesome point when $G = E_8(q)$ and p is associated with φ_{10}. Namely, we must rule out the case $\bar{J}/Z(\bar{J}) \cong PSU(10, q)$. Suppose that this occurs. Then $\bar{J}$ has a parabolic subgroup involving $L_5(q^2)$ and containing an element $e \in E^{\#}$. It follows that e is in a parabolic subgroup, P, of $E_8(q)$ involving either $E_7(q)$ or $D_7(q)$. We claim that $C_{G_0}(e)$ contains a conjugate of V. In the $E_7(q)$ case this follows from the fact that $Z(O^{r'}(P))$ is a long root subgroup. For the $D_7(q)$ case use (17.14)(ii) of [3] to see that the Levi factor of P acts on $O_r(P)'$ as on the natural module for $O^+(14, q)'$ and root groups correspond to singular 1-spaces. Considering the action of e on the usual module, we have the claim. On the other hand $C_J(e)$ contains no conjugate of V, and this contradicts the definition of J. So in all cases we have the claim that $\bar{J} = \bar{X}$.

By (2.7)(iii) and (2.11) we may take $E_0 \leqslant \hat{H}$ and find $e \in E_0 - (E_1 \cup E_2)$ such that $C_{\hat{U}}(e) \neq 1$. It follows that $L_0 = O^{r'}(C_{G_0}(e))$ is a nontrivial central product of Chevalley groups defined over extension fields of $\mathbf{F}_q$ (see (2.6)). Since $И(r, E_0) = \{1\}$, $E_0 \cap E(L_0) \neq 1$. In particular $E(L_0) \neq 1$ and $E(L_0)$ induces a group of inner automorphisms on $\bar{X}$. But $И(r, E_0) = \{1\}$ implies $E(L_0) \leqslant J$ and so $\overline{E(L_0)} \leqslant \bar{X}_1\bar{X}_2$. However $C(e) \cap \bar{X}_1\bar{X}_2$ is an r'-group, so this is a contradiction.

At this point we have dealt with all cases except those for which either p is "small" or G_0 has Lie rank 2.

(5.9) Let $G = F_4(q)$ or ${}^2E_6(q)$. If $G = {}^2E_6(q)$ and $p = 3$, then suppose $p \mid q - 1$. One of the following holds:

(i) $D_0 = G_0$.

(ii) $p = 3$, $G_0 = F_4(2)$ or $F_4(4)$, and for suitable choice of E_0, $C^0_{G_0}(E_0) \leqslant N_{G_0}(E_0)$.

PROOF. By Table (5.2) and the results of (5.3)–(5.8), the only cases to consider are $p = 5 \mid q + 1$ with $G = {}^2E_6(q)$ and $p = 3 \mid q \pm 1$. First consider the case $p = 5$. Then $G = {}^2E_6(q)$ and from the extended Dynkin diagram of G we find a subgroup, X, of G with X a central extension of $PSO^-(10, q)'$ and X generated by long root subgroups. Considering the action of H on X we then find a subgroup $H_1 \leqslant X$ with $[H_1, X] = 1$ and H_1 of order divisible by $(q+1)/(3, q+1)$. From order considerations we may take $E_0 \leqslant H_1X$. One then argues that $E_0 \leqslant O_p(H_1) \times F$, where F is the direct product of 5 copies of $O_p(Z_{q+1})$ and F acts on the usual module for $O^-(10, q)$ respecting a decomposition into five pairwise orthogonal, anisotropic 2-spaces. Hence there is a subgroup $E_1 \leqslant F$ with $E_0 \leqslant C(E_1)$, $E_1 \cong Z_p \times Z_p$, and E_1 centralizes a proper r-subgroup of X. So $C^r_{G_0}(E_1) = G_0$ by (2.3). Then minimality of G together with (2.16) and the results of §4 imply $G_0 = C^r_{G_0}(E_1) \leqslant D_0$. So from now on we may assume $p = 3$.

Suppose $G = F_4(q)$. From Table (3.3) we have a subgroup $L \leqslant G_0$ such that there exists $L_0 < L$ with $|L: L_0| = 3$ and $L_0 = L_1L_2$, the central product of two copies of $SL(3, q)$ or $SU(3, q)$, according to $3 \mid q - 1$ or $3 \mid q + 1$. If $3 \mid q - 1$, then we may take $L_2 = \langle U_{\pm\alpha_3}, U_{\pm\alpha_4}\rangle$ and $L_1 = \langle U_{\pm s}, U_{\pm\alpha_1}\rangle$ where s is the positive root of highest height. Now suppose $G = {}^2E_6(q)$ and $3 \mid q - 1$. We regard $F_4(q) \leqslant {}^2E_6(q)$ and write ${}^2E_6(q) = \langle U_{\pm\alpha_1}, U_{\pm\alpha_2}, \hat{U}_{\pm\alpha_3}, \hat{U}_{\pm\alpha_4}\rangle$, where $U_{\alpha_3} \leqslant \hat{U}_{\alpha_3}$, $U_{\alpha_4} \leqslant \hat{U}_{\alpha_4}$, etc. Here we have a subgroup $\hat{L} > L$ such that $\hat{L}$ contains $\hat{L}_0 = L_1\hat{L}_2$ as a normal subgroup of index 3, where $\hat{L}_2 = \langle \hat{U}_{\pm\alpha_3}, \hat{U}_{\pm\alpha_4}\rangle \cong SL(3, q^2)$. Then $L_0 < \hat{L}_0$, $L_2 = \hat{L}_2 \cap L$, and $[L_1, \hat{L}_2] = 1$. Let $\langle z\rangle = Z(L) = Z(\hat{L})$.

Suppose $q \neq 2, 4$ and $G_0 = F_4(q)$. We may assume that $z \in E_0$ (use (2.9)(a)). We claim that there exist 3-elements $x_1 \in L_1$ and $y_1 \in L_2$ such that $|x_1y_1| = 3$, $x_1y_1 \in L_0 - Z(L_0)$, and $x_1y_1 \in C(E_0)$. If $E_0 \cap L_0 \neq Z(L_0)$, then just choose $x_1y_1 \in E_0 - \langle z\rangle$. Suppose $E_0 \cap L_0 = \langle z\rangle$. Then $E_0 = \langle z\rangle \times \langle z'\rangle$ for $z' \in L - L_0$. Choose $x_1 \in L_1$, $y_1 \in L_2$ each of order 3 such that z' centralizes x_1 and y_1 mod $(\langle z\rangle)$. Then z' centralizes one of x_1, y_1, x_1y_1, or $x_1y_1^{-1}$ and the claim follows. If $E_0 \leqslant C(x_1)$ or if $E_0 \leqslant C(y_1)$, then set $E_1 = \langle x_1, z\rangle$, respectively $\langle y_1, z\rangle$. Then E_1 centralizes a component of L_0 so $C^r_{G_0}(E_1) = G_0$ by (2.3). By induction, (2.16), (2.6)(ii), (2.5), and §4 we have $D_0 = G_0$. So suppose neither x_1 nor y_1 is centralized by E_0.

Let $E_1 = \langle z, x_1y_1\rangle$. Then $E_0 \leqslant C(E_1)$ and if E_1 normalizes a proper r-subgroup of G_0, we can argue as above. Suppose this is not the case. Then $3 \mid q + 1$, $L_1 \cong L_2 \cong SU(3, q)$ and relabeling, if necessary, x_1 is in a conjugate, C, of $\langle U_{\pm s}\rangle$, where s is the root of highest height and where $C \leqslant L_1$. Then $E_1 < N(C)' = C \times I$, where $I \cong Sp(6, q)$. Considering elementary abelian 3-subgroups of $Sp(6, q)$ we argue that there exist conjugates C_1, C_2, and C_3 of C such that $E_1 < C \times C_1 \times C_2 \times C_3$. We claim that $I \leqslant C^r_{G_0}(E_1)$. This will follow from (3.1) once we show that the copies of $Sp(4, q)$ in I corresponding to $C_1 \times C_2$ and $C_2 \times C_3$ are in $C^r_{G_0}(E_1)$. Say, for example, I_0 is the copy of $Sp(4, q)$ with $C_1 \times C_2 \leqslant I_0$ and $[I_0, C_3] = 1$. Then $C_1 \times C_2 \leqslant C^r_{I_0}(E_1)$ and by (3.2) either $I_0 = C^r_{I_0}(E_1)$ or $C^r_{I_0}(E_1)$ normalizes $C_1 \times C_2$. A direct check shows the latter to be impossible. Thus $I_0 \leqslant C^r_{G_0}(E_1)$ and we conclude that $I \leqslant C^r_{G_0}(E_1)$. We may now assume that

$C_3 = \langle U_{\pm\alpha}\rangle$, for $\alpha = \alpha_2 + 2\alpha_3 + 2\alpha_4$. By symmetry $\tilde{I} = E(C_{G_0}(C_3)) \leqslant C^r_{G_0}(E_1)$. A check with root systems shows that $\langle I, \tilde{I}\rangle = G_0$, so $G_0 = C^r_{G_0}(E_1)$ and we argue as before that $G_0 = D_0$. Now suppose $G = {}^2E_6(q)$ with $3 \mid q-1$ and $q > 4$. Then we may take $E_0 < L < \hat{L}$, so by the above minimality of G and §4 we conclude $C^0_{G_0}(E_0) \geqslant \langle \hat{L}, F_4(q)\rangle = G_0$. So for these cases (i) holds.

Next suppose that $q = 2$ and $G = F_4(2)$. We will exhibit an elementary abelian subgroup, F, for which generation fails. Write $O_3(L_1) = \langle x_1, x_2\rangle$ and $O_3(L_2) = \langle y_1, y_2\rangle$, so that $\langle x_1, x_2\rangle \cong \langle y_1, y_2\rangle$ is extraspecial of order 3^3. Then $L_i = O_3(L_i)Q_i$ where $Q_i \cong Q_8$ for $i = 1, 2$. Finally, $L = L_0\langle e\rangle$, where $|e| = 3$. To see that e exists just observe that L contains a Sylow 3-subgroup of $F_4(2)$ and that $F_4(2)$ contains a direct product of four copies of S_3 (each generated by long root subgroups). So $F_4(2)$ has 3-rank at least 4, whereas L_0 has 3-rank 3. We may assume that $e \in C(x_1) \cap C(y_1)$, so setting $Z(L_0) = \langle z\rangle$ we let $F = \langle z, e, x_1, y_1\rangle$. By (3.4) and the structure of $SU(3,2)$ we may assume that the involution in Q_1 is a root involution of G_0, and that x_1 is a product of two root involutions. This is true for each $x \in \langle x_1, z\rangle - \langle z\rangle$. So for each such x, $C_{G_0}(x) = \langle x\rangle \times Sp(6,2)$. Let F_1 be maximal in F. If $x \in F_1$ for $x \in \langle x_1, z\rangle - \langle z\rangle$, then $C(F_1) \leqslant \langle x\rangle \times Sp(6,2)$ and we have $C_{G_0}(F_1) \leqslant N(F)$ (see (4.1) and the proof of (4.3)). From the work of Burgoyne [7] we have $L = C_{G_0}(z)$ and we can argue that L is normalized by a graph automorphism of G_0, which necessarily interchanges L_1 and L_2. Consequently, if F_1 contains an element of $\langle y_1, z\rangle - \langle z\rangle$ we again get $C_{G_0}(F_1) \leqslant N(F)$. So we may assume that F_1 contains z and an element $g = x_1^i y_1^j$ for $i = \pm 1$, $j = \pm 1$. Then $C_{G_0}(F_1) \leqslant C_{G_0}(z) \cap C_{G_0}(g) = C_L(g)$. But $C_L(g) \leqslant O_3(L)\langle e\rangle$ and it is easy to see that $C_L(g) \leqslant N(F)$. So (ii) holds.

Suppose $q = 4$ and $G = F_4(4)$ or ${}^2E_6(4)$. Write $L_0 = L_1L_2$ with $L_1 \cong L_2 \cong SL(3,4)$ and $\hat{L}_0 \cong L_1\hat{L}_2$ with $L_2 \leqslant \hat{L}_2 \cong SL(3,16)$. Let $J \in \mathrm{Syl}_3(L)$ and set $J_i = J \cap L_i$ for $i = 1, 2$. Then $J_1 \cong J_2$ is extraspecial of order 3^3 and as J has 3-rank 4 (as for $F_4(2)$) we may write $J = J_1J_2\langle e\rangle$ with $\langle e\rangle = 3$. Set $\langle z\rangle = Z(J) = Z(J_1) = Z(J_2)$ and note that $J \in \mathrm{Syl}_3(\hat{L})$. By (2.9) we may assume $z \in E_0$. Suppose that $a \in E_0$ with $a \in J_i - \langle z\rangle$. Then $E_0 \leqslant C_{G_0}(a)$. Now $a \in J$ where J is generated by two opposite root subgroups of G, so $C_{G_0}(a) = \langle a\rangle \times I$ where $I \cong Sp(6,q)$ if $G = F_4(q)$ and $I \cong SU(6,4)$ or $O^-(8,4)'$ if $G = {}^2E_6(4)$. Now argue that E_0 normalizes an r-subgroup of I (one can argue as in (4.2)). So in this case (i) holds by (2.3). Now assume that E_0 contains no such element, a, or any conjugate of such an element.

By Burgoyne [7] each $a \in E_0^\#$ is conjugate to z and $C_{G_0}(a) \cong C_{G_0}(z) = L$ (respectively $\hat{L}$). For $F_4(4)$ we will produce a subgroup E_0 for which (ii) holds, and for ${}^2E_6(4)$ we will show that (i) holds. First consider $G = F_4(4)$. As before write $J_1 = \langle x_1, x_2\rangle$ and $J_2 = \langle y_1, y_2\rangle$, where $[x_1, x_2] = z$ and $[y_1, y_2] = z^{-1}$. Set $E_0 = \langle z, x_1y_1, x_2y_2\rangle$. If $e \in E_0 - \langle z\rangle$, then $C_{G_0}(e) \cap C_{G_0}(z) = C_L(e)$ is solvable. By Burgoyne [7] it follows that $z \sim e$. So $E_0^\#$ is fused in G_0.

From the construction of L one can show that there exists $e \in J - J_1J_2$ such that $x_1^e = x_1$, $y_1^e = y_1$, $x_2^e = x_2x_1$, and $y_2^e = y_1y_2$. Let F be a hyperplane of E_0 with $z \in F$. As $N_L(E_0)$ is transitive on $(E_0/\langle z\rangle)^\#$, in order to show $C_{G_0}(F) \leqslant N_{G_0}(E_0)$ it will be enough to show $C_{G_0}(\langle z, x_1y_1\rangle) \leqslant N_{G_0}(E_0)$. But

$$C_{G_0}(\langle z, x_1y_1\rangle) = C_L(x_1y_1) = \langle x_1, x_1y_1, z, x_2y_2, e\rangle \leqslant N_{G_0}(E_0),$$

as needed. Now let F be an arbitrary hyperplane in E_0. Say $z^g \in F$. Then $E_0 \leqslant L^g$. If $F \leqslant L_0^g$, then F is L_0^g-conjugate to $\langle z^g, (x_1y_1)^g\rangle$, so $C_{G_0}(F)$ is L_0^g-conjugate to $C_L(x_1y_1)^g$, and as $C_L(x_1y_1)^g/\langle z^g, (x_1y_1)^g\rangle$ is abelian, we have $C_{G_0}(F) \leqslant N(E_0)$.

We claim that $N_{G_0}(E_0)$ is transitive on $E_0^{\#}$. We already have $N_{L_0}(E_0)$ transitive on $(E_0/\langle z\rangle)^{\#}$ and it follows that $E_0 - \langle z\rangle$ is fused in $N_{G_0}(E_0)$.

