TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 272, Number 1, July 1982

FINITE GROUPS CONTAINING AN INTRINSIC 2-COMPONENT OF CHEVALLEY TYPE OVER A FIELD OF ODD ORDER

BY

MORTON E. HARRIS[1]

ABSTRACT. This paper extends the celebrated theorem of Aschbacher that classifies all finite simple groups G containing a subgroup $L \cong \mathrm{SL}(2, q)$, q odd, such that L is subnormal in the centralizer in G of its unique involution. Under the same embedding assumptions, the main result of this work allows L to be almost any Chevalley group over a field of odd order and determines the resulting simple groups G. The results of this paper are an essential ingredient in the current classification of all finite simple groups. Major sections are devoted to deriving various properties of Chevalley groups that are required in the proofs of the three theorems of this paper. These sections are of some independent interest.

1. Introduction. Let L be a finite group. If $L = L'$ and $L/Z(L)$ is simple, then L is said to be quasisimple. If π is a set of prime integers, then $O^{\pi'}(L)$ is the subgroup of L generated by all π-elements of L and $O_\pi(L)$ is the maximal normal π-subgroup of L. Clearly $O^{\pi'}(L)$ is the intersection of all normal subgroups M of L such that $|L/M|_\pi = 1$. Also $O(L) = O_{2'}(L)$ is the maximal normal subgroup of L of odd order. If $L = L'$ and $L/O(L)$ is quasisimple, then L is said to be 2-quasisimple.

Let G denote a finite group. A subnormal quasisimple subgroup of G is said to be a component of G and a subnormal 2-quasisimple subgroup of G is said to be a 2-component of G. Clearly every component of G is a 2-component of G. Also $E(G)$ denotes the subgroup of G generated by all components of G, $L_{2'}(G)$ denotes the subgroup of G generated by all 2-components of G, $F^*(G) = F(G)E(G)$ where $F(G)$ is the Fitting subgroup of G, $S(G)$ denotes the maximal normal solvable subgroup of G, $\mathfrak{M}(G)$ denotes the Schur multiplier of G and $Z^*(G)$ denotes the full inverse image in G of $Z(G/O(G))$. A subnormal subgroup L of G such that $O(L) = O(G)$ and $L/O(L)$ is isomorphic to PSL(2, 3) or to SL(2, 3) is called a solvable 2-component of G. As in [**1**] for simplicity of terminology, when it is not necessary to distinguish between 2-components and solvable 2-components, we will refer to both as 2-components. Also, when advantageous, we will refer to a 2-component which definitely is 2-quasisimple as a perfect 2-component. If z is an involution of G and if J is a 2-component of $C_G(z)$ such that $z \in J$, then J is said to be intrinsic in $C_G(z)$.

Received by the editors January 27, 1981.

1980 *Mathematics Subject Classification*. Primary 20D05; Secondary 20D06.

[1]This research was partially supported by the National Science Foundation and the Institute for Advanced Study.

0002-9947/81/0000-1077/$17.25

Finite simple Chevalley groups over finite fields of odd characteristic are listed in [**16**, §17.1] and specifically exclude ${}^2G_2(3)' \cong \mathrm{PSL}(2, 8)$. We shall usually adhere to the notation of [**16**, §17.1]. Note that in the notation for the "twisted" groups, the field order parameter is always the order of the smaller field involved in the definition.

For any odd prime integer p, a finite group G is said to be a Chevalley group over a finite field of characteristic p if

(a) G is quasisimple and $G/Z(G)$ is a simple Chevalley group over a finite field of characteristic p; or

(b) $p = 3$ and $G \cong \mathrm{SL}(2, 3)$ or $G \cong \mathrm{PSL}(2, 3)$.

Unless mentioned to the contrary, all groups in this article are finite. As is standard in the theory of finite groups, a simple finite group is nonabelian.

In order to state efficiently the main results of this paper, we introduce

DEFINITION 1. Let p denote an odd prime integer. A finite group H will be said to be of $\mathfrak{M}(p)$-type if it satisfies the following three conditions.

(a) H is 2-quasisimple and $H/Z^*(H)$ is isomorphic to a simple Chevalley group over a finite field of characteristic p;

(b) $|Z^*(H)|$ is even; and

(c) if $H/Z^*(H)$ is isomorphic to $\mathrm{PSL}(2n, p^r)$ or to $\mathrm{PSU}(2n, p^r)$ for positive integers n and r, then $|Z^*(H)|_2 = |\mathfrak{M}(H/Z^*(H))|_2$.

Also a finite group H such that $H/O(H) \cong \mathrm{SL}(2, 3)$ will be said to be of $\mathfrak{M}(3)$-type.

Note that any finite 2-quasisimple group that satisfies conditions (a) and (b) and that does not satisfy the hypotheses of condition (c) is of $\mathfrak{M}(p)$-type.

We now state the three main results of this paper. The first result can be viewed as an extension of [**3**, Corollary III].

THEOREM 1. *Let G be a finite group such that $O^{2'}(G)$ is 2-quasisimple. Suppose that z is an involution of G such that $C_G(z)$ contains an intrinsic solvable or perfect 2-component L such that L is of $\mathfrak{M}(p)$-type for some odd prime integer p. Then $O^{2'}(G)/O(O^{2'}(G))$ is isomorphic to a Chevalley group over a finite field of characteristic p or $G/O(G)$ is isomorphic to M_{11}.*

THEOREM 2. *Let W be a 4-subgroup of the finite group G and let $W^\# = \{z_1, z_2, z_3\}$. Suppose that $C_G(W)$ contains solvable or perfect 2-components L_1 and L_2 such that $z_i \in L_i$ and L_i is of $\mathfrak{M}(p_i)$-type with p_i an odd prime integer, for $i = 1$ and 2. Then $O(G)O^{2'}(L_i)$ is subnormal in G or $O^{2'}(L_i)$ is contained in a unique perfect 2-component K_i of G such that $K_i/O(K_i)$ is isomorphic to M_{11} or to a Chevalley group over a finite field of characteristic p_i for $i = 1$ and 2.*

THEOREM 3. *Let G be a finite group such that $F^*(G)$ is simple. Suppose that G contains a 4-subgroup W such that $C_G(W)$ contains a perfect 2-component L such that $L \cap W \neq 1$ and L is of $\mathfrak{M}(p)$-type for some odd prime integer p. Let $w \in (L \cap W)^\#$. Then $\langle L^{L_{2'}(C_G(w))} \rangle$ is a single 2-component of $C_G(w)$ or $F^*(G)$ is a simple Chevalley group over a finite field of characteristic p.*

At this point, it is appropriate to discuss the significance of the hypotheses, the methods of proof and the importance of these results.

Suppose that K is a 2-quasisimple group such that $K/Z^*(K)$ is a simple Chevalley group over a finite field of odd prime characteristic p and assume also that there does not exist an involution t in K such that $C_K(t)$ contains a *perfect* intrinsic 2-component of $\mathfrak{M}(p)$-type. Then, by Lemma 5.3, $K/O(K)$ is isomorphic to

(i) $\mathrm{PSL}(2, p^n)$ for some positive integer n,

(ii) ${}^2G_2(3^{2n+1})$ for some positive integer n, or

(iii) a Chevalley group over a field of 3 elements and of Lie rank at most 4.

(In the cases of (iii) there is however an involution t in K such that $C_K(t)$ contains an intrinsic solvable 2-component by Lemma 5.2.) Consequently perfect intrinsic 2-components of $\mathfrak{M}(p)$-type of centralizers of involutions are, with these exceptions, available for such groups K. This observation and the results of this paper will be used in an inductive setting in **[26]** to show that a proof of the Unbalanced Group Conjecture and the $B(G)$-Conjecture and the classification of all finite groups G with $F^*(G)$ simple that contain an involution t such that $C_G(t)$ possesses a perfect 2-component K such that $K/Z^*(K)$ is isomorphic to any simple Chevalley group over any finite field of odd characteristic depends on the solution of a few specific "standard component problems" related to the exceptions (iii) above.

This paper and **[26]** include an alternate approach to the results of J. H. Walter in **[38]** and **[39]**.

Theorem 2 is a consequence of Theorem 1 and Theorem 3 is a consequence of Theorems 1 and 2. The proof of Theorem 1 is basically a combination of the fundamental results of M. Aschbacher in **[3]**, of the ideas of M. Aschbacher, J. G. Thompson and J. H. Walter contained in **[37]** and of the insights of the author that accrued from the research for **[24]**.

In order to illustrate the significance of condition (c) in the definition of groups of $\mathfrak{M}(p)$-type and with **[3**, Corollary III] in mind, consider a finite simple group G with an involution t such that $C_G(t)$ possesses a perfect intrinsic 2-component K with $K/Z^*(K)$ isomorphic to a simple Chevalley group over a finite field of order $q = p^n$ with p an odd prime and n a positive integer that is not $\mathrm{PSL}(2, q)$. By a fundamental property of such a group K (cf. Lemma 5.4), there is an involution z in K such that $z \neq t$ and $C_K(z)$ contains both an intrinsic 2-component J_1 with $J_1/O(J_1) \cong \mathrm{SL}(2, q)$ and at least one other 2-component J_2 of $\mathfrak{M}(p)$-type. Set $H = C_G(z)$. By **[3**, Corollary III], we may assume that $O(H)J_1$ is not subnormal in H. Also assume for simplicity of the present discussion that $O(H) = 1$ and $q \neq 3$; in which case both J_1 and J_2 are perfect.

Suppose that K is of $\mathfrak{M}(p)$-type. Then the critical condition (c) in the definition of groups of $\mathfrak{M}(p)$-type (cf. Lemma 5.4) enables one to choose J_2 such that t or tz lies in $Z(J_2)$. Straightforward arguments using L-Balance **[18**, Theorem 3.1] and properties of 2-components imply that J_1 and J_2 both lie in the same intrinsic component X of $E(H) = L_{2'}(H)$ with $Z(X) \cap \langle t, z \rangle = \langle z \rangle$. Set $\bar{X} = X/Z(X)$. Then $\bar{J}_2$ is an intrinsic 2-component of $C_{\bar{X}}(\bar{t})$ of $\mathfrak{M}(p)$-type and we conclude, by induction, that $\bar{X}$ is a simple Chevalley group over a finite field of characteristic p.

Since $\bar{J}_1 \cong \mathrm{PSL}(2, q)$ and $\bar{J}_1$ is a component of $C_{\bar{X}}(\bar{t})$, it follows from the known possibilities for $\bar{X}$ that $\bar{X} \cong P\Omega(m, q_1)$ with $m \geq 7$ and q_1 a power of p. By repeating this argument, if necessary, we reduce to the case in which $K/O(K) \cong \mathrm{Spin}(7, q)$.

Suppose on the other hand, that K is not of $\mathfrak{M}(p)$-type. In this case, if $\langle J_1, J_2 \rangle$ is contained in a single component X of $E(H)$ (which is not even necessarily the case), then $\bar{J}_2$ is neither of $\mathfrak{M}(p)$-type in $\bar{X} = X/Z(X)$ nor intrinsic in $C_{\bar{X}}(\bar{t})$ since $J_2 \cap \langle t, z \rangle = \langle z \rangle$ and $|Z^*(\bar{J}_2)|_2 < |Z^*(J_2)|_2$. Consequently a wider inductive setting seems required in order to identify X under these conditions and so the treatment of this particular problem is postponed to **[26]**.

Finally we remark that the bulk of the proof of Theorem 1 is devoted to treating the cases in which $L/Z^*(L)$ is a simple Chevalley group over a field of 3 elements and in which $L/O(L) \cong \mathrm{Spin}(7, q)$ for an odd prime power q.

In §2, we present various results that are required in our proofs of Theorems 1–3. Some of these lemmas are of independent interest. In §3, we utilize **[12]** to survey the conjugacy classes of involutions and semi-involutions and their centralizers in the classical linear groups over finite fields of odd order. These results are required at various points in our proofs in this paper, in **[26]** and are also of independent interest. In §4, we apply the theory of linear algebraic groups to survey the conjugacy classes of involutions and their centralizers in various Chevalley groups and their automorphism groups over finite fields of odd order. In some of these lemmas, since the machinery is available and for the sake of completeness, we derive more information than is actually required in this paper. However all of these results are required in **[26]** and are also of independent interest. In §5, we utilize our previous work to derive additional results that are required in our proofs of Theorems 1–3. Finally §§6–8 are devoted to proving Theorems 1–3, respectively.

Our notation is fairly standard and tends to follow the notation of **[16]**. In particular, if X is a group and $Y \subseteq X$, then $\mathfrak{I}(Y)$ denotes the set of involutions of Y. Also if X is a group such that $(|X/X'|, |\mathfrak{M}(X)|) = 1$, then X has a universal covering group and it is denoted by $\mathrm{Cov}(X)$ (cf. **[21]**).

Finally, the author would like to thank Professors Daniel Gorenstein and Michael O'Nan of Rutgers University, Professor Nicholas Burgoyne of the University of California at Santa Cruz and Professor Edward Cline of Clark University for stimulating discussions about this paper and the referee for excellent suggestions.

2. Preliminary results. In this section, we present several lemmas that are required at various points in our proofs of Theorems 1–3. Some of these results are of independent interest.

The first lemma is well known and is presented without a (trivial) proof.

LEMMA 2.1. *Let π be a set of prime integers. Let G be a group and let N be a subnormal subgroup of G such that $|G : N|_{\pi'} = 1$. Then $O^{\pi}(G) = O^{\pi}(N)$ and $O_{\pi'}(G) = O_{\pi'}(O^{\pi}(G)) = O_{\pi'}(N)$.*

LEMMA 2.2. *Let X and M be subgroups of the group G with $M \trianglelefteq G$. Let π be a set of prime integers and set $\bar{G} = G/M$. Then the following two conditions hold.*

(a) $\overline{O_{\pi}(X)} \leq O_{\pi}(\bar{X})$; *and*

(b) $\overline{O^{\pi}(X)} = O^{\pi}(\bar{X})$.

PROOF. Clearly $MO_\pi(X) \trianglelefteq MX$ and $(MO_\pi(X))/M \cong O_\pi(X)/(M \cap O_\pi(X))$ is a π-group; thus (a) holds. Also $MO^\pi(X) \trianglelefteq MX$ and $(MX)/(MO^\pi(X)) \cong X/(O^\pi(X)(M \cap X))$ which is an epimorphic image of $X/O^\pi(X)$. Thus $O^\pi(\bar{X}) \leqslant \overline{O^\pi(X)}$. On the other hand, if x is a π'-element of X, then $\bar{x}$ is a π'-element of $\bar{X}$ and hence $\overline{O^\pi(X)} \leqslant O^\pi(\bar{X})$. Thus (b) also holds and we are done.

LEMMA 2.3. *Let H and K be subgroups of a group G and set $M = \langle H, K \rangle$. Note that $[H, K] \trianglelefteq M$ (cf. [16, Theorem 2.2.1(iii)]) and set $\bar{M} = M/[H, K]$. Then the following two conditions hold.*

(a) $\bar{M} = \bar{H} * \bar{K}$; *and*

(b) $\langle K^H \rangle = [K, H]K \trianglelefteq M$ *and* $\langle H^K \rangle = [K, H]H \trianglelefteq M$.

PROOF. Since $\bar{M} = \langle \bar{H}, \bar{K} \rangle$ and $[\bar{H}, \bar{K}] = \overline{[H, K]} = 1$, (a) holds. For (b), note that $[K, H] \leqslant \langle K^H \rangle$ and $\overline{\langle K^H \rangle} = \langle \bar{K}^{\bar{H}} \rangle = \langle \bar{K} \rangle$. Thus (b) also holds.

LEMMA 2.4. *Let G be a not necessarily finite group such that $G = H \times K$ for subgroups H and K. Let α be an endomorphism of G such that α^2 leaves H invariant and $H^\alpha = K$. Then α^2 leaves K invariant, $K^\alpha \leqslant H$ and $C_G(\alpha) = \{hk \mid h \in C_H(\alpha^2)$ and $k = h^\alpha \in K\} \cong C_H(\alpha^2)$.*

PROOF. Clearly $K^{\alpha^2} = H^{\alpha^3} \leqslant H^\alpha = K$, so that α^2 leaves K invariant and $K^\alpha \leqslant H^{\alpha^2} \leqslant H$. Let $h \in H$ and $k \in K$ and suppose that $(hk)^\alpha = hk$. Then $h^\alpha = k$, $k^\alpha = h$, $h^{\alpha^2} = k^\alpha = h$ and the lemma follows.

LEMMA 2.5. *Let G be a not necessarily finite group with a nontrivial subgroup H of index 2 such that $H = K_1 \times K_2$ for subgroups K_1 and K_2. Assume that K_1 and K_2 are conjugate in G. Then the following two conditions hold.*

(a) *There is an involution $t \in G - H$ such that $K_1^t = K_2$ and $C_H(t) = \langle k_1 k_1^t \mid k_1 \in K_1 \rangle \cong K_1$; and*

(b) $\mathcal{I}(G - H) = t^G = t^H$.

PROOF. Let $x \in G - H$. Then $x^2 = k_1 k_2$ where $k_i \in K_i$ for $i = 1, 2$ and $K_1^x = K_2$. Hence $k_1^x = k_2$, $k_2^x = k_1$, $(k_1^{-1}x)^2 = 1$ and (a) holds. Assume that $\gamma = u_1 u_2 t \in \mathcal{I}(G - H)$ where $u_i \in K_i$ for $i = 1, 2$. Then $1 = \gamma^2 = u_1 u_2 u_1^t u_2^t$; hence $u_1^t = u_2^{-1}$, $u_2 t u_2^{-1} = u_2 u_1 t = \gamma$ and (b) also holds.

LEMMA 2.6. *Let α be an endomorphism of the group G such that $g^\alpha \in gZ(G)$ for all $g \in G$. Then the following two conditions hold.*

(a) *α is the identity on G'; and*

(b) *the function $\bar{\alpha}$: $G \to Z(G)$ defined by $g^{\bar{\alpha}} = g^{-1}g^\alpha$ is a group homomorphism.*

PROOF. If $g, h \in G$, then $[g, h]^\alpha = [g^\alpha, h^\alpha] = [g, h]$. Thus (a) holds. If $g, h \in G$, then $(gh)^{-1}(gh)^\alpha = h^{-1}g^{-1}g^\alpha h^\alpha = (g^{-1}g^\alpha)(h^{-1}h^\alpha)$. Thus (b) holds and we are done.

LEMMA 2.7. *Let G be a group such that $(|G/G'|, |\mathfrak{M}(G)|) = 1$. Let H be a subgroup of G such that $H \leqslant G' \cap Z(G)$ and $(|G/G'|, |H|) = 1$, let $M = \mathrm{Cov}(G)$ and set $\bar{G} = G/H$. Then $(|\bar{G}/\bar{G}'|, |\mathfrak{M}(\bar{G})|) = 1$ and M is a universal covering group of $\bar{G}$.*

PROOF. Let α: $M \to G$ and π: $G \to \bar{G}$ denote the canonic epimorphisms. Let $\beta = \pi \circ \alpha$. Thus β: $M \to \bar{G}$ is an epimorphism, $\mathrm{Ker}(\beta) = \alpha^{-1}(H)$ and $\mathrm{Ker}(\alpha) \cong \mathfrak{M}(G)$. Note that $\alpha^{-1}(G') = M'\,\mathrm{Ker}(\alpha) = M'$ since $\mathrm{Ker}(\alpha) \leqslant M' \cap Z(M)$. Thus $\mathrm{Ker}(\beta) \leqslant M'$ and $\bar{G}/\bar{G}' \cong G/G' \cong M/M'$. Also $\alpha([M, \mathrm{Ker}(\beta)]) = [G, H] = 1$, so that $[M, \mathrm{Ker}(\beta)] \leqslant \mathrm{Ker}(\alpha) \leqslant M' \cap Z(M)$. Fix $k \in \mathrm{Ker}(\beta)$ and define τ: $M \to \mathrm{Ker}(\alpha)$ by $m^\tau = [m, k]$ for $m \in M$. If $m_1, m_2 \in M$, then

$$(m_1m_2)^\tau = [m_1m_2, k] = [m_1, k][m_1, k, m_2][m_2, k] = [m_1, k][m_2, k] = m_1^\tau m_2^\tau$$

by [**16**, Lemma 2.2.4(i)] since $[m_1, k] \in Z(M)$. Thus τ is a homomorphism. Since $(|M/M'|, |\mathrm{Ker}(\alpha)|) = (|G/G'|, |\mathfrak{M}(G)|) = 1$, we conclude that $\mathrm{Im}(\tau) = 1$. Hence $k \in Z(M)$ and $\ker(\beta) \leqslant M' \cap Z(M)$. Now [**21**, Theorem 3(ii) and Corollary 1.2] imply that M is a covering group of $\bar{G}$. Thus $\ker(\beta) \cong \mathfrak{M}(\bar{G})$ by [**21**, Lemma 1(i)]. Since $\ker(\alpha) \leqslant \mathrm{Ker}(\beta)$ and $\alpha(\mathrm{Ker}(\beta)) = H$, it follows that $\mathrm{Ker}(\beta)/\mathrm{Ker}(\alpha) \cong H$. Thus $|\mathfrak{M}(\bar{G})| = |\mathrm{Ker}(\beta)| = |H|\,|\mathrm{Ker}(\alpha)|$ and $|\mathfrak{M}(\bar{G})|$ is relatively prime to $|G/G'| = |\bar{G}/\bar{G}'|$. The result now follows from a celebrated result of I. Schur (cf. [**21**, Theorem 3]).

COROLLARY 2.7.1. *If G is a quasisimple group, then G is a homomorphic image of* $\mathrm{Cov}(G/Z(G))$. *If G is a 2-quasisimple group, then $G/O(G)$ is a homomorphic image of* $\mathrm{Cov}(G/Z^*(G))$.

LEMMA 2.8. SL(2, 3) *is a universal covering group of* SL(2, 3) *and of* PSL(2, 3).

PROOF. By [**28**, V, Satz 25.5; **21**, Theorem 3(i)], SL(2, 3) is a universal covering group of SL(2, 3). Then Lemma 2.7 implies that SL(2, 3) is a universal covering group of PSL(2, 3).

The next result is obvious.

LEMMA 2.9. *Suppose that L is a solvable 2-component of the group G. Then $O(L) = O(G)$, $O^{2'}(L) < O^{3'}(L)$ and $O(L)O^{2'}(L) < L = O(L)O^{3'}(L) = O^2(L)$.*

LEMMA 2.10. *Let G be a group with $O(G) = 1$, let $H = O^2(G)$, let $N \leqslant Z(G)$ and set $\bar{G} = G/N$. Suppose that $\bar{G}$ is isomorphic to* PSL(2, 3) *or* SL(2, 3). *Then the following two conditions hold.*

(a) *$G = N * H$ and H is isomorphic to* PSL(2, 3) *or to* SL(2, 3); *and*

(b) *if $\bar{G} \cong$* SL(2, 3), *then $H \cong$* SL(2, 3).

PROOF. Clearly $Z(G) \leqslant O_2(G)$ and (b) follows from (a). Also $O^2(\bar{G}) = \overline{O^2(G)} = \bar{H}$ by Lemma 2.2. Thus $G = N * H$ and we may assume that $G = H = O^2(G)$. Set $Q = O_2(G)$ and observe that $|G| = 3|Q|$ and $G = Q\langle\rho\rangle$ for some element ρ of order 3. By [**16**, Theorem 5.3.5 and Theorem 2.2.1], we have $Q = [Q, \langle\rho\rangle]C_Q(\rho)$ and $[Q, \langle\rho\rangle] \lhd G$. Set $\bar{G} = G/[Q, \langle\rho\rangle]$. Then $\bar{G} = \overline{C_Q(\rho)} * \langle\bar{\rho}\rangle = O^2(\bar{G})$ and hence $C_Q(\rho) \leqslant [Q, \langle\rho\rangle] = Q = G'$. Consider the natural epimorphism π: $G \to \bar{G} = G/N$. Since $\mathrm{Ker}(\pi) = N \leqslant Z(G) \cap G'$, it follows from Lemma 2.8 and [**21**, Lemma 1(ii)] that G is a homomorphic image of SL(2, 3). Thus (a) holds and we are done.

LEMMA 2.11. *Let G be a group. Then the following six conditions hold.*

(a) $E(G)$ *is the irredundant central product of the distinct perfect components of* G *and* J *is a perfect component of* G *if and only if* J *is a minimal perfect normal subgroup of* $E(G)$;

(b) $Z(E(G)) \leq Z(F(G)) = C_G(F^*(G))$ *and* $[S(G), E(G)] = 1$;

(c) *every perfect 2-component is normal in* $L_{2'}(G)$, $L_{2'}(G)$ *is the irredundant product of the distinct perfect 2-components of* G *and* J *is a perfect 2-component of* G *if and only if* J *is a minimal perfect normal subgroup of* $L_{2'}(G)$;

(d) *if* J *is a perfect 2-component of* G, *then* $J = O^{2'}(O(G)J) = (O(G)J)^{(\infty)}$ *and* $[S(G), J] \leq O(J)$;

(e) *if* J *and* K *are distinct perfect or solvable 2-components of* G, *then* $[J, K] \leq O(J) \cap O(K)$; *and*

(f) *if* J *and* K *are distinct perfect 2-components of* G, *then the following three conditions are equivalent*: (i) $[J, K] = 1$, (ii) $[J, K] \leq Z(J)$, (iii) $[J, K] \leq Z(K)$.

PROOF. Clearly (a) and (b) follow from [5, §1], (c)–(e) follow from [**18**, §2] and (f) follows from Lemma 2.6(a).

LEMMA 2.12. *Let* N *be a normal subgroup of the group* G, *let* H *be a 2-quasisimple subgroup of* G *such that* $H \nleq N$ *and set* $\overline{G} = G/N$. *Then the following three conditions hold.*

(a) $\overline{H}$ *is 2-quasisimple and* $\overline{H}/Z^*(\overline{H}) \cong H/Z^*(H)$;

(b) $\overline{O(H)} = O(\overline{H})$; *and*

(c) $\overline{Z^*(H)} = Z^*(\overline{H})$.

PROOF. Clearly $N \cap H \leq Z^*(H)$, $\overline{H}' = \overline{H'} = \overline{H}$ and $\overline{H}/\overline{Z^*(H)} \cong H/Z^*(H)$ is simple. Also $\overline{O(H)} \leq O(\overline{H})$ by Lemma 2.2(a). Thus $\overline{Z^*(H)} \leq Z^*(\overline{H})$ and both (a) and (c) hold. Since $\overline{Z^*(H)}/\overline{O(H)}$ is a 2-group, (b) also holds and we are done.

LEMMA 2.13. *Let* G *be a group such that* $O^{2'}(G)$ *is 2-quasisimple. Set* $M = O^{2'}(G)$ *and* $\overline{G} = G/S(G)$. *Then the following three conditions hold.*

(a) $S(G) = O(G)Z^*(M) = C_G(M/O(M)) = C_G(M/Z^*(M))$;

(b) $Z^*(M) = S(G) \cap M$ *and* $O^2(S(G)) = O(G)$; *and*

(c) $\overline{M} = O^{2'}(\overline{G}) = F^*(\overline{G})$ *and* $\overline{M} \cong M/Z^*(M)$.

PROOF. Clearly

$$S(G) \cap M = Z^*(M)$$

and

$$O(G)Z^*(M) \leq S(G) \leq C_G(M/O(M)) = C_G(M/Z^*(M)).$$

Also $C_G(M/O(M)) \cap M = Z^*(M) \trianglelefteq G$, $|C_G(M/O(M))/Z^*(M)|$ is odd and $[C_G(M/O(M)), Z^*(M)] \leq O(M)$. Thus

$$C_G(M/O(M)) = O(G)Z^*(M) = S(G) \quad \text{and} \quad O^2(S(G)) = O(G).$$

Clearly $\overline{M} = O^{2'}(\overline{G}) \cong M/Z^*(M)$ and $C_{\overline{G}}(\overline{M}) = 1$. Thus $F^*(\overline{G}) = \overline{M}$ and the proof is complete.

For the convenience of the reader, we restate [**18**, Theorem 3.1; **22**, Proposition 1; **23**, Proposition 1].