It will suffice to show that $U^{\#}$ is fused in $N_{G_0}(E_0)$, where $U = \langle z, x_1y_1\rangle$. Let $P = \langle z, x_1, y_1, x_2, y_2, e\rangle \in \mathrm{Syl}_3(G)$. One checks that $U \trianglelefteq P$, $C_P(U)/U$ is elementary of order 3^3, and $U = C_P(U)'$. Let $z^g \in U - \langle z\rangle$. Then $C_P(U)$ has index 3 in a Sylow 3-subgroup of $C_G(z^g)$ and it follows that $N_G(U)/C_G(U)$ contains $SL(2,3)$. Also, $C_G(U) = C_L(U) = C_P(U)$. If w_0 is the long word in W, then w_0 induces a graph automorphism on L_1 and on L_2, so notation may be chosen so that w_0 inverts U, $w_0 \in N(E_0)$, and w_0 centralizes E_0/U. It follows that $E_0/U = C(w_0) \cap C_G(U)/U$ and so $E_0 \trianglelefteq N_G(U)$. This proves the claim. So if F is any hyperplane of E_0, letting $z^g \in F$, then we may take $g \in N(E_0)$. This gives $F \leqslant E_0 \leqslant L_0^g$ so by the above $C_{G_0}(F) \leqslant N(E_0)$. Therefore (ii) holds and the case of $F_4(q)$ is complete.

Finally, consider ${}^2E_6(4)$. By previous arguments we may take $E_0 < L < \hat{L}$ and assume that each element of $E_0^{\#}$ is conjugate to z and that E_0 normalizes no proper r-subgroup. Say $z \in E_0$ and set $D_0 = C_{G_0}^0(E_0)$. By §4 $\hat{L}_2 \leqslant D_0$, so $L_2 \leqslant D_0 \cap F_4(q)$. This holds for each element of $E_0^{\#}$. Now L_2 is generated by short root subgroups of $F_4(q)$, but as $q = 4$ we can apply (2.8) to get the structure of $\langle L_2^{D_0 \cap F_4(q)}\rangle = X$. Since $z \in L_2 \leqslant X$, we have $E_0 \leqslant X$. Analysis of the possible choices for X and using $\textit{И}(r, E_0) = \{1\}$ leads to $X = F_4(q)$. But then $D_0 \geqslant \langle F_4(q), \hat{L}_2\rangle = G_0$, so (i) holds. This completes the proof of (5.9).

(5.10) Let $G = E_6(q)$, or ${}^2E_6(q)$ and $p = 3 \mid q+1$. Then one of the following holds:

(i) $D_0 = G_0$.

(ii) $p = 3$, $G = E_6(2)$, $E_6(4)$, or ${}^2E_6(2)$, and there exists E_0 such that $C_{G_0}^0(E_0) \leqslant N_{G_0}(E_0)$.

Proof. By Table (5.2) together with (5.3)–(5.8) either $G = {}^2E_6(q)$ and $p = 3 \mid q+1$ or $G = E_6(q)$ and $p = 3$ or 5 with $p \mid q \pm 1$. First assume that $p \mid q+1$ and $G = E_6(q)$. Then $G_0 \geqslant \langle U_{\pm s}\rangle \times I_0$, where s is the positive root of highest height and $I_0 \cong SL(6,q)$. So $G_0 \geqslant X$, where $X \cong SL(2,q) \times (SL(2,q) \operatorname{wr} Z_3)$. If $p = 5$, then X contains a Sylow 5-subgroup of G_0 so we may take $E_0 \leqslant X$ and argue as in (5.5) to get $D_0 = G_0$. Suppose $p = 3$. By Table (5.2) we may assume $E_0 < F_4(q) < E_6(q)$ and $E_0 < L$, where L is the second group listed in Table (3.3) for the group $E_6(q)$. In fact, from the proof of (3.3) we may assume that $F_4(q) \cap L$ is the second group listed for $F_4(q)$. If $q > 2$, use (5.9), (4.1), and (4.5) to get $D_0 \geqslant \langle L, F_4(q)\rangle$ and then apply (2.8) to obtain $\langle L, F_4(q)\rangle = G_0$. Suppose $q = 2$. Here we choose $E_0 < L_0 \cap F_4(2) < L_0 \cap F_4(4)$ as it was constructed in the $F_4(4)$ case of (5.9)(ii). The same arguments show that $D_0 \leqslant N_{G_0}(E_0)$; hence (ii) holds.

If $p = 5 \mid q-1$ we again may take $E_0 \leqslant \langle U_{\pm s}\rangle \times I_0$. Then E_0 projects to an abelian subgroup of $I_0 \cong SL(6,q)$. Consequently E_0 acts reducibly on the usual module for $SL(6,q)$ and $\textit{И}(r, E_0) \neq \{1\}$. So here (2.3) implies that (i) holds. At this point we may take $p = 3$. If $G = E_6(q)$, then $3 \mid q-1$ and if $G = {}^2E_6(q)$, then $3 \mid q+1$.

By (3.3) G_0 contains a subgroup L such that L contains a normal subgroup, L_0, of index 3 and $L_0 = L_1L_2L_3$, a central product of three copies of $SL(3, q)$ or $SU(3, q)$ depending on whether $G_0 \cong E_6(q)$ or ${}^2E_6(q)$. Also $Z(L_0) = Z(L) \cong Z_3 \times Z_3$. We write $\langle x_i \rangle = Z(L_i)$. Then $x_i \neq 1$ for $i = 1, 2, 3$ and we may choose notation so that $x_1 = x_2x_3^{-1}$ and $z = x_1x_2$ generates $Z(G_0)$. By Table (5.2) a Sylow 3-subgroup of L has index 9 in a Sylow 3-subgroup of G_{00}. Here G_{00} denotes the preimage in $\hat{G}$ of $\bar{G}_\sigma$. We claim that $N_{G_{00}}(L)$ contains a Sylow 3-subgroup of G_{00}. To see this we go back to the proof of Table (3.3), where L was constructed. We had X_1, X_2, X_3 commuting copies of $SL(3, K)$ with centers $\langle z_1 \rangle, \langle z_2 \rangle, \langle z_3 \rangle$, respectively. For each i we constructed $a_i \in X_i$ with $a_i^\sigma = a_iz_i$. Then $a_1a_2^{-1} \in G_{00} - G_0$ and $a_1a_2^{-1} \in N(L)$. Also there was an element $g \in G_0$ interchanging X_1 and X_2, while normalizing X_3. Similarly, we can construct $g' \in G_0$ interchanging X_2 and X_3, normalizing X_1. So G_0 contains a 3-element x transitive on $\{X_1, X_2, X_3\}$. Then $L\langle x, a_1a_2^{-1} \rangle = \hat{L}$ contains a Sylow 3-subgroup of G_{00} and $L < \hat{L}$. We may assume $E_0 < \hat{L}$.

It is possible that $E_0 \nleqslant C(x_1)$, but we may assume $z \in E_1$ by (2.5), so $[x_1, E_0] \leqslant \langle z \rangle \leqslant E_0$. A slight extension of (2.9) allows us to take $x_1 \in E_0$. We may assume that there exists $y \in E_0$ with $[x_1, y] = 1$ and $y \notin \langle x_1, z \rangle$. To see this argue as follows. If $E_0 \cap C_{\hat{L}}(x_1) > \langle x_1, z \rangle$, just choose y in the intersection. Otherwise, $E_0 = \langle x_1, z, y \rangle$ for any $y \in E_0 - \langle x_1, z \rangle$. But then y is transitive on $\{X_1, X_2, X_3\}$, $\mathit{И}(r, E_0) \neq \{1\}$, and by (2.3), (i) holds.

Suppose $q > 4$ if $G = E_6(q)$ and $q > 2$ if $G = {}^2E_6(q)$. Then $L_0 \leqslant D_0$. Let V be a long root subgroup of G_0 and set $I = \langle V^G \cap D_0 \rangle$. By (3.4) $L_0 \leqslant I$ and by (2.8) I is a central product of finite groups of Lie type each defined over $\mathbf{F}_q$. Using order arguments we conclude that $I = L_0$. Also, $L_0 = O^{r'}(C_{G_0}(x_1))$.

Since $C_{\hat{G}}(x_1)$ is connected we can embed $\langle x_1 \rangle \times \langle y \rangle$ in a maximal torus $\hat{T}_1$ of $\hat{G}$. Let $\hat{T}_1 \leqslant \hat{B}_1$, where $\hat{B}_1$ is a Borel subgroup of $\hat{G}$ and let $\hat{U}_1$ be the unipotent radical of $\hat{B}_1$. Then $\hat{U}_1$ is the product of 36 root subgroups and since $L_0 = O^{r'}(C_{G_0}(x_1))$ we see that x_1 centralizes precisely 9 of these root subgroups. So some element of $\langle x_1, y \rangle - \langle x_1 \rangle$ must centralize at least 9 root subgroups of $\hat{U}_1$ and we may take this element to be y. Using this information together with basic properties of the root system of type E_6 we have the following possibilities for the Dynkin diagram of $E(C_{\hat{G}}(y))$: $A_2 \cup A_2 \cup A_2$, $A_1 \cup A_4$, $A_1 \cup A_5$, A_5, D_4, D_5. As in the proof of (2.6) we see that the components of $C_{G_0}(y)$ are central extensions of some of the following groups: $L_i(q)$, $2 \leqslant i \leqslant 5$, $U_i(q)$, $2 \leqslant i \leqslant 5$, $L_3(q^3)$, $U_3(q^3)$, $U_5(q)$, $D_5(q)$, ${}^2D_5(q)$, ${}^2D_4(q)$, $D_4(q)$, ${}^3D_4(q)$. Moreover, q^9 divides the order of $E(C_{G_0}(y))$.

Let J be a component of $C_{G_0}(y)$ and let $F = N_{E_0}(J)$. Suppose we knew that for any such J, $J = C_J^0(F)$. Then (2.16) implies that $E(C_{G_0}(y)) \leqslant D_0$. However, this contradicts the fact that $L_0 = I = E(D_0)$. Therefore, there is some component J with $J \neq C_J^0(F)$. By (4.1), (4.5) and induction we conclude that $F/C_F(J)$ is not contained in the subgroup of $\mathrm{Aut}(J)$ generated by inner and diagonal automorphisms. To handle the remaining cases we appeal to (6.1), (6.3), (6.4) and (6.5) of the next section. The proof of (6.1) does not require Theorem 1. The proofs of (6.3), (6.4) and (6.5) make use of Theorem 1, but since we are in a minimal situation these applications are valid. Therefore, we conclude that $J = C_J^0(F)$, which is a contradiction.

Suppose $G = {}^2E_6(2)$. Then G_0 has Sylow 3-subgroup of order 3^{10}. Let s be the positive root of highest height. Then $I = E(C_{G_0}(\langle U_{\pm s}\rangle)) \cong SU(6,2)$. Now I contains an elementary abelian subgroup, F_0, of order 3^5 which is normal of index 9 in a Sylow 3-subgroup of I. Set $\langle v\rangle = O_3(\langle U_{\pm s}\rangle)$ and $E_0 = \langle v\rangle \times F_0$. We claim that $C^0_{G_0}(E_0) \leqslant N_{G_0}(E_0)$. Let E_1 be a hyperplane in E_0 with $v^g \in E_1$ for some $g \in G_0$. Then $E_0 \leqslant C_{G_0}(E_1) \leqslant C_{G_0}(v) \cong Z_3 \times SU(6,2)$. So E_0 projects to $E(C_{G_0}(v))$ as an elementary abelian subgroup of order 3^5. No proper parabolic subgroup of $SU(6,2)$ contains such a subgroup. By (4.1) and the proof of (4.2), $C_{G_0}(E_1) \leqslant N(E_0)$.

Now suppose E_1 is a hyperplane of E_0 containing no G_0-conjugate of v. Then $E_0 = \langle v\rangle \times E_1$ and E_1 projects to F_0 in I. Let M be the natural module for $SU(6,2)$ and let $\{v_1,\dots,v_6\}$ be an orthonormal basis for M. We may asume F_0 is diagonal in the basis and $F_0 = \langle x_1\rangle \times \cdots \times \langle x_5\rangle$, where $(v_i)x_i = \alpha v_i$, $(v_{i+1})x_i = \alpha^{-1}v_{i+1}$, and $(v_j)x_i = v_j$ if $j \neq i, i+1$. Here $\alpha \in \mathbf{F}_4$ is an element of order 3. Each x_i is G_0-conjugate to v, so for each i either vx_i or $v^{-1}x_i$ is an element of E_1. Say $vx_1 \in E_1$. If $v^{-1}x_2 \in E_1$, then $x_1x_2 \in E_1$. However, we have $x_1x_2 \sim x_1$ in I, a contradiction. So $vx_2 \in E_1$. Similarly, $vx_3 \in E_1$. Then $(vx_1)(vx_2)(vx_3) = x_1x_2x_3 \in E_1$ and again we have $E_1 \cap v^{G_0} \neq \varnothing$. This is a contradiction. So $D_0 \leqslant N(E_0)$.

Finally, we consider $G = E_6(4)$. Let $\tilde{L} = L\langle x\rangle$. We claim that there exists $e \in \tilde{L} - L$ with $|e| = 3$. To see this consider the natural embedding of G_0 in $E_7(4)$. The group N of the Tits' system for $E_7(4)$ splits over the split torus, T. To see this just observe that the long word in the fundamental generators of the Weyl group of $E_7(4)$ inverts T. The splitting of N over T forces a similar splitting in G_0, and this guarantees the existence of e.

We may assume $X_1^e = X_2$ and $X_2^e = X_3$. So $E(C_{L_0}(e)) \cong PSL(3,4)$. Let $\langle b, c\rangle \in \mathrm{Syl}_3(E(C_{L_0}(e)))$ and set $E_0 = \langle x_1, x_2, b, c\rangle$. We fix a hyperplane, F, of E_0 and claim that $C_{G_0}(F) \leqslant N_G(E_0)$. Clearly, we may assume $z \in F$.

Suppose, $x_1 \in F$. As $N_G(E_0)$ is transitive on $(E_0/\langle x_1, x_2\rangle)^\#$, we may assume $F = \langle z, x_1, b\rangle = \langle x_1, x_2, b\rangle$. Then

$$C_{G_0}(F) \leqslant C_{G_0}(x_1) = L.$$

Write $b = b_1b_2b_3$ and $c = c_1c_2c_3$, where $b_i, c_i \in L_i$ for $i = 1,2,3$, and let $V = \langle x_1, x_2, b_1, b_2, b_3, c\rangle = C_{L_0}(b)$. Then $C_{G_0}(F) = C_L(b) = V\langle d\rangle$ for $d \in L - L_0$. Now e normalizes $V\langle d\rangle$ and we consider $V\langle e, d\rangle/\langle x_1, x_2\rangle$. The group $V/\langle x_1, x_2\rangle$ is elementary abelian and d normalizes $V/\langle x_1, x_2\rangle \cap C(e) = \langle b, c, x_1, x_2\rangle/\langle x_1, x_2\rangle$. On the other hand, $\langle [d, c_i]\rangle\langle x_i\rangle = \langle b_i, x_i\rangle$, for $i = 1,2,3$ and this implies that $[d, c] \leqslant \langle b, x_1, x_2\rangle = F$. We conclude that $F = C_{G_0}(F)' = C_L(b)'$. In particular, $E_0 \trianglelefteq C_L(b) = C_{G_0}(F)$.

Viewing $C_{G_0}(b) \cap C_{G_0}(x_1)$ first as a subgroup of L and then as a subgroup of $C_{G_0}(b)$, we conclude from [7] that $b \sim x_1$ in G_0. Let $U = \langle z, x_1, b\rangle$ and $D = N_{G_0}(U) \cap C(U/\langle z\rangle)$. Then D is a 3-group of index 3 in a Sylow 3-subgroup of $C_{G_0}(\langle x_1, z\rangle/\langle z\rangle)$ and in a Sylow 3-subgroup of $C_{G_0}(\langle b, z\rangle/\langle z\rangle)$. It follows that $O^{2'}(N_{G_0}(U))$ induces $SL(2,3)$ on $U/\langle z\rangle$.

We have $C_{G_0}(U) = \langle U, b_1, b_2, c, d\rangle$. Also,

$$\langle c_1, c_2, c_3, z\rangle\langle e\rangle = \langle c_1, c_1^e, c_1^{e^2}, z\rangle\langle e\rangle, \quad \langle b_1, b_2, b_3, z\rangle\langle e\rangle = \langle b_1, b_1^e, b_1^{e^2}, z\rangle\langle e\rangle,$$

and both groups are isomorphic to $Z_2 \times (Z_3 \operatorname{wr} Z_3)$. Set $b_0 = [b_1, e]$ and $c_0 = [c_1, e]$. Then $D = C_{G_0}(U)\langle e, c_0\rangle$. We will use the fact that $\langle z, b, b_0\rangle$ is the unique hyperplane of $\langle z, b_1, b_2, b_3\rangle$ that contains z and is normalized by e.

Now $D/C_{G_0}(U)$ acts on $U \cong Z_3 \times Z_3 \times Z_3$ as the group of transvections in $SL(3,3)$ with fixed direction (namely $\langle z\rangle$). Hence, $O^{2'}(N_{G_0}(U))/C_{G_0}(U)$ is isomorphic to $O^{2'}(P)$, for P a parabolic subgroup of $SL(3,3)$. In particular, $O^{2'}(N_{G_0}(U))$ is transitive on $(D/C_{G_0}(U))^{\#}$. Consider the action of D on $C_{G_0}(U)/U = \langle \bar{b}_1\rangle \times \langle \bar{b}_2\rangle \times \langle \bar{c}\rangle \times \langle \bar{d}\rangle$, where bars denote images modulo U. Then c_0 induces a transvection on this elementary group having centralizer $\langle \bar{b}_1, \bar{b}_2, \bar{c}\rangle$ and commutator space in $\langle \bar{b}_1, \bar{b}_2\rangle$. This commutator space is normalized by e, so earlier remarks imply that $[C_{G_0}(U)/U, \langle c_0\rangle] = \langle \bar{b}_0\rangle$. We must have e inducing a transvection on $C_{G_0}(U)/U$, and since $[b_1, e] = b_0$, we conclude that $[C_{G_0}(U)/U, \langle e\rangle] = \langle \bar{b}_0\rangle$, as well. It follows that $[D, C_{G_0}(U)]\, U/U = \langle \bar{b}_0\rangle \trianglelefteq O^{2'}(N_{G_0}(U))$. Moreover, $O^{2'}(N_G(U))$ normalizes $Z(D/U) \cap C_{G_0}(U)/U = \langle \bar{b}_0, \bar{c}\rangle$. Therefore, $O^{2'}(N_{G_0}(U)) \leqslant C(\langle b_0, c, U\rangle/U)$, so $O^{2'}(N_{G_0}(U)) \leqslant N(E_0)$. Consequently, there is an element $g \in N(E_0)$ such that $x_1 \in F^g$. By the previous case, $C_{G_0}(F^g) > N(E_0)$; whence $C_{G_0}(F) \leqslant N(E_0)$. This proves the claim. Therefore, (ii) holds and the proof of (5.10) is complete.

(5.11) Suppose $G = E_7(q)$ or $E_8(q)$. Then one of the following holds:

(i) $D_0 = G_0$.

(ii) $q = 2$ and $p = 3$.

(iii) $q = 4$, $p = 3$, and $G_0 = E_8(4)$.

Moreover, if $p = 3$ and $G_0 = E_7(2)$, $E_8(2)$, or $E_8(4)$, then there is an elementary abelian 3-subgroup $E_0 \leqslant G_0$ such that $C^0_{G_0}(E_0) \leqslant N_{G_0}(E_0)$.

PROOF. Since $p \nmid |Z(G_0)|$, $E_0 \cong E$ is elementary abelian. By Table (5.2) and (5.3)–(5.8) we need only consider the cases $p = 3, 5, 7$ and $p \mid q \pm 1$. Regard $I = E_7(q) \leqslant E_8(q) = G_0$ in the obvious way. Let $J = \langle U_{\pm s}\rangle$ for s the positive root of highest height in $G_0 = E_8(q)$. Then $I = E(C_{G_0}(J))$.

Suppose $p = 7$. Then Table (5.2) implies that JI contains a Sylow p-subgroup of G_0. If $p \mid q - 1$ then a Sylow p-subgroup of I is contained in a parabolic subgroup of I with a section of type $L_7(q)$. So here (2.3) gives (i). Suppose $p \mid q + 1$. By Table (3.3) and (3.4) $E_8(q)$ contains a central extension of $U_9(q) = X$ with X generated by root subgroups of $E_8(q)$. Then X contains a conjugate, J^g, of J such that $C_X(J^g) \geqslant O^{7'}(GU(7,q))$. So $I \geqslant O^{7'}(GU(7,q))$ and has Sylow 7-subgroups isomorphic to $O_7(Z_{q+1}) \operatorname{wr} Z_7$. Let $S \in \mathrm{Syl}_7(I)$ or $S \in \mathrm{Syl}_7(JI)$, according to $G_0 = E_7(q)$ or $E_8(q)$, and let S_0 be the unique subgroup of S with S_0 the direct product of 7 (respectively, 8) copies of $O_7(Z_{q+1})$. By (2.3) and the argument used to prove (4.2) we may assume that $E_0 \leqslant S_0$. On the other hand, I contains the direct product, $J_1 \times \cdots \times J_7$, of seven conjugates of J. Therefore, we may assume $E_0 \leqslant J_1 \cdots J_7$ (respectively, $JJ_1 \cdots J_7$) and argue as in (5.5) to get (i).

Suppose $G_0 = E_7(q)$ and $p \mid q - 1$. Here $p = 3$ or 5 and a Sylow p-subgroup of G_0 is contained in a conjugate of the parabolic subgroup of G_0 having an $E_6(q)$ section. So $\mathcal{U}(r, E_0) \neq \{1\}$ and (i) holds by (2.3).

Next suppose $G_0 = E_7(q)$ and $p \mid q + 1$, where $p = 3$ or 5. If $p = 5$, then let J_1 and $L \cong O^{5'}(GU(7,q))$ be as in the case $p = 7$. By orders we may assume that

$E_0 \leqslant J_1 L$, and arguing as before we have E_0 conjugate to a subgroup of $J_1 \cdots J_7$, where the product is a commuting product of conjugates of J_1. Once again argue as in (5.5) to get (i). Suppose $p = 3$. By Table (3.3) G_0 contains a subgroup R such that $E(R)$ is the covering group of ${}^2E_6(q)$ and $R/E(R) \cong Z(R) \cong Z_3$. Then R contains a Sylow 3-subgroup of G_0, so we may take $E_0 < R$. Also $E_0 \cong E$ is elementary abelian. Recall the subgroup L discussed in (5.10) and regard $E_0 \leqslant N_{G_0}(L)$. By (2.9) we may take $z \in E_0$, where $\langle z \rangle = Z(L)$. Assume $q > 2$. Let $e \in E_0 - \langle z \rangle$ and set $E_1 = \langle e, z \rangle$. By (2.16), (2.6)(ii), (2.5), (4.1) and results of this section, $O^{r'}(C_{G_0}(t)) \leqslant D_0$ for each $t \in E_1^{\#}$. By (5.10) and (3.4) $E(R) \leqslant I$, where $I = \langle V^{G_0} \cap D_0 \rangle$ and V is a root subgroup of G_0. Then, assuming (i) fails, (2.8) and a check yield $E(R) = I = O^{r'}(E(D_0))$. We then must have $O^{r'}(C_{G_0}(t)) \leqslant E(R)$ for each $t \in E_1^{\#}$. Now argue (see (2.7)(ii) and (2.11) for example) that $t \in E_1 - \langle z \rangle$ may be chosen with $O^{r'}(C_R(t)) \neq 1$. Then $И(r, E_1) \neq \{1\}$, $C^r_{G_0}(E_1) = G_0$ by (2.3), and we have a contradiction. This completes $G_0 = E_7(q)$ except for $q = 2, p = 3$.

Suppose $G_0 = E_8(q)$ and $p = 5$. By Table (3.3) G_0 contains a subgroup Y such that $E(Y)$ is the central product of two copies of $SL(5, q)$ or $SU(5, q)$, according to $5 \mid q - 1$ or $5 \mid q + 1$. Also $Y/E(Y) \cong Z_5$ and, by Table (5.2), Y contains a Sylow 5-subgroup of G_0. We take $E_0 < Y$, and let J and I be as before. Since a Sylow 5-subgroup of IJ has index 5 in one for G_0, we may assume IJ contains a hyperplane, say F, of E_0. Suppose F contains $F_0 \cong Z_5 \times Z_5$. As in (5.10) there are commuting G_0-conjugates $J_1, \ldots, J_7$ of J such that $F_0 \leqslant JJ_1 \cdots J_7 = J_0$. Clearly $J_0 = C^r_{J_0}(F_0)$, so by (2.16), (4.1), and the results of this section, we have $J_0 \leqslant D_0$. But, also, (4.1) implies $E(Y) \leqslant D_0$. We can now use (2.8) to argue that $D_0 = G_0$. So suppose $F \cong Z_5$, that is $E_0 \cong Z_5 \times Z_5$. By (2.9) we may assume $z \in E_0$, where $\langle z \rangle = Z(Y)$. From the structure of $SL(5, q)$ and $SU(5, q)$ it is clear that E_0 acts on an elementary abelian subgroup of $E(Y)$ of order 5^7. It follows that E_0 can be embedded in an elementary subgroup of order 5^3. So using (2.9) we reduce to the previous case, and obtain $D_0 = G_0$. From now on we have $p = 3$.

Let $G_0 \cong E_8(q)$ and use Table (3.3) to get a subgroup $L_1 \leqslant G_0$ such that $E(L_1) = A_0 B_1$, a central product, where $A_0 \cong SL(3, q)$ or $SU(3, q)$, $B_1 \cong \hat{E}_6(q)$ or ${}^2\hat{E}_6(q)$ and in each case the choice is determined by whether $q \equiv 1$ or $-1 \pmod 3$. Also $Z(L_1) = \langle z \rangle \cong Z_3$ and $L_1/E(L_1) \cong Z_3$. By Table (5.2) L_1 contains a Sylow 3-subgroup of G_0, so we may take $E_0 \leqslant L_1$. There is an element $x_1 \in B_1$ such that $|x_1| = 3$ and $E(C_{B_1}(x)) = A_1 A_2 A_3$, a central product of three G_0-conjugates of A_0 (see Table (3.3)). We set $A = A_0 A_1 A_2 A_3$ and choose notation so that $x_1 \in A_1$. Then $Z(A) = \langle z \rangle \times \langle x_1 \rangle$.

We also have $N_{L_1}(A)$ containing a subgroup $\hat{A} > A$ with $\hat{A}/A \cong Z_3 \times Z_3$ and $\hat{A}$ normalizing each A_i. Finally, there is a 3-element $e_1 \in N_{L_1}(A)$ such that $\hat{A} = \hat{A}^{e_1}$ and e_1 permutes $\{A_1, A_2, A_3\}$ as the 3-cycle $(1, 2, 3)$. By orders $\hat{A}\langle e_1 \rangle$ contains a Sylow 3-subgroup of G_0, so we choose $E_0 \leqslant \hat{A}\langle e_1 \rangle$.

First suppose $q > 4$. If $E_0 \leqslant \hat{A}$, then let $E_1 = \langle z, x_1 \rangle$. If $E_0 \nleqslant \hat{A}$, let $E_1 = \langle z, e \rangle$, where e is any element in $E_0 - \hat{A}$. Then $E_0 \leqslant C(E_1)$. In either case it is clear that E_1 centralizes a proper r-subgroup of G_0, so (2.3) gives $C^r_{G_0}(E_1) = G_0$. But the assumption $q > 4$ together with (4.1), (4.5), and the results of this section give $C^r_{G_0}(E_1) \leqslant D_0$. So (i) holds in this case.

Suppose, now, that $q = 4$, $p = 3$, and $G_0 = E_8(4)$. We will produce a subgroup E_0 such that (iii) holds. Retain the above notation. Thus $E(L_1) \cong SL(3,4) \cdot \hat{E}_6(4)$. Set $x_2 = x_1^{e_1}$ and $x_3 = x_2^{e_1}$. Then $x_1x_2x_3 = 1$, so $x_3 = x_1^{-1}x_2^{-1}$ and we may assume that $z = x_1x_2^{-1}$. Set $S \in \mathrm{Syl}_3(\hat{A}\langle e_1\rangle)$ with $e_1 \in S$, and write $S_i = S \cap A_i$ for $i = 0, 1, 2, 3$. Then $S_i = \langle a_i, b_i\rangle$, where $[a_i, b_i] = x_i$ for $i = 1, 2, 3$, $[a_0, b_0] = z$, and e_1 stabilizes $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$. We now set $E_0 = \langle z, x_1, a_0a_1a_2a_3, b_1b_2b_3, b_0b_1^{-1}b_2\rangle$ and observe that E_0 is elementary of order 3^5. We will show that $C_{G_0}^0(E_0) \leqslant N_G(E_0)$.

Let F be a hyperplane in E_0 and assume $z \in F$. First, suppose $x_1 \in F$. Then $C_{G_0}(F) \leqslant C_{G_0}(x_1) \cap C_{G_0}(z) = C_{L_1}(x_1) = \hat{A}$. Calculation within $\hat{A}$ yields $C_{G_0}(F) \leqslant N_{G_0}(E_0)$. Suppose F does not contain x_1, so that $E_0 = F \times \langle x_1\rangle$. Using the results of [7] one checks that $\langle b_1b_2b_3, x_1\rangle^{\#}$ is fused in $B_1 \cong \hat{E}_6(4)$. Let $\langle x\rangle = F \cap \langle b_1b_2b_3, x_1\rangle$. Then $E_0 \leqslant C_{L_1}(x) \sim C_{L_1}(x_1)$ and

$$C_{G_0}(F) \leqslant C_{G_0}(\langle z, x\rangle) = \hat{A}^g$$

for some $g \in B_1$. Since $x_1 \notin F$, F projects onto a Sylow 3-subgroup of $A_0/Z(A_0)$, and it follows that $C_{G_0}(F) \leqslant (A\langle g_1\rangle)^g$, where $g_1 \in \hat{A} \cap B_1$.

Let $Q = \langle z, x_1, a_1a_2a_3, b_1b_2b_3\rangle$. As in the $E_6(4)$ case, we have $(Q/\langle z\rangle)^{\#}$ fused in $N_{B_1}(Q)$. So we may take $g \in N_{B_1}(Q)$ such that $x_1^g = x$ and $Q \leqslant (A_1A_2A_3)^g$. We claim that $C_{G_0}(F)$ is contained in a Sylow 3-subgoup, Y, of $(A\langle g_1\rangle)^g$. It will suffice to show that $C_{(A_1A_2A_3)^g}(\tilde{F})$ is a 3-group, where $\tilde{F}$ is the projection of $F \cap A^g$ to $(A_1A_2A_3)^g$. However, $C_{A_1A_2A_3}(\langle x_1, z, q_1\rangle)$ is a 3-group for any $q_1 \in Q - \langle x_1, z\rangle$ and $\tilde{F} \cap Q^g > \langle x_1^g, z\rangle$. This proves the claim.