LEMMA 2.14. *Let G be a group, let M be a normal subgroup of G, let B be a 2-subgroup of G, let K be a 2-component of $C_G(B)$ and set $\bar{G} = G/M$. Then the following three conditions hold.*

(a) $L_{2'}(N_G(B)) = L_{2'}(C_G(B)) \leq L_{2'}(G)$;

(b) *if K is perfect, then $K \leq M$ or $\bar{K}$ is a perfect 2-component of $C_{\bar{G}}(\bar{B})$; and*

(c) *if K is solvable, then $O^{2'}(K) \leq M$ or $O(C_{\bar{G}}(\bar{B}))\bar{K}$ is a solvable 2-component of $C_{\bar{G}}(\bar{B})$.*

LEMMA 2.15. *Let K be a perfect 2-component of the group G and let B be a 2-subgroup of G such that $[K, B] \leq Z^*(K)$. Then $[K, B] \leq O(K)$, $O^{2'}(C_K(B)) = C_K(B)^{(\infty)}$, $C_K(B)^{(\infty)}$ is a perfect 2-component of $C_G(B)$ such that*

$$K = O(K)\big(C_K(B)^{(\infty)}\big) = \big(O(G)\big(C_K(B)^{(\infty)}\big)\big)^{(\infty)}$$

and

$$K/O(K) \cong C_K(B)^{(\infty)}/O\big(C_K(B)^{(\infty)}\big).$$

PROOF. Set $\bar{G} = G/O(G)$. Then $\bar{K}$ is a perfect component of $\bar{G}$ and $[\bar{K}, \bar{B}] \leq Z(\bar{K})$. Thus $[\bar{K}, \bar{B}] = 1$ by Lemma 2.6(a) and hence $\bar{K}$ is a component of $C_{\bar{G}}(\bar{B})$ and $[K, B] \leq O(K)$. Set $X = O(G)K$. As $K \trianglelefteq \trianglelefteq X$, we have $O^{2'}(X) = X^{(\infty)} = K$. Hence

$$X = O(G)C_K(B) = O(G)\big(C_K(B)^{(\infty)}\big),$$
$$C_X(B)^{(\infty)} = C_K(B)^{(\infty)} = O^{2'}(C_X(B)) = O^{2'}(C_K(B)),$$
$$K = O(K)C_K(B)^{(\infty)}$$

and $C_K(B)^{(\infty)}$ is a perfect 2-component of $C_G(B)$. Thus

$$K = O(K)\big(C_K(B)^{(\infty)}\big) = \big(O(G)C_K(B)^{(\infty)}\big)^{(\infty)},$$
$$K/O(K) \cong C_K(B)^{(\infty)}/O\big(C_K(B)^{(\infty)}\big)$$

and we are done.

In the next three lemmas, let B denote a 2-subgroup of the group G, let N and H be normal subgroups of the group G with $N \leq Z^*(H)$ and set $\bar{G} = G/N$. Also let M denote the full inverse image in H of $C_{\bar{H}}(\bar{B})$. Thus $\bar{M} = C_{\bar{H}}(\bar{B})$, $N \leq M = \{h \in H \mid [h, B] \leq N\}$, $C_H(B) \trianglelefteq C_G(B)$, $C_N(B) \trianglelefteq C_G(B)$ and $C_N(B) \leq C_H(B) = C_M(B)$.

LEMMA 2.16. *The following four conditions hold.*

(a) $[H, N] \leq O(N) \trianglelefteq G$;

(b) $O^2(M) = O(N)O^2(C_H(B))$;

(c) $O(M) = O(N)O(C_H(B))$; *and*

(d) $\overline{O(C_H(B))} = O(C_{\bar{H}}(\bar{B})) = O(\bar{M})$.

PROOF. Clearly $[H, N] \leqslant O(H) \cap N = O(N) \trianglelefteq G$, so that (a) holds. Also $NC_H(B) \leqslant M = \{h \in H \mid [B, h] \leqslant N\}$ and N has a normal 2-complement. Set $\tilde{G} = G/O(N)$ and note that $O^2(\overline{M}) = \overline{O^2(M)}$, $O^2(\tilde{M}) = \widetilde{O^2(M)}$, $\tilde{N} \leqslant O_2(Z(\tilde{H}))$ and $\overline{C_H(B)} = C_{\tilde{H}}(\tilde{B})$ by [16, Theorem 6.2.2]. Consequently, to prove (b), it suffices to assume that $O(N) = 1$. Let $\pi \in M$ with $|\pi|$ odd. Then π stabilizes the chain $BN \geqslant N \geqslant 1$. Hence $\pi \in C_H(BN) = C_H(B)$ by [16, Theorem 5.3.2] and (b) holds. Then

$$O(M) = O(O^2(M)) = O(N)O(O^2(C_H(B))) = O(N)O(C_H(B))$$

and (c) holds. For (d), let L denote the inverse image in H of $O(C_{\overline{H}}(\overline{B})) = O(\overline{M})$. Then $NO(C_H(B)) = NO(M) \leqslant L \trianglelefteq M$, $|L/N|$ is odd and $[L, N] \leqslant O(N)$. By [16, Theorem 7.4.3], we conclude that $L = O(L)N$. Since $[B, O(L)] \leqslant O(L) \cap N = O(N)$, we have $O(L) = O(N)C_{O(L)}(B)$. As $O(C_H(B)) \leqslant C_{O(L)}(B) \trianglelefteq C_H(B)$, we have $O(C_H(B)) = C_{O(L)}(B)$. Thus (d) holds and we are done.

LEMMA 2.17. *Let K be a 2-component of $C_H(B)$. Then the following two conditions hold.*

(a) *$\overline{K}$ is a 2-component of $C_{\overline{H}}(\overline{B})$ and $K/Z^*(K) \cong \overline{K}/Z^*(\overline{K})$; and*

(b) *$K = C_{NK}(B)^{(\infty)}$ if K is perfect and $K = O^2(C_{NK}(B))$ if K is solvable.*

PROOF. As $\overline{K} \cong K/(K \cap N)$ and $K \cap N \leqslant Z^*(K)$, we have

$$K/Z^*(K) \cong \overline{K}/Z^*(\overline{K}).$$

Clearly $K \trianglelefteq \trianglelefteq C_H(B) \trianglelefteq C_G(B)$, $O^{2'}(K) \nleqslant N$ and $O(C_{\overline{H}}(\overline{B})) = \overline{O(C_H(B))} \leqslant \overline{K}$ if $\overline{K}$ is solvable. Thus (a) follows from Lemma 2.14. Set $X = C_{NK}(B) = C_N(B)K$. Since $C_N(B) \trianglelefteq C_G(B)$, we have $C_N(B) \leqslant Z^*(C_H(B))$ and $[K, C_N(B)] \leqslant O(K)$. Thus $X^{(\infty)} = K$ if K is perfect. Suppose that K is solvable. Since

$$O^2(C_N(B)) = O(C_N(B)) \leqslant O(C_H(B)) \leqslant K = O^2(K) \trianglelefteq X,$$

we have $K = O^2(X)$ and we are done.

LEMMA 2.18. *Let $\bar{J}$ be a 2-component of $C_{\overline{H}}(\overline{B})$ and let K denote the full inverse image of $\bar{J}$ in H. Set $L = C_K(B)^{(\infty)}$ if $\bar{J}$ is perfect and $L = O^2(C_K(B))$ if $\bar{J}$ is solvable. Then the following four conditions hold.*

(a) *$\overline{L} = \bar{J}$ and $L/Z^*(L) \cong \bar{J}/Z^*(\bar{J})$;*

(b) *L is a perfect 2-component of $C_H(B)$ if $\bar{J}$ is perfect;*

(c) *L is a solvable 2-component of $C_H(B)$ if $\bar{J}$ is solvable; and*

(d) *if $|N|$ is odd, then $L/O(L) \cong \bar{J}/O(\bar{J})$, $L = O^{2'}(C_K(B))$ if $\bar{J}$ is perfect and $O^{2'}(C_K(B)) < L = C_K(B)$ if $\bar{J}$ is solvable.*

PROOF. Note that $[K, B] \leqslant N \lhd K \trianglelefteq \trianglelefteq M$ and $K = NO^2(K)$ since $O^2(\bar{J}) = \bar{J}$. Thus $O(N) \leqslant O^2(K) \trianglelefteq \trianglelefteq O^2(M) = O(N)O^2(C_H(B))$ by Lemma 2.16. Hence

$$O^2(K) = O(N)(O^2(K) \cap O^2(C_H(B))),$$

$$[O^2(K), B] \leqslant O(N), \qquad \bar{J} = \overline{O^2(K)},$$

and

$$O^2(K) \cap C_G(B) = O^2(K) \cap O^2(C_H(B)).$$

Suppose that $\bar{J}$ is perfect and $L = C_K(B)^{(\infty)}$. Then $L \leqslant O^2(C_H(B)) \cap O^2(K)$, $L \trianglelefteq\ \trianglelefteq C_M(B) = C_H(B) \trianglelefteq C_G(B)$ and $L = (O^2(C_H(B)) \cap O^2(K))^{(\infty)}$. Hence

$$\bar{J} = \bar{J}^{(\infty)} = \overline{O^2(K) \cap O^2(C_H(B))}^{(\infty)} = \bar{L}$$

and $K = NL$. Let Y denote the subgroup of L such that $N \cap L \leqslant Y \lhd L$ and $\bar{Y} = O(\bar{L})$ and let $T \in \mathrm{Syl}_2(N \cap L)$. Then $N \cap L = O(N \cap L)T$, $|Y/N \cap L|$ is odd, $[L, T] \leqslant O(N \cap L) \leqslant O(L)$ and $T \in \mathrm{Syl}_2(Y)$. Applying [**16**, Theorem 7.4.3], we conclude that $Y = O(Y)T$. Hence $O(Y) = O(L)$, $[L, Y] \leqslant O(Y) \trianglelefteq\ \trianglelefteq C_G(B)$ and $Y \leqslant Z^*(L) \leqslant S(L)$. As $L/Y \cong \bar{J}/O(\bar{J})$, which is quasisimple, $S(L)/O(L)$ is a 2-group and $[L, S(L)] \leqslant Y$. Then L stabilizes the chain $S(L) \geqslant Y \geqslant O(L)$. Since $L = O^2(L)$, we have $[L, S(L)] \leqslant O(L)$ by [**16**, Theorem 5.3.2]. Thus L is 2-quasisimple, $S(L) = Z^*(L)$ and $L/Z^*(L) \cong \bar{J}/Z^*(\bar{J})$.

Suppose that $\bar{J}$ is solvable and $L = O^2(C_K(B))$. Then $L \trianglelefteq\ \trianglelefteq C_M(B) = C_H(B) \trianglelefteq C_G(B)$, $L \leqslant O^2(K) \cap C_G(B) = O^2(K) \cap O^2(C_H(B)) \trianglelefteq C_K(B)$ and hence

$$L = O^2\big(O^2(K) \cap O^2(C_H(B))\big).$$

Thus $\bar{J} = O^2(\bar{J}) = \overline{O^2(K)} = O^2\big(\overline{O^2(K)}\big) = \bar{L}$. Let Y denote the subgroup of L such that $N \cap L \leqslant Y \lhd L$ and $\bar{Y} = O(\bar{L})$. Then, as above, $Y \trianglelefteq\ \trianglelefteq C_G(B)$, $Y = O(L)T$ where $T \in \mathrm{Syl}_2(N \cap L)$ and $[L, Y] \leqslant O(L)$. However $O(L) \leqslant O(C_H(B)) = O(C_G(B)) \cap H \leqslant C_K(B)$ since $O(C_{\bar{H}}(\bar{B})) = \overline{O(C_H(B))} \leqslant \bar{J}$. Thus $O(L) = O(C_H(B))$. Note that $L/Y \cong \bar{J}/O(\bar{J})$, which is isomorphic to PSL(2, 3) or SL(2, 3). Set $\tilde{L} = L/O(L)$. Then $O^2(\tilde{L}) = \widetilde{O^2(L)} = \tilde{L}$, $\tilde{T} \leqslant O_2(Z(\tilde{L}))$ and Lemma 2.10 implies that $\tilde{L}$ is isomorphic to PSL(2, 3) or SL(2, 3). Thus (a)–(c) hold.

For (d), assume that $|N|$ is odd. Then $K = NL$ implies that $C_K(B) = C_N(B)L$. Here both $|C_N(B)|$ and $|C_K(B)/L|$ are odd. Thus $L = O^{2'}(C_K(B))$ if $\bar{J}$ is perfect and $L = C_K(B)$ if $\bar{J}$ is solvable. Also $L \cap N \leqslant O(L)$, $\bar{J} = \bar{L} \cong L/(L \cap N)$ and (d) is clear. The proof of this lemma is now complete.

LEMMA 2.19. *Let z be an involution of the group G and set $H = C_G(z)$. Let L be a perfect 2-component of G and let J be a perfect 2-component of H. Then the following three conditions hold*:

(a) *if $L^z = L$, then every perfect 2-component of $C_L(z)$ is a perfect 2-component of H*;

(b) *if $L^z \neq L$, then $C_{LL^z}(z)^{(\infty)}$ is a perfect 2-component of H,*

$$O(C_{LL^z}(z)) = O(G) \cap (C_{LL^z}(z))$$

and $C_{LL^z}(z)^{(\infty)}/O(C_{LL^z}(z)^{(\infty)})$ is a homomorphic image of $L/O(L)$; and

(c) *there is a perfect 2-component K of G such that either* (i) *$K^z = K$ and J is a perfect 2-component of $C_K(z)$, or* (ii) *$K^z \neq K$ and $J = C_{KK^z}(z)^{(\infty)}$. Also, in either case, $[J, KK^z] = [J, L_{2'}(G)] = \langle J^{L_{2'}(G)} \rangle = KK^z$.*

PROOF. Suppose that $L^z = L$. Then $L \trianglelefteq\ \trianglelefteq G$, $C_L(z) \trianglelefteq\ \trianglelefteq H$ and (a) holds. Assume that $L^z \neq L$ and set $M = LL^z$ and $\bar{G} = G/O(G)$. Note that $M \trianglelefteq\ \trianglelefteq G$, $\bar{z} \in \mathcal{I}(\bar{G})$, $\bar{H} = C_{\bar{G}}(\bar{z})$ and $L = O^{2'}(O(G)L) = (O(G)L)^{(\infty)}$ by Lemma 2.18. Clearly

$\bar{L}$ and $\bar{L}^{\bar{z}}$ are distinct components of $\bar{G}$, $\bar{M} = \overline{LL^z} = \bar{L}*\bar{L}^{\bar{z}} \trianglelefteq\, \trianglelefteq \bar{G}$ and $\overline{C_M(z)} = C_{\bar{M}}(\bar{z}) \trianglelefteq\, \trianglelefteq \bar{H}$. Moreover $L_{2'}(C_{\bar{M}}(\bar{z})) = C_{\bar{M}}(\bar{z})^{(\infty)}$ and is a homomorphic image of $\bar{L} \cong L/O(L)$ by [**17**, Lemma 2.1] and $L_{2'}(C_{\bar{M}}(\bar{z}))$ is a component of $\bar{H}$. Set $\mathcal{J} = C_M(z)^{(\infty)}$. Then $\bar{\mathcal{J}} = C_{\bar{M}}(\bar{z})^{(\infty)}$ and hence $O(G)\mathcal{J}$ is the full inverse image in G of $L_{2'}(C_{\bar{M}}(\bar{z}))$. Since $\mathcal{J} \trianglelefteq\, \trianglelefteq (O(G)\mathcal{J}) \cap C_G(z) = C_{O(G)}(z)\mathcal{J}$, we have $\mathcal{J} = (O(G)\mathcal{J} \cap C_G(z))^{(\infty)}$. Thus $\mathcal{J}$ is a 2-component of H by Lemma 2.18. Also $\bar{\mathcal{J}} = L_{2'}(C_{\bar{M}}(\bar{z}))$ and $\bar{\mathcal{J}}$ is quasisimple. Thus $O(C_{\bar{M}}(\bar{z})) \leqslant C_{\bar{M}}(\bar{\mathcal{J}})$, so that $O(C_{\bar{M}}(\bar{z})) \leqslant Z(\bar{M}) = Z(\bar{L})Z(\bar{L}^{\bar{z}})$ by [**17**, Lemma 2.2]. Since $Z(\bar{L})$ is a 2-group, $O(C_M(z)) = O(G) \cap C_M(z)$ and (b) holds. Clearly the first part of (c) follows from (a), (b) and [**18**, Lemma 2.18 and Corollary 3.2]. Also $J = J' \leqslant KK^z \trianglelefteq L_{2'}(G)$, so that $J \leqslant [J, KK^z] \leqslant [J, L_{2'}(G)] = \langle J^{L_{2'}(G)} \rangle \trianglelefteq KK^z$ by Lemma 2.3. Thus $J \leqslant X = [J, KK^z] \trianglelefteq KK^z = Y$ and X is not solvable. Set $\bar{Y} = Y/S(Y)$. Clearly $K \trianglelefteq Y$, $K^z \trianglelefteq Y$, $S(K) = Z^*(K) = S(Y) \cap K$, $S(K^z) = Z^*(K^z) = S(Y) \cap K^z$, $K = (S(Y)K)^{(\infty)}$, $K^z = (S(Y)K^z)^{(\infty)}$ and $1 \neq \bar{X} \trianglelefteq \bar{Y} = \overline{KK^z}$ where $\bar{K} \cong K/Z^*(K)$ and $\overline{K^z} \cong K^z/Z^*(K^z)$ are simple. Suppose that $XS(Y) = KK^z$. Then $(KK^z)^{(\infty)} = KK^z = X^{(\infty)} \leqslant X$ and $X = KK^z$. Thus to conclude the proof of the lemma, it suffices to assume that $K \neq K^z$ and $\bar{Y} = \bar{K} \times \overline{K^z}$. However if $\bar{X} = \bar{K}$, then $J = J^{(\infty)} \leqslant (KS(Y))^{(\infty)} = K$ and hence $K^z = K$. Similarly $\bar{X} = \overline{K^z}$ is impossible and the proof of this lemma is complete.

Our next result sharpens [**1**, Theorem 2(2)].

LEMMA 2.20. *Let G be a group, let $z \in \mathcal{I}(G)$ and let K be a 2-component of $C_G(z)$. Suppose that L is a perfect 2-component of G such that $L^z \neq L$. Then exactly one of the following two conditions holds.*

(a) $[K, L] \leqslant O(L)$ *and* $[K, L^z] \leqslant O(L^2)$; *or*

(b) $K = C_{LL^z}(z)^{(\infty)}$.

PROOF. Assume that G is a counterexample of minimal order to the lemma. Applying [**1**, Theorem 2] and Lemmas 2.17–2.19, we conclude that $O(G) = 1$, $G = (LL^z)(O(C_G(z)) \times \langle z \rangle)$, $G^{(\infty)} = E(G) = L * L^z$, $K \leqslant O^2(G) = LL^zO(C_G(z)) = N_G(L) = N_G(L^z)$ and $|G/O^2(G)| = 2$. Thus K is solvable by Lemma 2.19. Clearly $O_2(G) = Z(L) * Z(L^z) = Z(E(G)) = C_G(E(G))$ and $F^*(G) = E(G)$. Set $J = C_{E(G)}(z)^{(\infty)}$ and $\bar{G} = G/O_2(G)$. Thus J is a perfect component of $C_G(z)$ with $O(J) = 1$ by Lemma 2.19 and hence $[K, J] = 1$. But $J_1 = \langle xx^z \mid x \in L \rangle \leqslant C_{E(G)}(z)$ and J_1 is a homomorphic image of L. Thus $J_1 \leqslant J$ and hence $[K, J_1] = 1$. Now [**2**, Lemma 2.5] implies that $K = O^2(K) \leqslant C_G(E(G)) = O_2(G)$. This contradiction completes the proof.

The next result is a slight refinement of [**1**, Theorem 2(4)].

LEMMA 2.21. *Let z be an involution of the group G, let $H = C_G(z)$, let L be a 2-component of G and let K be an intrinsic 2-component of $H = C_G(z)$. Then $[K, L] \leqslant O(L)$ or $O^{2'}(K) \leqslant L$.*

PROOF. By [**1**, Theorem 2(4)], we have $[K, L] \leqslant O(G)$ or $O^{2'}(K) \leqslant L$. Suppose that $[K, L] \leqslant O(G)$ and $O(G) \neq O(L)$. Then L is perfect and $K \leqslant N_G(LO(G)) \leqslant N_G(L)$ since $L = O^{2'}(LO(G))$. Hence $[K, L] \leqslant O(G) \cap L = O(L)$ and we are done.

The next lemma utilizes J. G. Thompson's concept of a critical subgroup of a p-group (cf. [**16**, pp. 185–186]) and extends [**39**, Lemma 4.1].

LEMMA 2.22. *Let G be a group with $O(G) = 1$. Let $z \in \mathfrak{I}(G)$ and let K be a solvable 2-component of $C_G(z)$. Then exactly one of the following two conditions holds*:

(a) $O^{2'}(K) \leqslant E(G)$, $[O^{2'}(K), E(G)] \neq 1$ *and* $K \leqslant C_G(O_2(G))$; *or*

(b) $O^{2'}(K) = [C_P(z), K]$ *for every critical subgroup P of $O_2(G)$.*

PROOF. Let $S \in \mathrm{Syl}_2(K)$ and let ρ be a 3-element of $N_K(S) - O(K)$. Thus

$$O(X_G(z))S \lhd K = O(C_G(z))\langle \rho^K \rangle = O(C_G(z))S\langle \rho \rangle.$$

By [**16**, Theorem 5.3.4], we have $[O(C_G(z)), O_2(G)] = 1$. First, we suppose that $O^{2'}(K) \leqslant C_G(E(G))$. Then $O_2(G) \neq 1$ since $F^*(G) = O_2(G)E(G)$ and $C_G(F^*(G)) = Z(O_2(G))$. Let P be a critical subgroup of $O_2(G)$. Suppose that $[P, \langle \rho \rangle] = 1$. Then $\rho \in C_G(O_2(G))$, $K \leqslant C_G(O_2(G))$, $O^{2'}(K) \leqslant C_G(F^*(G)) = Z(O_2(G))$ and $O^{2'}(K) \leqslant \Omega_1(Z(O_2(G))) \cap C_G(z)$. Since $Z(O_2(G)) \leqslant P$, this is impossible. Thus $[P, \langle \rho \rangle] \neq 1$. Then $[C_P(z), \langle \rho \rangle] \neq 1$ by [**16**, Theorem 5.3.4] and (b) holds by [**23**, Lemma 2.8]. Finally, suppose that $[O^{2'}(K), E(G)] \neq 1$. Then $O^{2'}(K) \leqslant E(G)$ by [**1**, Theorem 2] and $O^{2'}(K) \nleqslant O_2(G)$. Hence $[C_{O_2(G)}(z), K] = 1$ by [**23**, Lemma 2.8] and $[O_2(G), O^2(K)] = [O_2(G), K] = 1$. Thus (a) holds and we are done.

LEMMA 2.23. *Let G be a group with $O(G) = 1$. Let $z \in \mathfrak{I}(G)$ and let K be an intrinsic solvable 2-component of $C_G(z)$ such that $O^{2'}(K) \leqslant O_2(G)$. Then the following three conditions hold*:

(a) $O(C_G(z)) = 1$ *and* $E(G) = E(C_G(z))$;

(b) $K \trianglelefteq\, \trianglelefteq G$; *and*

(c) *if M is a solvable or perfect 2-component of $C_G(z)$, then $M \trianglelefteq\, \trianglelefteq G$.*

PROOF. Let P be a critical subgroup of $O_2(G)$ and let Q be the unique Sylow 2-subgroup of K. Then $z \in Q' \leqslant P' \leqslant Z(P)$, $P \leqslant C_G(Z(P)) \leqslant C_G(z)$, $L_{2'}(C_G(z)) = E(G) = E(C_G(z))$ by Lemma 2.19 and $[O(C_G(z)), P] = 1$. Thus

$$O(C_G(z)) \leqslant C_G(O_2(G)E(G)) = Z(O_2(G)),$$

(a) holds and $[K, E(G)] = 1$. Let ρ be an element of order 3 in K. Thus $\rho \notin C_G(O_2(G))$ and hence $Q = [P, K] = [P, \langle \rho \rangle]$ and $[\Omega_1(Z(P)), \rho] = 1$ by [**23**, Lemma 2.6]. Hence $[Z(P), \langle \rho \rangle] = 1$, $K \leqslant C_G(Z(P)) \leqslant C_G(z)$ and (b) holds. For the proof of (c), it suffices to consider a solvable 2-component M of $C_G(z)$ by (a). But then $[E(C_G(z)), M] = [E(G), M] = 1$ and $O^{2'}(M) \trianglelefteq P \leqslant C_G(z)$ by Lemma 2.22. Let R be the unique Sylow 2-subgroup of M and let ν be an element of M of order 3. Suppose that $R' \neq 1$. Then $[Z(P), \langle \nu \rangle] = [Z(P), M] = 1$ by [**23**, Lemma 2.6] and hence $M \leqslant C_G(Z(P))$ and $M \trianglelefteq\, \trianglelefteq G$. Suppose that $R' = 1$. Then $P = R \times C_P(M)$ by [**23**, Lemma 2.5] and hence $P' \leqslant C_P(M)$. Since $z \in P' \leqslant Z(P)$, we have $M \leqslant C_G(P') \leqslant C_G(z)$ and $C_G(P') \trianglelefteq G$. Thus $M \trianglelefteq\, \trianglelefteq G$ and we are done.

LEMMA 2.24. *Let G be a group, let $z \in \mathfrak{I}(G)$, let L be a 2-component of G such that $L^z \neq L$ and let K be a 2-component of $C_G(z)$ such that $[L, O^{2'}(K)] \leqslant O(L)$. Then $[L, K] \leqslant O(L)$.*

PROOF. Assume that $[L, K] \nleq O(L)$. Since $K = O^{2'}(K)$ if K is perfect, it follows that K is solvable. Also [**1**, Theorem 2] implies that $K \leq LL^z O(C_G(z))$ and $O^{2'}(K) \leq LL^z$. Thus $K = (K \cap (LL^z))O(C_G(z))$ and

$$O^{2'}(K) \leq C_K(L/O(L)) \cap C_K(L^z/O(L^z)) \cap (LL^z) = K \cap (Z^*(L)Z^*(L^z)).$$

It follows that a Sylow 2-subgroup of K is abelian and K acts trivially on $O^{2'}(K)O(K)/O(K)$. This contradiction establishes the lemma.

DEFINITION 2.1. A simple group K is said to be θ-balanced if every group H such that $F^*(H) = K$ has the property that $(|O(C_H(t))|, 3) = 1$ for all $t \in \mathcal{I}(H)$.

Note that if a simple group K is balanced (in the terminology of [7]) or is a simple Chevalley group over a field of characteristic 3 (cf. [**9**, Lemma]), then K is θ-balanced.

The next result was suggested by situations arising in [**24**].

LEMMA 2.25. *Let G be a group and let $z \in \mathcal{I}(G)$. Let K be a solvable 2-component of $C_G(z)$ and let L be a perfect 2-component of G. Suppose that the simple group $L/Z^*(L)$ is θ-balanced. Also suppose that $K \leq N_G(L)$, $[K, O^{2'}(K)] \leq O(L)$ and that $O^{2'}(K)$ is not contained in L. Then $O^{3'}(K) \leq C_K(L/O(L))$.*

PROOF. By Lemma 2.24, we may assume that $L^z = L$. Set $H = L(K\langle z\rangle)$. Then $C_H(z) \trianglelefteq \trianglelefteq C_G(z)$, $O(C_H(z)) \leq O(C_G(z)) = O(K)$ and K is a solvable 2-component of $C_H(z)$. Thus we may assume that $G = H = L(K\langle z\rangle)$.

Suppose that $O(G) = 1$. Clearly $O^{2'}(K) \leq C_G(L) = C_G(L/Z(L)) = S(G)$ and $S(G) \cap L = Z(L)$. If $z \in S(G)$, then L is a perfect 2-component of $C_G(z)$ and hence $[L, K] = 1$. Thus we may assume that $z \notin S(G)$. Set $\overline{G} = G/S(G)$. Then $F^*(\overline{G}) = \overline{L}$ is simple, $\bar{z} \in \mathcal{I}(\overline{G})$ and $|\overline{K}|$ is odd. The inverse image of $C_{\overline{G}}(\bar{z})$ in G is

$$N_G(\langle z\rangle S(G)) = N_L(\langle z\rangle Z(L))K\langle z\rangle = N_G(\langle z\rangle Z(L)).$$

Also $[Z(L), O^{2'}(K)O(K)] = 1$ and $[C_{Z(L)}(z), K] = 1$ since $O^{2'}(K) \nleq Z(L)$ by [**23**, Lemma 2.8]. Hence $[Z(L), K] = 1$ by [**16**, Theorem 5.3.4]. This implies that $K \trianglelefteq \trianglelefteq C_G(\langle z\rangle Z(L)) \leq N_G(\langle z\rangle Z(L))$. Thus $S(G)K \trianglelefteq \trianglelefteq N_G(\langle z\rangle S(G))$ and $\overline{K} \leq O(C_{\overline{G}}(\bar{z}))$ since $|\overline{K}|$ is odd. Since $\overline{L}$ is θ-balanced, it follows that $O^{3'}(K) \leq S(G)$ and we are done in this case.

Suppose that $O(G) \neq 1$ and set $\overline{G} = G/O(G)$. Then $\bar{z} \in \mathcal{I}(\overline{G})$, $\overline{L}$ is a perfect 2-component of $\overline{G}$ and is a solvable 2-component of $C_{\overline{G}}(\bar{z}) = \overline{C_G(z)}$ by Lemma 2.17. Since $O^{2'}(\overline{K}) = \overline{O^{2'}(K)}$, $O^{3'}(\overline{K}) = \overline{O^{3'}(K)}$ and $|O(G)Z^*(L)/Z^*(L)|$ is odd, it is clear that the lemma follows from the above.

The next result was suggested by the proof of [**24**, Theorem 1].

LEMMA 2.26. *Let G be a group and let $z \in \mathcal{I}(G)$. Suppose that $C_G(z)$ contains 2-components L_1 and L_2 with $z \in L_2$. Assume also that $O(G)L_1$ is not subnormal in G if L_1 is perfect and that $O(G)O^{3'}(L_1)$ is not subnormal in G if L_1 is solvable. Then $O^{2'}(L_2)$ is contained in a unique perfect 2-component K of G. Also $z \notin Z^*(K)$, K is $C_G(z)$-invariant, $L_1 \leq K$ if L_1 is perfect and $[K, L^{3'}(L_1)] = K$ if L_1 is solvable.*

PROOF. First suppose that $O(G) = 1$. Clearly $O^{2'}(L_2) \leqslant E(G)$ and $[L_2, O_2(G)] = 1$ by Lemmas 2.22 and 2.23. If $z \in Z(E(G))$, then $L_{2'}(C_G(z)) = E(C_G(z)) = E(G)$, L_1 is solvable and $L_1 \leqslant C_G(E(G)) \leqslant C_G(z)$. Since L_1 is not subnormal in $C_G(E(G))$, it follows that $z \notin Z(E(G))$. Hence [**1**, Theorem 2(4)] implies that there is a unique perfect component K of G such that $O^{2'}(L_2) \leqslant K$. Clearly $z \notin Z(K)$ and $C_G(z)$ normalizes K. If $[L_1, K] = 1$, then $L_1 \leqslant C_G(E(G)) \leqslant C_G(z)$ since all components of $E(G)$ with the exception of K are components of $C_G(z)$ by Lemma 2.19. Since L_1 is not subnormal in $C_G(E(G))$, we have $[L_1, K] = K$. Thus $L_1 \leqslant K$ if L_1 is perfect by Lemma 2.19. Suppose that L_1 is solvable and that $[K, O^{3'}(L_1)] = 1$. Then $O^{3'}(L_1) \leqslant C_G(E(G)) \leqslant C_G(z)$ and $O^{3'}(L_1) \trianglelefteq\, \trianglelefteq G$, a contradiction. Thus $O(G) \neq 1$. Set $\bar{G} = G/O(G)$ and let $J_i = O(G)L_i$ for $i = 1, 2$. Thus $\bar{J}_1$ and $\bar{J}_2$ are 2-components of $C_{\bar{G}}(\bar{z})$, $\bar{z} \in \mathcal{Y}(\bar{J}_2)$, $\bar{J}_1$ is not subnormal in $\bar{G}$ if $\bar{J}_1$ is perfect and $O^{3'}(\bar{J}_1) = O^{3'}(\bar{L}_1) = \overline{O^{3'}(L_1)}$ is not subnormal in $\bar{G}$ if $\bar{J}_1$ is solvable. Hence, by the above, $O^{2'}(\bar{J}_2) = O^{2'}(\bar{L}_2) = \overline{O^{2'}(L_2)}$ is contained in a unique perfect component $\bar{K}$ of $\bar{G}$, etc. Let K_1 denote the inverse image of $\bar{K}$ in G and set $K = K_1^{(\infty)}$. Then $K = O^{2'}(K_1)$ is a perfect 2-component of G, $K_1 = O(G)K$, $O^{2'}(L_2) \leqslant O^{2'}(J_2) \leqslant K$, $z \notin Z^*(K)$, K is $C_G(z)$ invariant and is the unique perfect 2-component of G that contains $O^{2'}(L_2)$. If L_1 is perfect, then $L_1 \leqslant K_1$ and hence $L \leqslant K = K_1^{(\infty)}$. Suppose that L_1 is solvable. Then $\bar{K} = [O^{3'}(\bar{L}_1), \bar{K}] = [\overline{O^{3'}(L_1)}, \bar{K}]$ and hence $[O^{3'}(L_1), K] = K$. The proof of this lemma is now complete.

The next result of this section is a compilation of results of various authors. For references, see [**13**, §2; **19**].

LEMMA 2.27. *Let X be a simple Chevalley group over a finite field of order q where $q = p^n$ for some odd prime p and positive integer n. Then X has a universal covering group,* $\mathrm{Cov}(X)$, *and* $S(\mathrm{Cov}(X)) = Z(\mathrm{Cov}(X))$. *Set* $Y = \mathrm{Cov}(X)/O(\mathrm{Cov}(X))$. *Then exactly one of the following* 16 *conditions holds.*

(1) $X \cong \mathrm{PSL}(m, q) \cong A_{m-1}(q)$ *for some integer* $m \geqslant 2$ *with* $(m, q) \neq (2, 3)$; $Z(\mathrm{SL}(m, q)) \cong Z_{(m, q-1)}$; *if* $(m, q) \neq (2, 9)$, *then* $\mathrm{Cov}(X) \cong \mathrm{SL}(m, q)$ *and if* $(m, q) = (2, 9)$, *then* $O(\mathrm{Cov}(X)) \cong Z_3$ *and* $Y \cong \mathrm{SL}(2, 9)$;

(2) $X \cong \mathrm{PSU}(m, q) \cong {}^2A_{m-1}(q)$ *for some integer* $m \geqslant 3$; $Z(\mathrm{SU}(m, q)) \cong Z_{(m, q+1)}$; *if* $(m, q) \neq (4, 3)$, *then* $\mathrm{Cov}(X) \cong \mathrm{SU}(m, q)$ *and if* $(m, q) = (4, 3)$, *then* $O(\mathrm{Cov}(X)) \cong Z_3 \times Z_3$ *and* $Y \cong \mathrm{SU}(4, 3)$;

(3) $X \cong \mathrm{PSp}(2m, q) \cong C_m(q)$ *for some integer* $m \geqslant 2$; $Z(\mathrm{Sp}(2m, q)) \cong Z_2$ *and* $\mathrm{Cov}(X) \cong \mathrm{Sp}(2m, q)$;

(4) $X \cong P\Omega(2m+1, q) \cong B_m(q)$ *for some integer* $m \geqslant 3$; $Z(\mathrm{Spin}(2m+1, q)) \cong Z_2$; *if* $(2m+1, q) \neq (7, 3)$, *then* $\mathrm{Cov}(X) \cong \mathrm{Spin}(2m+1, q)$ *and if* $(2m+1, q) = (7, 3)$, *then* $O(\mathrm{Cov}(X)) \cong Z_3$ *and* $Y \cong \mathrm{Spin}(7, 3)$;

(5) $X \cong P\Omega(2m, q, 1) \cong D_m(q)$ *for some even integer* $m \geqslant 4$, $Z(\mathrm{Spin}(4m, q, 1)) \cong E_4$ *and* $\mathrm{Cov}(X) \cong \mathrm{Spin}(4m, q, 1)$;

(6) $X \cong P\Omega(2m, q, -1) \cong {}^2D_m(q)$ *for some even integer* $m \geqslant 4$, $Z(\mathrm{Spin}(2m, q, -1)) \cong Z_2$ *and* $\mathrm{Cov}(X) \cong \mathrm{Spin}(2m, q, -1)$;

(7) $X \cong P\Omega(2m, q, 1) \cong D_m(q)$ *for some odd integer* $m \geqslant 5$; $Z(\mathrm{Spin}(2m, q, 1)) \cong Z_{(4, q-1)}$ *and* $\mathrm{Cov}(X) \cong \mathrm{Spin}(2m, q, 1)$;

(8) $X \cong P\Omega(2m, q, -1) \cong {}^2D_m(q)$ *for some odd integer* $m \geqslant 5$, $Z(\mathrm{Spin}(2m, q, -1)) \cong Z_{(4,q+1)}$ *and* $\mathrm{Cov}(X) \cong \mathrm{Spin}(2m, q, -1)$;
(9) $X \cong E_6(q)$ *and* $Z(\mathrm{Cov}(X)) \cong Z_{(3,q-1)}$;
(10) $X \cong E_7(q)$ *and* $Z(\mathrm{Cov}(X)) \cong Z_2$;
(11) $X \cong E_8(q) \cong \mathrm{Cov}(X)$;
(12) $X \cong F_4(q) \cong \mathrm{Cov}(X)$;
(13) $X \cong G_2(q)$; *if* $q \neq 3$, *then* $X = \mathrm{Cov}(X)$ *and if* $q = 3$, *then* $Z(\mathrm{Cov}(X)) \cong Z_3$;
(14) $X \cong {}^3D_4(q) \cong \mathrm{Cov}(X)$;
(15) $X \cong {}^2E_6(q)$ *and* $Z(\mathrm{Cov}(X)) \cong (3, q+1)$; *or*
(16) $p = 3$, *n is odd*, $n \geqslant 3$ *and* $X \cong {}^2G_2(q) \cong \mathrm{Cov}(X)$.

LEMMA 2.28. *Let N be a normal subgroup of the group G, let H be a subgroup of G that is of* $\mathfrak{M}(p)$*-type and set* $\overline{G} = G/N$. *Then* $\mathcal{I}(Z^*(H)) \subseteq N$ *or* $\overline{H}$ *is of* $\mathfrak{M}(p)$*-type.*

PROOF. Assume that $t \in \mathcal{I}(Z^*(H)) - N$. Suppose that $H/O(H) \cong \mathrm{SL}(2,3)$. Then $N \cap H \trianglelefteq H$ and hence $N \cap H \leqslant O(H)$ and $\overline{H}/O(\overline{H}) \cong \mathrm{SL}(2,3)$. Thus we may assume that H is 2-quasisimple. Then, by Lemma 2.12, $\overline{H}$ is 2-quasisimple, $\overline{H}/Z^*(\overline{H}) \cong H/Z^*(H)$, $\bar{t} \in Z^*(\overline{H}) = \overline{Z^*(H)}$ and $\bar{t} \in \mathcal{I}(\overline{G})$. Suppose that $\overline{H}$ is not of $\mathfrak{M}(p)$-type. Then $H/Z^*(H)$ is isomorphic to $\mathrm{PSL}(2n, p^r)$ or $\mathrm{PSU}(2n, p^r)$ for some positive integers n and r, $Z^*(H)$ has cyclic Sylow 2-subgroups and $|\overline{Z^*(H)}|_2 < |Z^*(H)|_2 = |\mathfrak{M}(H/Z^*(H))|_2$. As $Z^*(H) \cap N \trianglelefteq Z^*(H)$, $|Z^*(H) \cap N|$ is even and $t \notin N$, we have a contradiction and the lemma holds.

3. Centralizers of involutions and semi-involutions in the classical linear groups over finite fields of odd characteristic. In this section, we shall review the survey of the conjugacy classes of involutions and semi-involutions and their centralizers in the classical linear groups over finite fields of odd order as presented in [**12**, Chapitre I, §§3, 4, 13 and 14] and add a few observations that we shall require at various points in the proofs in this paper.

Throughout this section, let k denote a finite field of order $q = p^n$ where p is an odd prime integer and n is a positive integer. Also let V be a finite dimensional vector space over k with $\dim(V/k) = m$.

Suppose that $m = 1$. Then $\mathrm{GL}(V/k) = \langle \lambda I_V \mid \lambda \in k^\times \rangle \cong k^\times$, $\mathrm{SL}(V/k) = 1$ and consequently we shall usually assume that $m > 1$.

Before discussing the classical linear groups, we present two lemmas that we shall need in subsequent discussions.

LEMMA 3.1. *Suppose that* $m > 1$. *Let H be a finite group and let G be a subgroup of index 2 of H such that* $G \cong \mathrm{GL}(V/k)$; *thus* $G' \cong \mathrm{SL}(V/k)$ *and* $Z(G) \cong k^\times$. *Assume that there is an element* $\tau \in H - G$ *such that* $\tau^2 \in Z(G)$ *and* τ *acting by conjugation on G induces transpose-inverse on G with respect to the basis* $B = \{v_1, \ldots, v_m\}$ *of* V/k. *Then the following two conditions hold.*

(a) *If* $m = 2$ *and* $\alpha \in G$ *has matrix* $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ *with respect to B, then* $C_H(G') = \langle Z(G), \alpha\tau \rangle$ *and* $(\alpha\tau)^2 = (-I_v)\tau^2$; *and*
(b) *if* $m > 2$, *then* $C_H(G') = Z(G)$.

PROOF. Let $M = C_H(G')$. By [**25**, Proposition 2], we have $M \cap G = C_G(G') = Z(G) = \langle \lambda I_V \mid \lambda \in k^\times \rangle$, so that $|M/Z(G)| \leq 2$. First suppose that $m = 2$ and let α be as in (a). Then $\alpha\tau \in M - Z(G)$ and it is clear that (a) holds. Consequently we may suppose that $m > 2$ and $|M/Z(G)| = 2$. Thus there is an element $\beta \in G$ such that $\beta\tau \in M$.

For each integer $1 \leq i < m$, let u_i denote the unique element of G such that: $u_i(v_j) = v_j$ if $j \notin \{i, i+1\}$, $u_i(v_i) = v_{i+1}$ and $u_i(v_{i+1}) = -v_i$. Clearly $\langle u_i \mid 1 \leq i < m \rangle \leq G' \cap C_G(\tau) \leq C_G(\tau, \beta)$. Also, for each $1 \leq i < m$, let t_i denote the unique element of G such that $t_i(v_j) = v_j$ if $j \notin \{i, i+1\}$, $t_i(v_i) = -v_i$ and $t_i(v_{i+1}) = -v_{i+1}$. Clearly $t_i^\tau = t_i \in G'$ for all $1 \leq i < m$ and hence $\beta \in C_G(\langle t_i \mid 1 \leq i < m \rangle) = \bigcap_{i=1}^m \mathrm{Stab}_G(kv_i)$. Since $\beta \in C_G(\langle u_i \mid 1 \leq i < m \rangle)$, we conclude that $\beta \in Z(G)$ and hence $M = \langle Z(G), \tau \rangle$. Since $\tau \notin C_H(G')$, we have a contradiction and the proof of this lemma is complete.

LEMMA 3.2. *Suppose that $m > 1$ and n is even. Let H be a finite group and let G be a subgroup of index 2 of H such that $G \cong \mathrm{GL}(V/k)$; thus $G' \cong \mathrm{SL}(V/k)$ and $Z(G) \cong k^\times$. Let $\sigma \in \mathscr{I}(\mathrm{Aut}(k))$ and suppose that there is an element $\tau \in H - G$ such that $\tau^2 \in Z(G)$ and τ acting by conjugation on G induces a unitary automorphism (transpose-inverse-automorphism induced by σ) on G with respect to the basis $B = \{v_1, \ldots, v_m\}$ of V/k. Then $C_H(G') = Z(G)$.*

PROOF. Let $M = C_H(G')$, so that $M \cap G = C_G(G') = Z(G) = \langle \lambda I_V \mid \lambda \in k^\times \rangle$ and $|M/Z(G)| \leq 2$. Assume that there is an element $\beta \in G$ such that $\beta\tau \in M$.

For each integer $1 \leq i < m$, let u_i and t_i be as in Lemma 3.1. First suppose that $m > 2$. Then, as in Lemma 3.1, it follows that $\beta \in Z(G)$ and $M = \langle Z(G), \tau \rangle$. Since $\tau \notin C_H(G')$ we have a contradiction. Consequently, $m = 2$. Let $C_k(\sigma) = k_0$, $q_0 = |k_0|$ and let $N: k^\times \to k_0^\times$ denote the norm mapping of k/k_0. Clearly $q_0^2 = q$ and $|\mathrm{Ker}(N)| = q_0 + 1$ by [**28**, Lemma 8.5]. Hence there is an element $c \in \mathrm{Ker}(N)$ with $c \notin \{1, -1\}$. Let $x \in G$ be such that $x(v_1) = cv_1$ and $x(v_2) = c^{-1}v_2$. Then $x \in G'$ and $x^\tau = x$. Hence $\beta \in C_G(x) = \mathrm{Stab}_G(kv_1) \cap \mathrm{Stab}_G(kv_2)$. On the other hand, $\beta \in C_G(u_1)$ and hence $\beta \in Z(G)$. Thus $M = \langle Z(G), \tau \rangle$ and since $\tau \notin C_H(G')$, we have a contradiction to complete the proof.

3A. *The general linear groups*. Let $G = \mathrm{GL}(V/k)$, $H = \mathrm{SL}(V/k)$ and $Z = Z(G) = \langle \lambda I_V \mid \lambda \in k^\times \rangle$. Also set $\bar{G} = G/Z$. Clearly $G/H \cong k^\times$, $G' = H$, $Z \cap H = \langle \lambda I_V \mid \lambda \in k^\times \text{ and } \lambda^m = 1 \rangle \cong Z_{(q-1,m)}$ and $\bar{G}/\bar{H} \cong k^\times/((k^\times)^m)$ since the inverse image of $\bar{H}$ in G is $Z * H = \langle x \in G \mid \det(x) \in (k^\times)^m \rangle$.

Let $u \in G - Z$ be such that $u^2 = \gamma I_V$ for some $\gamma \in k^\times$, so that $|\bar{u}| = 2$. Let $M = N_G(\langle u, Z \rangle)$. Thus M is the inverse image in G of $C_{\bar{G}}(\bar{u})$. Setting $U = \langle u, Z \rangle$, we have $M = N_G(U)$, U is abelian and $C_G(U) = C_G(u)$. Clearly $U \leq Z * H$ if and only if $|(U \cap H)/(Z \cap H)| = 2$ and [**12**, I, §4(4)] implies that $|M/C_G(U)| \leq 2$.

First assume that U is not cyclic. Then $U = Z \times \langle w \rangle$ where $w \in \mathscr{I}(G - Z)$, $w \neq -I_V$, $C_G(U) = C_G(w)$, $w^M \subseteq \{w, (-I_V)w\}$ and $|w^M| = |M/C_G(U)| \leq 2$. Also as in [**12**, I, §§3 and 4], we have $V = V^+ \oplus V^-$ where

$$V^+ = C_V(w) \quad \text{and} \quad V^- = \{v \in V \mid w(v) = -v\} = [V, w].$$

Also $1 \leqslant \dim(V^+/k) < m$, $\det(w) = (-1)^{\dim(V^-/k)}$ and $U \nleqslant Z * \hat{H}$ if and only if $\dim(V^-/k)$ is odd and $-1 \notin (k^\times)^m$. Clearly $G = HC_G(w)$, $w^G = w^H$ and there is an isomorphism α: $C_G(U) \to \mathrm{GL}(V^+/k) \times \mathrm{GL}(V^-/k)$ with $\alpha(Z) = \langle(\lambda I_{V^+}, \lambda I_{V^-}) \mid \lambda \in k^\times\rangle$ and $\alpha(w) \in Z(\mathrm{GL}(V^-/k))$. Clearly

$$(\mathrm{GL}(V^+/k) \times \mathrm{GL}(V^-/k))' = \mathrm{SL}(V^+/k) \times \mathrm{SL}(V^-/k)$$

and hence

$$\alpha\Big(C_{C_G(U)}\big(C_G(U)'\big)\Big) = \big\langle(\lambda I_{V^+}, \delta I_{V^-}) \mid \lambda, \delta \in k^\times\big\rangle \cong k^\times \times k^\times$$

by [**25**, Proposition 2]. Suppose that $|M/C_G(U)| = 2$. Then $w \sim (-I_V)w$ in M and hence there is an involution $g \in M$ such that g: $C_V(w) \leftrightarrow [V, w]$, $w^g = (-I_V)w$ and $M = C_G(U)\langle g\rangle \cong \mathrm{GL}(V^+/k) \operatorname{wr} Z_2$.

The above discussion shows that if $x, y \in \mathcal{I}(G)$, then the following three conditions are equivalent: (i) $x \sim y$ in G; (ii) $\dim(C_V(x)/k) = \dim(C_V(y)/k)$; (iii) $x \sim y$ via H. Also for any integer p with $1 \leqslant p < m$, there is an involution $z \in G - Z$ such that $\dim(C_V(z)/k) = p$.

Next assume that U is cyclic. Then $U = \langle Z, w\rangle$ where $w^2 = \gamma I_V$ for some $\gamma \in k^\times - (k^\times)^2$. Hence $X^2 - \gamma$ is irreducible in the polynomial ring $k[X]$ and is the minimal polynomial of w. Thus m is even, $(X^2 - \gamma)^{m/2}$ is the characteristic polynomial of w and $\det(w) = (-\gamma)^{m/2}$. Consequently $U \nleqslant Z * H$ if and only if $(-\gamma)^{m/2} \notin (k^\times)^m$. As in [**12**, I, §3], let K be a quadratic extension field of k such that $K = k(\rho)$ where $\rho^2 = \gamma$, so that $\{1, \rho\}$ is a basis of K/k. As in [**12**, I, §3], for $v \in V$ and $a, b \in k$, set $v(a + b\rho) = va + w(v)b$. Then V becomes a vector space over K of dimension $\frac{m}{2}$ and $C_G(U) = \mathrm{GL}(V/K)$. Let $B = \{v_1, \ldots, v_{m/2}\}$ be a basis of V/K and let $\mathrm{Gal}(K/k) = \langle\tau\rangle$, so that $B_1 = \{v_1, \ldots, v_{m/2}, v_1\rho, \ldots, v_{m/2}\rho\}$ is a basis of V/k, for any $1 \leqslant i \leqslant \frac{m}{2}$, $w(v_i) = v_i\rho$ and $w(v_i\rho) = v_i\gamma$, $\tau \in \mathrm{Aut}(k)$, $|\tau| = 2$, $\tau(\rho) = -\rho$ and $C_K(\tau) = k$. For any $v = \sum_{i=1}^{m/2} v_i d_i$ with $d_i \in K$ for $1 \leqslant i \leqslant \frac{m}{2}$, set $x(v) = \sum_{i=1}^{m/2} v_i\tau(d_i)$. Then $x \in \mathrm{GL}(V/k) = G$, $\det(x) = (-1)^{m/2}$, $|x| = 2$ and $w^x = -w$ since $xw(v) = x(v\rho) = v\tau(\rho) = -x(v)\rho = -(wx(v))$. Thus $M = C_G(U)\langle x\rangle \leqslant \Gamma L(V/K)$ and x acts like a field automorphism of order 2 on $C_G(U) = \mathrm{GL}(V/K)$. Since the norm N: $K^\times \to k^\times$ is epimorphic, it is easy to see that $G = C_G(U)H$ and hence $w^G = w^H$. Also it is obvious that for every $\delta \in (K^\times) - (K^\times)^2$, there is an element $w_1 \in wZ = U - Z$ such that $w_1^2 = \delta I_V$. Consequently all cyclic subgroups X of G such that $Z \leqslant X$ and $|X/Z| = 2$ are conjugate via H because of the basis B_1 of V/k. Moreover, if m is even, there are cyclic subgroups X of G such that $Z \leqslant X$ and $|X/Z| = 2$ since there are elements $w \in G$ such that $w^2 = \gamma I_V$ for any $\gamma \in k^\times - (k^\times)^2$.

3B. *The symplectic groups*. Suppose, in this section, that f: $V \times V \to k$ is a bilinear symplectic scalar product that is nonsingular (cf. [**25**, §1; **29**, §7; etc.]). Thus $f(v_1, v_2) = -f_2(v_2, v_1)$ for all v_1, $v_2 \in V$ and $m = \dim(V/k)$ is even. Let $G = \mathrm{GSp}(V/k)$ and $H = \mathrm{Sp}(V/k)$ be as defined in [**12**, I, §9]. Thus $Z = \langle\lambda I_V \mid \lambda \in k^\times\rangle \leqslant G$, $H = G'$, $Z(G) = Z$ by [**25**, Proposition 3] and $Z \cap H = \langle -I_V\rangle$. For each $u \in G$, there is a unique element $r_u \in k^\times$ such that $f(u(v_1), u(v_2)) = r_u f(v_1, v_2)$ for all v_1, $v_2 \in V$, r_u is called the multiplicator of u and the mapping γ: $G \to k^\times$ defined by $\gamma(u) = r_u$ for $u \in G$ is an epimorphism with $\mathrm{Ker}(\gamma) = H$ (cf. [**25**, Proposition 3]). Set $\bar{G} = G/Z$. Then $|\bar{G}/\bar{H}| = 2$ and the inverse image of $\bar{H}$ in G is $Z * H = \{u \in G \mid r_u \in (k^\times)^2\}$.

For $m = 2$, we have, by [**28**, II, 9.12], $G = \mathrm{GL}(V/k)$, $\gamma = \det$ and $H = \mathrm{SL}(V/k)$.

Let $u \in G - Z$ be such that $u^2 = \gamma I_V$ for some $\gamma \in k^\times$, so that $|\bar{u}| = 2$. Let $U = \langle u, Z\rangle$ and $M = N_G(U)$. Thus U is abelian, $\bar{U} = \langle \bar{u}\rangle \cong U/Z$, $C_G(U) = C_G(U)$, M is the inverse image in G of $C_{\bar{G}}(\bar{u})$, $U \leqslant Z * H$ if and only if $|(U \cap H)/(Z \cap H)| = 2$ and $|M/C_G(U)| \leqslant 2$ as in §3A.

First assume that U is not cyclic. Then $U = Z \times \langle w\rangle$ where $w \in \mathcal{I}(G - Z)$, $w \neq -I_V$, $C_G(U) = C_G(w)$, $w^M \subseteq \{w, (-I_V)w\}$, $|w^M| = |M/C_G(U)| \leqslant 2$ and $r_w^2 = 1$.

Suppose that $r_w = 1$. Then $w \in H$, $U \leqslant Z * H$, $V = V^+ \perp V^-$ where $V^+ = C_V(w)$, $V^- = \{v \in V \mid w(v) = -v\} = [V, w]$ and the restrictions of f to V^+/k and V^-/k yield nonsingular symplectic vector spaces. Thus $\dim(V^+/k)$ is even, $2 \leqslant \dim(V^+/k) < m$, $C_G(U) \cong \{(w_1, w_2) \in (\mathrm{GSp}(V^+/k)) \times (\mathrm{GSp}(V^-/k)) \mid r_{w_1} = r_{w_2}\}$, $G = HC_G(U)$ and $w^G = w^H$. Suppose that $|M/C_G(U)| = 2$. Then $w \sim (-I_V)w$ in M and [**29**, Proposition 9.13] implies that there is an involution $g \in M \cap H$ such that $g\colon V^+ \leftrightarrow V^-$, $w^g = (-I_V)w$ and $M = C_G(U)\langle g\rangle$.

It now is clear that if $x, y \in \mathcal{I}(H)$, then the following three conditions are equivalent: (i) $x \sim y$ in G; (ii) $\dim(C_V(x)/k) = \dim(C_V(y)/k)$; (iii) $x \sim y$ in H. Also for any even integer p with $2 \leqslant p < m$, there is an involution $z \in G - Z$ such that $\dim(C_V(z)/k) = p$.

Suppose that $r_w = -1$. Then $w \notin H$, $U \leqslant Z * H$ if and only if $q \equiv 1 \pmod 4$, $V^+ = C_V(w)$ and $V^- = \{v \in V \mid w(v) = -v\} = [V, w]$ are totally isotropic subspaces of V with $V = V^+ \oplus V^-$ and $\dim(V^+/k) = \dim(V^-/k) = \frac{m}{2}$. Also there are bases $\{v_i \mid 1 \leqslant i \leqslant \frac{m}{2}\}$ of V^+/k and $\{v_{j+m/2} \mid 1 \leqslant j \leqslant \frac{m}{2}\}$ of V^-/k such that $f(v_i, v_{j+m/2}) = \delta_{ij}$ for all $1 \leqslant i, j \leqslant \frac{m}{2}$. Suppose that $x \in C_G(U)$. Then x leaves invariant both V^+ and V^- and $f(x(v_1), x(v_2)) = r_x f(v_1, v_2)$ for all $v_1 \in V^+$ and $v_2 \in V^-$. Conversely, if $x \in \mathrm{GL}(V/k)$ is such that x leaves invariant both V^+ and V^- and $f(x(v_1), x(v_2)) = sf(v_1, v_2)$ for all $v_1 \in V^+$, $v_2 \in V^-$ and for some fixed $s \in k$, then $x \in C_G(U)$ and $r_x = s$. For $(\lambda_1, \lambda_2) \in k^\times \times k^\times$, let $(\lambda_1 I_{V^+}, \lambda_2 I_{V^-})$ denote the element of $C_G(U)$ such that $(\lambda_1 I_{V^+}, \lambda_2 I_{V^-})(v_1 + v_2) = \lambda_1 v_1 + \lambda_2 v_2$ for all $v_1 \in V^+$ and $v_2 \in V^-$. Then

$$\gamma\big((\lambda_1 I_{V^+}, \lambda_2 I_{V^-})\big) = \lambda_1\lambda_2 \quad \text{and} \quad X = \big\langle (\lambda_1 I_{V^+}, \lambda_2 I_{V^-}) \mid \lambda_1, \lambda_2 \in k^\times \big\rangle \leqslant C_G(U).$$

Also for each $g \in \mathrm{GL}(V^+/k)$, there is a unique element $g^* \in \mathrm{GL}(V^-/k)$ such that $f(g(v_1), g^*(v_2)) = f(v_1, v_2)$ for all $v_1 \in V^+$ and $v_2 \in V^-$. Hence the mapping α: $\mathrm{GL}(V^+/k) \to C_H(U)$ defined by $\alpha(g)(v_1 + v_2) = g(v_1) + g^*(v_2)$ for all $v_1 \in V^+$, $v_2 \in V^-$ and $g \in \mathrm{GL}(V^+/k)$ induces an isomorphism of $\mathrm{GL}(V^+/k)$ onto $C_H(U)$. Consequently

$$C_G(U) = C_H(U)X = C_H(U) * X = C_H(U) \times \big\langle (I_{V^+}, \lambda I_{V^-}) \mid \lambda \in k^\times \big\rangle$$

since $C_H(U) \cap X = \langle (\lambda I_{V^+}, \lambda^{-1} I_{V^-}) \mid \lambda \in k^\times \rangle$. Note that $C_G(U)' \cong \mathrm{SL}(V^+/k)$ and $Z < X = C_{C_G(U)}(C_G(U)')$. Let $g \in \mathrm{GL}(V/k)$ be such that $g\colon v_i \leftrightarrow v_{i+m/2}$ for all $1 \leqslant i \leqslant \frac{m}{2}$. Then $g \in \mathcal{I}(G)$ with $r_g = -1$, $gw = (-I_V)wg \in H$, $(gw)^2 = -I_V$, $M = C_G(U)\langle g\rangle$ and it is easy to see that conjugation by g induces transpose inverse on

$C_H(U) \cong \mathrm{GL}(V^+/k)$ and that g: $\langle (\lambda I_{V^+}, I_{V^-}) \mid \lambda \in k^\times \rangle \leftrightarrow \langle (I_{V^+}, \lambda I_{V^-}) \mid \lambda \in k^\times \rangle$. Clearly $G = HC_G(U)$, $w^G = w^H$ and it follows from the existence of the bases of V^+/k and of V^-/k described above that all involutions w of G with $r_w = -1$ are conjugate under H. Moreover, it is easy to see from [**29**, Proposition 9.13], that there are involutions $w \in G$ such that $r_w = -1$.