Suppose $Y = Y_0\langle e\rangle$ for some $e \in F$, where $Y_0 = Y \cap A^g$. If $y \in C_Y(F)$, then $y = e^iy_0$ for some $y_0 \in Y_0$. So

$$E_0^y = E_0^{y_0} = (\langle e\rangle(E_0 \cap Y_0))^{y_0} = \langle e\rangle^{y_0}(E_0 \cap Y_0)^{y_0} = \langle e\rangle(E_0 \cap Y_0) = E_0$$

and $y \in N_{G_0}(E_0)$. Suppose no such element, e, exists. Then $F \leqslant Y_0$. The projection of E_0 to B_1 is nonabelian, so the projection of F to B_1 is nonabelian. This forces $C_Y(F) \leqslant Y_0$. In particular, $E_0 \leqslant Y_0$ and so $C_Y(F) \leqslant N(E_0)$. We have now shown that $C_{G_0}(F) \leqslant N_{G_0}(E_0)$, whenever F is a hyperplane of E_0 containing z.

If F is a hyperplane of E_0 with $E_0 = F\langle z\rangle$, then F contains $x_1z^i = x$, for some i. As $A \leqslant C(x)$ we use [7] to conclude that $x \sim z$. So $C_{G_0}(x) \sim C_{G_0}(z)$ and $A \leqslant C_G(x)$. In view of the embedding of E_0 in A we can argue as above that $C(F) \leqslant N(E_0)$. This completes the proof that $C_{G_0}^0(E_0) \leqslant N_{G_0}(E_0)$ for this case.

We are left with the cases $G_0 = E_7(2)$ or $E_8(2)$ and $p = 3$. Let s be the positive root of highest height and $J = \langle U_{\pm s}\rangle \cong S_3$. Then $C_{G_0}(O_3(J)) = O_3(J) \times E(C_{G_0}(J))$ and $E(C_{G_0}(J)) \cong O^+(12,2)'$ or $E_7(2)$, according to whether $G_0 \cong E_7(2)$ or $E_8(2)$. In the former case $E(C_{G_0}(J))$ contains $J_1 \times \cdots \times J_6$, a direct product of 6 G_0-conjugates of J. To see this just consider the direct product of three copies of $O^+(4,2)$ in $O^+(12,2)$. If $G_0 = E_8(2)$ let J_0 be a conjugate of J in $E(C_{G_0}(J))$. Then G_0 contains $J \times J_1 \times \cdots \times J_6$ or $J \times J_0 \times J_1 \times \cdots \times J_6$, depending on whether $G_0 = E_7(2)$ or $E_8(2)$. Let $O_3(J) = \langle x\rangle$ and $O_3(J_i) = \langle x_i\rangle$ for $i = 1, \ldots, 6$ ($i = 0, \ldots, 6$ if $G_0 \cong E_8(2)$).

We claim that $C^0_{G_0}(E_0) \leq N_{G_0}(E_0)$, where $E_0 = \langle x, x_1, \ldots, x_6 \rangle$ or $\langle x, x_0, x_1, \ldots, x_6 \rangle$, depending on whether $G_0 \cong E_7(2)$ or $E_8(2)$. First suppose $G_0 \cong E_7(2)$. Let F be maximal in E_0 with $x \in F$. Then by (4.1) we have $C_{G_0}(F) = C_{G_0}(F) \cap C_{G_0}(x) \leq N_{G_0}(E_0)$. Now suppose $E_0 = \langle x \rangle \times F$. As above we may assume $\langle x_i \rangle \nleq F$ for $i = 1, 2$. So $x^m x_1, x^n x_2 \in F$ for $m, n \in \{1, -1\}$ and $a \in F$ for $a = x_1 x_2$ or $x_1 x_2^{-1}$. Now $C(a) \cap E(C_{G_0}(J)) = \langle a \rangle \times O^-(10, 2)'$, so $C_G(a) \cap C(J) = \langle a \rangle \times O^-(10, 2)'$. From Burgoyne [7] we conclude that $C_{G_0}(a) = \langle a \rangle \times J \times O^-(10, 2)'$. Again we use (4.1) to argue that $C_{G_0}(F) \leq N_{G_0}(E_0)$. So $C^0_{G_0}(E_0) \leq N_{G_0}(E_0)$.

Now suppose $G_0 \cong E_8(2)$. Here we argue as above. First let F be maximal in E_0 with $x \in F$. Then $C_{G_0}(F) \leq C_{G_0}(x) = \langle x \rangle \times E_7(2)$, so from the above paragraph we conclude that $C_{G_0}(F) \leq N_{G_0}(E_0)$. To complete the proof argue as above, using [7] to show that $E(C_{G_0}(a)) \cong O^-(14, q)$. This yields $C^0_{G_0}(E_0) \leq N_{G_0}(E_0)$ and completes the proof of (5.11).

(5.12) Let $G = G_2(q)$. Then one of the following holds:

(i) $D_0 = G_0$.

(ii) $q = 2$, $p = 3$, $G_0 \cong \mathrm{Aut}(PSU(3,3))$ and D_0 normalizes a Sylow 3-subgroup of G_0, for any proper 3-subgroup, E_0, of G_0.

Proof. By (5.3), (5.8), and Table (5.2) we may assume $p = 3$. From Table (3.3) we see that G_0 contains a subgroup L with $L \cong SL(3, q)$ or $SU(3, q)$, according to whether $3 \mid q - 1$ or $3 \mid q + 1$. Then L contains a Sylow 3-subgroup of G_0, and we take $E_0 < L$. If $3 \mid q - 1$, then $L \cong SL(3, q)$ has just one class of elementary groups of order $3^n > 3$. They are isomorphic to $Z_3 \times Z_3$ and are conjugate to a subgroup of H. So (i) holds by (2.3).

Now suppose $3 \mid q + 1$ and $q > 2$. Again $E_0 \cong Z_3 \times Z_3$ and $Z(L) < E_0$. So $L \leq D_0$. For $e \in E_0 - Z(L)$, e is contained in $\langle U_{\pm s} \rangle^g \leq L$ for some $g \in G$, where s is the positive (long) root of highest height (to see this check the action of e on the usual module for $SU(3, q)$). Then E_0 acts on $E(C_{G_0}(\langle U_{\pm s} \rangle^g)) = \langle U_{\pm \alpha_1} \rangle^g$. Consequently $\langle U^g_{\pm \alpha_1} \rangle \leq D_0$, and usual arguments (e.g. apply (2.8)) show that $G_0 = \langle L, \langle U_{\pm \alpha_1} \rangle^g \rangle$. Again (i) holds.

For $q = 2$, just note that $G_2(q) \cong \mathrm{Aut}(U_3(3))$, so (ii) holds. This proves (5.12).

(5.13) Let $G_0 = {}^3D_4(q)$. Then $D_0 = G_0$.

Proof. By (5.3), (5.7) and Table (5.2) we may assume that p is associated with Φ_3 or Φ_6, or that $p = 3$. Assume $q \neq 2$. By Table (3.3) G_0 contains a subgroup L with the following properties: $O^{2'}(L) \cong SL(3, q)$ if $3 \mid q - 1$ or if p is associated with $q^3 - 1$; $O^{2'}(L) \cong SU(3, q)$ if $3 \mid q + 1$ or if p is associated with $q^6 - 1$; $L = O^{2'}(L)C$ where $C \cong q^2 + q + 1$ or $q^2 - q + 1$ depending on whether $O^{2'}(L) \cong SL(3, q)$ or $SU(3, q)$; $[L, C] = 1$ and $L \cap C = Z(L)$.

Suppose p is associated with $q^3 - 1$ or $q^6 - 1$. Then L contains a Sylow p-subgroup of G_0 and we take $E_0 \leq L$. Then $E_0 \cong Z_p \times Z_p$ and $E_0 = (E_0 \cap E(L)) \times (E_0 \cap C)$. At this point we can argue as in (5.8) to conclude $D_0 = G_0$.

Suppose, then, that $p = 3$. Again using Table (3.3) we see that $L < \bar{L}$ with $\bar{L}/L \cong Z_3$. Then $\bar{L}$ contains a Sylow 3-subgroup of G_0 and we take $E_0 < \bar{L}$. Also G_0

contains $\langle U_{\pm\alpha_2}\rangle \times \langle U_{\pm s}\rangle$, for s the positive root of highest height, and this group is isomorphic to $SL(2, q) \circ SL(2, q^3)$. Thus a Sylow 3-subgroup of G_0 contains a group isomorphic to $O_3(Z_{q\pm 1}) \times O_3(Z_{q^3\pm 1})$. It follows from these facts that G_0 has 3-rank 2 and $E_0 = \langle z, y\rangle$ for $\langle z\rangle = Z(L)$. In particular $\bar{L} \leqslant D_0$.

Let $I = \langle V^G \cap D_0\rangle$, where V is a root subgroup for a long root. Then $O^{2'}(L) \leqslant I < D_0$. Using (2.8) together with the fact that $I \leqslant D_0 \leqslant N(I)$, we have $I = O^{2'}(L)$. So $O^{2'}(L) \trianglelefteq D_0$ and hence $D_0 \leqslant N(\langle z\rangle)$. Also (2.7) and (2.11) imply that we may choose y such that $K = E(C_{\hat{G}}(y)) \neq 1$. Then $K_\sigma \leqslant D_0 \leqslant N(\langle z\rangle)$. Consequently, (2.3) implies the result, unless $S_3 \cong O^{2'}(K_\sigma) \nleqslant C(z)$. However, from Burgoyne [7] we see that $O^{2'}(K_\sigma) \cong S_3$ is impossible.

The remaining case is $q = 2$. We have $G_0 \geqslant L \cong SU(3, 2)$ and $G_0 \geqslant X \cong L_2(2) \times L_2(8)$. Let $S_0 = O_3(L)$. Then S_0 has index 3 in a Sylow 3-subgroup S of G_0 and $S \leqslant N_{G_0}(L)$. For $s \in S - S_0$, $S = S_0\langle s\rangle$ and since S has exponent 9, $|s| = 9$. Thus $E_0 < S_0$ and G_0 contains one class of subgroups isomorphic to $Z_3 \times Z_3$. We may take $E_0 \leqslant X \cap L$ so that $\langle X, L\rangle \leqslant D_0$. Moreover, $E_0 \leqslant L$ implies that $E_0 - \langle z\rangle$ is fused, so $L_2(8) \cong E(C_{G_0}(e)) \leqslant D_0$ for each $e \in E_0 - \langle z\rangle$. One can now use (2.8) to conclude that $D_0 = G_0$.

(5.14) Let $G_0 = {}^2F_4(q)'$. Then one of the following holds:

(i) $q > 2$ and $D_0 = G_0$.

(ii) $q = 2, p = 5$, $E_0 \cong Z_5 \times Z_5$, and $C^0_{G_0}(E_0) \leqslant N_{G_0}(E_0)$.

(iii) $q = 2$, $p = 3$, $E_0 \cong Z_3 \times Z_3$, and $C^0_{G_0}(E_0) \cong L_3(3)\langle\gamma\rangle$, where γ is the graph automorphism of $L_3(3)$. In this case if we set $G_1 = {}^2F_4(2)$, then $C^0_{G_1}(E_0) = G_1$.

Proof. By (5.3) we may assume that p is associated with Φ_1, Φ_2, or Φ_4. Suppose the latter holds and $q \neq 2$. Then G_0 contains a subgroup $X = (X_1 \times X_2)\langle t\rangle$, where $X_2 = X_1^t$, $X_1 \cong Sz(q)$, and $t^2 = 1$. By (5.2) we may take $E_0 \leqslant X_1 \times X_2$ and $E_0 = \Omega_1(P)$ for $P \in \mathrm{Syl}_p(G_0)$. Clearly $(X_1 \times X_2)\langle t\rangle \leqslant D_0$. However the arguments used in (5.8) do not work here because of the lack of root involutions. We proceed as follows.

Let $S_1 \in \mathrm{Syl}_2(X_1)$ and set $S_2 = S_1^t$. Choose $1 \neq x \in \Omega_1(S_1)$. Then $S_2 \leqslant C_{G_0}(x)$ so x is 2-central in G_0 (see e.g. (18.6) of [3]). Now $C_{D_0}(x) \geqslant S_1 \times X_2$ and $C_{G_0}(x) = O^{2'}(P)$ for P a maximal parabolic subgroup of G_0. The structure of $C_{G_0}(x)$ is given explicitly in §10 of [12]. We check that if $C_{D_0}(x) > S_1 \times X_2$, then $C_{D_0}(x) \geqslant S_1 \cdot O_2(P)'$, and it is easy to check that this group and X_1 generate G_0. So suppose $C_{D_0}(x) = S_1 \times X_2$. Let $S = (S_1 \times S_2)\langle t\rangle$. Clearly $Z(S_1) \times Z(S_2)$ is weakly closed in S, and hence $S_1 \times S_2 = C_S(Z(S_1)Z(S_2))$ is characteristic in S. By the Krull-Schmidt theorem (p. 120 of [16]) the pair $\{S_1Z(S_2), Z(S_1)S_2\}$ is characteristic in S, and taking squares, we conclude that $\{Z(S_1), Z(S_2)\}$ is characteristic in S. From this and the above we see that $S \in \mathrm{Syl}_2(D_0)$. Also Corollary 4 of [13] shows that $Z(S_1)Z(S_2)$ is strongly closed in S. But now consider $C_{X_1X_2}(t) \cong Sz(q)$. By (18.6) of [3] t is G_0-conjugate to an involution in $Z(S_1)$. Applying Sylow's theorem in $C_{G_0}(t)$ to $\langle e\rangle = E \cap C_{X_1X_2}(t)$ we see that $Y = O^{2'}(C_G(e)) \cong Sz(q)$. But then $t \in Y \leqslant D_0$ and t is a square in D_0. This contradicts the fact that $Z(S_1)Z(S_2)$ is strongly closed in S. So $D_0 = G_0$ in this case.

If $q = 2$ and $p = 5$, then $E_0 = O_5(X_1X_2)$ (notation as above) and $E_0 \in \mathrm{Syl}_5(G_0)$. If $E_i = O_5(X_i)$, then $C_{G_0}(E_i) = E_i \times X_j$, where $\{i, j\} = \{1,2\}$. Also it is easy to use the above arguments to check that $N_{G_0}(E_0)$ is transitive on $E_0^\#$. It follows that $C_{G_0}^0(E_0) \leqslant N(E_0)$ and (ii) holds.

We now suppose $p \mid q \pm 1$. If $p \mid q - 1$, then by (5.2) $E_0^g \leqslant H$ for some $g \in G$, and so (i) holds by (2.3). So, assume $p \mid q + 1$. Suppose $p > 3$. Then by (5.2) we may assume

$$E_0 \leqslant \langle U_{\pm\alpha_1}\rangle \times \langle U_{\pm\alpha_1}^{s_2 s_1 s_2}\rangle \cong SL(2,q) \times SL(2,q).$$

This implies $E_0 \leqslant C(E_1)$ where $E_1 \cong Z_3 \times Z_3$. If $x \in E_1^\#$, then by [7] $C_{G_0}(x) \cong SU(3,q)$. By (4.1) it will suffice to show that $G_0 = C_{G_0}^0(E_1)$. That is, we may now assume $p = 3$.

Next we will show that it suffices to look at the case $q = 2$. First note that ${}^2F_4(2)$ is a subgroup of ${}^2F_4(q)$. Choose $E_0 \leqslant \langle U_{\pm\alpha_1}\rangle \times \langle U_{\pm\alpha_1}^{s_2 s_1 s_2}\rangle = X$. With the natural embedding of $Y = {}^2F_4(2) \leqslant G_0$ we may take $E_0 \leqslant {}^2F_4(2)'$. Each involution in ${}^2F_4(2)$ is in ${}^2F_4(2)'$ (see 18.6 of [3]). Since the Weyl group of ${}^2F_4(2)$ and G_0 are the same, we may consider the Weyl group of G_0 as a subgroup of Y. Let $I = \langle X, Y\rangle \leqslant D_0$. Then I contains all subgroups $\langle U_{\pm\alpha}\rangle$ of G_0 for $U_\alpha \sim U_{\alpha_1}$. From the commutator relations (see §10 of [**12**]) we see that if β is any root with $\langle U_{\pm\beta}\rangle \cong Sz(q)$, then $\Omega_1(U_\beta) \leqslant I$. For such a root β we have $I \cap \langle U_{\pm\beta}\rangle \geqslant \langle \Omega_1(U_{\pm\beta})\rangle = \langle U_{\pm\beta}\rangle$, for $q > 2$. So $I = G_0$ and it suffices to look at the case $q = 2$.