Next assume that U is cyclic. Then $U = \langle Z, w \rangle$ where $w^2 = \gamma I_V$ for some $\gamma \in k^\times - (k^\times)^2$. Since $r_w^2 = \gamma^2$, we have $r_w = \pm\gamma$. Also, let $K = k(\rho)$, ρ, τ, V/k, etc. be as in §3A and for any $v_1, v_2 \in V$, set $f_0(v_1, v_2) = \rho f(v_1, v_2) + f(v_1, w(v_2))$. Thus $f_0(v_2, v_1) = -\rho f(v_1, v_2) - r_w\gamma^{-1} f(v_1, w(v_2))$ for any $v_1, v_2 \in V$.

Suppose that $r_w = \gamma$. Then $f_0(v_1, v_2) = -f_0(v_2, v_1)$ for all $v_1, v_2 \in V$ and f_0: $V \times V \to K$ is a nonsingular bilinear symplectic scalar product on V/K. Hence $\dim(V/K) = \frac{m}{2}$ is even and $U \not\leqslant Z * H$ since $r_w = \gamma \notin (k^\times)^2$. It readily follows that $C_G(U) = \{x \in \mathrm{GSp}(V/K) \mid r_x \in k^\times\}$ and hence $G = HC_G(U)$ and $w^G = w^H$ by [**25**, Proposition 3]. Also by [**29**, Proposition 9.13], V/K has a basis $B = \{v_1^{(i)}, v_2^{(i)} \mid 1 \leqslant i \leqslant \frac{m}{4}\}$ such that for all $1 \leqslant i, j \leqslant \frac{m}{4}$ and $1 \leqslant r, s \leqslant 2$, we have

$$f_0\big(v_r^{(i)}, v_s^{(j)}\big) = 0 \quad \text{if } i \neq j \text{ or } r = s \quad \text{and} \quad f_0\big(v_1^{(i)}, v_2^{(i)}\big) = 1 = -f_0\big(v_2^{(i)}, v_1^{(i)}\big).$$

Consequently $B_1 = \{v_1^{(i)}, v_2^{(i)}, v_1^{(i)}\rho, v_2^{(i)}\rho \mid 1 \leqslant i \leqslant \frac{m}{4}\}$ is a basis of V/k such that for all $1 \leqslant i, j \leqslant \frac{m}{4}$ and $1 \leqslant r, s \leqslant 2$, we have

$$f\big(v_r^{(i)}, v_s^{(j)}\big) = f\big(v_r^{(i)}, v_s^{(j)}\rho\big) = f\big(v_r^{(i)}\rho, v_s^{(j)}\rho\big) = 0 \quad \text{if } i \neq j \text{ or } r = s,$$

$$f\big(v_1^{(i)}, v_2^{(i)}\big) = f\big(v_1^{(i)}\rho, v_2^{(i)}\rho\big) = 0 \quad \text{and} \quad f\big(v_1^{(i)}, v_2^{(i)}\rho\big) = f\big(v_1^{(i)}\rho, v_2^{(i)}\big) = +1.$$

Clearly $w(v_r^{(i)}) = v_r^{(i)}\rho$ and $w(v_r^{(i)}\rho) = v_r^{(i)}\gamma$ for all $1 \leqslant r \leqslant 2$ and $1 \leqslant i \leqslant \frac{m}{4}$. Let $x \in \Gamma L(V/K)$ be induced by τ with respect to the basis B of V/K as in §3A. Then $xw = (-I_V)wx$, $x \in \Gamma\mathrm{Sp}(V/K)$, $f(x(v_1), x(v_2)) = -f(v_1, v_2)$ for all $v_1, v_2 \in V$, $x \in G$, $|x| = 2$ and $M = C_G(U)\langle x \rangle \leqslant \Gamma\mathrm{Sp}(V/K)$. Clearly every element $w_1 \in wZ$ is such that $w_1^2 = r_{w_1} I_V$ with $r_{w_1} \in k^\times - (k^\times)^2$. Next suppose that X is a cyclic subgroup of G such that $Z \leqslant X$, $|X/Z| = 2$ and such that $X - Z$ contains an element z with $z^2 = r_z I_V$. Then $r_z \notin (k^\times)^2$ since X is cyclic and we may assume that $r_z = \gamma$. The existence of the basis B_1 of V/k above implies that X and U are conjugate via H. Moreover, when $\frac{m}{2}$ is even, there are such subgroups X of G. To see this, let $\gamma \in k^\times - (k^\times)^2$, let $K = k(\rho)$ where $\rho^2 = \gamma$ as above, let W/K be a vector space with $\dim(W/K) = \frac{m}{2}$ and let g: $W \times W \to K$ be a K-bilinear nonsingular symplectic scalar product (cf. [**29**, Proposition 9.13]). Since $K = k + k\rho$, we conclude that $g = \rho g_1 + g_2$ where g_i: $W \times W \to k$ is a k-bilinear scalar product on W/k for $i = 1, 2$. For each $0 \neq v \in W$, $g(v, W) = K = g(W, v)$ and $g(v, v) = 0$. Thus g_1 and g_2 are nonsingular symplectic scalar products on W/k. Since g is K-bilinear, it follows that $g_1(v_1\rho, v_2) = g_1(v_1, v_2\rho)$ and hence $g_1(v_1\rho, v_2\rho) = \gamma g_1(v_1, v_2)$ for all $v_1, v_2 \in W$. Thus, if m denotes multiplication by ρ on W/k, then $m^2 = \gamma I_W$, $m \in \mathrm{GSp}(W/k)$ and $r_m = \gamma$ with respect to the symplectic space $(W/k, g_1)$. Since $(V/k, f)$ and $(W/k, g_1)$ are isometric, our proof of the existence of such subgroups X of G is complete.

Suppose that $r_W = -\gamma$. Then $f_0(v_2, v_1) = \tau(f_0(v_1, v_2))$ for all $v_1, v_2 \in V$ and f_0: $V \times V \to K$ is a nonsingular τ-bilinear Hermitian scalar product on V/K. Clearly

$U \leqslant Z * H$ if and only if $q \equiv -1 \pmod 4$. Since the multiplicators of all elements of $GU(V/K)$ lie in $k^\times$, it follows that $C_G(U) = GU(V/K)$. Clearly $G = HC_G(U)$ and $w^G = w^H$ by [**25**, Proposition 4]. Also, by [**29**, Proposition 8.8], V/K has a basis $B = \{v_i \mid 1 \leqslant i \leqslant \frac{m}{2}\}$ such that $f_0(v_i, v_j) = \delta_{ij}$ for all $1 \leqslant i, j \leqslant \frac{m}{2}$. Consequently $B_1 = \{v_i, v_i\rho \mid 1 \leqslant i \leqslant \frac{m}{2}\}$ is a basis of V/K such that for all $1 \leqslant i, j \leqslant \frac{m}{2}$ we have

$$f(v_i, v_j) = f(v_i, v_j\rho) = f(v_i\rho, v_j\rho) = 0 \quad \text{if } i \neq j$$

and

$$f(v_i, v_i) = f(v_i\rho, v_i\rho) = 0 \quad \text{and} \quad f(v_i, v_i\rho) = +1.$$

Clearly $w(v_i) = v_i\rho$ and $w(v_i\rho) = v_i\gamma$ for all$1 \leqslant i \leqslant \frac{m}{2}$. Let $x \in \Gamma L(V/K)$ be induced by τ with respect to the basis B of V/K as above. Then $xw = (-I_V)wx$, $x \in \Gamma U(V/K)$, $f(x(v_1), x(v_2)) = -f(v_1, v_2)$ for all $v_1, v_2 \in V$, $x \in G$, $|x| = 2$ and $M = C_G(U)\langle x \rangle \leqslant \Gamma U(V/K)$. As above all cyclic subgroups X of G such that $Z \leqslant X$, $|X/Z| = 2$ and $X - Z$ contains an element z with $z^2 = -r_z I_V$ are conjugate under H. Moreover such subgroups X of G always exist. To see this, let $\gamma \in k^\times - (k^\times)^2$, let $K = k(\rho)$ where $\rho^2 = \gamma$ and let $\mathrm{Gal}(K/k) = \langle \tau \rangle$ be as above. Let W/K be a vector space of dimension $\frac{m}{2}$ and let $g\colon W \times W \to K$ be a nonsingular τ-bilinear Hermitian scalar product on W/K. Since $K = k + k\rho$, we have $g = \rho g_1 + g_2$ where $g_i\colon W \times W \to k$ is a k-bilinear scalar product on W/k for $i = 1, 2$. As above, it follows that g_1 is a nonsingular symplectic scalar product on W/k and that $g_1(v_1\rho, v_2) = -g_1(v_1, v_2\rho)$ and hence $g_1(v_1\rho, v_2\rho) = -\gamma g_1(v_1, v_2)$ for all $v_1, v_2 \in W$. The existence of such subgroups X of G now follows as above.

3C. *The unitary groups*. Assume in this section that n is even, so that $q = q_0^2$ where $q_0 = p^{n/2}$ and let $\sigma \in \mathrm{Aut}(k)$ with $|\sigma| = 2$. Set $k_0 = C_k(\sigma)$, let $N\colon k^\times \to k_0^\times$ denote the norm mapping of k/k_0, and let $f\colon V \times V \to k$ be a σ-linear Hermitian scalar product that is nonsingular (cf. [**25**, §1; **29**, §7; etc.]). Thus $f(v_1, v_2) = \sigma(f(v_2, v_1))$ for all $v_1, v_2 \in V$. Let $G = GU(V/k)$ and $H = U(V/k)$ be as defined in [**12**, I, §9]. Thus $Z = \langle \lambda I_V \mid \lambda \in k^\times \rangle = Z(G)$ by [**25**, Proposition 4]. For each $u \in G$, there is a unique element $r_u \in k_0^\times$ such that $f(u(v_1), u(v_2)) = r_u f(v_1, v_2)$ for all $v_1, v_2 \in V$, r_u is called the multiplicator of u and the mapping $\gamma\colon G \to k_0^\times$ defined by $\gamma(u) = r_u$ for $u \in G$ is an epimorphism with $\mathrm{Ker}(\gamma) = H$, cf. [**29**, Proposition 4]. Also [**29**, Lemma 8.5] implies that $G = H * Z$.

Note that $U(V/k) \cong U(m, q_0)$, $U(V/k) \cap \mathrm{SL}(V/k) = \mathrm{SU}(V/k) \cong \mathrm{SU}(m, q_0)$ and $\mathrm{PSU}(V/k) \cong \mathrm{PSU}(m, q_0)$, etc. by our notational convention that adheres to [**16**, §17.1].

Suppose that $m = 2$. Then V/k has a basis $B = \{v_1, v_2\}$ such that $f(v_1, v_1) = f(v_2, v_2) = 1$ and $f(v_1, v_2) = 0$. With matrices relative to the basis B, set

$$X = \left\{ \begin{pmatrix} a & b \\ -\sigma(b) & \sigma(a) \end{pmatrix} \middle| \, a, b \in k \text{ and } N(a) + N(b) = \det\begin{pmatrix} a & b \\ -\sigma(b) & \sigma(a) \end{pmatrix} \neq 0 \right\}.$$

Then, as in [**28**, II, 8.8], it follows that $G = Z * X$ where $\gamma|_X = \det$ and $X \cap H = \mathrm{SU}(V/k) \cong \mathrm{SL}(2, q_0)$.

Consequently, since $G = H * Z$, for greater simplicity we restrict our attention to $H = U(V/k)$. Note that $H' = \mathrm{SU}(V/k)$. Set $Z_1 = Z \cap H = \langle \lambda I_V \mid \lambda \in \mathrm{Ker}(N) \rangle \cong \mathrm{Ker}(N)$ and $\overline{H} = H/Z_1$. Then $|\overline{H}/\overline{H}'| = (m, q_0 + 1)$, $\overline{H}' \cong H'/(Z_1 \cap H') \cong \mathrm{PSU}(V/k)$ and the inverse image of $\overline{H}'$ in H is $H' * Z_1 = \{u \in H \mid \det(u) \in (\mathrm{Ker}(N))^m\}$.

Let $u \in H - Z_1$ be such that $u^2 = \gamma I_V$ for some $\gamma \supseteq \mathrm{er}(N)$, so that $|\bar{u}| = 2$. Let $U = \langle Z_1, u \rangle$ and $M = N_H(U)$. Thus U is abelian, $\overline{U} = \langle \bar{u} \rangle$, $C_H(U) = C_H(u)$, M is the inverse image in H of $C_{\overline{H}}(\bar{u})$, $U \leqslant H' * Z$, if and only if

$$|(U \cap H')/(Z_1 \cap H')| = 2 \quad \text{and} \quad |M/C_H(U)| \leqslant 2$$

as in §3A.

First assume that U is not cyclic. Then $U = Z_1 \times \langle w \rangle$ where $w \in \mathcal{I}(H - Z_1)$, $w \neq -I_V$, $C_H(U) = C_H(w)$, $w^M \subseteq \{w, (-I_V)w\}$ and $|w^M| = |M/C_H(U)|$. As above, we get $V = V^+ \perp V^-$ where $V^+ = C_V(w)$, $V^- = \{v \in V \mid w(v) = -v\} = [V, w]$ and the restrictions of f to V^+/k and V^-/k yield nonsingular unitary vector spaces. Thus there is an isomorphism

$$\alpha\colon C_H(U) \to U(V^+/k) \times U(V^-/k), \qquad H = H'C_H(U) \text{ and } w^H = w^{H'}.$$

Clearly $U \leqslant Z_1 * H'$ if and only if $(-1)^{\dim(V^-/k)} \in (\mathrm{Ker}(N))^m$. If $|M/C_H(U)| = 2$, then **[29**, Proposition 8.8(b)] implies that there is an involution $g \in M$ such that $g\colon V^+ \leftrightarrow V^-$, $w^g = (-I_V)w$ and $M = C_H(U)\langle g \rangle \cong U(V^+/k) \mathrm{wr}\, Z_2$.

It now is clear that if $x, y \in \mathcal{I}(H)$, then the following three conditions are equivalent: (i) $x \sim y$ in $G = GU(V/k)$; (ii) $\dim(C_V(x)/k) = \dim(C_V(y)/k)$; (iii) $x \sim y$ in H. Also [**29**, Proposition 8.8] implies that for any integer p with $1 \leqslant p < m$, there is an involution $z \in H - Z_1$ such that $\dim(C_V(z)/k) = p$.

Next assume that U is cyclic. Then $u^2 = \gamma I_V$ where $\gamma \in \mathrm{Ker}(N) - (\mathrm{Ker}(N))^2$. Now $|k^\times| = q - 1 = (q_0)^2 - 1$, $|\mathrm{Ker}(N)| = q_0 + 1$ and $2 \mid (q_0 - 1)$. Hence there is an element $\lambda \in k^\times$ such that $\lambda^2 = \gamma$. Set $u_0 = u(\lambda^{-1} I_V)$, so that $(u_0)^2 = I_V$, $u_0 \in GU(V/k)$, $C_H(U) = C_H(u) = C_H(u_0)$ and $N_H(U) = N_H(\langle u_0 \rangle \times Z_1)$.

Clearly $r_{u_0} = N(\lambda^{-1}) \neq 1$ and $N(\lambda)^2 = N(\gamma) = 1$, so that $r_{u_0} = N(\lambda^{-1}) = -1$. Hence m is even, $V = V^+ \oplus V^-$ where $V^+ = C_V(u_0)$ and $V^- = \{v \in V \mid u_0(v) = -v\} = [V, u_0]$ are totally isotropic subspaces of V with $\dim(V^+/k) = \dim(V^-/k) = \frac{m}{2}$. Since $\det(u) = (-\gamma)^{m/2}$ it follows that $U \leqslant Z_1 * H'$ if and only if $m_2 > |q_0 + 1|_2$ if $|m|_2 > 2$ and $q_0 \equiv 1 \pmod 4$ if $|m|_2 = 2$. As in §3B, it follows that there is an isomorphism $\alpha\colon \mathrm{GL}(V^+/k) \to C_H(u_0) = C_H(U)$ and an involution $g \in H$ such that $gu = (-I_V)ug$, $N_H(U) = C_H(u)\langle g \rangle$ and conjugation by g induces a unitary automorphism on $C_H(U) \cong \mathrm{GL}(V^+/k)$. Note that $u(v) = \lambda v$ if $v \in V^+$ and $u(v) = -\lambda v$ if $v \in V^-$. Next suppose that X is an arbitrary cyclic subgroup of H such that $Z_1 \leqslant X$ and $|X/Z_1| = 2$. Then there is an element $z \in X - Z_1$ such that $z^2 = \gamma I_V$ and it is now clear that X and U are conjugate via H. Moreover, when m is even, such subgroups X of H exist since then V has complementary totally isotropic subspaces each of dimension $\frac{m}{2}$ by [**29**, Proposition 9.14].

3D. *The orthogonal groups.* Assume in this section that $f\colon V \times V \to k$ is a bilinear symmetric (orthogonal) scalar product that is nonsingular (cf. [**25**, §1; **29**, §7; etc.]).

Thus $f(v_1, v_2) = f(v_2, v_1)$ for all $v_1, v_2 \in V$. Also let $g\colon V \to k$ be the associated quadratic form on V/k so that $g(v) = \frac{1}{2} f(v, v)$ and $f(v_1, v_2) = g(v_1 + v_2) - g(v_1) - g(v_2)$ for all $v_1, v_2 \in V$ (cf. [**29**, §10]).

In this setting, there is associated (to V/k and f) a unique element $D(V/k) \in k^\times/(k^\times)^2$ called the discriminant (of $(V/k, f)$) (cf. [**29**, Definition 7.5]). Clearly if $c \in k^\times$, then $cf\colon V \times V \to k$ is a bilinear nonsingular symmetric scalar product with discriminant $c^m D(V/k)$.

Note that for any given dimension $m = \dim(V/k)$ and suitable f, $D(V/k)$ can be either of the two elements of $k^\times/(k^\times)^2$ and two orthogonal spaces over k are isometric if and only if they have the same dimension and discriminant (cf. [**29**, Proposition 8.9]).

Let $G = GO(V/k)$ and $H = O(V/k)$ be as defined in [**12**, I, §9]. Thus $Z = \langle \lambda I_V \mid \lambda \in k^\times \rangle \leqslant Z(G)$ and $Z \cap H = \langle -I_V \rangle$. Also for each $u \in G$, there is a unique element $r_u \in k^\times$ such that $f(u(v_1), u(v_2)) = r_u f(v_1, v_2)$ for all $v_1, v_2 \in V$ and r_u is called the multiplicator of u. The mapping $\gamma\colon G \to k^\times$ defined by $\gamma(u) = r_u$ for $u \in G$ is such that $\operatorname{Ker}(\gamma) = H$, $\gamma^{-1}(k^\times)^2 = Z * H$, γ maps G onto $k^\times$ if m is even and onto $(k^\times)^2$ if m is odd by [**25**, Proposition 5(b)]. Set $\Omega = H'$, $K = SO(V/k)$ and $\overline{G} = G/Z$. Then $\overline{H} \cong H/\langle -I_V \rangle$, $\Omega \leqslant K \leqslant H$ and det maps H onto $\langle -1 \rangle$ so that $|H/K| = 2$ by [**29**, Proposition 8.10].

By [**29**, Corollary 14.6], all maximal isotropic subspaces of V/k are conjugate under $H = O(V/k)$ and the dimension of any such subspace of V/k is called the index of V, $\operatorname{ind}(V/k)$. As in [**29**, Example 14.7], if m is odd, then $\operatorname{ind}(V/k) = (m-1)/2$ and if m is even then either $\operatorname{ind}(V/k) = \frac{m}{2}$ and $D(V/k) = (-1)^{m/2}(k^\times)^2$ or $\operatorname{ind}(V/k) = \frac{m}{2} - 1$ and $D(V) = (-1)^{m/2}c(k^\times)^2$ where $c \in k^\times - (k^\times)^2$. Thus the index distinguishes the two types of orthogonal vector spaces of the same even dimension.

From our notational convection (cf. [**16**, §17.11]), if m is even, we have $P\Omega(V/k) \cong P\Omega(m, q, 1)$ if $\operatorname{ind}(V/k) = \frac{m}{2}$ and $P\Omega(V/k) \cong P\Omega(m, q, -1)$ if $\operatorname{ind}(V/k) = \frac{m}{2} - 1$.

As in [**29**, Proposition 20.2], there is a homomorphism $\sigma\colon H = O(V/k) \to k^\times/(k^\times)^2$ called the Spinornorm. The proof of [**29**, Proposition 20.10] yields $\sigma(-I_V) = D(V/k)2^{-m}(k^\times)^2$. If $\operatorname{ind}(V/k) > 0$, then $\sigma(K) = k^\times/(k^\times)^2$ and $\operatorname{Ker}(\sigma) \cap K = \Omega$ by [**29**, Proposition 20.3 and Theorem 20.8]. If $m \geqslant 3$, then $C_G(\Omega) = Z$ by [**25**, Proposition 5] and $\operatorname{ind}(V) > 0$ and $K' = H' = \Omega$ by [**29**, Propositions 9.2(b) and 20.9].

Suppose that $m = \dim(V/k)$ is odd for the moment. Then $H = \langle -I_V \rangle \times K$ and $G = Z * H = Z \times K$. Also, if $c \in k^\times - (k^\times)^2$, then $\{(V/k, f), (V/k, cf)\}$ represents the two classes of nonisometric orthogonal spaces over k of dimension m and we have $GO((V/k, f)) = GO((V/k, cf))$, $O(V/k, f) = O(V/k, cf)$, etc.

Next suppose that $m = \dim(V/k)$ is even and $m \geqslant 4$. Then $|G/(Z * H)| = 2$, $Z(K) = \langle -I_V \rangle$, $-I_V \in \Omega$ if and only if $D(V/k) \in (k^\times)^2$, $K = \langle -I_V \rangle \times \Omega$ and $H = \langle -I_V \rangle \times \operatorname{Ker}(\sigma)$ when $D(V/k) \notin (k^\times)^2$ and $Z(\Omega) = \langle -I_V \rangle$ when $D(V/k) \in (k^\times)^2$ by [**25**, Proposition 5].

We shall now discuss the cases with $1 \leqslant m \leqslant 6$.

When $m = 1$, we have $G = Z$, $H = \langle -I_V \rangle$, $K = \Omega = 1$ and $\sigma(-I_V) = 2^{-1}D(V/k)(k^\times)^2$.

Suppose that $m = 2$ and $\operatorname{ind}(V/k) > 0$. Then [29, Proposition 9.14] implies that V/k has a basis $B = \{v_1, v_2\}$ such that $f(v_1, v_1) = f(v_2, v_2) = 0$ and $f(v_1, v_2) = 1$ (i.e. V/k is a hyperbolic plane). Then $D(V/k) = (-1)(k^\times)^2$, the involution t of $\mathrm{GL}(V/k)$ such that $t\colon v_1 \leftrightarrow v_2$ lies in $H - K$, $G = M\langle t\rangle$ where $M = \mathrm{Stab}_{\mathrm{GL}(V/k)}(kv_1) \cap \mathrm{Stab}_{\mathrm{GL}(V/k)}(kv_2)$, $\gamma\colon G \to k^\times$ is an epimorphism, $\sigma\colon H \to k^\times/(k^\times)^2$ is an epimorphism by [29, Proposition 9.9], $K \cong Z_{q-1}$, t inverts K and $H = K\langle t\rangle$ is dihedral. Also, passing to matrices with respect to the basis B, it is easy to see that

$$H = \left\langle \begin{pmatrix} 0 & a \\ a^{-1} & 0 \end{pmatrix} \middle| a \in k^\times \right\rangle, \qquad K = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| a \in k^\times \right\}$$

and that if $a \in k^\times$, then

$$\sigma\left(\begin{pmatrix} 0 & a \\ a^{-1} & 0 \end{pmatrix}\right) = -a(k^\times)^2 \quad \text{and} \quad \sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) = a(k^\times)^2.$$

Suppose that $m = 2$ and $\operatorname{ind}(V/k) = 0$. Then, as in the proof of [29, Lemma 15.1(b)], there is an element $c \in k^\times - (k^\times)^2$ such that $D(V/k) = -c(k^\times)^2$ and V/k has a basis $B = \{v_1, v_2\}$ such that $g(a_1v_1 + a_2v_2) = a_1^2 - ca_2^2$ for all $a_1, a_2 \in k$. Also if K/k is a quadratic extension field such that $K = k(\rho)$ where $\rho^2 = c$ and if $N\colon K^\times \to k^\times$ denotes the norm mapping of K/k, then there is a k-linear isomorphism $\varepsilon\colon V \to K$ such that $N(\varepsilon(v)) = g(v)$ for all $0 \neq v \in V$. Let τ denote the involution of $\mathrm{Aut}(K)$, so that $C_K(\tau) = k$. Focusing attention on $(K/k, N)$, it follows that $\tau \in O(K/k) - SO(K/k)$, $|\tau| = 2$, τ inverts $SO(K/k) = \langle \lambda I_K \mid \lambda \in \operatorname{Ker}(N)\rangle \cong Z_{q+1}$, $H = O(K/k) = SO(K/k)\langle\tau\rangle$ is dihedral, $G = GO(K/k) = \langle \lambda I_K \mid \lambda \in K^\times\rangle\langle\tau\rangle$, $Z(G) = Z$, $\gamma\colon G \to k^\times$ is an epimorphism and $\sigma\colon H = O(K/k) \to k^\times/(k^\times)^2$ is an epimorphism since $N\colon K^\times \to k^\times$ is an epimorphism. Also, passing to matrices with respect to the bases $B = \{1, \rho\}$ of K/k, it is easy to see that

$$SO(K/k) = \left\{ \begin{pmatrix} a_1 & a_2 \\ ca_2 & a_1 \end{pmatrix} \middle| a_1, a_2 \in k \text{ and } a_1^2 - ca_2^2 = 1 \right\},$$

$$\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\begin{aligned} O(K/k) - SO(K/k) &= SO(K/k)\tau \\ &= \left\{ \begin{pmatrix} a_1 & -a_2 \\ ca_2 & -a_1 \end{pmatrix} \middle| a_1, a_2 \in k \text{ and } a_1^2 - ca_2^2 = 1 \right\}. \end{aligned}$$

Let $w = a_1 + a_2\rho \in K^\times$ with $\{0\} \neq \{a_1, a_2\} \subseteq k$, so that $N(w) = a_1^2 - ca_2^2 \neq 0$. Let R_w denote the reflection corresponding to w so that R_w has matrix

$$\frac{1}{N(w)} \begin{pmatrix} -a_1^2 - ca_2^2 & -2a_1a_2 \\ 2a_1a_2c & a_1^2 + ca_2^2 \end{pmatrix}$$

with respect to the basis $B = \{1, \rho\}$ of K/k and $\sigma(R_w) = N(w)(k^\times)^2$. Let $\{b_1, b_2\} \subseteq k$ be such that $b_1^2 - cb_2^2 = 1$, so that

$$T = \begin{pmatrix} b_1 & -b_2 \\ cb_2 & -b_1 \end{pmatrix} \in O(K/k) - SO(K/k).$$

If $b_1 = 1$, then $b_2 = 0$, $T = \tau = R_w$ for all $w = a_1 + a_2\rho$ with $a_1 = 0 \neq a_2$ and hence $\sigma(T) = \sigma(\tau) = -c(k^\times)^2$. If $b_1 \neq 1$, then it is easy to see that $\sigma(T) = 2(1 - b_1)(k^\times)^2$. Consequently

$$\left(\begin{pmatrix} b_1 & b_2 \\ cb_2 & b_1 \end{pmatrix}\right) = -2c(1 - b_1)(k^\times)^2$$

for all $\{b_1, b_2\} \subseteq k$ with $b_1^2 - cb_2^2 = 1$ and $b_1 \neq 1$. Note that σ maps

$$SO(K/k) = \left\{\begin{pmatrix} a_1 & a_2 \\ ca_2 & a_1 \end{pmatrix} \middle| a_1, a_2 \in k \text{ and } a_1^2 - ca_2^2 = 1\right\}$$

onto $k^\times/(k^\times)^2$. For $\sigma((\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix})) = -c(k^\times)^2$ and $c \notin (k^\times)^2$. Moreover, if $-c \in (k^\times)^2$, then $\sigma(O(K/k)) = \sigma(SO(K/k)) = k^\times/(k^\times)^2$ and the assertion is proved.

If $m = 3$, then $H = \langle -I_V \rangle \times K$, $K \cong \mathrm{PGL}(2, q)$ and $\Omega \cong \mathrm{PSL}(2, q)$ by [**29**, Proposition 24.1].