Now $G_0 = {}^2F_4(2)'$, $E_0 \cong Z_3 \times Z_3$, and by Burgoyne [7] each $e \in E_0^\#$ satisfies $C(e) \cap {}^2F_4(2) \cong SU(3,2)$. We then have $C_{G_0}(e)$ a subgroup of index 2 in $SU(3,2)$. Write $E_0 = \langle a\rangle \times \langle b\rangle$. In $G_1 = {}^2F_4(2)$ there is just one class of such subgroups and it follows that $G_0 \geqslant X_1 \times X_2 \cong S_3 \times S_3$, with $\langle a\rangle = O_3(X_1)$ and $\langle b\rangle = O_3(X_2)$. So $C_{G_0}(a)$ contains an involution, t_1, that inverts b. Let $S = O_3(C_{G_1}(b))$. We now have $N_{G_1}(\langle b\rangle)$ an extension of S by a semidihedral group of order 16, and $N_{G_0}(\langle b\rangle)$ an extension of S by D_8. Let T be a Sylow 2-subgroup of $N_{G_0}(\langle b\rangle)$, with $t_1 \in T$. Then $E_0^T = \{E_0, E_1\}$ where $E_1 = \langle b, c\rangle$ and $E_0E_1 = S$. T contains a klein group $T_0 = \langle t_1\rangle \times \langle t_2\rangle$ with the following properties: t_1 inverts c, t_2 centralizes c, t_2 inverts a, and t_2 inverts b. So $T_0 = N_T(E_0) = N_T(E_1)$ and $T = T_0\langle j\rangle$, where j is an involution with $t_i^j = t_2$ and $a^j = c$.

Let $P_0 = N_{G_0}(E_0)$ and $P_1 = N_{G_0}(E_1)$. Looking at $P_0 \cap C(a)$ and $P_0 \cap C(b)$ we easily have P_0 being the semidirect product of E_0 with $GL(2,3)$. Similarly for P_1. Set $X = \langle P_0, P_1\rangle$. Then $X = X^j$ and we claim that $X \cong L_3(3)$.

The claim is proved by first giving a (B, N)-pair in X. Let $Y = P_0 \cap P_1 = S\langle t_1, t_2\rangle = ST_0$. Y will be a Borel subgroup of X and T_0 a Cartan subgroup. P_0 is 2-transitive on the cosets of Y, so there is an involution $j_0 \in N_{P_0}(T_0)$ such that $P_0 = Y \cup Yj_0Y$. Similarly $P_1 = Y \cup Yj_1Y$, where we may take $j_1 = j_0^j$. Now $S = \langle a\rangle\langle b\rangle\langle c\rangle$ and we then have

$$\begin{aligned} j_0Sj_1 &= j_0\langle a\rangle\langle b\rangle\langle c\rangle j_1 = (\langle a\rangle\langle b\rangle)^{j_0} j_0 j_1 \langle c\rangle^{j_1} \\ &\subseteq \langle a, b\rangle j_0 j_1 \langle b, c\rangle \subseteq S j_0 j_1 S. \end{aligned}$$

This implies $j_0Yj_1 \subseteq Yj_0j_1Y$. To get additional relations observe that since j_0 normalizes T_0, j_0 must permute the proper subgroups of E_0 that are T_0-invariant, namely $\langle a\rangle$ and $\langle b\rangle$. As $j_0 \notin N(\langle b\rangle)$, j_0 interchanges $\langle a\rangle$ and $\langle b\rangle$. Similarly j_1

interchanges $\langle b\rangle$ and $\langle c\rangle$. Thus,

$$\begin{aligned} j_0 S j_1 j_0 &= j_0 E_0\langle c\rangle j_1 j_0 = E_0 j_0\langle c\rangle j_1 j_0 = E_0 j_0 j_1\langle b\rangle j_0 \\ &= E_0 j_0 j_1 j_0\langle a\rangle \subseteq S j_0 j_1 j_0 S. \end{aligned}$$

At this point the additional relations are obtained by proving that $\langle j_0, j_1\rangle \cong S_3$. Now $\langle j_0, j_1\rangle$ induces S_3 on $\langle t_1, t_2\rangle$ and $\langle j_0, j_1\rangle \leqslant \langle j_0, j\rangle$ as a subgroup of index 2. From the known information on centralizers of elements of order 3 and 5 we see that if $\langle j_0, j_1\rangle \not\cong S_3$, then $\langle j_0, j\rangle$ is dihedral of order divisible by 24, and an element of order 4 centralizes $\langle t_1, t_2\rangle$. To see that this is impossible argue as follows. We may take $X_1 = \langle U_{\pm\alpha_1}\rangle$ and $X_2 = \langle U_{\pm\alpha_1}^{s_2 s_1 s_2}\rangle$. Then we have $\langle t_1, t_2\rangle \sim \langle U_{\alpha_1}, U_{\alpha_1}^{s_2 s_1 s_2}\rangle$, and using (18.6)(ii) of **[3]** together with the relations in §10 of **[12]**, we see that $C(\langle t_1, t_2\rangle)$ is elementary abelian. So $\langle j_0, j_1\rangle \cong S_3$ and $X \cong L_3(3)$ by Theorem 2 of **[15]**. Thus $X\langle j\rangle \cong \mathrm{Aut}(L_3(3))$.

From the structure of $\mathrm{Aut}(L_3(3))$ we see that $C_{G_0}(e) \leqslant X\langle j\rangle$ for each $e \in E_0^{\#}$. Also $C_X^0(E_0)$ is a maximal parabolic subgroup of X. Indeed $C_X^0(E_0) = N_X(E_0)$. Since $C_{X\langle j\rangle}(b) > C_X(b)$ we have $C_{G_0}^0(E_0) = C_{G_0}^0(E_1) = X\langle j\rangle$. So $D_0 = X\langle j\rangle$ for this choice of E_0. Now, in $G_1 = {}^2F_4(2)$ there is just one class of subgroups of type $Z_3 \times Z_3$. So $C_{G_0}^0(E_0) \cong \mathrm{Aut}(L_3(3))$ for any such subgroup $E_0 \leqslant G_0$.

If $D_1 = C_{G_1}^0(E_0)$ for E_0 as above, then we easily see that D_1 is transitive on elements of order 3 in D_1 and D_1 is strongly 3-embedded in G_1. So $|G_1 : D_1| \equiv 1 \pmod{27}$. But with $D_0 < D_1$ this is impossible. The proof of (5.14) is now complete.

With the results of this section we have now completed the proof of Theorem 1.

6. Automorphisms and Theorem 2. In this section we deal with the case G simple, $E \leqslant \mathrm{Aut}(G)$, $E \cong E_{p^n}$ with $n > 1$, but E not contained in the subgroup of $\mathrm{Aut}(G)$ generated by inner and diagonal automorphisms of G. Since p is odd this usually means that some element of E induces a field automorphism of G. Let $G_0 = \hat{G}_\sigma$ and let G_1 be the inverse image in $\hat{G}$ of $\overline{G}_\sigma$. Then $G_1/Z(G_1)$ is isomorphic to G extended by all diagonal automorphisms of G. Let G_2 be the extension of G_1 by field automorphisms of G_0.

We first assume that E is contained in $G_2/Z(G_0)$. There is a p-subgroup $E_0 \leqslant G_2$ such that $E_0 Z(G_0)/Z(G_0) = E$. Set $D_0 = C_{G_0}^0(E_0)$ and $D = C_G^0(E)$. Then $D_0 = G_0$ implies that $D = G$. Note that the only cases not included under the above restriction on E_0 is $G \cong D_4(q)$ or ${}^3D_4(q)$.

(6.1) Suppose $E_0 \leqslant G_2$, but $E_0 \nleqslant G_1$.

(i) If $p \geqslant 7$, then $D_0 = G_0$.

(ii) Suppose $m_p(E) > 2$ and $D_0 < G_0$. For $e \in E_0 - (E_0 \cap G_1)$ set $L = C_{G_0}(e)$ and $E_1 = E_0 \cap G_1$. Then there exists e such that $C_L^0(E_1) < L$. So the pair (L, p) is given in Theorem 1.

(iii) If $m_p(E) = 2$, either $D_0 = G_0$, or $p = 5$ and $G \cong Sz(2^5)$, or $p = 3$ and $G \cong L_2(8)$.

Proof. Suppose $D_0 < G_0$. By Lang's theorem **[18]** each element in $E_0 - (E_0 \cap G_1)$ induces a field automorphism on G_0. Set $E_1 = E_0 \cap G_1$. If $m_p(E) = 2$, then $C_{G_0}(e) \leqslant D_0$ for each $e \in E_0 - E_1$. Suppose $m_p(E) > 2$. If (ii) fails, then $C_{G_0}(e) \leqslant D_0$ for each $e \in E_0 - E_1$. So in all cases we have $C_{G_0}(e) \leqslant D_0$ for each $e \in E_0 - E_1$.

First suppose that for $e \in E_0 - E_1$, $C_{G_0}(e) \not\cong S_3, A_4$, or $Sz(2)$. Then Theorem 1 of [**8**] implies that $O^{r'}(C_G(e)) \leqslant D \leqslant C_G(e)$ for each $e \in E_0 - E_1$ (we are using the fact that $O^{r'}(C_G(e)) = C_{G_0}(e)Z(G_0)/Z(G_0)$). Since E is generated by such elements e, we have $O^{r'}(C_G(e)) \leqslant C_G(E)$ for $e \in E_0 - E_1$. This contradicts (2.3).

Consider the three exceptional cases. If $O^{r'}(C_G(e)) \cong S_3$, then $p = 3$ and $G \cong L_2(8)$. If $O^{r'}(C_G(e)) \cong Sz(2)$, then $p = 5$ and $G \cong Sz(2^5)$. For these cases (iii) holds. Finally, assume $O^{r'}(C_G(e)) \cong A_4$. Here $p = 3 = r$, against our standing hypothesis. We have now proved (6.1).

The pairs (G, p) appearing in (i), (ii), and (iv) of Theorem 1 each give rise to larger configurations where generation fails. These correspond to the situation of (6.1)(ii).

(6.2) Suppose (G_0, E_0, p) is one of the cases constructed in (4.1) or (5.1) where generation fails, and suppose that the pair (G, p) satisfies (i), (ii), or (iv) of Theorem 1. Let $G_0 = \hat{G}_\sigma$, $\tilde{G}_0 = \hat{G}_{\sigma^p}$ and $\tilde{E}_0 = E_0\langle\tau\rangle$, where τ is the field automorphism of order p that σ induces on $\tilde{G}_0$. Then $C^0_{\tilde{G}_0}(\tilde{E}_0) < \tilde{G}_0$.

PROOF. We are assuming that G_0 is one of the groups $O^{\pm}(2n,2)'$, $Sp(2n,2)$, $SU(n,2)$, $E_6(2)$, ${}^2E_6(2)$, $F_4(2)$, $E_7(2)$, $E_8(2)$, $G_2(2)'$, or ${}^2F_4(2)'$. In the last case $p = 5$; otherwise $p = 3$. Let E_0 be as constructed in the proof of (4.1) or (5.1). In each of these cases it is clear from the definition of E_0 that $\tilde{G}_0$ contains a subgroup I such that I is a τ-invariant homocyclic p-group satisfying $E_0 = \Omega_1(I) = [I, \tau]$. For example, in many of these cases $E_0 = O_3(J_1 \times \cdots \times J_k)$, where each $J_i \cong SL(2,2)$ and each J_i is generated by conjugates of long root subgroups. Here we take $E_0 < I$ with $I \in \mathrm{Syl}_3(\tilde{J}_1 \times \cdots \tilde{J}_k)$, where $J_i < \tilde{J}_i \cong SL(2,2^3)$ and $\tilde{J}_i$ is generated by long root subgroups of $\tilde{G}_0$.

For the moment exclude the configuration of (4.1)(iii). Then in each of the remaining cases we showed that $C^0_{G_0}(E_0) \leqslant N_{G_0}(E_0)$, and we claim that $C^0_{\tilde{G}_0}(\tilde{E}_0) < N_{\tilde{G}_0}(E_0)$. To see this let F be a maximal subgroup of $\tilde{E}_0$. If $F = E_0$, then clearly $C(F) \leqslant N(E_0)$. Suppose $F \neq E_0$, so $F = F_1\langle e\tau\rangle$, where $F_1 = F \cap E_0$ and $e \in E_0$. There exists $g \in I$ such that $\tau^g = e\tau$. Then $C_{\tilde{G}_0}(F) = C_{\tilde{G}_0}(F_1\langle e\tau\rangle) = C_{\tilde{G}_0}(F_1\langle\tau\rangle)^g = C_{G_0}(F)^g \leqslant N_{G_0}(E_0)^g \leqslant N_{\tilde{G}_0}(E_0)$. This proves the claim.

If (4.1)(iii) holds, then $G_0 \cong Sp(2n,2)$, $\tilde{G}_0 \cong Sp(2n,8)$, and $D_0 \cong O^{\pm}(2n,2)$. Here embed D_0 in $\tilde{D}_0 \cong O^{\pm}(2n,8)$, such that $\tilde{D}_0^\tau = \tilde{D}_0$. We can then choose $I < \tilde{D}_0$ and use the above argument to get $C^0_{\tilde{G}_0}(\tilde{E}_0) \leqslant \tilde{D}_0$.

(6.3) Suppose (6.1)(ii) holds with the pair (L, p) given in Theorem 1(iii). Then $D_0 = G_0$.

PROOF. We have $p = 3$ and $L \cong SL(3^k,4)$, ${}^2F_4(2)'$, $F_4(4)$, $\hat{E}_6(4)$, or $E_8(4)$. By (6.1)(iii) we may take $m_p(E) \geqslant 3$. The case $L \cong {}^2F_4(2)'$ is out since $L = C_{G_0}(e)$ (see (5.14)(iii)). Let $E_1 = E_0 \cap G_1$.

First assume $L \cong SL(3,4)$, so $G_0 \cong SL(3,64)$. By (4.1) we may write $E_1 = \langle a, b\rangle$ where $[a, b] = z$ and $\langle z\rangle = Z(G_0)$. Write $E_0 = E_1\langle e\rangle$ and choose e so that $[a, e] = z$. Then $T = C_{G_0}(e) \cong SL(3,4)$ and a induces an outer automorphism of order 3 on T. Then $C_T(a) \cong Z_{21}$ as otherwise $C_T(a) \cong Z_3 \times L_2(4)$ and E_1 normalizes a 2-group L, contradicting (2.3). Thus, $C_{G_0}(F) = C_T(a) \cong Z_{21}$ where $F = \langle a, e, z\rangle$. Now letting a vary we conclude that $T \leqslant C^0_{G_0}(E_0)$. Next, let e vary to conclude that $D_0 = G_0$.

Suppose $L \cong SL(3^k, 4)$ with $k > 1$. Let $E_0 = E_1\langle e\rangle$ and choose $a, b \in E_1$ such that $\langle a\rangle Z(G_0) \neq \langle b\rangle Z(G_0)$. Then consideration of the usual module for G_0 shows that $\langle a, b\rangle$ normalizes a proper parabolic subgroup of G_0. By (2.3) $C^r_{G_0}(\langle a, b\rangle) = G_0$. Now (2.16), Theorem 1, (6.1), induction and the above paragraph yield $D_0 = G_0$. Essentially the same argument works for the cases $L = F_4(4)$, $\hat{E}_6(4)$, or $E_8(4)$. Namely, choose $\langle a, b\rangle \leqslant E_1$ with $\langle a\rangle Z(G_0) \neq \langle b\rangle Z(G_0)$ and use [7] to conclude that $\langle a, b\rangle$ stabilizes a proper parabolic subgroup of G_0. Then proceed as before. This proves (6.3).