If $m = 4$ and $\mathrm{ind}(V/k) = 1$, then $H' = \Omega \cong \mathrm{PSL}(2, q^2)$ and $H = \langle -I_V \rangle \times \mathrm{Ker}(\sigma)$ where $\mathrm{Ker}(\sigma)$ is isomorphic to $\mathrm{PSL}(2, q^2)$ extended by a field automorphism of order 2 (cf. [**29**, Proposition 24.12]).

To discuss the case $m = 4$ and $\mathrm{ind}(V/k) = 2$, we shall apply the methods of [**29**, Lemma 24.10 and Proposition 24.11]. Thus let W, W' be two vector spaces over k of dimension 2 with bases $B = \{w_1, w_2\}$ and $B' = \{w_1', w_2'\}$ respectively and let $V = W \otimes_k W'$ so that $B^* = \{v_{ij} = w_i \otimes w_j' \mid 1 \leqslant i,\ j \leqslant 2\}$ is a basis of V/k. Define $g(\sum_{i,j=1}^{2} c_{ij}v_{ij}) = c_{11}c_{22} - c_{12}c_{21}$ for all $\{c_{ij} \mid 1 \leqslant i, j \leqslant 2\} \subseteq k$. Then V/k becomes a nonsingular orthogonal vector space of index 2 and $V = (kv_{11} + kv_{22}) \perp (k(-v_{12}) + kv_{21})$ is an orthogonal sum of hyperbolic planes. Let $A \in \mathrm{GL}(W/k)$ and $A' \in \mathrm{GL}(W'/k)$, so that $g((A \otimes A')(v)) = \det(A)\det(A')g(v)$ for all $v \in V$ as in [**29**, Lemma 24.10]. Moreover, as in this reference, if $H = \{(A, A') \mid A \in \mathrm{GL}(W/k),\ A' \in \mathrm{GL}(W'/k)$ and $\det(A)\det(A') = 1\}$, then the mapping γ such that $\gamma((A, A')) = A \otimes_k A'$ is an epimorphism of H onto $SO(V/k)$ with $\mathrm{Ker}(\gamma) = \{(aI_W, a^{-1}I_{W'}) \mid a \in k^\times\}$ and such that $\gamma(\mathrm{SL}(W/k) \times \mathrm{SL}(W'/k)) = \Omega(V/k) \cong \mathrm{SL}(2, q) * \mathrm{SL}(2, q)$. Let T, T' have matrix $(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix})$ with respect to the bases B, B' of W/k and W'/k respectively and let x and y have matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with respect to the bases B^* of V/k. Then

$$\gamma(T \otimes T') = x \in \mathcal{I}(SO(V/k) - \Omega(V/k)),$$

$y \in \mathcal{I}(O(V/k) - SO(V/k))$, $xy = yx$, $O(V/k) = \Omega(V/k)\langle x, y \rangle$, y interchanges the two 2-components of $\Omega(V/k)$ and $SO(V/k) = \Omega(V/k)\langle x \rangle$ normalizes the two 2-components of $\Omega(V/k)$.

If $m = 5$, then $\Omega \cong \mathrm{PSp}(4, q)$ by [**29**, Proposition 24.13].

If $m = 6$ and $\mathrm{ind}(V/k) = 3$, then $D(V) = -1(k^\times)^2$ and $\Omega(V/k)$ is isomorphic to $\mathrm{SL}(4, k)/A$ where A is the unique central subgroup of $\mathrm{SL}(4, k)$ of order 2 by [**29**, Proposition 24.15].

If $m = 6$ and $\operatorname{ind}(V/k) = 2$, then $D(V) = -c(k^\times)^2$ where $c \in k^\times - (k^\times)^2$ and $\Omega(V/k)$ is isomorphic to $SU(4, k)/B$ where B is the unique central subgroup of $SU(4, k)$ of order 2 by [**29**, Proposition 24.15].

For the remainder of this section, assume that $m \geqslant 7$. Thus $K' = H' = \Omega = \operatorname{Ker}(\sigma) \cap K$, $C_H(\Omega) = \langle -I_V \rangle$, σ maps K onto $k^\times/(k^\times)^2$ and $H/\Omega \cong E_4$.

Let $u \in G - Z$ be such that $u^2 = \gamma I_V$ for some $\gamma \in k^\times$, so that $|\bar{u}| = 2$. Let $U = \langle u, Z \rangle$ and $M = N_H(U)$. Thus U is abelian, $\bar{U} = \langle \bar{u} \rangle \cong U/Z$, $C_G(u) = C_G(U)$, M is the inverse image in H of $C_{\bar{H}}(\bar{u})$, and $|M/C_H(U)| \leqslant 2$.

First assume that U is not cyclic. Then $U = Z \times \langle w \rangle$ where $w \in \mathfrak{I}(G - Z)$, $w \neq -I_V$, $C_G(U) = C_G(w)$, $w^M \subseteq \{w, (-I_V)w\}$, $|w^M| = |M/C_H(U)| \leqslant 2$ and $r_w^2 = 1$.

Suppose that $r_w = 1$. Then $w \in H$, $U \cap H = \langle -I_V, w \rangle$, $U \leqslant Z * H$, $V = V^+ \perp V^-$ where $V^+ = C_V(w)$, $V^- = \{v \in V \mid w(v) = -v\} = [V, w]$ and the restrictions of f to V^+/k and V^-/k yield nonsingular orthogonal vector spaces such that $D(V/k) = D(V^+/k)D(V^-/k)$. Thus

$$C_G(U) \cong \{(w_1, w_2) \in (GO(V^+/k)) \times (GO(V^-/k)) \mid r_{w_1} = r_{w_2}\},$$
$$C_H(U) \cong O(V^+/k) \times O(V^-/k),$$
$$H = \Omega C_H(w) \quad \text{and} \quad w^H = w^\Omega.$$

Also $w \in \Omega$ if and only if $\dim(V^-/k)$ is even and $D(V^-/k) \in (k^\times)^2$ by [**29**, Lemma 20.6]. If $|M/C_H(U)| = 2$, then there is an involution $g \in M$ such that g: $V^+ \leftrightarrow V^-$, $w^g = (-I_V)w$ and $M = C_H(U)\langle g \rangle \cong O(V^+/k) \operatorname{wr} Z_2$. Also [**25**, Proposition 5] implies that if m is odd, then $G = HC_G(U)$ and if m is even, then $G = HC_G(U)$ if $\dim(V^+/k)$ is even and $|G\colon (HC_G(U))| = 2$ if $\dim(V^+/k)$ is odd. Also if $h \in G = GO(V/k)$, then $w_1 = w^h \in \mathfrak{I}(H)$, $C_V(w_1) = h^{-1}(V^+)$, $D(C_V(w_1)/k) = (r_h)^{\dim(V^+/k)} D(V^+/k)$, $[V, w_1] = h^{-1}(V^-)$ and $D([V, w_1]/k) = (r_h)^{\dim(V^-/k)} D(V^-/k)$. If $x, y \in \mathfrak{I}(H)$, then it is clear that the following three conditions are equivalent: (i) $x \sim y$ in $H = O(V/k)$, (ii) $x \sim y$ via Ω; (iii) $\dim(C_V(x)/k) = \dim(C_V(y)/k)$ and $D(C_V(x)/k) = D(C_V(y)/k)$. Also for any integer p with $1 \leqslant p < m$ and any element $a \in k^\times/(k^\times)^2$ it is clear that there is an involution $z \in H - Z$ such that $\dim(C_V(z)/k) = p$ and $D(C_V(z)/k) = a$.

Suppose that $r_w = -1$. Then $w \notin H$, $U \leqslant Z * H$ if and only if $q \equiv 1 \pmod 4$, $V^+ = C_V(w)$ and $V^- = \{v \in V \mid w(v) = -v\} = [V, w]$ are totally isotropic subspaces of V with $V = V^+ \oplus V^-$, m is even, $\dim(V^+/k) = \dim(V^-/k) = \frac{m}{2}$ and $\operatorname{ind}(V) = \frac{m}{2}$.

Suppose that $q \equiv 1 \pmod 4$ and let $\nu \in k^\times$ be such that $\nu^2 = -1$. Also, as in §3B, choose bases $\{v_i \mid 1 \leqslant i \leqslant \frac{m}{2}\}$ and $\{v_{i+m/2} \mid 1 \leqslant i \leqslant \frac{m}{2}\}$ of V^+/k and V^-/k respectively such that $f(v_1, v_{j+m/2}) = \delta_{ij}$ for all $1 \leqslant i, j \leqslant \frac{m}{2}$. Set $H_i = kv_i + kv_{i+m/2}$ so that H_i is a hyperbolic plane for all $1 \leqslant i \leqslant \frac{m}{2}$ and $V = H_1 \perp \cdots \perp H_{m/2}$. Then $U \cap H = \langle (\nu I_V)w \rangle$ where $((\nu I_V)w)^2 = -I_V$, $((\nu I_V)w)(v_i) = \nu v_i$ and $((\nu I_V)w)(v_{i+m/2}) = -\nu v_{i+m/2}$ for all $1 \leqslant i \leqslant \frac{m}{2}$. Thus $\sigma((\nu I_V)w) = \nu^{m/2}(k^\times)^2$ by [**29**, Example 20.4 and Lemma 20.6]. Hence $U \leqslant Z * \Omega$ if and only if $\nu^{m/2} \in (k^\times)^2$.

As in §3B, we have $C_H(U) \cong GL(V^+/k)$, $M = C_H(U)\langle g \rangle$ where $g \in \mathfrak{I}(H)$ and conjugation by g induces transpose inverse on $C_H(U) \cong GL(V^+/k)$, $G = HC_G(U)$, $w^G = w^H$ and all involutions with $r_w = -1$ are conjugate under H. Moreover, when

m is even and $\operatorname{ind}(V) = \frac{m}{2}$, the existence of complementary totally isotropic subspaces of V each of dimension $\frac{m}{2}$ by [29, Proposition 9.15] implies the existence of such subgroups U of G.

Next assume that U is cyclic. Thus $U = \langle Z, w\rangle$ where $w^2 = \gamma I_V$ for some $\gamma \notin (k^\times)^2$. Thus $r_w = \pm\gamma$ since $r_u^2 = \gamma^2$ and m is even. Also let $K = k(\rho)$, ρ, τ, V/k, N: $K^\times \to k^\times$, etc. be as in §3A and, for any $v_1, v_2 \in V$, set $f_0(v_1, v_2) = \rho f(v_1, v_2) + f(v_1, w(v_2))$. Consequently

$$f_0(v_2, v_1) = \rho f(v_1, v_2) + \gamma r_w^{-1} f(v_1, w(v_2)).$$

Suppose that $r_w = \gamma$. Then $f_0(v_2, v_1) = f_0(v_1, v_2)$ for all $v_1, v_2 \in V$ and f_0: $V \times V \to K$ is a nonsingular K-bilinear symmetric scalar product on V/K. Also $U \nleq Z * H$ since $r_w = \gamma \notin (k^\times)^2$. It readily follows that $C_G(U) = \{x \in GO(V/K) \mid r_x \in k^\times\}$, $C_H(U) = O(V/K)$, $G = HC_G(U)$ by [25, Proposition 5(b)] since $k^\times \leqslant (K^\times)^2$ and $w^G = w^H$. Let $\alpha = a + b\rho$ be any element of $D(V/K)$. Then [29, Proposition 8.9] implies that V/K has a basis $B = \{v_i \mid 1 \leqslant i \leqslant \frac{m}{2}\}$ such that

$$f_0(v_i, v_j) = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } 1 \leqslant i = j < \dfrac{m}{2}, \\ \alpha & \text{if } i = j = \dfrac{m}{2}. \end{cases}$$

Consequently $B_1 = \{v_i, v_i\rho \mid 1 \leqslant i \leqslant \frac{m}{2}\}$ is a basis of V/k such that

$$f(v_i, v_j) = f(v_i, v_j\rho) = f(v_i\rho, v_j\rho) = 0 \quad \text{if } i \neq j,$$
$$f(v_i, v_i) = f(v_i\rho, v_i\rho) = 0 \quad \text{and} \quad f(v_i, v_i\rho) = 1 \quad \text{if } 1 \leqslant i < \tfrac{m}{2},$$
$$f(v_{m/2}, v_{m/2}) = b, \qquad f(v_{m/2}\rho, v_{m/2}\rho) = \gamma b$$

and $f(v_{m/2}, v_{m/2}\rho) = a$. Calculating the discriminant of V/k using B_1 yields $(-1)^{m/2}N(\alpha) \in D(V/k)$. However $(K^\times)^2$ is the inverse image of $(k^\times)^2$ in $K^\times$ under N. Thus $D(V/K) = \alpha(K^\times)^2$ is uniquely determined. Clearly $w(v_i) = v_i\rho$ and $w(v_i\rho) = v_i\gamma$ for all $1 \leqslant i \leqslant \frac{m}{2}$. Let $x \in \Gamma L(V/K)$ be induced by τ with respect to the basis B of B/K. Then $xw = (-I_V)wx$, $x \in \Gamma O(V/K)$, $f(x(v_1), x(v_2)) = -f(v_1, v_2)$ for all $v_1, v_2 \in V$, $x \in G$, $|x| = 2$ and $N_G(U) = C_G(U)\langle x\rangle$. Suppose that X is a cyclic subgroup of G such that $Z \leqslant X$, $|X/Z| = 2$ and such that $X - Z$ contains an element with $z^2 = r_z I_V$. Then $r_z \notin (k^\times)^2$ since X is cyclic and we may assume that $r_z = \gamma$. The existence of the basis B_1 of V/k and the discussion above imply that X and U are conjugate via H. Moreover, when m is even, such subgroups X of G always exist. To see this, let $\gamma \in k^\times - (k^\times)^2$ and let $K = k(\rho)$ with $\rho^2 = \gamma$, τ, N: $K^\times \to k^\times$, etc. be as above. Let W/K be a vector space of dimension $\frac{m}{2}$ and let g: $W \times W \to K$ be a nonsingular orthogonal scalar product on W/K such that if $\alpha \in D(W/K)$, then $(-1)^{m/2}N(\alpha) \in D(V/k)$. Since $K = k + k\rho$, we have $g = \rho g_1 + g_2$ where g_i: $W \times W \to k$ is a k-bilinear nonsingular orthogonal scalar product on W/k for $i = 1, 2$. Also $g_1(v_1\rho, v_2) = g_1(v_1, v_2\rho)$ and hence $g_1(v_1\rho, v_2\rho) = \gamma g_1(v_1, v_2)$ for all $v_1, v_2 \in W$. Also, as above, we conclude that $D((W/k, g_1)) =$

$D(V/k)$. Since $\dim(W/k) = m$, we conclude that $(W/k, g_1)$ and $(V/k, f)$ are isometric. Then the existence of such subgroups X of G follows as in §3B.

Suppose that $r_w = -\gamma$. Then $f_0(v_2, v_1) = -\tau(f_0(v_1, v_2))$ and $\rho^{-1}f_0$: $V \times V \to K$ is a nonsingular Hermitian scalar product on V/K. Clearly $U \leqslant Z * H$ if and only if $q \equiv -1 \pmod 4$ and since the multiplicators of all elements of $GU(V/K)$ lie in $k^\times$, it follows that $C_G(U) = GU(V/K)$, $G = HC_G(U)$ and $w^H = w^H$. Also [**29**, Proposition 8.8] implies that V/K has a basis $B = \{v_i \mid 1 \leqslant i \leqslant \frac{m}{2}\}$ such that $\rho^{-1}f_0(v_i, v_j) = \delta_{ij}$ for all $1 \leqslant i, j \leqslant \frac{m}{2}$. Consequently $B_1 = \{v_i, v_i\rho \mid 1 \leqslant i \leqslant \frac{m}{2}\}$ is a basis of V/k such that $f(v_i, v_j) = f(v_i, v_j\rho) = f(v_i\rho, v_j\rho) = 0$ for all $i \neq j$ with $1 \leqslant i, j \leqslant \frac{m}{2}$, $f(v_i, v_i) = 1$, $f(v_i\rho, v_i\rho) = r_w$ and $f(v_i, v_i\rho) = 0$ for all $1 \leqslant i \leqslant \frac{m}{2}$. Thus $(-\gamma)^{m/2} \in D(V/k)$, $\operatorname{ind}(V/k) = \frac{m}{2}$ if $\frac{m}{2}$ is even and $\operatorname{ind}(V/k) = \frac{m}{2} - 1$ if $\frac{m}{2}$ is odd. Let $x \in \Gamma L(V/K)$ be induced by τ with respect to the basis B of V/K. Then $xw = (-I_V)wx$, $x \in H = O(V/k)$, $|x| = 2$ and $N_G(U) = C_G(U)\langle x\rangle \leqslant \Gamma U(V/K)$. It is easy to see that all cyclic subgroups X of G such that $Z \leqslant X$, $|X/Z| = 2$ and $X - Z$ contains an element z with $z^2 = -r_z I_V$ are conjugate under H. Finally, when m is even, $\operatorname{ind}(V/k) = \frac{m}{2}$ if $\frac{m}{2}$ is even and $\operatorname{ind}(V/k) = \frac{m}{2} - 1$ if $\frac{m}{2}$ is odd, such subgroups X of G always exist. To see this, let $\gamma \in k^\times - (k^\times)^2$, $K = k(\rho)$ with $\rho^2 = \gamma$, τ, N: $K^\times \to k^\times$, etc. be as above. Also let W/K be a vector space of dimension $\frac{m}{2}$ and let g: $W \times W \to K$ be a nonsingular skew-Hermitian scalar product on V/K. Let $g = \rho g_1 + g_2$ where g_i: $W \times W \to k$ is a k-bilinear nonsingular scalar product on W/k for $i = 1, 2$. Then $g_1(v_1, v_2) = g_1(v_2, v_1)$, $g_1(v_1\rho, v_2) = -g(v_1, v_2\rho)$ and hence $g_1(v_1\rho, v_2\rho) = -\gamma g_1(v_1, v_2)$ for all $v_1, v \in W$. As above, we conclude that $(W/k, g_1)$ is a nonsingular orthogonal vector space with $\dim(W/k) = m$ and $(-\gamma)^{m/2} \in D((W/k, g_1))$. Then by hypothesis, $D(V/k) = D(W/k)$. Thus $(V/k, f)$ and $(W/k, g_1)$ are isometric by [**29**, Proposition 8.9] and the existence of such subgroups X of G follows as in §3B.

This concludes §3.

4. Applications of the theory of linear algebraic groups. In this section, we apply the theory of linear algebraic groups to survey the conjugacy classes of involutions and their centralizers in various Chevalley groups and their automorphism groups over finite fields of odd order. In some cases, since the machinery is at hand and for completeness, we derive more information than is actually required in this paper. However all of these results are utilized in [**26**] and are of independent interest.

We begin this section with some results on endomorphisms of linear algebraic groups that are slight reformulations of some results in [**33**, **35**]. Then we combine these results with the methods and results of [**30**, **35**]. The remainder of this section presents our applications of this material to the Chevalley groups over finite fields of odd order.

Our first results concern the following situation.

$\overline{G}$ is a linear algebraic group and σ is an endomorphism of $\overline{G}$ onto $\overline{G}$ such that $\overline{G}_\sigma = C_{\overline{G}}(\sigma) = \{g \in \overline{G} \mid \sigma(g) = g\}$ is finite.

The basic results about the structure of $\overline{G}_\sigma$, conjugacy, etc. are contained in [**35**, §§10–15]. The first result in this context that we specifically mention is a consequence of [**35**, Corollary 10.9].

LEMMA 4.1. *Let $\overline{G}$ be a connected linear algebraic group and let σ be an endomorphism of $\overline{G}$ onto $\overline{G}$ such that $\overline{G}_\sigma$ is finite and* $\mathrm{Ker}(\sigma) = 1$. *Let $H = \overline{G}\langle\sigma\rangle$ (the semidirect product) and let $g \in \overline{G}$. Then $g\sigma$ and σ are conjugate via an element of $\overline{G}$.*

LEMMA 4.2. *Let $\overline{G}$ be a connected linear algebraic group and let $\sigma\colon \overline{G} \to \overline{G}$ be an endomorphism of $\overline{G}$ such that σ is an automorphism of the underlying group. Let n be a positive integer, set $F = \overline{G}_{\sigma^n} = \{g \in \overline{G} \mid \sigma^n(g) = g\}$ and assume that F is finite. Thus $\overline{G}_\sigma = \{g \in \overline{G} \mid \sigma(g) = g\}$ is finite, $\sigma_0 = \sigma|_F$ induces an automorphism of F and $\sigma_0^n = 1$. Set $H = F\langle\sigma_0\rangle$ (the semidirect product). Let $f\sigma_0 \in H$ with $f \in F$ and $|f\sigma_0| = n$. Then $f\sigma_0$ and σ_0 are conjugate via an element of F.*

PROOF. Since $(f\sigma_0)^n = (f\sigma(f)\sigma^2(f) \cdots \sigma^{n-1}(f))\sigma_0^n = 1$, we have $f\sigma(f) \cdots \sigma^{n-1}(f) = 1$. By [**35**, Theorem 10.1], there is an element $x \in \overline{G}$ such that $x\sigma(x)^{-1} = f$. Hence $\sigma(x^{-1}) = x^{-1}f$, $\sigma^2(x^{-1}) = x^{-1}f\sigma(f)$, etc. and $\sigma^n(x^{-1}) = x^{-1}f\sigma(f) \cdots \sigma^{n-1}(f) = x^{-1}$. Thus $x \in F$, $x\sigma_0x^{-1} = x\sigma(x)^{-1}\sigma_0 = f\sigma_0$ and we are done.

Suppose that A is a not necessarily finite group and that σ is an endomorphism of A. Then $H^1(\sigma, A)$ denotes A modulo the equivalence relation: $a \sim b$ if $a = cb\sigma(c)^{-1}$ for some $c \in A$. As an example, if σ is the identity on A, then $H^1(\sigma, A)$ is the set of conjugacy classes of A.

For the next two results, as above, we let $\overline{G}$ be a linear algebraic group and let σ be an endomorphism of $\overline{G}$ onto $\overline{G}$ such that $\overline{G}_\sigma$ is finite. Consequently $\sigma(\overline{G}^0) = \overline{G}^0$ by [**36**, §1.13, Proposition 2(b)] (where $\overline{G}^0$ denotes the irreducible component of $\overline{G}$ that contains the identity of $\overline{G}$). For convenience of the reader, we restate [**33**, I, 2.6].

LEMMA 4.3. *Suppose that $\overline{G}$ is connected and that A is a (closed) subgroup of $\overline{G}$ fixed by σ. Then the natural map from $H^1(\sigma, A)$ into $H^1(\sigma, A/A^0)$ is bijective.*

The second result in this context that we present is a slight refinement of [**33**, I, 3.4(b)].

LEMMA 4.4. *Suppose that $m \in \overline{G}$ is such that $\sigma(m) = m$ and $\overline{G} = \langle\overline{G}^0, m\rangle = \overline{G}^0\langle m\rangle$. Let $M = \mathrm{ccl}_G(m)$ and let $A = C_{\overline{G}^0}(m)$, so that $\sigma(A) \leqslant A$, $A = C_{\overline{G}}(m) \cap \overline{G}^0$ is a closed subgroup of $\overline{G}$, A/A^0 is a finite group and $C_{\overline{G}}(m) = A\langle m\rangle$. Let $\mathfrak{A}$ be a set of representatives in A of the cosets in a representative choice from the equivalence classes of $H^1(\sigma, A/A^0)$ and suppose that $\mathfrak{A} = \{\alpha_i \in A \mid 1 \leqslant i \leqslant n\}$ where $|\mathfrak{A}| = n$. For each $\alpha_i \in \mathfrak{A}$ with $1 \leqslant i \leqslant n$, choose (by* [**33**, I, *Theorem* 2.2]) *an element $g_i \in \overline{G}^0$ such that $g_i\sigma(g_i)^{-1} = \alpha_i$. Then the following three conditions hold.*

(a) *$m^{g_i} \in \overline{G}_\sigma$ for all $1 \leqslant i \leqslant n$;*

(b) *$\{m^{g_i} \mid 1 \leqslant i \leqslant n\}$ is a set of representatives for the orbits of $\overline{G}_\sigma$ on $M_\sigma = \{x \in M \mid \sigma(x) = x\}$; and*

(c) *$C_{\overline{G}_\sigma}(m^{g_i}) = C_{(\overline{G}^0)_\sigma}(m^{g_i})\langle m^{g_i}\rangle$ and*

$$C_{(\overline{G}^0)_\sigma}(m^{g_i}) = \left(C_{\overline{G}^0}(m^{g_i})\right)_\sigma = (A^{g_i})_\sigma = \left(\langle x \in A \mid \alpha_i\sigma(x)\alpha_i^{-1} = x\rangle\right)^{g_i} = \left(A_{\beta_i\sigma}\right)^{g_i}$$

where β_i denotes the inner automorphism of $\overline{G}$ induced by α_i^{-1}, for all $1 \leqslant i \leqslant n$.

PROOF. Note that $C_{\bar{G}}(m)$ and hence $A = C_{\bar{G}}(m) \cap \bar{G}^0$ are closed subgroups of $\bar{G}$ by [**6**, I, (1.7), Proposition (c)]. Clearly $\sigma(m^{g_i}) = \sigma(g_i)^{-1}m\sigma(g_i) = g_i^{-1}\alpha_i m\alpha_i^{-1}g_i = m^{g_i}$ and (a) holds. Since $C_{\bar{G}}(m) = A\langle m\rangle$, it follows that $\bar{G}^0$ acts transitively (by conjugation) on M. Suppose that $m^{g_i} = m^{g_j h}$ for some $h \in \bar{G}_\sigma$ and $1 \leqslant i, j \leqslant n$. Since $\bar{G}_\sigma = (\bar{G}^0)_\sigma\langle m^{g_j}\rangle$, we may assume that $h \in (\bar{G}^0)_\sigma$. Then $g_j h g_i^{-1} \in A = C_{\bar{G}^0}(m)$ and $g_j h = a g_i$ for some $a \in A$. Thus $(g_j h)\sigma(g_j h)^{-1} = g_j\sigma(g_j)^{-1} = \alpha_j = a g_i\sigma(g_i)^{-1}\sigma(a)^{-1} = a\alpha_i\sigma(a)^{-1} \sim \alpha_i$ and hence $i = j$. Now [**33**, I, 3.4(b)] yields (b). Choose any i with $1 \leqslant i \leqslant n$. Clearly

$$C_{\bar{G}_\sigma}(m^{g_i}) = C_{(\bar{G}^0)_\sigma}(m^{g_i})\langle m^{g_i}\rangle \quad\text{and}\quad C_{(\bar{G}^0)_\sigma}(m^{g_i}) = \left(C_{\bar{G}^0}(m^{g_i})\right)_\sigma = (A^{g_i})_\sigma.$$

Let $x \in A$. Since the following three conditions are equivalent: (i) $x^{g_i} \in \bar{G}_\sigma$, (ii) $g_i\sigma(g_i)^{-1}\sigma(x)\sigma(g_i)g_i^{-1} = x$ and (iii) $\alpha_i\sigma(x)\alpha_i^{-1} = x$, we also have (c). The proof of this lemma is now complete.

Next, we introduce some (standard) notation and results from [**30**, **7**].

Let p be a prime integer, let K be an algebraic closure of $Z/(pZ)$, let $\mathfrak{G}$ denote a complex semisimple Lie algebra and let π denote a faithful representation of $\mathfrak{G}$. Let $\bar{G} = G_{\pi,K}$ denote the Chevalley group obtained from the triple $(\mathfrak{G}, \pi, K)$ (cf. [**7**, §3]). In this construction and notation, $\bar{G}$ is a (connected) semisimple linear algebraic group, $\bar{B}$ is a Borel subgroup of $\bar{G}$, $\bar{H}$ is a maximal torus of $\bar{G}$, $\bar{U} = \bar{B}_u$ (the unipotent radical of $\bar{B}$), $\bar{B} = \bar{U}\bar{H}$, $\bar{N} = N_{\bar{G}}(\bar{H})$, $W = \bar{N}/\bar{H}$, etc. (cf. [**34**, §5; **7**, §3]). Clearly, since $\bar{H}$ is abelian, W acts on $\bar{H}$ by conjugation.

Let $P(\pi)$ denote the set of weights of π and let Γ_π denote the Z-module generated by $P(\pi)$. Then $\bar{H}$ can be described as follows.

(4.1) for $\chi \in \operatorname{Hom}(\Gamma_\pi, K^\times)$, associate to χ the automorphism of V, (the representation space of π), defined by: $h(\chi)v = \chi(m)v$ for each $v \in V_m$ and $m \in P(\pi)$.

Then the mapping $\chi \to h(\chi)$ is an isomorphism of $\operatorname{Hom}(\Gamma_\pi, K^\times)$ onto $\bar{H}$.

Moreover, letting Φ denote the root system of $\mathfrak{G}$, we have

(4.2) if $\chi \in \operatorname{Hom}(\Gamma_\pi, K^\times)$, $\alpha \in \Phi$ and $u \in K$, then $h(\chi)x_\alpha(u)h(\chi)^{-1} = x_\alpha(\chi(\alpha)u)$.