We will have completed the proof of Theorem 2 once we deal with the case $E_0 \nleqslant G_2$. This requires the existence of a graph automorphism of order 3, so $G_0 \cong D_4(q)$ or ${}^3D_4(q)$.

(6.4) Suppose $G_0 \cong {}^3D_4(q)$ and $E_0 \nleqslant G_2$. Then either $C^0_{G_0}(E_0) = G_0$ or $q = 2$, $p = 3$ and there exists $E_0 \cong Z_3 \times Z_3 \times Z_3$ such that $C^0_{G_0}(E_0) < G_0$.

PROOF. First suppose $q > 2$ and note that $E_0 \leqslant G_0\langle\tau\rangle$, where τ denotes the graph automorphism of $D_4(q^3)$, restricted to G_0. Then $G_2(q) \cong C_{G_0}(\tau)$. By Table (3.3) $C_{G_0}(\tau)$ contains a group $L \cong SL(3, q)$ or $SU(3, q)$, according to $3 \mid q - 1$ or $3 \mid q + 1$.

Let $S \in \mathrm{Syl}_3(L)$. By Table (5.2) S has index 3 in a Sylow 3-subgroup of G_0, so $S \times \langle\tau\rangle$ has index 3 in a Sylow 3-subgroup of $G_0\langle\tau\rangle$. Therefore, we may assume that $E_0 \leqslant N(S \times \langle\tau\rangle) \leqslant N(Z(S \times \langle\tau\rangle)) = N(\langle x, \tau\rangle)$, where $\langle x\rangle = Z(L) = Z(S)$. Since $C_{G_0}(\tau)$ does not contain a Sylow 3-subgroup of G_0, $\tau \sim \tau x^i$ for $i = 0, 1, 2$. Now apply Theorem 2 of [8] to conclude that $G_0 = \langle O^{r'}(C_{G_0}(\tau x^i)) \mid i = 0, 1, 2\rangle$. So if $E_0 \leqslant C(\langle x, \tau\rangle)$, then $G_0 = C^0_{G_0}(E_0)$ by Theorem 1. Hence we may assume $E_0 \nleqslant C(\tau)$.

By (2.9) we may assume $x \in E_0$. Suppose $q > 4$. Then $L \leqslant D_0$ so $L \leqslant \langle V^G \cap D_0\rangle$, where V is a long root subgroup of G_0. Using (2.8) and Table (5.2) we conclude that $E(D_0) = L$, $D_0 = G_0$, or $E(D_0) \cong G_2(q)$. Suppose $E(D_0) = G_2(q)$. Then one checks that $O^{r'}(D_0E_0) \cong G_2(q) \times Z_3$, the Z_3 factor being generated by a conjugate of τ. By (2.9) we may assume $\tau^G \cap E_0 \neq \varnothing$, which is not the case. Suppose $E(D_0) = L$. For $e \in E_0 - \langle x\rangle$ one can argue within $\hat{G}$ to conclude that $O^{r'}(C_{G_0}(e)) \neq 1$. But $q > 4$ implies $O^{r'}(C_{G_0}(e)) \leqslant D_0 \leqslant N(L)$. Since $L = E(N_{G_0}(L))$, $O^{r'}(C_{G_0}(e)) \leqslant L$. In particular $\langle e, x\rangle$ centralizes a nontrivial r-subgroup of G. By (2.3), $C^r_{G_0}(\langle e, x\rangle) = G_0$, while the above arguments applied to $e_1 \in \langle e, x\rangle$ show $C^r_{G_0}(\langle e, x\rangle) \leqslant N(L)$. This is a contradiction. Hence $D_0 = G_0$.

Next, suppose $q = 4$. Then $L \cong SL(3, 4)$. If $m_3(E_0) = 2$, then $L = O^{r'}(C(x)) \leqslant D_0$ and we argue as above to get $D_0 = G_0$. Suppose $m_3(E_0) \geqslant 3$. Then $m_3(E_0) = 3$ and we write $E_0 = \langle x, e, a\tau\rangle$ for some $e \in G_0$ such that $\tau^e \neq \tau$. Since $S\langle\tau\rangle = S \times \langle\tau\rangle$ we may take $a \in S$. Then $e \notin S \times \langle\tau\rangle$. Now, $C_L(e) \cong Z_3 \times L_2(4)$ or Z_{21}. In the former case, E_0 normalizes a proper 2-subgroup of G_0, so (2.3) implies $D_0 = G_0$. Assume $C_L(e) \cong Z_{21}$. Then $L = \langle C_L(v)\colon v \in \langle e, a\tau\rangle - \langle a\tau\rangle\rangle$, so $L \leqslant C^0_{G_0}(E_0) = D_0$. As above, either $D_0 = G_0$ or $L \trianglelefteq D_0$. So suppose $L \trianglelefteq D_0$. Let $y \in \langle x, e\rangle^{\#}$. By [7] either $O^{r'}(C(y)) \cong L_2(64)$ or $y \sim x$. In the former case we have $O^{r'}(C(y)) \leqslant D_0$, against $L = E(D_0)$. So $\langle x, e\rangle^{\#}$ is fused in G_0. In particular, $E(C_{G_0}(e)) \cong SL(3, 4)$. Since $C(e) \cap C(x)$ contains 7-elements, x induces an outer automorphism on $E(C_{G_0}(e))$ and we can then argue that $E(C_{G_0}(e)) \leqslant D_0 \leqslant N(L)$, a contradiction.

Finally, let $q = 2$ and retain the above notation. Set $E_0 = \langle x, a, \tau\rangle$, where $a \in S - \langle x\rangle$. Then $S \trianglelefteq C_{G_0}(x)$, as $L \cong SU(3,2)$, so $S \trianglelefteq C_{G_0}(F)$ for any hyperplane, F, of E_0 with $x \in F$. Suppose F is a hyperplane of E_0 with $F \cap S = \langle ax^j\rangle$ for $j = 0, 1$, or 2. Then $\tau x^i \in F$ for some $i = 0, 1, 2$, and $\tau \sim \tau x^i$ implies that $C_{G_0}(\tau) \cong G_2(2) \cong \mathrm{Aut}(U_3(3))$. Now $S \in \mathrm{Syl}_3(C_{G_0}(\tau x^i))$ and it follows that $C_{G_0}(F) \leqslant N_{G_0}(S)$. We have now proved $C^0_{G_0}(E_0) \leqslant N_{G_0}(S)$, so (6.4) is proved.

Similar arguments will be used for $G_0 = D_4(q)$.

(6.5) Suppose $G = D_4(q)$ and $E_0 \nleqslant G_2$. Then one of the following holds:

(i) $C^0_{G_0}(E_0) = G_0$.

(ii) $q = 2$ or 4, $p = 3$, and $E_0 \leqslant D_4(q)\langle\sigma\rangle$ for σ a graph automorphism (of order 3) of G_0.

(iii) $q = 8$ and $E_0G_0 = O^{3'}(\mathrm{Aut}(D_4(8)))$.

Moreover if $q = 2, 4$ or 8, there exists $E_0 < \mathrm{Aut}(G_0)$ for which $C^0_{G_0}(E_0) \leqslant N_{G_0}(E_0)$.

PROOF. Suppose E_0 contains an element in the coset of G_1 given by a field or graph-field automorphism. Then we can argue as in (6.1) to get $C^0_{G_0}(E_0) = G_0$, except when $q = 8$ and (iii) holds or $E_0 \cap G_0 = 1$. Suppose $E_0 \cap G_0 = 1$. Then $E_0 \cong Z_3 \times Z_3$ and by Lang's theorem E_0 contains both a field automorphism and a graph-field automorphism of G_0. Here $C^0_{G_0}(E_0) = G_0$ by Theorem 1 of [**8**]. Suppose $q = 8$. Let σ be a field automorphism of G_0 of order 3, τ a graph automorphism of order 3, and $[\sigma, \tau] = 1$. Then $C_{G_0}(\sigma) \cap C_{G_0}(\tau) \cong G_2(2)$ and we let E_1 be an elementary subgroup of $C_{G_0}(\langle\sigma, \tau\rangle)$ of order 9. Set $E_0 = E_1\langle\sigma, \tau\rangle$.

Let $J = J_1 \times J_2 \times J_3 \times J_4$ with each J_i generated by opposite root subgroups of G_0. We may assume that σ induces a field automorphism on each J_i and that $\langle\tau\rangle$ is transitive on $\{J_1, J_2, J_3\}$ while fixing J_4. Let $E_0 = \langle a, b\rangle$ with $a \in J_4$ and $b \in J_1J_2J_3$. We claim $C^0_{G_0}(E_0) \leqslant N(\langle b\rangle)$. Let F be a hyperplane in E_0. If $b \in F$, then clearly $C_{G_0}(F) \leqslant N(\langle b\rangle)$, so suppose $b \notin F$. Since a, ab, ab^{-1} are all conjugate in $C_{G_0}(\sigma) \cap C_{G_0}(\tau) \cap N(E_0)$, we may assume $a \in F$. From here the claim is easily checked.

From now on we assume $E_1 = E \cap G_0$ is a hyperplane of E_0 and $E_0 \leqslant G_1\langle\tau\rangle$ for τ a graph automorphism of G_0 such that $|\tau| = 3$. From Table (3.3) and its verification we conclude that G_0 contains a subgroup X such that there exists $L \trianglelefteq X$ with $L \cong SL(3, q)$ or $SU(3, q)$, according to $q \equiv 1, -1 \pmod 3$. Also, $Z = Z(X) \cong Z_{q\mp 1} \times Z_{q \mp 1}$ and $|X: LZ| = 3$. Finally, $X^\tau = X$ and $L = C_X(\tau)$. By Table (5.2) we may assume $E_0 \leqslant X\langle\tau\rangle$ and by (2.9) we may assume $x \in E_1$, where $\langle x\rangle = Z(L)$. Assume $q > 4$. Then (4.1) implies $L \leqslant C^0_{G_0}(E_0)$.

Suppose $\tau \in E_0$. Then $E_1 \leqslant C_{G_0}(\tau) \cong G_2(q)$, and so E_1 is contained in a Sylow 3-subgroup of $C_{G_0}(\tau)$. Table (5.2) implies that L contains a Sylow 3-subgroup of $C_{G_0}(\tau)$, so assume $E_1 < L$. Therefore $Z(X) \leqslant C(E_1) \leqslant C^0_{G_0}(E_0)$. By (5.1) $G_2(q) \cong C_{G_0}(\tau) \leqslant C^0_{G_0}(E_0)$ and so $C^0_{G_0}(E_0) \geqslant \langle C_{G_0}(\tau), Z\rangle = G_0$ (use (2.8) and the fact that $C_{G_0}(\tau)$ is self-normalizing in G_0). From now on assume E_0 contains no conjugate of τ.

The results in [**9**] (see (3.5)) imply that we may assume $(\tau a)^g \in E_0$, where $a \neq 1$ and $a \in H \cap \langle U_{\pm\alpha_2}\rangle$. Also, using the Bruhat decomposition we see that $C_{\hat{G}}(\tau a)^g \cong PSL(3, K)$ (or see (4.3) of [**9**]) and then $C_{G_0}(\tau a)^g = C_{\hat{G}_0}(\tau a)^g \cong PGL(3, q)$

or $PGU(3, q)$, according to $q \equiv 1, -1 \pmod 3$. By (4.1) $O^{r'}(C^0_{G_0}(\tau a))^g \leqslant C^0_{G_0}(E_0)$, so $C^0_{G_0}(E_0) \geqslant \langle L, O^{r'}(C_{G_0}(\tau a))^g \rangle$. Now let I be the normal closure of L in $C^0_{G_0}(E_0)$ and argue, using (2.8) (extended slightly to cover the case $E_0 \nleqslant G_1$), that $C^0_{G_0}(E_0) = G_0$. This requires a check that embeddings such as $PSL(3, q) \leqslant PSL(4, q) \leqslant D_4(q)$ if $3 \mid q - 1$ and $PSU(3, q) \leqslant PSU(4, q) \leqslant D_4(q)$ if $3 \mid q + 1$ are impossible. For this one can use representation theory (e.g. the tensor product theorem of Steinberg [**24**]).

We are left with the cases $q = 2, 4$. Here we construct an example E_0 for which (ii) holds. Let a be an element of order 3 in $\langle U_{\pm \alpha_2} \rangle$ and x be as above. We may assume $\langle U_{\pm \alpha_2} \rangle \leqslant L = C_X(\tau)$. Let $E_{00} = \langle x, \tau a, y \rangle$, where $y = y_1 y_2$ with $y_1 \in L$, $y_2 \in Z - L$, $|y_1| = |y_2| = 3$, $a^{y_1} = ax$, and $\tau^{y_2} = \tau x^{-1}$. Then E_{00} is elementary abelian.

Let $P = O^3(C_{G_0}(\tau a)) \cong L_3(4)$ or $U_3(2)$. Now $\langle x \rangle = [E_{00}, \langle a \rangle]$ and $a \in C_{G_0}(\tau a)$, so $x \in P$. Also $C_{G_0}(a) \cap C_{G_0}(\tau) \geqslant SL_2(q)$ (as $C_{G_0}(\tau) \cong G_2(q)$). Thus $SL_2(q) \leqslant C_{G_0}(\tau a) \cap C_{G_0}(a)$, and a induces outer automorphism on P. Let $\langle a, y \rangle \cap P = \langle x \rangle \times \langle y^1 \rangle$ for some $y^1 \notin L$. Set $E_0 = \langle x, y^1, \tau a \rangle$. We claim that $C^0_{G_0}(E_0) \leqslant N_{G_0}(E_0)$.

Let F be a hyperplane of E_0. If $F = \langle x, y^1 \rangle$, then it is easy to see that $C_{G_0}(F) \leqslant N_{G_0}(E_0)$. Suppose $F \neq \langle x, y^1 \rangle$. If $x \in F$, we again check that $C_{G_0}(F) \leqslant N_{G_0}(E_0)$. So suppose $x \notin F$. Then F contains $\tau a x^i$ for some $i = 0, 1, 2$. Now $\tau a x^i \sim \tau a$ by an element of L normalizing E_0. Therefore, $E_0 \cap G_0 \leqslant O^3(C_{G_0}(\tau a x^i))$ and we easily see that $C_{G_0}(F) \leqslant N_{G_0}(E_0)$. This proves the claim and completes the proof of (6.5).

We have now completed the proof of Theorem 2.

7. Additional results. In this section we will prove Theorems 3–7. Theorem 3 is an immediate consequence of Theorems 1 and 2 so we concentrate on the notions of "layer generation" and "balance". Notation will be as in §2. Let G_1 be the group G extended by all diagonal automorphisms of G. As in (2.1) we may prove Theorems 4 and 6 for the group G_0.

(7.1) Assume $Z_p \times Z_p \cong E < G_1$. Then one of the following holds:
(i) $C^0_{G_0}(E_0) < G_0$.
(ii) $C^r_{G_0}(E_0) = G_0$.
(iii) $p \mid q - 1$, $G_0 \cong SL(p, q)$, and E_0 is nonabelian.
(iv) $p \mid q + 1$, $G_0 \cong SU(p, q)$, and E_0 is nonabelian.

PROOF. Suppose $C^0_{G_0}(E_0) = G_0$. Let $e \in E_0 - Z(G_0)$ and let $\hat{D}_1, \ldots, \hat{D}_k$ be the products of the $\langle \sigma \rangle$-orbits of components of $C_{\hat{G}}(e)$. Let $T = Z(C_{\hat{G}}(e))$ and for $i = 1, \ldots, k$ set $L_i = O^{r'}((\hat{D}_i)_\sigma)$. Then $O^{r'}(C_{G_0}(e)) = L_1 \cdots L_k$ (see (2.5)).