For $\alpha \in \Phi$, set $\bar{\mathfrak{X}}_\alpha = \langle x_\alpha(u) \mid u \in K\rangle$, so that $\bar{H} \leqslant N_{\bar{G}}(\bar{\mathfrak{X}}_\alpha)$ and the mapping $u \to x_\alpha(u)$ is an isomorphism of $(K, +)$ onto $\bar{\mathfrak{X}}_\alpha$. Then $\bar{U} = \langle \bar{\mathfrak{X}}_\alpha \mid \alpha \in \Phi^+\rangle$ and $\bar{U}^- = \langle \bar{\mathfrak{X}}_\alpha \mid \alpha \in \Phi^-\rangle$ is the unipotent radical of the Borel subgroup that is "opposite" to $\bar{B}$ relative to $\bar{H}$ (cf. [**6**, IV, §14]).

Note that every semisimple element of $\bar{G}$ is conjugate in $\bar{G}$ to an element of $\bar{H}$ by [**36**, §§2.12 and 2.13] and W controls the $\bar{G}$-fusion of elements of $\bar{H}$ by [**35**, 6.3]. Also if $\chi \in \operatorname{Hom}(\Gamma_\pi, K^\times)$, then [**30**, Proposition 1, (i)–(iii)] holds and

$$C_{\bar{G}}(h(\chi))/\left(C_{\bar{G}}(h(\chi))^0\right) \cong W_\chi/\left(W_\chi^0\right)$$

where W_χ and W_χ^0 are as in [**30**, §2].

Let E denote the Q-module generated by Φ and fix a W-invariant inner product $(*, *)$ on E and for any Z-submodule Γ of E set $\Gamma^\perp = \{x \in E \mid (x, y) \in Z$ for all $y \in \Gamma\}$. Also, as in [**30**, §2], ad denotes the adjoint representation of $\mathfrak{G}$ and sc denotes the simply connected representation of $\mathfrak{G}$. Thus $\Gamma_{ad} \leqslant \Gamma_\pi \leqslant \Gamma_{sc} \leqslant E$, Γ_{ad} is the Z-submodule of E generated by Φ, Γ_{sc} is the Z-submodule of E generated by all weights of $\mathfrak{G}$ and W stabilizes Γ_{sc}, Γ_π and Γ_{ad}. Also $\Gamma_{sc}^\perp$, $\Gamma_\pi^\perp$ and $\Gamma_{ad}^\perp$ are W-stable and $\Gamma_{sc}^\perp \leqslant \Gamma_\pi^\perp \leqslant \Gamma_{ad}^\perp$. With this action of W on Γ_π and with trivial action of W on $K^\times$, the isomorphism of (4.1) becomes a W-isomorphism.

As a standard, if $0 \neq \beta \in E$, set $\beta^* = 2\beta/(\beta, \beta)$ and for $t \in K^\times$ and $\alpha \in \Phi$, let

(4.3) $h_\alpha(t) = h(\chi)$ where χ is the unique element of $\mathrm{Hom}(\Gamma_\pi, K^\times)$ such that $\chi(m) = t^{(m,\alpha^*)}$ for any $m \in \Gamma_\pi$.

We shall frequently be concerned with the following situation.

τ is an (algebraic group) endomorphism of $\overline{G} = G_{\pi,K}$ onto itself that leaves $\overline{H}$ and $\overline{U}$ invariant and such that (i) $\overline{G}_\tau$ is finite or (ii) τ is an automorphism of $\overline{G}$ as an algebraic group.

Suppose that τ satisfies (i). Then, as noted above, [**35**, §§10–15] presents many basic results in this situation. Note also that if γ is any (algebraic group) endomorphism of $\overline{G}$ onto $\overline{G}$ such that $\overline{G}_\gamma$ is finite then γ is conjugate via $\mathrm{Inn}(\overline{G})$ to an endomorphism of $\overline{G}$ that leaves $\overline{H}$ and $\overline{U}$ invariant by [**35**, Corollary 10.10].

Next suppose that τ, as above, satisfies (ii). Then [**35**, §§7–9] presents many basic results in this situation. In particular, $C_{\overline{G}}(\tau)$ is closed, $C_{\overline{G}}(\tau)^0$ is reductive and contains every unipotent element of $C_{\overline{G}}(\tau)$ and the structure of $C_{\overline{G}}(\tau)/C_{\overline{G}}(\tau)^0$ is given by [**35**, Lemma 9.2]. Also, if $\overline{G}$ is simply connected, then $C_{\overline{G}}(\tau) = C_{\overline{G}}(\tau)^0$ by [**35**, Theorem 8.2 and Corollary 9.4]. Clearly τ fixes $\overline{U}^-$ and $\overline{N}$ and $C_{\overline{G}}(\tau)^0 = \langle \overline{U}_\tau, \overline{H}_\tau, (\overline{U}^-)_\tau \rangle$ by [**35**, Lemma 9.2(a) and the proof of Theorem 8.2] and $C_{\overline{G}}(\tau) = \langle \overline{U}_\tau, \overline{N}_\tau \rangle$ by uniqueness in [**35**, 6.3]. For example, when τ is the inner automorphism of $\overline{G}$ induced by the element $h(\chi) \in \overline{H}$ where $\chi \in \mathrm{Hom}(\Gamma_\pi, K^\times)$, then the structures of $C_{\overline{G}}(\tau)$, $C_{\overline{G}}(\tau)^0$ and $C_{\overline{G}}(\tau)/(C_{\overline{G}}(\tau)^0)$ are readily apparent (cf. [**35**, 8.3(c); **30**, Proposition 1; **30**, §8]). Note also that if γ is any semisimple automorphism of $\overline{G}$, then γ is conjugate via $\mathrm{Inn}(\overline{G})$ to a semisimple automorphism of $\overline{G}$ that leaves $\overline{H}$ and $\overline{U}$ invariant by [**35**, Theorem 7.5].

Let λ denote the element of $\mathrm{Aut}(K)$ such that $\lambda(u) = u^p$ for all $u \in K$, let n be an arbitrary positive integer and set $q = p^n$, $\sigma = \lambda^n$ and $k = C_K(\lambda)$. Then $C_K(\lambda)$ is the prime subfield of K and k is the unique subfield of K of order q.

Since $\overline{G}$, $\overline{H}$, $\overline{N}$, $\overline{U}$ and $\overline{U}^-$ are all defined over $C_K(\lambda)$, both λ and σ induce, in a natural way, endomorphisms of $\overline{G}$ that leave invariant $\overline{H}$, $\overline{N}$, $\overline{U}$, $\overline{U}^-$ and $\overline{B} = \overline{U}\overline{H}$ and that we shall also denote by λ and σ, respectively. Thus $\lambda^n = \sigma$ as endomorphisms of $\overline{G}$, λ and σ are automorphisms of $\overline{G}$ as a group, $C_{\overline{G}}(\lambda) = \overline{G}_\lambda \leqslant C_{\overline{G}}(\sigma) = \overline{G}_\sigma$, $\overline{G}_\sigma$ is finite and λ-invariant, etc. Let $G = \langle \overline{U}_\sigma, (\overline{U}^-)_\sigma \rangle$. Then G is the Chevalley group associated with the triple $(\mathfrak{G}, \pi, k)$, $G = O^{p'}(\overline{G}_\sigma)$, $\overline{G}_\sigma = G(\overline{H}_\sigma)$, $\overline{N}_\sigma = (\overline{H}_\sigma)(\overline{N}_\lambda)$, $\overline{N} = \overline{H}(\overline{N}_\lambda)$, $\overline{H}_\sigma$ is the image of $\mathrm{Hom}(\Gamma_\pi, k^\times)$ under the isomorphism of (4.1), $\mathrm{Hom}(\Gamma_k, k^\times) = \{\chi \in \mathrm{Hom}(\Gamma_\pi, K^\times) \mid \chi^{q-1} = 1\}$ and $\overline{H}_\sigma = \{h \in \overline{H} \mid h^{q-1} = 1\}$, etc. (cf. [**35**, §12; **7**, §3]).

As on [30, p. F-5], we fix a generator $\mathcal{K}$ of $k^\times$ and define a homomorphism $\Gamma_\pi^\perp \to \operatorname{Hom}(\Gamma_\pi, k^\times)$ by

(4.4) $$\lambda \to \chi_\lambda \quad \text{where } \chi_\lambda(\xi) = \mathcal{K}^{(\lambda,\xi)} \text{ for } \xi \in \Gamma_\pi.$$

Clearly this yields an exact sequence of W-modules

(4.5) $$1 \to (q-1)\Gamma_\pi^\perp \to \Gamma_\pi^\perp \to \operatorname{Hom}(\Gamma_\pi, k^\times) \to 1.$$

Next we impose the additional assumption that $\mathfrak{G}$ is a simple Lie algebra. Let $B = \{\alpha_1, \ldots, \alpha_l\}$ be a set of simple roots and let α_0 denote the highest root of Φ. Then $\overline{G}$ is a simple linear algebraic group and $\alpha_0 = \sum_{i=1}^l m_i\alpha_i$ where each m_i is a positive integer. As in [30, §3], set

(4.6) $$\mathfrak{D} = \{\xi \in E \mid 0 \leqslant (\alpha_i, \xi) \text{ for all } 1 \leqslant i \leqslant l \text{ and } (\alpha_0, \xi) \leqslant q-1\}.$$

Then the remaining definitions and results of [30, §§3–5] yield the following three lemmas.

LEMMA 4.5. *Assume that $\mathfrak{G}$ is a simple Lie algebra and that m is a positive divisor of $q-1$. Then every element of $\overline{G}$ of order m is conjugate in $\overline{G}$ to an element of $\overline{H}_\sigma$ of the form $h(\chi_\lambda)$ for some $\lambda \in \mathfrak{D} \cap \Gamma_\pi^\perp$.*

LEMMA 4.6. *Assume that $\mathfrak{G}$ is a simple Lie algebra and let $\beta, \delta \in \mathfrak{D} \cap \Gamma_\pi^\perp$. Then the following four conditions are equivalent.*

(a) *$h(\chi_\beta) \sim h(\chi_\delta)$ in $\overline{G}$;*

(b) *$h(\chi_\beta) \sim h(\chi_\delta)$ via G;*

(c) *$\beta \sim \delta$ via $\mathfrak{F}_\pi$; and*

(d) *$\beta \sim \delta$ via Ω_π.*

LEMMA 4.7. *Assume that $\mathfrak{G}$ is a simple Lie algebra. Then the following two conditions hold.*

(a) *$\Omega_\pi \cong \mathfrak{F}_\pi/\gamma \cong \Gamma_\pi^\perp/\Gamma_{\mathrm{sc}}^\perp \cong \Gamma_{\mathrm{sc}}/\Gamma_\pi$; and*

(b) *if $\beta \in \mathfrak{D} \cap \Gamma_\pi^\perp$, then $\Omega_{\pi,\beta} \cong C_{\overline{G}}(h(\chi_\beta))/(C_{\overline{G}}(h(\chi_\beta))^0)$.*

Also, as in [30, §7], let $\{\mathfrak{S}_1, \ldots, \mathfrak{S}_l\}$ be the Z-basis of $\Gamma_{\mathrm{ad}}^\perp$ such that $(\mathfrak{S}_i, \alpha_j) = \delta_{ij}$ for all $1 \leqslant i, j \leqslant l$. Then we have the following elementary extension of [30, Proposition 7].

LEMMA 4.8. *Assume that $\mathfrak{G}$ is a simple Lie algebra, let $\delta \in \mathfrak{D} \cap \Gamma_\pi^\perp$ and let m be a positive divisor of $q-1$. Then $h(\chi_\delta)$ is an element of $\overline{H}$ of order m if and only if the following four conditions hold.*

(a) *$\delta = ((q-1)/m)(\sum_{i=1}^l a_i\mathfrak{S}_i)$ for nonnegative integers a_i with $1 \leqslant i \leqslant l$;*

(b) *$\sum_{i=1}^l a_i m_i \leqslant m$;*

(c) *$m(\delta, \Gamma_\pi) \leqslant (q-1)Z$; and*

(d) *if f is a proper divisor of m, then $f(\delta, \Gamma_\pi) \nleqslant (q-1)Z$.*

PROOF. Clearly $|h(\chi_\delta)| = m$ if and only if (i) $\mathcal{K}^{m(\delta,\mu)} = 1$ for all $\mu \in \Gamma_\pi$ and (ii) if f is a proper divisor of m, then $\mathcal{K}^{f(\delta,\mu)} \neq 1$ for some $\mu \in \Gamma_\pi$. Clearly (a)–(d) imply

$\delta \in \mathfrak{D} \cap \Gamma_\pi^\perp$ and $|h(\chi_\delta)| = m$. Conversely, since $\delta \in \Gamma_\pi^\perp \leqslant \Gamma_{\mathrm{ad}}^\perp$, we have $\delta = \sum_{i=1}^{l} c_i \mathfrak{E}_i$ for integers c_i with $1 \leqslant i \leqslant l$. Then $(q-1) \mid mc_i$ for all $1 \leqslant i \leqslant l$ since $\Gamma_{\mathrm{ad}} \leqslant \Gamma_\pi$. From the fact that $\delta \in \mathfrak{D}$, we conclude (a) and (b). Since (c) and (d) follow from (i) and (ii), we are done.

Now utilizing the facts noted above, we shall derive various results that we require about Chevalley groups over finite fields of odd order. Consequently, for the remainder of this section, we assume that p is odd. Also let $k_1 = C_K(\sigma^2)$ so that $k \leqslant k_1$ and k_1 is the unique subfield of K of order q^2.

As observed before, by our notational convention, if m is an even integer and V/k is an orthogonal finite dimensional vector space with $\dim(V/k) = m$, then $P\Omega(V/k) \cong P\Omega(m, q, 1)$ if $\mathrm{ind}(V/k) = \frac{m}{2}$ and $P\Omega(V/k) \cong P\Omega(m, q, -1)$ if $\mathrm{ind}(V/k) = \frac{m}{2} - 1$ (cf. §3D). Also, as is standard and throughout the remainder of this paper, we set $P\Omega(6, q, 1) = \mathrm{PSL}(4, q)$, $\mathrm{Spin}(6, q, 1) = \mathrm{SL}(4, q)$, $P\Omega(6, q, -1) = \mathrm{PSU}(4, q)$, $\mathrm{Spin}(6, q, -1) = \mathrm{SU}(4, q)$, $P\Omega(5, q) = \mathrm{PSp}(4, q)$, $\mathrm{Spin}(5, q) = \mathrm{Sp}(4, q)$, $P\Omega(4, q, 1) = \mathrm{PSL}(2, q) \times \mathrm{PSL}(2, q)$, $\mathrm{Spin}(4, q, 1) = \mathrm{SL}(2, q) \times \mathrm{SL}(2, q)$, $P\Omega(4, q, -1) = \mathrm{PSL}(2, q^2)$ and $\mathrm{Spin}(4, q, -1) = \mathrm{SL}(2, q^2)$.

LEMMA 4.9. *Let $X = \mathrm{Cov}(E_7(q))$. Then the following six conditions hold.*

(a) *$Z(X) = \langle z \rangle$ where z is an involution*;

(b) *there is a unique conjugacy class $\mathfrak{K}$ of involutions of X such that if $\tau \in \mathfrak{K}$, then $C_X(\tau)$ contains a 2-component J with $\tau \in J$ and $J \cong \mathrm{SL}(2, q)$*;

(c) *if $\mathfrak{K}$, τ and J are as in* (b), *then $C_X(\tau)$ contains, besides J, precisely one other 2-component J_1 and $J_1 \cong \mathrm{Spin}(12, q, 1)$ with $Z(J_1) = \langle \tau, z \rangle$.*

(d) *$\{\tau, z, \tau z\}$ is a set of representatives of the conjugacy classes of involutions in X*;

(e) *there are elements γ_1, γ_2 of X such that $\gamma_i^2 = z$ and $L_{2'}(C_X(\gamma_i)) = E(C_X(\gamma_i))$ for $i = 1, 2$, $E(C_X(\gamma_1))$ is a quotient of* $\mathrm{SL}(8, q)$ *if $q \equiv 1 \pmod 4$ and of* $\mathrm{SU}(8, q)$ *if $q \equiv -1 \pmod 4$ and $E(C_X(\gamma_2))$ is a quotient of* $\mathrm{Cov}(E_6(q))$ *if $q \equiv 1 \pmod 4$ and of* $\mathrm{Cov}({}^2E_6(q))$ *if $q \equiv -1 \pmod 4$; and*

(f) *all elements γ of X such that $\gamma^2 = z$ are conjugate in X to γ_1 or γ_2.*

PROOF. Let $\mathfrak{G} = E_7$, let π denote the simply connected representation of $\mathfrak{G}$, so that $\overline{G}$ is a simply connected linear algebraic group, and let $B = \{\alpha_1, \ldots, \alpha_7\}$ be as in [**8**, Planche VI]. Then $\Gamma_{\mathrm{ad}} = \sum_{i=1}^{7} Z\alpha_i$ and $\Gamma_\pi = \Gamma_{\mathrm{sc}} = \sum_{i=1}^{7} Z\overline{\omega}_i$ where $\{\overline{\omega}_1, \ldots, \overline{\omega}_7\}$ is as in [**8**, Planche VI]. Also $G = \overline{G}_\sigma \cong X$ by [**35**, Theorem 12.4] and $C_{\overline{G}}(h)$ is connected for all $h \in \overline{H}$ by [**35**, Theorem 8.1]. Thus, we have $\Omega_\pi = 1$, $\mathfrak{E}_i = \overline{\omega}_i$ for all $1 \leqslant i \leqslant 7$, $\Gamma_\pi^\perp = \Gamma_{\mathrm{ad}}$, $\Gamma_{\mathrm{ad}}^\perp = \Gamma_\pi$ and $\alpha_0 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$. Then Lemmas 4.4–4.8 imply that $\overline{G}$ and G each have three conjugacy classes of involutions represented by $h(\chi_{\lambda_i})$ for $i = 1, 2, 3$ where

$$\lambda_1 = \frac{q-1}{2}\overline{\omega}_1, \quad \lambda_2 = \frac{q-1}{2}\overline{\omega}_6 \quad \text{and} \quad \lambda_3 = \frac{q-1}{2}(2\overline{\omega}_7) = (q-1)\overline{\omega}_7.$$

Also (4.4) yields

$$h(\chi_{\lambda_1}) = h_{\alpha_3}(-1)h_{\alpha_5}(-1)h_{\alpha_7}(-1) = h_{\alpha_0}(-1),$$
$$h(\chi_{\lambda_2}) = h_{\alpha_2}(-1)h_{\alpha_5}(-1) \quad \text{and}$$
$$h(\chi_{\lambda_3}) = h_{\alpha_2}(-1)h_{\alpha_5}(-1)h_{\alpha_7}(-1).$$

Setting $\bar{L}_1 = \langle \bar{\mathfrak{X}}_{\pm\alpha_0} \rangle$, $\bar{L}_2 = \langle \bar{\mathfrak{X}}_{\pm\alpha_i} \,|\, 2 \leq i \leq 7 \rangle$, $\bar{L}_3 = \langle \bar{\mathfrak{X}}_{\pm\alpha_7} \rangle$ and $\bar{L}_4 = \langle \bar{\mathfrak{X}}_{\pm\alpha_0}, \bar{\mathfrak{X}}_{\pm\alpha_i} \,|1 \leq i \leq 5 \rangle$ [**30**, Proposition 8], Lemma 4.7(b) and (4.2) imply: $Z(\bar{G}) = Z(G) = \langle h(\chi_{\lambda_3}) \rangle$, $C_{\bar{G}}(h(\chi_{\lambda_1})) = (\bar{L}_1 * \bar{L}_2)\bar{H}$, $C_{\bar{G}}(h(\chi_{\lambda_2})) = (\bar{L}_3 * \bar{L}_4)\bar{H}$, $\bar{L}_i \trianglelefteq C_{\bar{G}}(h(\chi_{\lambda_1}))$ for $i = 1, 2$, $\bar{L}_j \trianglelefteq C_{\bar{G}}(h(\chi_{\lambda_2}))$ for $j = 3, 4$, $\bar{L}_1 \cong \bar{L}_3 \cong \mathrm{SL}(2, K)$, $Z(\bar{L}_1) = \langle h(\chi_{\lambda_1}) \rangle$, $Z(\bar{L}_3) = \langle h_{\alpha_7}(-1) \rangle$, $\bar{L}_2 \cong \bar{L}_4 \cong \mathrm{Spin}(12, K)$, $Z(\bar{L}_2) = \langle h(\chi_{\lambda_3}), h(\chi_{\lambda_1}) \rangle \cong E_4$ and $Z(\bar{L}_4) = \langle h(\chi_{\lambda_3}), h(\chi_{\lambda_2}) \rangle \cong E_4$. Also

$$C_{\bar{G}}\big(h(\chi_{\lambda_1})\big) = C_{\bar{G}}\big(h(\chi_{\lambda_1})h(\chi_{\lambda_3})\big) \quad \text{and} \quad Z\big(C_{\bar{G}}\big(h(\chi_{\lambda_1})\big)\big) = \langle h(\chi_{\lambda_1}), h(\chi_{\lambda_3}) \rangle.$$

Thus $h(\chi_{\lambda_1})h(\chi_{\lambda_3}) \sim h(\chi_{\lambda_2})$ in both $\bar{G}$ and G. Moreover, setting $L_1 = C_{\bar{L}_1}(\sigma)$ and $L_2 = C_{\bar{L}_2}(\sigma)$, we have $L_1 \cong \mathrm{SL}(2, k)$, $Z(L_1) = \langle h(\chi_{\lambda_1}) \rangle$, $L_2 \cong \mathrm{Spin}(12, k, 1)$, $Z(L_2) = \langle h(\chi_{\lambda_1}), h(\chi_{\lambda_3}) \rangle \cong E_4$ and

$$C_G\big(h(\chi_{\lambda_1})\big) = C_G\big(h(\chi_{\lambda_1})h(\chi_{\lambda_3})\big) = C_{\bar{G}}\big(h(\chi_{\lambda_1})\big)_\sigma = (L_1 * L_2)\big(\bar{H}_\sigma\big).$$

Thus (a)–(d) hold.

For (e) and (f), let ν be an element of order 4 in $K^\times$. Then Lemma 4.8 implies that $\bar{G}$ contains precisely two conjugacy classes of elements γ such that $\gamma^2 = z$ and these two conjugacy classes have representatives

$$\gamma_1 = h(\chi_{\lambda_4}) = h_{\alpha_2}(\nu^3)h_{\alpha_5}(\nu)h_{\alpha_6}(-1)h_{\alpha_7}(\nu^3)$$

and

$$\gamma_2 = h(\chi_{\lambda_5}) = h_{\alpha_1}(-1)h_{\alpha_2}(\nu^3)h_{\alpha_4}(-1)h_{\alpha_5}(\nu)h_{\alpha_7}(\nu^3)$$

where

$$\lambda_4 = \frac{q-1}{4}(2\bar{\omega}_2) = \frac{q-1}{2}\bar{\omega}_2 \quad \text{and} \quad \lambda_5 = \frac{q-1}{4}(2\bar{\omega}_7) = \frac{q-1}{2}\bar{\omega}_7.$$

Also, as above, we have $C_{\bar{G}}(\gamma_1) = \bar{L}_1\bar{H}$ and $C_{\bar{G}}(\gamma_2) = \bar{L}_2\bar{H}$ where $\bar{L}_1 = \langle \bar{\mathfrak{X}}_{\pm\alpha_i}, \bar{\mathfrak{X}}_{\pm\alpha_0} \,|\, 1 \leq i \leq 7 \text{ and } i \neq 2 \rangle$ and $\bar{L}_2 = \langle \bar{\mathfrak{X}}_{\pm\alpha_i} \,|\, 1 \leq i \leq 6 \rangle$. Note that if $u \in K^\times$, then

$$h_{\alpha_0}(u) = h_{\alpha_1}(u^2)h_{\alpha_2}(u^2)h_{\alpha_3}(u^3)h_{\alpha_4}(u^4)h_{\alpha_5}(u^3)h_{\alpha_6}(u^2)h_{\alpha_7}(u).$$

Thus $\bar{L}_1$ is a quotient of $\mathrm{SL}(8, K)$, $z \in \bar{L}_1$, $\gamma_1 \in \bar{L}_1$ if and only if $8 \,|\, (q-1)$, $\bar{L}_2/Z(\bar{L}_2) \cong E_6(K)$ and $z \notin \bar{L}_2$. Suppose that $q \equiv 1 \pmod 4$. Then $\gamma \in k$, $\langle \gamma_1, \gamma_2 \rangle \leq \bar{G}_\sigma$ and we have (e) and (f) in this case.

Finally assume that $q \equiv -1 \pmod 4$. Then $\nu \notin k$ and σ inverts both γ_1 and γ_2. Let w be an element of $\bar{N}_\lambda$ such that the w induced automorphism $\tilde{w}$ of Φ satisfies $\tilde{w}$: $\alpha_1 \leftrightarrow \alpha_6$, $\alpha_3 \leftrightarrow \alpha_5$, $\alpha_7 \leftrightarrow -\alpha_0$ and fixes α_2 and α_4. Then $w^2 \in \bar{H}$ and w inverts γ_1 and γ_2. Letting β denote the inner automorphism of $\bar{G}$ induced by w, it follows that $\beta\sigma$ fixes both γ_1 and γ_2' and $\beta\sigma$ is conjugate via $\mathrm{Inn}(\bar{G})$ to σ by Lemma 4.1. Since $(\bar{L}_1)_{\beta\sigma}$ is a quotient of $\mathrm{SU}(8, q)$ and $(\bar{L}_2)_{\beta\sigma}$ is a quotient of $\mathrm{Cov}({}^2E_6(q))$ by [**35**, §11], Lemma 4.4 implies (e) and (f) in this case also and we are done.

LEMMA 4.10. *Let $X = \mathrm{Spin}(7, q)$ with $Z(X) = \langle z \rangle$ and let $S \in \mathrm{Syl}_2(X)$. Then the following six conditions hold.*

(a) *$Z(S) = \langle z \rangle \cong Z_2$, $\Omega_1(S) = S$ and S has a normal 4-subgroup*;

(b) *all involutions of $X - Z(X)$ are conjugate in X*;

(c) *if* $t \in \mathfrak{I}(X - \langle z \rangle)$, *then* $|X\colon C_X(t)|_2 = 2$, $C_X(t)$ *contains precisely three* 2-*components* J_1, J_2 *and* J_3. *These* 2-*components may be indexed so that* $\mathrm{SL}(2, q) \cong J_i \lhd C_X(t)$ *for all* $1 \leqslant i \leqslant 3$, $Z(J_1) = \langle z \rangle$, $Z(J_2) = \langle t \rangle$, $Z(J_3) = \langle tz \rangle$ *and* $C_X(J_1 * J_2 * J_3) = \langle z, t \rangle$. *Also there is an involution* $\tau \in C_X(t) - (J_1 * J_2 * J_3)$ *such that* $C_X(t) = (J_1 * J_2 * J_3)\langle \tau \rangle$ *and* $J_i\langle \tau \rangle$ *has semidihedral Sylow* 2-*subgroups for all* $1 \leqslant i \leqslant 3$;

(d) *there are two elements* γ_1, γ_2 *of* X *such that* $\gamma_i^2 = z$ *and* $L_{2'}(C_X(\gamma_i)) = E(C_X(\gamma_i))$ *for* $i = 1, 2$, $E(C_X(\gamma_1)) \cong \mathrm{Sp}(4, q)$ *with* $Z(E(C_X(\gamma_1))) = \langle z \rangle$, $E(C_X(\gamma_2)) \cong \mathrm{SL}(4, q)$ *if* $q \equiv 1 \pmod 4$, $E(C_X(\gamma_2)) \cong \mathrm{SU}(4, q)$ *if* $q \equiv -1 \pmod 4$ *and* $Z(E(C_X(\gamma_2))) = \langle \gamma_2 \rangle$;

(e) *all elements* γ *of* X *such that* $\gamma^2 = z$ *are conjugate in* X *to* γ_1 *or* γ_2; *and*

(f) *if* E *is a normal* 4-*subgroup of* S, *then* $T = C_S(E)$ *is a maximal subgroup of* S, $\Omega_1(T) = T$, $Z(T) = E$ *and* $T \in \mathrm{Syl}_2(C_X(E))$.

PROOF. Let $\mathfrak{G} = B_3$, let π denote the simply connected representation of $\mathfrak{G}$, so that $\overline{G}$ is a simply connected linear algebraic group, and adopt the notation of [**8**, Planche II]. Let $B = \{\alpha_1, \alpha_2, \alpha_3\}$, so that $\Gamma_{\mathrm{ad}} = \sum_{i=1}^3 Z\alpha_i$, $\Gamma_\pi = \Gamma_{\mathrm{sc}} = \sum_{i=1}^3 Z\overline{\omega}_i$ where $\{\overline{\omega}_1, \overline{\omega}_2, \overline{\omega}_3\}$ are as in [**8**, Planche II], $\alpha_0 = \alpha_1 + 2\alpha_2 + 2\alpha_3$, $\mathcal{E}_i = \overline{\omega}_i$ for $i = 1, 2$, $\mathcal{E}_3 = 2\omega_3$, $\Gamma_{\mathrm{ad}}^\perp = \sum_{i=1}^3 Z\mathcal{E}_i < \Gamma_\pi$, $\Gamma_\pi^\perp = \sum_{i=1}^2 Z\alpha_i + Z(2\alpha_3)$, $\Omega_\pi = 1$, $G = \overline{G}_\sigma \cong X$ and we may assume that $X = G$. Then, as in Lemma 4.9, $\overline{G}$ and G have precisely two conjugacy classes of involutions represented by $h(\chi_{\lambda_1})$, $h(\chi_{\lambda_2})$ where

$$\lambda_1 = \frac{q-1}{2}(2\overline{\omega}_1) \quad \text{and} \quad \lambda_2 = \frac{q-1}{2}\overline{\omega}_2.$$

Also setting $z = h(\chi_{\lambda_1})$ and $t = h(\chi_{\lambda_2})$, we have $z = h_{\alpha_3}(-1)$, $t = h_{\alpha_0}(-1) = h_{\alpha_1}(-1)h_{\alpha_3}(-1)$, $Z(\overline{G}) = Z(G) = \langle z \rangle$ and $C_{\overline{G}}(t) = (\overline{J}_0 * \overline{J}_1 * \overline{J}_3)\overline{H}$ where $\overline{J}_i = \langle \mathfrak{X}_{\pm\alpha_i} \rangle \cong \mathrm{SL}(2, K)$ for all $i \in \{0, 1, 3\}$, $Z(\overline{J}_0) = \langle t \rangle$, $Z(\overline{J}_3) = \langle z \rangle$ and $Z(\overline{J}_1) = \langle tz \rangle$. Then, as in [**30**, §8], $C_G(t) = C_{\overline{G}}(t)_\sigma = (J_0 * J_1 * J_3)(\overline{H}_\sigma)$ where $\mathrm{SL}(2, k) \cong (\overline{J}_i)_\sigma = J_i \trianglelefteq C_G(t)$ for all $i \in \{0, 1, 3\}$, $Z(J_0) = \langle t \rangle$, $Z(J_3) = \langle z \rangle$ and $Z(J_1) = \langle tz \rangle$. Also

$$|G| = q^9(q^2 - 1)(q^4 - 1)(q^6 - 1)$$

by [**34**, §9], $(\overline{H}_\sigma) \cap J_1 = \langle h_{\alpha_1}(\mathcal{K}) \rangle$, $(\overline{H}_\sigma) \cap J_3 = \langle h_{\alpha_3}(\mathcal{K}) \rangle$, $(\overline{H}_\sigma) \cap J_0 = \langle h_{\alpha_0}(\mathcal{K}) \rangle$ where $h_{\alpha_0}(u) = h_{\alpha_1}(u)h_{\alpha_2}(u^2)h_{\alpha_3}(u)$ for all $u \in k^\times$, $|C_G(t)\colon (J_0 * J_1 * J_3)| = 2$ and $|C_G(t)| = q^3(q^2 - 1)^3$. Thus (b) holds, $C_G(t) = (J_0 * J_1 * J_3)\langle h_{\alpha_2}(\mathcal{K}) \rangle$ and $|G\colon C_G(t)|_2 = 2$.

Set $M = C_X(t)$ and let $\{i, j, k\} = \{0, 1, 3\}$. Then

$$J_j * J_k = J_j \times J_k \leqslant C_M(J_i) \leqslant C_M(\mathfrak{X}_{\alpha_i}) \trianglelefteq N_M(\mathfrak{X}_{\alpha_i}) = J_j J_k \mathfrak{X}_{\alpha_i} \overline{H}_\sigma.$$

Since $C_{\overline{H}_\sigma}(x_{\alpha_i}(1)) = ((\overline{H}_\sigma) \cap J_j) \times ((\overline{H}_\sigma) \cap J_k)$, we have $C_M(\mathfrak{X}_{\alpha_i}) = J_j \times J_k \times \mathfrak{X}_{\alpha_i}$ and $C_M(J_i) = J_j \times J_k$. Thus $C_X(J_0 * J_1 * J_3) = \langle t, z \rangle = Z(M)$ and $M/C_M(J_i) \cong \mathrm{PGL}(2, q)$. For $r \in \{0, 1, 3\}$, let ω_{α_r} be as in [**34**, p. 30, (R5)]. Then $\langle \omega_{\alpha_0}, \omega_{\alpha_1}, \omega_{\alpha_3} \rangle$ is abelian, $\omega_{\alpha_0}^2 = t$, $\omega_{\alpha_1}^2 = tz$ and $\omega_{\alpha_3}^2 = z$. Hence $\lambda = \omega_{\alpha_0}\omega_{\alpha_1}\omega_{\alpha_3}$ is an involution and λ inverts $h_{\alpha_2}(\mathcal{K})$. Thus $\tau = h_{\alpha_2}(\mathcal{K})\lambda \in \mathfrak{I}(M)$ and $M = (J_0 * J_1 * J_3)\langle \tau \rangle$. Moreover, for $r \in \{0, 1, 3\}$, we have $(J_r\langle \tau \rangle)/Z(J_r) \cong \mathrm{PGL}(2, q)$ and [**27**, Lemma 2.2] readily implies that $J_r\langle \tau \rangle$ has semidihedral Sylow 2-subgroups. Thus (c) holds. Next let $T \in \mathrm{Syl}_2(M)$ and let $T < V \in \mathrm{Syl}_2(X)$. Then $T = C_V(t)$, $Z(T) = \langle t, z \rangle$, $\Omega_1(T) = T$, $|V\colon T| = 2$ and $Z(V) = \langle z \rangle$. Thus $\langle t, z \rangle \trianglelefteq V$, $T = C_V(\langle t, z \rangle)$ and (f) holds.

Clearly $N_X(\langle t, z\rangle) = MV$, $J_3 \trianglelefteq MV$ and $J_0^{MV} = J_0^V = J_1^{MV} = J_1^V = \{J_0, J_1\}$ since $t^V = t\langle z\rangle$. By the discussion in §3D, there is an element $\alpha \in MV - M$ such that $\alpha^2 \in \langle z\rangle$ and $[\alpha, J_3] \leqslant \langle z\rangle$. Hence $[\alpha, J_3] = 1$ and a Sylow 2-subgroup of $\langle J_3, \alpha\rangle$ is not quaternion. Thus $J_3\alpha$ contains an involution. Since $MV = M\langle\alpha\rangle$, (a) also holds.

For (d) and (e), let ν be an element of order 4 in $K^\times$. Then Lemmas 4.5–4.7 imply that $\bar{G}$ contains precisely two conjugacy classes of elements γ such that $\gamma^2 = z$ and these two conjugacy classes have representatives

$$\gamma_1 = h(\chi_{\lambda_1}) = h_{\alpha_1}(-1)h_{\alpha_3}(\nu^3), \quad \text{where } \lambda_1 = \frac{q-1}{4}(2\mathfrak{E}_3) = \frac{q-1}{4}(4\bar{\omega}_3)$$

and

$$\gamma_2 = h(\chi_{\lambda_2}) = h_{\alpha_1}(-1)h_{\alpha_2}(-1)h_{\alpha_3}(\nu), \quad \text{where } \lambda_2 = \frac{q-1}{4}(2\mathfrak{E}_1) = \frac{q-1}{4}(2\bar{\omega}_1).$$

Also by [**30**, Proposition 8], we have $C_{\bar{G}}(\gamma_2) = \bar{L}_1\bar{H}$ and $C_{\bar{G}}(\gamma_2) = \bar{L}_2\bar{H}$, where $\gamma_1 \in \bar{L}_1 = \langle \bar{\mathfrak{X}}_{\pm\alpha_0}, \bar{\mathfrak{X}}_{\pm\alpha_1}, \bar{\mathfrak{X}}_{\pm\alpha_2}\rangle \cong \mathrm{SL}(4, K)$ and $z \in \bar{L}_2 = \langle \bar{\mathfrak{X}}_{\pm\alpha_2}, \bar{\mathfrak{X}}_{\pm\alpha_3}\rangle \cong \mathrm{Sp}(4, K)$. Suppose that $q \equiv 1 \pmod 4$. Then $\nu \in k$, $\gamma_1 \in C_{\bar{L}_1}(\sigma) \cong \mathrm{SL}(4, k)$, $z \in C_{\bar{L}_2}(\sigma) \cong \mathrm{Sp}(4, k)$ and we have (d) and (e) in this case. Finally, suppose that $q \equiv -1 \pmod 4$. Then $\nu \notin k$ and σ inverts both γ_1 and γ_2. Let $\delta = \alpha_1 + \alpha_2 + \alpha_3$. Thus $\delta \in \Phi$ and ω_δ, as defined on [**34**, p. 30, (R5)], has coefficients in the prime subfield of K and is such that $\omega_\delta \in \bar{N}$ and $(\omega_\delta)^2 = h_{\alpha_3}(-1)$. Set $w = \omega_\delta$. Then the w induced automorphism $\tilde{w}$ of Φ satisfies: $\tilde{w}$: $\alpha_1 \leftrightarrow -\alpha_0$ and $\tilde{w}(\alpha_j) = \alpha_j$ for $j = 2, 3$. Thus w also inverts γ_1 and γ_2.

Letting β denote the inner automorphism of $\bar{G}$ induced by w, it follows that $\beta\sigma$ fixes both γ_1 and γ_2 and $\beta\sigma$ is conjugate via $\mathrm{Inn}(\bar{G})$ to σ by Lemma 4.1. Since $(\bar{L}_1)_{\beta\sigma} \cong \mathrm{SU}(4, k)$ and $(\bar{L}_2)_{\beta\sigma} \cong \mathrm{Sp}(4, k)$ by [**35**, §11], we have (d) and (e) in this case also by Lemma 4.4 and our proof is complete.

LEMMA 4.11. *Let $X = \mathrm{Spin}(7, q)$ and $Z(X) = \langle z\rangle$ where z is an involution. Let $\mathfrak{A} = \mathrm{Aut}(X)$, $\mathfrak{B} = \mathrm{Inn}(X)$ and let $\mathfrak{C}$ be an arbitrary subgroup of $\mathfrak{A}$ with $\mathfrak{B} \leqslant \mathfrak{C}$. Then the following nine conditions hold.*

(a) *$P\Omega(7, q) \cong \Omega(7, q) \cong \mathfrak{B} = \mathfrak{A}'$ and $\mathfrak{C} \trianglelefteq \mathfrak{A} \cong P\Gamma O(7, q)$;*

(b) *$\mathfrak{A}/\mathfrak{B} \cong Z_2 \times Z_n$;*

(c) *there are involutions α_1, α_2, α_3 in $\mathfrak{A} - \mathfrak{B}$ such that $\mathfrak{B}\alpha_1 = \mathfrak{B}\alpha_2 = \mathfrak{B}\alpha_3$, $\mathfrak{B}\langle\alpha_1\rangle \cong SO(7, q) \cong PGO(7, q)$, $E(C_X(\alpha_i)) = L_{2'}(C_X(\alpha_i))$ and $Z(E(C_X(\alpha_i))) = \langle z\rangle$ for $i = 1, 2$ and 3, $E(C_X(\alpha_1)) \cong \mathrm{SU}(4, q)$ if $q \equiv 1 \pmod 4$, $E(C_X(\alpha_1)) \cong \mathrm{SL}(4, q)$ if $q \equiv -1 \pmod 4$, $E(C_X(\alpha_2)) \cong \mathrm{Sp}(4, q)$ and $E(C_X(\alpha_3))$ contains precisely two components J_1 and J_2, $Z(J_1) = Z(J_2) = \langle z\rangle$ and J_1 and J_2 may be indexed so that $J_1 \cong \mathrm{SL}(2, q)$ and $J_2 \cong \mathrm{SL}(2, q^2)$;*

(d) *all involutions in $\mathfrak{B}\alpha_1$ are conjugate via an element of $\mathfrak{B}$ to α_1, α_2 or α_3;*

(e) *$\mathfrak{B}$ contains precisely one conjugacy class of involutions $\mathfrak{K}$ such that if $\tau \in \mathfrak{K}$, then $C_{\mathfrak{B}}(\tau)$ contains an intrinsic 2-component J with $J/O(J) \cong \mathrm{SL}(2, q)$;*

(f) *if $\mathfrak{K}$, τ and J are as in (e), then $O(C_{\mathfrak{A}}(\tau)) = O(C_{\mathfrak{C}}(\tau)) = 1$ and $C_{\mathfrak{B}}(\tau)$ contains, besides J, two other 2-components J_1 and J_2 such that $\tau \in J_1 \cong \mathrm{SL}(2, q)$, $J_2 \cong \mathrm{PSL}(2, q)$ and $C_{\mathfrak{A}}(J * J_1 * J_2) = \langle\tau\rangle$;*

(g) *if n is odd, then $O^{2'}(\mathfrak{A}) = \mathfrak{B}\langle\alpha_1\rangle$;*

(h) *if n is even, then there is an involution $\tau \in \mathfrak{A} - (\mathfrak{B}\langle\alpha_1\rangle)$ such that $C_X(\tau) \cong \mathrm{Spin}(7, \sqrt{q})$, $Z(C_X(\tau)) = \langle z\rangle$, all involutions of $\mathfrak{A} - \mathfrak{B}\langle\alpha_1\rangle$ are conjugate via an element of $\mathfrak{B}\langle\alpha_1\rangle$ to τ and $\mathfrak{I}(\mathfrak{B}\tau\alpha_1) = \varnothing$; and*

(i) *if $4 \mid |\mathfrak{C}/\mathfrak{B}|$, then n is even and $\mathfrak{B}\langle\tau\rangle \leqslant \mathfrak{C}$.*

PROOF. Observe that the natural epimorphism of X onto $X/Z(X) \cong \Omega(7, q) \cong P\Omega(7, q)$ induces an isomorphism of $\mathrm{Aut}(X)$ onto $\mathrm{Aut}(X/Z(X))$ by [**21**, Corollary 4.1].

Adopt the notation of the previous lemma. Note that $Z(\bar{G}) = \langle z\rangle$, set $\tilde{\bar{G}} = \bar{G}/\langle z\rangle$ and observe that [**6**, II, Theorem 6.8] implies that $\tilde{\bar{G}}$ is a connected linear algebraic group and the natural epimorphism of $\bar{G}$ onto $\tilde{\bar{G}} = \bar{G}/\langle z\rangle$ is a morphism of linear algebraic groups. Also λ and σ induce endomorphisms of $\tilde{\bar{G}}$, $\tilde{\bar{G}}_\sigma = \tilde{X}\tilde{K}$ where $\tilde{K} = \{h \in \bar{H} \mid \sigma(h)h^{-1} \in \langle z\rangle\}$, $\tilde{X} = X/Z(X) \cong P\Omega(7, q) \cong \Omega(7, q)$, $\tilde{X} = O^{p'}(\tilde{\bar{G}}_\sigma) = (\tilde{\bar{G}}_\sigma)'$ and $|\tilde{X}\tilde{K}/\tilde{X}| = 2$. Moreover λ leaves $\tilde{\bar{G}}_\sigma$ invariant. Letting λ^* denote the restriction of λ to $\tilde{\bar{G}}_\sigma$, it follows that λ^* induces an automorphism of $\tilde{\bar{G}}_\sigma$ of order n, $C_{\tilde{\bar{G}}_\sigma\langle\lambda^*\rangle}(\tilde{X}) = 1$ and $\tilde{\bar{G}}_\sigma\langle\lambda^*\rangle \cong \mathrm{Aut}(X)$ with $\tilde{X}$ corresponding to $\mathrm{Inn}(X)$ (cf. [**11**, §12.5]). By [**25**, Proposition 5], $|P\Gamma O(7, q)| = |\mathfrak{A}|$ and $P\Gamma O(7, q)$ is isomorphic to a subgroup of $\mathfrak{A} = \mathrm{Aut}(X) \cong \mathrm{Aut}(X/Z(X))$. Thus (a) and (b) hold. Also Lemma 4.10 implies that $\tilde{\bar{G}}$ possesses three conjugacy classes of involutions which are represented by

$$\tilde{t} = h_{\alpha_1}(-1), \quad \tilde{\gamma}_1 = \widetilde{h_{\alpha_1}(-1)}\widetilde{h_{\alpha_3}(\nu^3)} \quad \text{and} \quad \tilde{\gamma}_2 = \widetilde{h_{\alpha_1}(-1)}\widetilde{h_{\alpha_2}(-1)}\widetilde{h_{\alpha_3}(\nu)}.$$

Let w be as in Lemma 4.10. Then $w \in \bar{N}_{\lambda}$, w inverts γ_1 and γ_2, $\tilde{w}^2 = 1$ and $t^w = tz$. Thus for any $j \in \{t, \gamma_1, \gamma_2\}$, we have $\tilde{j} \in \tilde{\bar{G}}_\sigma$ and

$$C_{\tilde{\bar{G}}}(\tilde{j}) = \widetilde{C_{\bar{G}}(j)}\langle\tilde{w}\rangle = \widetilde{N_{\bar{G}}(\langle z, j\rangle)}, \quad \text{where } C_{\tilde{\bar{G}}}(\tilde{j})^0 = \widetilde{C_{\bar{G}}(j)}.$$

Choose $g \in \bar{G}$ such that $g\sigma(g)^{-1} = w$, so that $\tilde{g}\sigma(\tilde{g})^{-1} = \tilde{w}$.

Lemma 4.4 implies that $\tilde{\bar{G}}_\sigma$ has 6 conjugacy classes of involutions represented by $\{\tilde{t}, \tilde{t}^{\tilde{g}}, \tilde{\gamma}_1, \tilde{\gamma}_1^{\tilde{g}}, \tilde{\gamma}_2, \tilde{\gamma}_2^{\tilde{g}}\}$ of which only three conjugacy classes lie in $\tilde{X}$ by Lemma 4.10. Clearly $J_0^w = J_1$ and (e) holds. Also it is easy to see from §3D and [**25**, Proposition 5(d)] applied to $P\Gamma O(7, q)$ that (e) and (f) hold. Alternatively (f) also follows easily from the proof of Lemma 4.10. For, with t, $\bar{J}_0$, J_0, etc. as in Lemma 4.10, we have $N_{\tilde{\bar{G}}_\sigma\langle\lambda^*\rangle}(\bar{J}_0 * \bar{J}_1 * \bar{J}_3) = C_{\bar{G}_\sigma}(t)\langle\lambda^*\rangle$ and hence it suffices to study $C_X(t)\langle\lambda^*\rangle = (J_0 * J_1 * J_3)\bar{H}_\sigma\langle\lambda^*\rangle$. Set $L \equiv C_X(t)\langle\lambda^*\rangle$ and let $\{i, j, k\} = \{0, 1, 3\}$. Then, as in Lemma 4.10,

$$J_j * J_k = J_j \times J_k \leqslant C_L(J_i) \leqslant C_L(\mathfrak{X}_{\alpha_i}) \trianglelefteq N_L(\mathfrak{X}_{\alpha_i}) = J_j J_k \mathfrak{X}_{\alpha_i}\bar{H}_\sigma\langle\lambda^*\rangle.$$

Also, since $C_{\bar{H}_\sigma}(x_{\alpha_i}(1)) = (\bar{H}_\sigma \cap J_j) \times (\bar{H}_\sigma \cap J_k)$, we have $C_L(\mathfrak{X}_{\alpha_i}) = (J_j \times J_k)\mathfrak{X}_{\alpha_i}$. Consequently $C_L(J_i) = J_j \times J_k$ and $C_L(J_0 * J_1 * J_3) = \langle t, z\rangle$. Now Lemma 2.7 implies (f). Let β denote the inner automorphism of $\bar{G}$ induced by w^{-1}. Note that $\beta\sigma = \sigma\beta$, $\beta^2 = 1$ and $(\beta\sigma)^2 = \sigma^2$ as endomorphisms of $\bar{G}$.

First let $\bar{A} = C_{\bar{G}}(t)$. Then $\bar{A} = C_{\bar{G}}(\langle t, z\rangle)$ and both β and σ leave $\bar{A}$ invariant. As in Lemma 4.10, $\{J_0, J_1, J_3\}$ is the set of 2-components of $C_X(t) = \bar{A}_\sigma$. Also $C_X(t^g) = C_{\bar{G}_\sigma}(t^g) = ((C_{\bar{G}}(t))_{\beta\sigma})^g$, $(\beta\sigma)(\bar{J}_0) = \bar{J}_1$ and $(\beta\sigma)(\bar{J}_3) = \bar{J}_3$. Then Lemma 2.5 and [**30**, §8; **35**, §11.6] imply that $C_{\bar{G}}(t)_{\beta\sigma}$ contains precisely two 2-components

$\mathfrak{J}_1$ and $\mathfrak{J}_2$ and by suitable indexing we may assume that $\mathfrak{J}_1 = (\bar{J}_3)_{\beta\sigma} \cong \mathrm{SL}(2, k)$, $Z(\mathfrak{J}_1) = \langle z \rangle$, $\mathfrak{J}_2 = C_{(\bar{J}_0 \times \bar{J}_1)}(\beta\sigma) \cong \mathrm{SL}(2, k_1)$ and $Z(\mathfrak{J}_2) = \langle z \rangle$.

Next let $\bar{A} = C_{\bar{G}}(\gamma_1) = C_{\bar{G}}(\langle \gamma_1 \rangle)$. Thus β and σ both leave $\bar{A}$ invariant and, as above, $(\bar{L}_1)_\sigma \cong \mathrm{SL}(4, k)$, $Z((\bar{L}_1)_\sigma) = \langle \gamma_1 \rangle$ if $4 \mid (q-1)$, $Z((\bar{L}_1)_\sigma) = \langle z \rangle$ if $4 \nmid (q-1)$ and $(\bar{L}_1)_\sigma$ is the unique 2-component of $C_X(\gamma_1) = \bar{A}_\sigma$. Also $C_X(\gamma_1^g) = (A_{\beta\sigma})^g$, $(\beta\sigma)(\bar{L}_1) = \bar{L}_1$ and, as above, $(\bar{L}_1)_{\beta\sigma} \cong \mathrm{SU}(4, k)$, $Z((\bar{L}_1)_{\beta\sigma}) = \langle \gamma_1 \rangle$ if $4 \nmid (q-1)$ and $Z((\bar{L}_1)_{\beta\sigma}) = \langle z \rangle$ if $4 \mid (q-1)$.

Next let $\bar{A} = c_{\bar{G}}(\gamma_2) = C_{\bar{G}}(\langle \gamma_2 \rangle)$. Thus β and σ both leave $\bar{A}$ invariant, and, as above $(\bar{L}_2)_\sigma \cong (\bar{L}_2)_{\beta\sigma} \cong \mathrm{Sp}(4, k)$ and $Z((\bar{L}_2)_\sigma) = Z((\bar{L}_2)_{\beta\sigma}) = \langle z \rangle$.

Clearly (g) holds. Suppose that n is even, let $m = \frac{n}{2}$ and set $\tau^* = (\lambda^*)^m$. Thus τ^* is the restriction of λ^m to $\tilde{\bar{G}}_\sigma$, $(\lambda^m)^2 = \sigma$ and $C_X(\tau^*) = C_{\bar{G}}(\lambda^m) \cong \tilde{\mathrm{Spin}}(7, \sqrt{q})$. Also Lemma 4.2 implies that all involutions of $(\bar{G}_\sigma)\tau^*$ are conjugate via $\tilde{\bar{G}}_\sigma$ to τ^*. Thus (h) and (i) hold. Finally (c) and (d) follow from the above and §3D and the proof of this lemma is complete.

LEMMA 4.12. *Let $X = \mathrm{Spin}(2m+1, q)$ with $m \geq 4$. Then the following five conditions hold.*

(a) *$Z(X) = \langle z \rangle$ where z is an involution;*

(b) *there is an involution $t \in X - Z(X)$ such that $C_X(t)$ contains 2-components J_1 and J_2 with $J_1 \cong J_2 \cong \mathrm{SL}(2, q)$, $Z(J_1) = \langle t \rangle$ and $Z(J_2) = \langle tz \rangle$;*

(c) *m is odd if and only if there is a conjugacy class $\mathfrak{K}$ of involutions in X such that if $\tau \in \mathfrak{K}$, then $C_X(\tau)$ possesses a 2-component J with $z \in J$ and $J/O(J) \cong \mathrm{SL}(2, q_1)$ with $q_1 = p^r$ for some positive integer r;*

(d) *if m is odd and $\mathfrak{K}$, τ and J are as in* (c), *then $\mathfrak{K}$ is unique, $q_1 = q$, $O(J) = 1$ and $C_X(\tau)$ possesses, besides J, precisely one other 2-component $\mathfrak{J}$ and $\mathfrak{J} \cong \mathrm{Spin}(2(m-1), q, 1)$ with $Z(\mathfrak{J}) = \langle z, \tau \rangle$; and*

(e) *there is an involution $j \in X$ such that $C_X(j)$ contains a component J with $\langle z, j \rangle = Z(J)$, $J \cong \mathrm{Spin}(2m, q, 1)$ if m is even and $J \cong \mathrm{Spin}(2(m-1), q, 1)$ if m is odd.*

PROOF. Let $\mathfrak{G} = B_m$, let π denote the simply connected representation of $\mathfrak{G}$ so that $\bar{G}$ is a simply connected linear algebraic group and adopt the notation of [8, Planche II]. Thus $B = \{\alpha_1, \ldots, \alpha_m\}$, $\Gamma_{\mathrm{ad}} = \sum_{i=1}^m Z\alpha_i$, $\Gamma_\pi = \Gamma_{\mathrm{sc}} = \sum_{i=1}^m Z\bar{\omega}_i$, $\alpha_0 = \alpha_1 + \sum_{i=2}^m 2\alpha_i$, $\mathfrak{E}_i = \bar{\omega}_i$ for all $1 \leq i \leq m-1$, $\mathfrak{E}_m = 2\bar{\omega}_m$, $\Gamma_{\mathrm{ad}}^\perp = \sum_{i=1}^m Z\mathfrak{E}_i < \Gamma_\pi$, $\Gamma_\pi^\perp = \sum_{i=1}^{m-1} Z\alpha_i + Z(2\alpha_m)$, $X \cong G = \bar{G}_\sigma$ and $\Omega_\pi = 1$, etc. Then, as in Lemma 4.9, $\bar{G}$ and G have precisely $1 + [\frac{m}{2}]$ conjugacy classes of involutions which are represented by

$$\left\{ h(\chi_\lambda) \;\middle|\; \lambda = \frac{q-1}{2}\bar{\omega}_j \text{ with } j \text{ even and } 2 \leq j \leq m-1 \right\}$$
$$\cup \left\{ h(\chi_{\lambda_1}), h(\chi_{\lambda_2}) \;\middle|\; \lambda_1 = \frac{q-1}{2}(2\bar{\omega}_1) \text{ and } \lambda_2 = \frac{q-1}{2}(2\bar{\omega}_m) \text{ if } m \text{ is even} \right\}.$$

Then, as in Lemma 4.9, we have $Z(G) = \langle h(\chi_{\lambda_1}) \rangle$, $h(\chi_{\lambda_1}) = h_{\alpha_m}(-1)$ and $h_{\alpha_0}(-1) = h(\chi_\lambda)$ with $\lambda = \frac{q-1}{2}\bar{\omega}_2$. Also $t = h_{\alpha_0}(-1)$ satisfies (b) and it is easy to see, as in Lemma 4.9, that (c)–(e) hold.

LEMMA 4.13. *Let $X = \mathrm{Spin}(8, q, 1)$. Then the following three conditions hold.*

(a) *$Z(X) \cong E_4$ and $Z(X) = \langle t_1, t_2 \rangle$ where t_1 and t_2 are distinct commuting involutions*;

(b) *all involutions in $X - Z(X)$ are conjugate in X*; *and*

(c) *if $\tau \in \mathfrak{I}(X - Z(X))$, then $C_X(\tau)$ possesses precisely four 2-components J_1, J_2, J_3, J_4. Also the 2-components of $C_X(\tau)$ can be indexed so as to satisfy the following conditions*:

(i) *$J_1 \cong \mathrm{SL}(2, q)$ and $Z(J_1) = \langle \tau \rangle$*;

(ii) *$J_2 \cong \mathrm{SL}(2, q)$ and $Z(J_2) = \langle \tau t_1 \rangle$*;

(iii) *$J_3 \cong \mathrm{SL}(2, q)$ and $Z(J_3) = \langle \tau t_2 \rangle$*; *and*

(iv) *$J_4 \cong \mathrm{SL}(2, q)$ and $Z(J_4) = \langle \tau t_1 t_2 \rangle$.*

PROOF. Let $\mathfrak{G} = D_4$, let π denote the simply connected representation of $\mathfrak{G}$ so that $\overline{G}$ is a simply connected linear algebraic group and adopt the notation of [**8**, Planche IV]. Thus $B = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, $\Gamma_{\mathrm{ad}} = \sum_{i=1}^4 Z\alpha_i$, $\Gamma_\pi = \Gamma_{\mathrm{sc}} = \sum_{i=1}^4 Z\overline{\omega}_i$, $\alpha_0 = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$, $\mathfrak{S}_i = \overline{\omega}_i$ for all $1 \leqslant i \leqslant 4$, $\Gamma_{\mathrm{ad}}^\perp = \Gamma_\pi$, $G = \overline{G}_\sigma \cong X$ and $\Omega_\pi = 1$, etc. Then, as in Lemma 4.9, $\overline{G}$ and G have precisely 4 conjugacy classes of involutions represented by

$$\left\{ h(\chi_{\lambda_i}) \,|\, \lambda_1 = \frac{q-1}{2}\overline{\omega}_2, \lambda_2 = \frac{q-1}{2}(2\overline{\omega}_1), \lambda_3 = \frac{q-1}{2}(2\overline{\omega}_3), \lambda_4 = \frac{q-1}{2}(2\overline{\omega}_4) \right\}.$$

Also, as in Lemma 4.9, we have

$$X \cong G = \overline{G}_\sigma,\ h(\chi_{\lambda_1}) = h_{\alpha_0}(-1) = h_{\alpha_1}(-1)h_{\alpha_3}(-1)h_{\alpha_4}(-1),$$
$$h(\chi_{\lambda_2}) = h_{\alpha_3}(-1)h_{\alpha_4}(-1),\ h(\chi_{\lambda_3}) = h_{\alpha_1}(-1)h_{\alpha_4}(-1),$$
$$h(\chi_{\lambda_4}) = h_{\alpha_1}(-1)h_{\alpha_3}(-1),\ Z(G) = \langle h(\chi_{\lambda_2}), h(\chi_{\lambda_3}) \rangle$$

and (a) and (b) hold. Also, setting $\tau = h_{\alpha_0}(-1)$ and observing that $h_{\alpha_1}(-1) = \tau h_{\alpha_3}(-1)h_{\alpha_4}(-1)$, $h_{\alpha_3}(-1) = \tau h_{\alpha_1}(-1)h_{\alpha_4}(-1)$ and $h_{\alpha_4}(-1) = \tau h_{\alpha_1}(-1)h_{\alpha_3}(-1)$, it readily follows, as in Lemma 4.9, that (c) holds and we are done.