First suppose that E_0 is abelian, so $E_0 \leqslant C_{\hat{G}}(e)$ for each $e \in E_0 - Z(G_0)$. By (2.4), $C_{\hat{G}}(e) = \hat{D}_1 \cdots \hat{D}_k \hat{T}_0$. Choose $f \in E_0 - \langle e \rangle Z(G_0)$ and consider $Y = C(f) \cap C_{\hat{G}}(e)$. If $Y_i^0 = (Y \cap \hat{D}_i)^0$ is not a maximal torus of $\hat{D}_i$, then $(Y_i^0)_\sigma$ contains an element of order r, and (ii) follows from (2.3). So assume that Y_i^0 is a maximal torus of $\hat{D}_i$ for each $i = 1, \ldots, k$. Then Y^0 is a σ-invariant maximal torus of $\hat{G}$ with $E_0 \leqslant Y^0$. We conclude from (2.12) that $(C_{\hat{G}}(e))_\sigma = L_1 \cdots L_k Y_\sigma$ and from (2.11) that $C^r_{G_0}(E_0) \neq 1$. Then $(C_{\hat{G}}(e))_\sigma = L_1 \cdots L_k Y_\sigma \leqslant L_1 \cdots L_k C_{G_0}(E_0) \leqslant N_{G_0}(C^r_{G_0}(E_0))$. Since e was arbitrary we have $G_0 = C^0_{G_0}(E_0) \leqslant N(C^r_{G_0}(E_0))$ and (ii) holds.

Suppose E_0 is nonabelian. Then $p \mid |Z(G_0)|$ and $G \cong L_n(q)$, $U_n(q)$, $E_6(q)$, or ${}^2E_6(q)$. In the first case $p \mid (n, q-1)$, in the second case $p \mid (n, q+1)$, and in the other cases $p = 3$. If $G = L_n(q)$, then as in §4 we may assume E_0 acts on the usual module for $G_0 = SL(n, q)$. If E_0 acts reducibly, then E_0 is contained in a parabolic subgroup of G_0E_0 and (ii) follows from (2.3). If E_0 acts irreducibly, then elementary arguments from representation theory yield $n = p$, and (ii) holds.

Next, suppose $G_0 = SU(n, q)$. Then we may assume that E_0 is a subgroup of $GU(n, q)$. Let V be the usual module for $GU(n, q)$; V is an $\mathbf{F}_{q^2}$-space of dimension n. Let $e \in E_0 - Z(G_0)$ with $|e| = p$, and let $f \in E_0 - \langle e \rangle Z(G_0)$. If $\langle z \rangle = E_0 \cap Z(G_0)$, then we may assume $e^f = ez$. The eigenspaces of e on V are permuted transitively by $\langle f \rangle$ and each has dimension n/p. Let $V_1, \ldots, V_p$ be these eigenspaces. Using the fact that 1 is an eigenvalue of e, an easy check gives $V = V_1 \perp \cdots \perp V_p$. If $\dim(V) = p$, then (iv) holds. So suppose $\dim(V) > p$; that is, $\dim(V_i) > 1$ for $i = 1, \ldots, p$. Let $0 \neq v_1 \in V_1$ be an isotropic vector and $\langle v_1 \rangle, \ldots, \langle v_p \rangle$ the images of $\langle v_1 \rangle$ under $\langle f \rangle$. Then $\langle v_1, \ldots, v_p \rangle = \langle v_1 \rangle \perp \cdots \perp \langle v_p \rangle$ is an E_0-invariant isotropic subspace of V. Therefore E_0 is contained in a proper parabolic subgroup of $GU(n, q)$ and (ii) follows from (2.3).

Finally, we assume that either $G_0 = E_6(q)$ with $p = 3 \mid q - 1$ or $G_0 = {}^2E_6(q)$ and $p = 3 \mid q + 1$. Let $e, f \in E_0$ with $[e, f] \neq 1$. Then $f \notin C_{\hat{G}}(e)$, although $f \in N(C_{\hat{G}}(e))$. This is because $C_{\hat{G}}(e) = C_{\hat{G}}(e^f)$. By (2.3) we may assume that f normalizes no proper r-subgroup of $C_{G_0}(e)$. Then considering the $\langle f \rangle$-orbits on components in $C_{\hat{G}}(e)$ and then the action of σ on these orbits we see that f must normalize each component of $C_{\hat{G}}(e)$. Also, if we choose e such that $|e| = 3$, then it is easy to see that $E(C_{\hat{G}}(e)) \neq 1$. Indeed, just embed e in a maximal torus of $\hat{G}$ and consider the action of e on the unipotent radical of a Borel group. Let Y be a component of $C_{\hat{G}}(e)$. Then $Y = Y^f$ and there is a root subgroup, $V \leqslant Y$, with $V = V^f$. To see this just consider the isomorphism that f induces on Y. By (7.5) of Steinberg **[23]**, f stabilizes a Borel subgroup of Y, and we let V be the center of the unipotent radical of the Borel subgroup.

Now consider E_0 as a subgroup of $\hat{P} = N_{\hat{G}}(V)$. $\hat{P}$ is a parabolic subgroup of $\hat{G}$ with $\hat{P}/\hat{P}_u \cong SL(6, K)I$, where $I = I^0$ is a one-dimensional torus centralizing the $SL(6, K)$ factor. By (5.16) of **[22]**, E_0 stabilizes a maximal torus of $\hat{P}$ and we may assume $E_0 \leqslant N_{\hat{P}}(\hat{I})$, where $\hat{I}$ is a torus of $\hat{G}$ with $\hat{I}\hat{P}_u/\hat{P}_u = I$. So E_0 is in a Levi factor of $\hat{P}$ and it follows that $C_{\hat{P}}(E_0)$ contains a component. But $C(f) \cap C_{\hat{G}}(e)$ does not contain a component. This is impossible. This completes the proof of (7.1).

THEOREM 4 (LAYER GENERATION). *Let $Z_p \times Z_p \cong E \leqslant \mathrm{Aut}(G)$. Then one of the following holds*:

(i) $C^0_{G_0}(E_0) < G_0$.

(ii) $C^r_{G_0}(E_0) = G_0$.

(iii) $p \mid q - 1$, $G_0 \cong SL(p, q)$, $E \leqslant G_1$, *and* E_0 *is nonabelian.*

(iv) $p \mid q + 1$, $G_0 \cong SU(p, q)$, $E \leqslant G_1$, *and* E_0 *is nonabelian.*

PROOF. By (7.1) we may assume $E \nleqslant G_1$, and we may also assume that $C^0_{G_0}(E_0) = G_0$. First suppose that it is not the case that $p = 3$ and $G \cong D_4(q)$ or

${}^3D_4(q)$. Then Lang's theorem **[18]** implies that $E-(E\cap G_1)$ consists of field automorphisms of G. As in the proof of (6.1) we use Theorem 1 of **[8]** to get $C^r_{G_0}(E_0)=O^{r'}(C_{G_0}(e))$ for each $e\in E_0$ with $eZ(G_0)\notin G_1$. But then E_0 normalizes a proper parabolic subgroup of G_0, and (2.3) gives (ii).

Suppose $p=3$ and $G\cong D_4(q)$ or ${}^3D_4(q)$. Either $E_0\cap G_0\cong Z_3$ or $G\cong D_4(q)$ and $E\cap G_1=1$. In the latter case the argument of the first paragraph of the proof of (6.4) gives $C^r_{G_0}(E_0)=G_0$. So suppose $E_0\cap G_0=\langle e\rangle\cong Z_3$. In addition we may assume that $G_0E_0=G_0\langle\tau\rangle$, for τ a graph automorphism of G_0. For otherwise use the argument of the above paragraph.

Suppose $f\in E_0-\langle e\rangle$ and f induces a graph automorphism on G_0. Then $C_{G_0}(f)\cong G_2(q)$ and we consider $C_{G_0}(f)\leqslant G_2(K)=Y$, where $Y=\hat{G}_f$. Now embed e in a maximal torus of Y and then consider the action of e on the unipotent radical of a Borel subgroup of Y. It is easy to deduce that $E(C_Y(e))\neq 1$, so applying σ to this group and using (2.3), we again see that (ii) holds. From now on we assume that E_0 contains no conjugate of τ.

Suppose $E_0\leqslant C_{G_0}(\tau)\times\langle\tau\rangle\cong G_2(q)\times Z_3$. $C_G(\tau)$ contains a subgroup, L, such that $L\cong SL(3,q)$ or $SU(3,q)$ depending on whether $q\equiv 1,-1 \pmod 3$. By order considerations we see that L contains a Sylow 3-subgroup of $C_G(\tau)$ so we take $E_0\leqslant L\times\langle\tau\rangle$. As $\tau\notin E_0$, the projection of E_0 to L contains a representative of each class of elements of order 3 in L. However, it is easy to check that $\tau\sim\tau l$ for some element, l, of order 3 in L. This contradicts the above.

Suppose $G_0\cong{}^3D_4(q)$ or $G_0\cong D_4(q)$ with $q>4$. Then the argument in the fourth paragraph of the proof of (6.5) shows that $G_0=C^r_{G_0}(E_0)$, proving (ii). So now assume $G_0=D_4(q)$ with $q=2$ or 4.

We use the notation in the proof of (6.5). Write $E_0=\langle e\rangle\times\langle\tau a\rangle$, where $C_{G_0}(\tau a)\cong PGU(3,2)$ or $PGL(3,4)$, depending on whether $q=2$ or 4. If $e\in O^{2'}(C_{G_0}(\tau a))$, then e is 3-central in $C_{G_0}(\tau a)$. Since $\tau\in C_{G_0\langle\tau\rangle}(\tau a)$, this would imply that E_0 centralizes a conjugate of τ, which we have just seen to be impossible. Therefore $e\notin O^{2'}(C_{G_0}(\tau a))$ and $C_{G_0}(\tau a)=O^{2'}(C_{G_0}(\tau a))\langle e\rangle$. Each $f\in E_0-\langle e\rangle$ satisfies $f\sim\tau a$, so $C_{G_0}(f)=O^{2'}(C_{G_0}(f))\langle e\rangle$, for each $f\in E_0-\langle e\rangle$.

If we also have $C_{G_0}(e)=O^{2'}(C_{G_0}(e))\langle e\rangle$, then $C^0_{G_0}(E_0)\leqslant N_{G_0}(C^r_{G_0}(E_0))=X$, and the result follows. So suppose $C_{G_0}(e)\nleqslant X$. Then $C_{G_0}(e)>O^{2'}(C_{G_0}(e))\langle e\rangle$ and by [7] we have $e\sim x$, where x is as in the proof of (6.5). Consider $C_{G_0}(e)$. This group can be expressed $C_{G_0}(e)=Z\langle u,d\rangle$, where $Z=O^{2'}(C_{G_0}(e))\cong SU(3,2)$ or $SL(3,4)$, depending on whether $q=2$ or 4, $Z\langle u\rangle\cong GU(3,2)$ or $GL(3,4)$, $|d|=3$, and $C_{G_0}(e)=Z\langle u\rangle\times\langle d\rangle$. Now τa acts on $C_{G_0}(e)$ and $[\tau a,d]=e^{\pm 1}$. So $d\in N(E_0)\leqslant X$ and a Sylow 3-subgroup of X has index at most 3 in a Sylow 3-subgroup of G_0.

We claim that X contains a Sylow 3-subgroup of G_0. Suppose false. If $S\in\mathrm{Syl}_3(C_X(e))$ with $S^{\tau a}=S$, then $S\in\mathrm{Syl}_3(X)$, $S=\langle x_1,x_2\rangle\times\langle d\rangle$, where $\langle x_1,x_2\rangle=S\cap Z$. Let V be the natural module for G_0. We may assume $\dim(C_V(d))=6$ and Z acts irreducibly on $C_V(d)$, while centralizing $[V,d]$. It follows that e is fixed-point-free on $C_V(d)$, but elements of $\langle x_1,x_2\rangle-\langle e\rangle$ centralize a 2-space of $C_V(d)$. Now consider $C_{G_0}(\tau a)\cong PGU(3,2)$ or $PGL(3,4)$. We have seen that $e\notin C_{G_0}(\tau a)'$ and $\dim(C_V(e))=2$. Also $C_{G_0}(\tau a)'$ acts irreducibly on V and it is easy to see that each element of order 3 in $C_{G_0}(\tau a)'$ centralizes just a 2-space in V. So $C_{G_0}(\tau a)$ contains a subgroup of order 27 of the form $\langle e, j\rangle$, with $\langle e, j\rangle$ extraspecial. Apply Sylow's theorem to $X\langle \tau a\rangle$ and conclude that $(C_{G_0}(e)' \times \langle d\rangle)\langle \tau a\rangle$ contains a conjugate of $\langle e, j\rangle \times \langle \tau a\rangle$. This forces $\langle e, j\rangle \sim \langle x_1, x_2\rangle$ in G_0. But $\dim(C_V(e)) = 2$, while $\dim(C_V(w)) = 4$ for each $w \in \langle x_1, x_2\rangle - \langle e\rangle$. This is a contradiction. Thus X contains a Sylow 3-subgroup of G_0. Let $S < \bar{S} \in \mathrm{Syl}_3(X)$. Then $|\bar{S}: S| = 3$ and from the above considerations we see that $\langle e\rangle$ is weakly closed in $Z(S)$. So $\bar{S} \leq C_{G_0}(e)$ and we now have $C_{G_0}(e) \leq X$. This contradicts the earlier assumption. We have now proved Theorem 4.

Next, we will consider the notion of balance.

(7.2) Suppose $Z_p \times Z_p \cong E \leq G_1$. Then one of the following holds:

(i) $C_G^0(E) < G$.

(ii) $\bigcap_e O_{p'}(C_X(e)) = 1$ for any $G \leq X \leq \mathrm{Aut}(G)$, the intersection ranging over $e \in E^\#$.

(iii) $G \cong L_p(q)$, $p \mid q - 1$, and E_0 is nonabelian.

(iv) $G \cong U_p(q)$, $p \mid q + 1$, and E_0 is nonabelian.

PROOF. Suppose that (i), (iii), and (iv) are false. Then Theorem 4 and (2.1) imply that $C_G^r(E) = G$. So it will suffice to prove that if $1 \neq g \in O_{p'}(C_X(e))$ for each $e \in E^\#$ and some $G \leq X \leq \mathrm{Aut}(G)$, then $C_G^r(E) \leq C_G(g)$. Fix such an element g with g of prime order, and $e \in E^\#$.

Choose a p-element $e_1 \in \hat{G}$ such that e_1 projects to the element $e \in \bar{G}$. Then $Y = C_{\hat{G}}(e_1) = E_1 \cdots E_k T$, where the product is a commuting product of the connected torus T and the components $E_1, \ldots, E_k$ (see (2.5)). Passing to $\bar{G}$, let $\bar{Y} = \bar{E}_1 \cdots \bar{E}_k \bar{T}$. Then $C_{\bar{G}}(e)$ normalizes $\bar{Y}$ and $C_{\bar{G}}(e)/\bar{Y}$ is a finite p-group.

Let $\bar{D}_1, \ldots, \bar{D}_l$ be the orbit products of σ on $\{\bar{E}_1, \ldots, \bar{E}_k\}$ and set $L_i = O^{r'}((\bar{D}_i)_\sigma)$. Then $O^{r'}(C_G(e)) = L_1 \cdots L_l$. The element g permutes the components (solvable components) $L_1, \ldots, L_l$ of $C_G(e)$. Suppose $[L_i, g] \neq 1$ for some $i = 1, \ldots, l$. Since $g \in O_{p'}(C_X(e))$, we have $[L_i, g]$ a p'-group. It follows that L_i is a p'-group. This is clear if $L_i^g \neq L_i$. If $L_i^g = L_i$, then $[L_i, g] = L_i$ unless L_i is a solvable component, and in this case the assumption $p \neq 2, r$ gives the assertion. In particular, $p \nmid q^2 - 1$ unless L_i is a Suzuki group.

The condition $p \nmid q^2 - 1$ implies that $Z(G_0)$ has order prime to p and $E \cong E_0 \cong Z_p \times Z_p$. So $E_0 \leq C_{\hat{G}}(e_1)$ and it follows that $[L_i, E_0] \leq L_i$. Also, if $f \in E_0 - \langle e_1\rangle$, then f induces the product of an inner and diagonal automorphism on L_i. However, L_i is a p'-group and $p \nmid q^2 - 1$. It follows that $[L_i, f] = [L_i, E_0] = 1$. If L_i is a Suzuki group, then q is even and $\bar{G} \cong Sp(2n, K)$ or $F_4(K)$. Thus $Z(\hat{G}) = 1$ and we can use the same argument to obtain $[L_i, E_0] = 1$.

Choose $f \in E^\#$ and consider $L_i \leq O^{r'}(C_G(f))$. f acts on each component of $C_G(e)$ and we claim that $[f, \bar{D}_i] = 1$. Suppose $\bar{D}_i$ is the product of n components of $C_G(e)$. Then $\langle \sigma^n\rangle$ is the stabilizer of each of these components and we take a component $I \leq \bar{D}_i$. Then $I^{\sigma^n} = I$ and $[O^{r'}(I_{\sigma^n}), f] = 1$. From (2.1) of **[8]** we conclude that $[I, f] = 1$. The claim follows.

Since E_0 is abelian we choose $f \in E_0 - \langle e_1\rangle$ and embed f in a maximal torus, $\hat{H}$, of $C_{\hat{G}}(e_1) = C_{\hat{G}}(e_1)^0$. Then $E_0 < \hat{H}$ and we let $\bar{H} \leq \bar{B}$, where $E \leq \bar{H}$ is a maximal torus of $\bar{G}$ and $\bar{B}$ is a Borel subgroup of $\bar{G}$. By (4.1) of [22] the components of $C_{\bar{G}}(f)$ are generated by the root subgroups in $\bar{G}_u$ that are centralized by f. It follows that the normal closure of $\bar{D}_i$ in $C_{\bar{G}}(f)^0$, call it $\bar{K}_i$, is the product of a $\langle\sigma\rangle$-orbit of components of $C_{\bar{G}}(f)$. Set $X_i = O^{r'}((\bar{K}_i)_\sigma)$. Then $[\bar{L}_i, g] \neq 1$ implies $[X_i, g] \neq 1$.

Now rechoose the pair (L_i, e), if necessary, so that $|L_i|$ is maximal subject to the conditions that L_i is a component (solvable component) of $C_G(e)$ with $[L_i, g] \neq 1$. We then have $L_i = X_i \leqslant O^{r'}(C_G(f))$. So $N_G(L_i) \geqslant C_G^r(e) = G$, a contradiction. This proves that $[g, L_i] = 1$ for each component (solvable component), L_i, of $C_G(e)$ and for each $e \in E^{\#}$. Therefore $G = C_G^r(E) \leqslant C_G(g)$ which is impossible.

The next result handles case (i) of (7.2).

(7.3) Suppose $Z_p \times Z_p \cong E \leqslant G_1$ and $C_G^0(E) < G$. Then one of the following occurs:

(i) $\bigcap_{1 \neq e} O_{p'}(C_X(e)) = 1$, for any $G \leqslant X \leqslant \mathrm{Aut}(G)$.

(ii) $p = 3$ and $G = L_3(4)$.

PROOF. Apply Theorem 1 to restrict the possibilities for G and p. In each case $|G_1: G| = 1$ or p. Let $Y = \bigcap_{1 \neq e} O_{p'}(C_{G_1}(e))$. We will first show that $Y = 1$.

If $G = {}^2F_4(2)'$ with $p = 5$, then $E \in \mathrm{Syl}_5(G)$ and there is an element $e \in E^{\#}$ with $C_G(e) \cong Z_5 \times D_{10}$. This already implies $Y = 1$. So we now assume $p = 3$ and $q = 2$ or 4. We may then apply the results of [7]. There are only two possible configurations where some element $e \in E^{\#}$ satisfies $O_{3'}(C_G(e)) \neq 1$. The cases are $G \cong PSL(3^k, 4)$ and $G = E_6(4)$, and $O_{3'}(C_G(e)) \cong Z_7$ in either case.

So supposing $Y \neq 1$ we must have $Y = Z_7$ and $O_{3'}(C_G(e)) = O_{3'}(C_G(f))$ for each $f \in E^{\#}$. Also $C_G^0(E) \leqslant N_G(Y)$, so by (2.3), E is contained in no proper parabolic subgroup of G. For $G = PSL(3^k, 4)$ this implies that the preimage of E in $GL(3^k, 4)$ acts irreducibly on the usual module for $GL(3^k, 4)$. Then $E \cong Z_3 \times Z_3$ implies $k = 1$, and it is easy to see that $Y = 1$.

Suppose $G = E_6(4)$. By [7] we have $E(C_G(e)) \cong {}^3D_4(4)$ for each $e \in E^{\#}$. Fix $e \in E^{\#}$ and $f \in E - \langle e\rangle$. Then use [7] to see that $C(f) \cap E(C_G(e))$ has even order. So E is contained in a parabolic subgroup of G_1 and (2.3) implies that $C_G^0(E) = G$, against our hypothesis. Therefore $Y = 1$.

Set $Y_1 = \bigcap_{1 \neq e} O_{p'}(C_{\mathrm{Aut}(G)}(e))$ and suppose $Y_1 \neq 1$. By the above $Y_1 \cap G_1 = 1$. Considering the possibilities for G we see that Y_1 is quite restricted. For example if $G = {}^2F_4(2)'$, then $\mathrm{Aut}(G) = {}^2F_4(2)$ and there are no involutions in $\mathrm{Aut}(G) - G$. So this case is out. In the remaining cases $\mathrm{Aut}(G)/G_1 \cong 1, Z_2, Z_2 \times Z_2, Z_4$, or S_3 (the latter possible only if $G = D_4(2)$). So $Y_1 \cong Z_2, Z_4$, or $Z_2 \times Z_2$. Let y be an involution in Y_1.

The possible actions of y on G are determined in §19 of [3]. Namely, y induces a field automorphism, graph automorphism, graph-field automorphism, or $O_2(C_G(y)) \neq 1$. This last case contradicts (2.3), so this does not occur. Choose $e \in E^{\#}$ and let $\bar{L}_1, \ldots, \bar{L}_l$ be the components (solvable components) of $C_{\bar{G}}(e)$. By (2.3) И$(r, E) = \{1\}$, and it follows that $3 \mid |L_i|$ for $i = 1, \ldots, l$. However $[\bar{L}_i, y]$ is a $3'$-group for each $i = 1, \ldots, l$. Since $r = 2$ we necessarily have $[\bar{L}_i, y] = 1$ for $i = 1, \ldots, l$. The conclusion is that $C_G^2(E) = C_{C_G(y)}^2(E)$.

If $G = PSL(3^k, 4)$, then as before we get $k = 1$. So (ii) holds. We assume now that $G \neq PSL(3^k, 4)$. If $G = PSU(n, 2)$, then $E \leqslant C_G(y) = Sp(n, 2)$ or

$Sp(n-1,2)$, so the preimage of E in $GU(n,2)$ is elementary abelian. Here (4.1) and linear algebra show that the condition $C_G^2(E) = C_{C_G(y)}^2(E)$ is impossible, Similarly for $G = O^{\pm}(n,2)'$.

For the remaining cases use the tables in [7] to check that $C_G^2(E) \neq C_{C_G(y)}^2(E)$.

We now prove

THEOREM 5 (BALANCE). *If $Z_p \times Z_p \cong E \leqslant \mathrm{Aut}(G)$, then $\bigcap_{e\neq 1} O_{p'}(C_X(e)) = 1$ for any $G \leqslant X \leqslant G_1$ and one of the following holds:*

(i) $\bigcap_{e\neq 1} O_{p'}(C_X(e)) = 1$, *for any $G \leqslant X \leqslant \mathrm{Aut}(G)$.*

(ii) $p \mid q-1$, $G \cong L_p(q)$, $E \leqslant G_1$ *and E_0 is nonabelian.*

(iii) $p \mid q+1$, $G \cong U_p(q)$, $E \leqslant G_1$, *and E_0 is nonabelian.*

PROOF. Suppose $1 \neq g \in \bigcap_{1\neq e} O_{p'}(C_{\mathrm{Aut}(G)}(e))$. First assume that either (7.2)(iii) or (7.2)(iv) holds. Then E_0 is nonabelian and is absolutely irreducible on the usual module for G_0. It follows that $g \notin G_1$. This together with (7.2) and (7.3) allow us now to assume $E \nleqslant G_1$. Suppose each $e \in E - G_1$ induces a field automorphism on G. For $e \in E - G_1$, p divides the order of $C_G(e)$ and it follows that p divides the order of $O^{r'}(C_G(e))$. So $g \in \bigcap_{1\neq e} O_{p'}(C_{\mathrm{Aut}(G)}(e))$ implies $[O^{r'}(C_G(e)), g] = 1$. As in the proof of (6.1) we apply Theorems 1 and 2 of [**8**]. It is necessary to look at the cases $G \cong L_2(2^p)$, $L_2(3^p)$, and $Sz(2^p)$ individually, but this is straightforward. For all the other cases we have $O^{r'}(C_G(e)) \leqslant C_G(g)$ for each $e \in E - G_1$ and so $O^{r'}(C_G(e)) = O^{r'}(C_G(g))$ for each $e \in E - G_1$. So $E \leqslant C(O^{r'}(C_G(g)))$ and looking at $E \cap G_1$, we contradict (2.1) of [**8**].

So we now suppose that some $e \in E - G_1$ does not induce a field automorphism on G. Then $p = 3$, and $G \cong D_4(q)$ or ${}^3D_4(q)$. The above arguments allow us to assume that $E \leqslant G_1\langle\tau\rangle$, where τ is a graph automorphism of G with $|\tau| = 3$. As in the fourth paragraph of the proof of (6.5) we argue that for $e \in E - G_1$, $C_G(e) \cong G_2(q)$, $PGL(3,q)$, or $PGU(3,q)$. The first case occurs if e is conjugate to τ, and one of the other cases (according to $q \equiv 1, -1 \pmod 3$) otherwise. For $q > 2$ and $e \in E \cap G_1$, $O^{r'}(C_G(e)) = E(C_G(e))$ (see (2.5)). So under the assumption $q > 2$ we argue as before (e.g. as in (7.2)) that $[O^{r'}(C_G(e)), g] = 1$ for each $e \in E^{\#}$. Then $C_G^r(E) \leqslant C_G(g)$ and we have a contradiction if $C_G^r(E) = G$. By Theorems 1, 2, and 4 we are reduced to the cases $G = D_4(2)$, $D_4(4)$, or ${}^3D_4(2)$.

Choose $e \in E - G_1$. The possible choices (listed above) for $C_G(e)$ and the results of [7] show that $g \notin G_1$. Considering $\mathrm{Aut}(G)$ we see that $|g| = 2$, $G = D_4(4)$, and g induces a field automorphism on G. Then $[g, C_G(e)] = C_G(e)'$ ($= G_2(4)$ or $L_3(4)$), a contradiction. This completes the proof of Theorem 5.

THEOREM 6. *Suppose $p \geqslant 7$ and $(p, |\overline{W}|) = 1$, where $\overline{W}$ is the Weyl group of the associated algebraic group of G. Then $C_{G_0}^r(E_0) = G_0$.*

PROOF. First suppose $E \leqslant G_1$. Then (2.7) implies that E_0 is contained in a maximal torus of $\hat{G}$ and $C_{\hat{G}}(e) = C_{\hat{G}}(e)^0$ for each $e \in E_0^{\#}$. Choose $Z_p \times Z_p \cong F \leqslant E_0$. By Theorems 4 and 1 we have $C_{G_0}^r(F) = G_0$. For $f \in F^{\#}$ each component (solvable component) of $O^{r'}(C_{G_0}(f))$ has Weyl group with order dividing that of $|\overline{W}|$. This is because such a component is obtained as the fixed point group of σ on a product of components of $C_{\hat{G}}(f)$ and each of the components of $C_{\hat{G}}(f)$ is generated by the root groups in a subsystem of $\hat{\Sigma}$ (see (4.1) of [22]). So we can apply induction to get $O^{r'}(C_{G_0}(f)) \leqslant C^r_{G_0}(E_0)$. Then $G_0 = C^r_{G_0}(F) \leqslant C^r_{G_0}(E_0)$.

Suppose $E \nleqslant G_1$. Then $E - (E \cap G_1)$ consists of field automorphisms and the arguments of (6.1) give the result.

Our last result determines the strongly p-embedded subgroups of Chevalley groups. For this we drop the assumption $p \neq 2, r$.

THEOREM 7. *Let p be an arbitrary prime and suppose $X \leqslant \mathrm{Aut}(G)$ is a proper, strongly p-embedded subgroup of GX. Assume $m_p(X) \geqslant 2$. Then X is contained in the normalizer of a Sylow p-subgroup of G and one of the following holds*:

(i) $p = 3$ *and* $G \cong L_3(4)$.

(ii) $p = 5$ *and* $G \cong Sz(2^5)$ *or* ${}^2F_4(2)'$.

(iii) $G \cong L_2(q)$, $U_3(q)$, $Sz(q)$, *or* ${}^2G_2(q)'$, *where q is a power of p*.

PROOF. Assume X is strongly p-embedded in GX. Then X contains a Sylow p-subgroup, P, of G. If $p = 2$, apply Bender [4] to get the result. If $p = r$, then either G has Lie rank 1 (which gives (iii)) or G is generated by the proper parabolic subgroups of G that contain P. As these groups are p-local subgroups, we have $G \leqslant H$, a contradiction. So we may now assume $p \neq 2, r$.

For any $Z_p \times Z_p \cong E \leqslant X$ we have $C^0_G(E) \leqslant X$. So Theorems 1 and 2 restrict the possibilities for G. If $p = 5$, then (i) holds. So now assume $p = 3$ and q is a power of 2. If G has Lie rank at least 3, then from the Dynkin diagram it is easy to produce a proper parabolic subgroup of G containing two commuting copies of $L_2(q)$. But then there exists $Z_3 \times Z_3 \cong E \leqslant X \cap G$ with E in a proper parabolic subgroup of G and $C^0_G(E) < G$. This contradicts (2.3). Therefore G has Lie rank at most 2.

If G has Lie rank 1, then by Theorems 1 and 2, $GH \cong \mathrm{Aut}(Sz(2^5))$ or $\mathrm{Aut}(L_2(8))$, so that (ii) or (iii) holds. Suppose G has Lie rank 2. Then $m_3(G) \geqslant 2$ and there exists $Z_3 \times Z_3 \cong E \leqslant X \cap G$, satisfying $C^0_G(E) < G$. Theorem 1 now gives $G \cong L_3(4)$, $PSp(4,2)'$, $PSU(5,2)$, $G_2(2)'$, or ${}^2F_4(2)'$. If $G = G_2(2)'$, then $G \cong U_3(3)$ and (iii) holds. If $G \cong {}^2F_4(2)'$ with $X < G$, then $X \cong \mathrm{Aut}(L_3(3))$ and $X = \langle N_G(E) \colon Z_3 \times Z_3 \cong E \leqslant P \rangle$. On the other hand, choosing $g \in {}^2F_4(2) - {}^2F_4(2)'$ with $g \in N(P)$, we must have $X^g = X$. But then $C_{G_1}(X)$ contains an involution in $G_1 - G$, whereas $G_1 - G$ contains no involution.

If $G \cong PSU(5,2)$, a Borel subgroup of G has 3-rank 2. So this case is out by (2.3). Thus $G \cong L_3(4)$ or $PSp(4,2)' \cong L_2(9)$, and Theorem 7 is proved.

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