LEMMA 4.14. *Let $X = \mathrm{Spin}(8, q, -1)$. Then the following four conditions hold.*

(a) *$Z(X) = \langle z \rangle$ where z is an involution*;

(b) *all involutions in $X - Z(X)$ are conjugate in X*;

(c) *if $\tau \in \mathfrak{I}(X - Z(X))$, then $C_X(\tau)$ possesses precisely three 2-components J_1, J_2, J_3. Also the 2-components of $C_X(\tau)$ can be indexed so as to satisfy the following three conditions*:

(i) *$J_1 \cong \mathrm{SL}(2, q^2)$ and $Z(J) = \langle z \rangle$*;

(ii) *$J_2 \cong \mathrm{SL}(2, q)$ and $Z(J_2) = \langle \tau \rangle$*; *and*

(iii) *$J_3 \cong \mathrm{SL}(2, q)$ and $Z(J_3) = \langle \tau z \rangle$*; *and*

(d) *if $\gamma \in X$ is such that $\gamma^2 = z$, then $C_X(\gamma)$ contains a unique 2-component L with $L \cong \mathrm{SL}(4, q)$ or $L \cong \mathrm{SU}(4, q)$.*

PROOF. Assume the notation of Lemma 4.13 and let ρ denote the graph automorphism of order 2 of the root system of D_4 such that $\rho(\alpha_i) = \alpha_i$ for $i = 1, 2$ and $\rho(\alpha_3) = \alpha_4$. Then ρ induces a semisimple automorphism of $\overline{G}$, which we shall also denote by ρ, such that if $\alpha \in \Phi$ and $u \in K$, then $\rho(x_\alpha(u)) = x_{\rho(u)}(\alpha)$. Also, as endomorphisms of $\overline{G}$, we have $\sigma\rho = \rho\sigma$ and $(\sigma\rho)^2 = \sigma^2$ and we may take $\overline{X} = \overline{G}_{\sigma\rho} = C_{\overline{G}}(\sigma\rho)$. Then (a) with $z = h_{\alpha_3}(-1)h_{\alpha_4}(-1)$ and (b) follow from Lemma 4.13. Also $\tau = h_{\alpha_0}(-1) \in \mathfrak{I}(X - Z(X))$ and $C_X(\tau) = C_{\overline{G}}(\tau)_{\sigma\rho}$. Since $C_{\overline{G}}(\tau) = \overline{L}\overline{H}$ where $\overline{L} = \overline{J}_0 * \overline{J}_1 * \overline{J}_3 * \overline{J}_4$ and $\mathrm{SL}(2, K) \cong \overline{J}_i = \langle \overline{\mathfrak{X}}_{\alpha_i}, \overline{\mathfrak{X}}_{-\alpha_i} \rangle \trianglelefteq C_{\overline{G}}(t)$ for all $i \in \{0, 1, 3, 4\}$, the methods of [**30**, §§2 and 8] imply that $C_X(\tau) = (\mathfrak{J}_0 * \mathfrak{J}_1 * \mathfrak{J})H$ where $H = C_{\overline{H}}(\sigma\rho)$, $\mathrm{SL}(2, q) \cong C_{\overline{J}_0}(\sigma\rho) = C_{\overline{J}_0}(\sigma) = \mathfrak{J}_0$, $\mathrm{SL}(2, q) \cong C_{\overline{J}_1}(\sigma\rho) = C_{\overline{J}_1}(\sigma) = \mathfrak{J}_1$, $Z(\mathfrak{J}_0) = \langle \tau \rangle$, $Z(\mathfrak{J}_1) = \langle \tau z \rangle$, $\mathrm{SL}(2, q^2) \cong C_{\overline{J}_3\overline{J}_4}(\sigma\rho) = \mathfrak{J} = \langle x_{\alpha_3}(u)x_{\alpha_4}(\sigma(u)),\ x_{-\alpha_3}(u)x_{-\alpha_4}(\sigma(u)) \mid u \in k_1 \rangle$ and $Z(\mathfrak{J}) = \langle h_{\alpha_3}(-1)h_{\alpha_4}(-1) \rangle = \langle z \rangle$. Thus (c) holds.

For (d), let ν be an element of order 4 in $K^\times$. Lemma 4.8 implies that $\overline{G}$ contains precisely two conjugacy classes of elements γ of order 4 such that $\gamma^2 = z$ and these two conjugacy classes have representatives

$$\gamma_1 = h(\chi_{\lambda_1}) = h_{\alpha_1}(-1)h_{\alpha_3}(\nu^3)h_{\alpha_4}(\nu^3),$$

$$\text{where } \lambda_1 = \frac{q-1}{4}(2\overline{\omega}_3 + 2\overline{\omega}_4) = \frac{q-1}{2}(\overline{\omega}_3 + \overline{\omega}_4)$$

and

$$\gamma_2 = h(\chi_{\lambda_2}) = h_{\alpha_1}(-1)h_{\alpha_2}(-1)h_{\alpha_3}(\nu)h_{\alpha_4}(\nu), \quad \text{where } \lambda_2 = \frac{q-1}{4}(2\overline{\omega}_1) = \frac{q-1}{2}\overline{\omega}_1.$$

Also we have $C_{\overline{G}}(\gamma_1) = \overline{L}_1\overline{H}$ and $C_{\overline{G}}(\gamma_2) = \overline{L}_2\overline{H}$ where $\gamma_1 \in \overline{L}_1 = \langle \overline{\mathfrak{X}}_{\pm\alpha_0}, \overline{\mathfrak{X}}_{\pm\alpha_1}, \overline{\mathfrak{X}}_{\pm\alpha_2} \rangle \cong \mathrm{SL}(4, K)$, $\gamma_2 \notin \overline{L}_2 = \langle \overline{\mathfrak{X}}_{\pm\alpha_2}, \overline{\mathfrak{X}}_{\pm\alpha_3}, \overline{\mathfrak{X}}_{\pm\alpha_4} \rangle$ is of type A_3 and $z \in \overline{L}_2$. Suppose that $q \equiv 1 \pmod 4$. Then $\nu \in k$, $\gamma_1 \in C_{\overline{L}_1}(\rho\sigma) \cong \mathrm{SL}(4, q)$ and $z \in C_{\overline{L}_2}(\rho\sigma) \cong \mathrm{SU}(4, q)$ by [**35**, §11]. It follows that (d) holds in this case. Finally, suppose that $q \equiv -1 \pmod 4$. Then $\nu \notin k$ and $\rho\sigma$ inverts both γ_1 and γ_2. Also, from [**8**, Planche IV], it follows that there is an element $w \in \overline{N}$ with coefficients in the prime field of K such that $w^2 \in \overline{H}$ and such that the w induced automorphism $\tilde{w}$ of Φ satisfies $\tilde{w}$: $\alpha_1 \leftrightarrow -\alpha_0$, $\tilde{w}(\alpha_2) = \alpha_2$ and $\tilde{w}$: $\alpha_3 \leftrightarrow \alpha_4$. Then, by [**34**, p. 30], we have $\gamma_i^w = \gamma_i^{-1}$ for $i = 1, 2$. Letting β denote the inner automorphism of $\overline{G}$ induced by conjugation by w, it follows that $\beta\rho\sigma$ fixes both γ_1 and γ_2 and $\beta\rho\sigma$ is conjugate via $\mathrm{Inn}(\overline{G})$ to $\rho\sigma$ by Lemma 4.1. Since $(\overline{L}_1)_{\beta\rho\sigma} \cong \mathrm{SU}(4, q)$ and $(\overline{L}_2)_{\beta\rho\sigma} \cong \mathrm{SL}(4, q)$ by [**35**, §11], we have (d) in this case also.

Lemma 4.15. *Let* $X = \mathrm{Spin}(2m, q, 1)$ *for some even integer* $m \geqslant 6$. *Then the following three conditions hold.*

(a) $Z(X) \cong E_4$ *and* $Z(X) = \langle t_1, t_2 \rangle$ *where* t_1 *and* t_2 *are distinct commuting involutions of* X;

(b) *there is a unique conjugacy class of involutions* $\mathfrak{K}$ *of* X *such that if* $\tau \in \mathfrak{K}$, *then* $C_X(\tau)$ *possesses a* 2-*component* J *with* $\tau \in J$ *and* $J/O(J) \cong \mathrm{SL}(2, q)$; *and*

(c) *if* $\mathfrak{K}$ *and* τ *are as in* (b), *then* $O(J) = 1$, J *is the unique* 2-*component of* $C_X(\tau)$ *that contains* τ, $C_X(\tau)$ *contains precisely two other* 2-*components* J_1 *and* J_2 *such that, by appropriate indexing, we may assume*: $J_1 \cong \mathrm{SL}(2, q)$ *and* $J_2 \cong \mathrm{Spin}(2m-4, q, 1)$ *and, by appropriate indexing of* $Z(X)^\#$, *we may also assume*: $Z(J_1) = \langle \tau t_1 \rangle$ *and* $Z(J_2) = \langle \tau t_2, t_1 \rangle$.

Proof. Let $\mathfrak{G} = D_m$, let π denote the simply connected representation of $\mathfrak{G}$ so that $\overline{G}$ is a simply connected linear algebraic group and let $B = \{\alpha_1, \ldots, \alpha_m\}$ be as in [**8**, Planche IV]. Then $\Gamma_{\mathrm{ad}} = \sum_{i=1}^{m} Z\alpha_i$ and $\Gamma_\pi = \Gamma_{\mathrm{sc}} = \sum_{i=1}^{m} Z\overline{\omega}_i$ where $\{\overline{\omega}_i \mid 1 \leqslant i \leqslant m\}$ is as given in [**8**, Planche IV], $\varepsilon_i = \overline{\omega}_i$ for all $1 \leqslant i \leqslant m$, $\Gamma_{\mathrm{ad}}^{\perp} = \Gamma_\pi$, $G = \overline{G}_\sigma \cong X$, $\Omega_\pi = 1$ and $\alpha_0 = \alpha_1 + \sum_{j=2}^{m-2} 2\alpha_j + \alpha_{m-1} + \alpha_m$, etc. Then, as in Lemma 4.9, $\overline{G}$ and G have precisely $3 + (m-2)/2$ conjugacy classes of involutions represented by

$$\mathfrak{B} = \left\{ h(\chi_\lambda) \mid \lambda = \frac{q-1}{2}\overline{\omega}_i \text{ with } i \text{ even and } 2 \leqslant i \leqslant m-2 \right\}$$
$$\cup \left\{ h(\chi_{\lambda_i}) \mid \lambda_i = \frac{q-1}{2}(2\overline{\omega}_i) \text{ with } i \in \{1, m-1, m\} \right\}.$$

Also $Z(G) \cong E_4$, $h(\chi_{\lambda_1}) = h_{\alpha_{m-1}}(-1)h_{\alpha_m}(-1)$, $h(\chi_{\lambda_{m-1}}) = h_{\alpha_1}(-1)h_{\alpha_3}(-1) \cdots h_{\alpha_{m-3}}(-1)h_{\alpha_{m-1}}(-1)^{m/2}h_{\alpha_m}(-1)^{(m-2)/2}$, $h(\chi_{\lambda_m}) = h(\chi_{\lambda_{m-1}})h(\chi_{\lambda_1})$ and $Z(G) = \langle h(\chi_{\lambda_1}), h(\chi_{\lambda_{m-1}})\rangle$. In addition, it is easy to see that $\mu = h(\chi_\lambda)$ with $\lambda = \frac{q-1}{2}\overline{\omega}_2$ is the unique involution of $\mathfrak{B}$ satisfying the conditions required in (b) and that the conditions of (c) hold in $C_G(\mu) = (C_{\overline{G}}(\mu))_\sigma$.

LEMMA 4.16. *Let* $X = \mathrm{Spin}(2m, q, -1)$ *for some even integer* $m \geqslant 6$. *Then the following five conditions hold.*

(a) $Z(X) = \langle z \rangle$ *where* z *is an involution*;

(b) *there is a unique conjugacy class* $\mathfrak{K}_1$ *of involutions in* X *such that if* $\tau \in \mathfrak{K}_1$, *then* $C_X(\tau)$ *possesses a* 2*-component* J *with* $\tau \in J$ *and* $J/O(J) \cong \mathrm{SL}(2, q)$;

(c) *if* $\mathfrak{K}_1$, τ *and* J *are as in* (b), *then* $O(J) = 1$, J *is the unique* 2*-component of* $C_X(\tau)$ *that contains* τ *and* $C_X(\tau)$ *contains, besides* J, *precisely two other* 2*-components* J_1 *and* J_2 *which may be indexed so that* $J_1 \cong \mathrm{SL}(2, q)$, $J_2 \cong \mathrm{Spin}(2m-4, q, -1)$, $Z(J_1) = \langle \tau z \rangle$ *and* $Z(J_2) = \langle z \rangle$;

(d) *there is a unique conjugacy class* $\mathfrak{K}_2$ *of involutions in* X *such that if* $\tau \in \mathfrak{K}_2$, *then* $C_X(\tau)$ *possesses a* 2*-component* J *with* $z \in J$ *and* $J/O(J) \cong \mathrm{SL}(2, q_1)$ *where* $q_1 = p^r$ *for some positive integer* r; *and*

(e) *if* $\mathfrak{K}_2$, τ *and* J *are as in* (d), *then* $O(J) = 1$, $q_1 = q^2$ *and the following two conditions hold*:

(i) $C_X(\tau)$ *possesses, besides* J, *exactly one other* 2*-component* $\mathfrak{J}$; *and*

(ii) $\mathfrak{J} \cong \mathrm{Spin}(2m-4, q, 1)$ *and* $Z(\mathfrak{J}) = \langle \tau, z \rangle$.

PROOF. Assume the notation of Lemma 4.15 and let ρ denote the graph automorphism of order 2 of the root system of D_m (cf. [**11**, §12.2]). Then ρ induces a semisimple automorphism of $\overline{G}$, which we shall also denote by ρ, such that if $\alpha \in \Phi$ and $u \in K$, then $\rho(x_\alpha(u)) = x_{\rho(\alpha)}(u)$. Also as endomorphisms of $\overline{G}$, we have $\rho\sigma = \sigma\rho$ and $(\sigma\rho)^2 = \sigma^2$ and we may take $X = \overline{G}_{\sigma\rho}$. Note that $Z(\overline{G}) = \langle h(\chi_{\lambda_1}), h(\chi_{\lambda_{m-1}})\rangle$, $Z(\overline{G})_{\rho\sigma} = \langle h(\chi_{\lambda_1})\rangle$ and for each even integer i with $2 \leqslant i \leqslant m-2$ and for $\lambda_i = \frac{q-1}{2}\overline{\omega}_i$, we have

$$h(\chi_{\lambda_i}) = h_{\alpha_1}(-1)h_{\alpha_3}(-1) \cdots h_{\alpha_{i-1}}(-1)\left(\left(h_{\alpha_{m-1}}(-1)h_{\alpha_m}(-1)\right)^{i/2}\right).$$

Thus $Z(X) = \langle z \rangle$ where $z = h(\chi_{\lambda_1}) = h_{\alpha_{m-1}}(-1)h_{\alpha_m}(-1)$ and X possesses precisely $1 + (m-2)/2$ conjugacy classes of involutions represented by $\mathfrak{B} = \{h(\chi_\lambda) \mid \lambda = \frac{q-1}{2}\overline{\omega}_i$ with i even and $2 \leqslant i \leqslant m-2\} \cup \{z\}$. Thus (a) holds and it follows via arguments similar to those above that $h(\chi_{\lambda_2}) = h_{\alpha_0}(-1)$ is the unique involution of $\mathfrak{B}$ satisfying the conditions required in (b), that $h(\chi_{\lambda_{m-2}})$ is the unique involution of $\mathfrak{B}$ satisfying the conditions required in (d) and that (c) and (e) also hold.

LEMMA 4.17. *Let $X = \mathrm{Spin}(2m, q, 1)$ for some odd integer $m \geqslant 5$. Also let $Z(X) = \langle \gamma \rangle$ where $|\gamma| = (4, q-1)$ and let z be the unique involution in $Z(X)$. Then the following three conditions hold.*

(a) *There is a unique conjugacy class $\mathfrak{K}$ of involutions of X such that if $\tau \in \mathfrak{K}$, then $C_X(\tau)$ contains a 2-component J with $\tau \in J$ and $J/O(J) \cong \mathrm{SL}(2, q)$;*

(b) *if $\mathfrak{K}$, τ and J are as in* (a), *then $O(J) = 1$, J is the unique 2-component of $C_X(\tau)$ that contains τ and $C_X(\tau)$ contains, besides J, precisely two other 2-components J_1 and J_2. Also by suitable indexing we may assume $J_1 \cong \mathrm{SL}(2, q)$, $J_2 \cong \mathrm{Spin}(2m-4, q, 1)$, $Z(J_1) = \langle \tau z \rangle$, $Z(J_2) = \langle z \rangle$ if $q \equiv -1 \pmod 4$ and $Z(J_2) = \langle \tau\gamma \rangle$ if $q \equiv 1 \pmod 4$; and*

(c) *there does not exist an involution $\tau \in X$ such that $C_X(\tau)$ possesses a 2-component J with $z \in J$ and $J/O(J) \cong \mathrm{SL}(2, q^2)$.*

PROOF. Let $\mathfrak{G} = D_m$, let π denote the simply connected representation of $\mathfrak{G}$ so that $\overline{G}$ is a simply connected linear algebraic group and let $B = \{\alpha_1, \ldots, \alpha_m\}$ be as in [8, Planche IV]. Then $\Gamma_{\mathrm{ad}} = \sum_{i=1}^m Z\alpha_i$ and $\Gamma_\pi = \Gamma_{\mathrm{sc}} = \sum_{i=1}^m Z\overline{\omega}_i$ where $\{\overline{\omega}_i \mid 1 \leqslant i \leqslant m\}$ is as given in [8, Planche IV], $\varepsilon_i = \overline{\omega}_i$ for all $1 \leqslant i \leqslant m$, $\Gamma_{\mathrm{ad}}^{\perp} = \Gamma_\pi$, $G = \overline{G}_\sigma \cong X$, $\Omega_\pi = 1$ and $\alpha_0 = \alpha_1 + \sum_{j=2}^{m-2} 2\alpha_j + \alpha_{m-1} + \alpha_m$, etc. Let ν be an element of order 4 in $K^\times$. Then $Z(\overline{G}) = \langle h(\chi_\beta) \rangle \cong Z_4$ where $\beta = \frac{q-1}{4}(4\overline{\omega}_m) = (q-1)\overline{\omega}_m$,

$$h(\chi_\beta) = h_{\alpha_1}(-1)h_{\alpha_3}(-1) \cdots h_{\alpha_{m-1}}(\nu^{m-2})h_{\alpha_m}(\nu^m)$$

and $h(\chi_\beta)^2 = h_{\alpha_{m-1}}(-1)h_{\alpha_m}(-1)$. Also $Z(G) = Z(\overline{G})$ if $4 \mid (q-1)$ and $Z(G) = \langle h(\chi_\beta)^2 \rangle$ if $4 \nmid (q-1)$. Moreover, as in Lemma 4.9, $\overline{G}$ and G have precisely $2 + (m-3)/2$ conjugacy classes of involutions represented by

$$\mathfrak{B} = \left\{ h(\chi_\lambda) \mid \lambda = \frac{q-1}{2}\overline{\omega}_i \text{ with } i \text{ even and } 2 \leqslant i \leqslant m-3 \right\}$$
$$\cup \left\{ h(\chi_{\lambda_1}), h(\chi_{\lambda_2}) \mid \lambda_1 = \frac{q-1}{2}(2\overline{\omega}_1) \text{ and } \lambda_2 = \frac{q-1}{2}(\overline{\omega}_{m-1} + \overline{\omega}_m) \right\}.$$

Here $h(\chi_{\lambda_1}) = h_{\alpha_{m-1}}(-1)h_{\alpha_m}(-1) = h(\chi_\beta)^2$ and $\mu = h(\chi_\lambda)$ with $\lambda = \frac{q-1}{2}\overline{\omega}_2$ is the unique involution of $\mathfrak{B}$ satisfying the conditions required in (a). Finally, since $\mu = h_{\alpha_1}(-1)h(\chi_\beta)^2 = h_{\alpha_0}(-1)$, both (b) and (c) readily follow.

LEMMA 4.18. *Let $X = \mathrm{Spin}(2m, q, -1)$ for some odd integer $m \geqslant 5$. Also let $Z(X) = \langle \gamma \rangle$ where $|\gamma| = (4, q+1)$ and let z be the unique involution in $Z(X)$. Then the following three conditions hold.*

(a) *There is a unique conjugacy class of involutions $\mathfrak{K}$ of X such that if $\tau \in \mathfrak{K}$, then $C_X(\tau)$ contains a 2-component J with $\tau \in J$ and $J/O(J) \cong \mathrm{SL}(2, q)$;*

(b) *if $\mathfrak{K}$, τ and J are as in* (a), *then $O(J) = 1$, J is the unique 2-component of $C_X(\tau)$ that contains τ and $C_X(\tau)$ contains, besides J, precisely two other 2-elements J_1 and J_2. Also by suitable indexing we may assume $J_1 \cong \mathrm{SL}(2, q)$, $J_2 \cong \mathrm{Spin}(2m-4, q, -1)$, $Z(J_1) = \langle \tau z \rangle$, $Z(J_2) = \langle z \rangle$ if $q \equiv 1 \pmod 4$ and $Z(J_2) = \langle \tau\gamma \rangle$ if $q \equiv -1 \pmod 4$; and*

(c) *there does not exist an involution $\tau \in X$ such that $C_X(\tau)$ possesses a 2-component J with $z \in J$ and $J/O(J) \cong \mathrm{SL}(2, q^2)$.*

PROOF. Assume the notation of Lemma 4.17, let ρ denote the graph automorphism of order 2 of the root system of D_m (cf. [**11**, §12.2]) and assume that ρ induces an automorphism of $\bar{G}$ as in the previous lemmas, etc. Then we may take $X = \bar{G}_{\sigma\rho}$, $Z(X) = \langle h_{\alpha_{m-1}}(-1)h_{\alpha_m}(-1)\rangle$ if $q \equiv 1 \pmod 4$ and $Z(X) = \langle h(\chi_\beta)\rangle$ if $q \equiv -1 \pmod 4$ and both $\bar{G}$ and X have precisely $2 + (m-3)/2$ conjugacy classes of involutions with representatives as in the proof of Lemma 4.17. Also the methods of the proof of Lemma 4.14 and the information in the proof of Lemma 4.17 readily yield (a)–(c).

LEMMA 4.19. *Let $X = E_8(q)$, let $\mathfrak{A} = \mathrm{Aut}(X)$ and let $\mathfrak{B} = \mathrm{Inn}(X)$. Then the following five conditions hold.*

(a) $\mathfrak{A} = \mathfrak{B}A$ *where* $A \cap \mathfrak{B} = 1$, $X \cong \mathfrak{B} = \mathfrak{A}'$ *and A is the subgroup of $\mathfrak{A}$ induced by* $\mathrm{Aut}(k)$, *so that* $A \cong \mathrm{Aut}(k) \cong Z_n$;

(b) $\mathfrak{B}$ *contains involutions t and v such that $C_{\mathfrak{B}}(t)$ possesses precisely two 2-components J_1 and J_2 and by appropriate indexing we may assume that* $Z(J_1) = Z(J_2) = \langle t\rangle$, $J_1 \cong \mathrm{SL}(2, q)$, $J_2 \cong \mathrm{Cov}(E_7(q))$, $|C_{\mathfrak{B}}(t)/(J_1 * J_2)| = 2$ *and* $C_{\mathfrak{B}}(J_1 * J_2) = \langle t\rangle$ *and such that* $Z(C_{\mathfrak{B}}(v)') = \langle v\rangle$, $c_{\mathfrak{B}}(v)'$ *is a proper quotient of* $\mathrm{Spin}(16, q, 1)$, $|C_{\mathfrak{B}}(v)/C_{\mathfrak{B}}(v)'| = 2$ *and* $C_{\mathfrak{B}}(C_{\mathfrak{B}}(v)') = \langle v\rangle$;

(c) $\{t, v\}$ *is a set of representatives for the conjugacy classes of involutions of* $\mathfrak{B}$;

(d) *if n is odd, then* $O^{2'}(\mathfrak{A}) = \mathfrak{B}$; *and*

(e) *if n is even, then the unique involution $\tau \in A$ is such that* $C_{\mathfrak{B}}(\tau) \cong E_8(\sqrt{q})$ *and* $\mathfrak{J}(\mathfrak{A} - b) = \tau^{\mathfrak{B}}$.

PROOF. Clearly (a), (d) and (e) follow from Lemma 4.2, [**35**, Theorem 12.4; **11**, Theorem 12.5.1]. Also (b) and (c) follow from [**30**, §9 and Proposition 9(ii)] or the methods of this section.

The same references in the proof just above yield

LEMMA 4.20. *Let $X = F_4(q)$, let $\mathfrak{A} = \mathrm{Aut}(X)$ and let $\mathfrak{B} = \mathrm{Inn}(X)$. Then the following five conditions hold.*

(a) $\mathfrak{A} = \mathfrak{B}A$ *where* $A \cap \mathfrak{B} = 1$, $X \cong \mathfrak{B} = \mathfrak{A}'$ *and A is the subgroup of $\mathfrak{A}$ induced by* $\mathrm{Aut}(k)$, *so that* $A \cong \mathrm{Aut}(k) \cong Z_n$;

(b) $\mathfrak{B}$ *contains involutions t and v such that* $C_{\mathfrak{B}}(t) \cong \mathrm{Spin}(9, q)$ *with* $Z(C_{\mathfrak{B}}(t)) = \langle t\rangle$, $C_{\mathfrak{B}}(v)$ *has precisely two 2-components J_1 and J_2 and by appropriate indexing we may assume that* $Z(J_1) = Z(J_2) = \langle v\rangle$, $J_1 \cong \mathrm{SL}(2, q)$, $J_2 \cong \mathrm{Sp}(6, q)$, $|C_{\mathfrak{B}}(v)/(J_1 * J_2)| = 2$ *and* $C_{\mathfrak{B}}(J_1 * J_2) = \langle v\rangle$;

(c) $\{t, v\}$ *is a set of representatives for the conjugacy classes of involutions of* $\mathfrak{B}$;

(d) *if n is odd, then* $O^{2'}(\mathfrak{A}) = \mathfrak{B}$; *and*

(e) *if n is even, then the unique involution $\tau \in A$ is such that* $C_{\mathfrak{B}}(\tau) \cong F_4(\sqrt{q})$ *and* $\mathfrak{J}(\mathfrak{A} - \mathfrak{B}) = \tau^{\mathfrak{B}}$.

LEMMA 4.21. *Let $X = {}^3D_4(q)$, let $\mathfrak{A} = \mathrm{Aut}(X)$ and let $\mathfrak{B} = \mathrm{Inn}(X)$. Also let* $k_2 = C_K(\sigma^3)$, *so that* $|k_2| = q^3$. *Then the following four conditions hold.*

(a) $\mathfrak{A} = \mathfrak{B}A$ *where* $A \cap \mathfrak{B} = 1$, $X \cong \mathfrak{B} = \mathfrak{A}'$ *and A is the subgroup of $\mathfrak{A}$ induced by* $\mathrm{Aut}(k_2)$, *so that* $A \cong \mathrm{Aut}(k_2) \cong Z_{3n}$;

(b) $\mathfrak{B}$ *has one conjugacy class of involutions and if* $t \in \mathfrak{J}(\mathfrak{B})$, *then the structure of* $C_{\mathfrak{B}}(t)$ *is as given in* [**14**]; *in particular,* $C_{\mathfrak{B}}(t)$ *has exactly two 2-components J_1 and J_2*

which may be indexed so that $Z(J_1) = Z(J_2) = \langle t \rangle$, $J_1 \cong \mathrm{SL}(2, q)$, $J_2 \cong \mathrm{SL}(2, q^3)$, $|C_{\mathfrak{B}}(t)/(J_1 * J_2)| = 2$ *and* $C_{\mathfrak{B}}(J_1 * J_2) = \langle t \rangle$;

(c) *if n is odd, then* $O^{2'}(\mathfrak{A}) = \mathfrak{B}$; *and*

(d) *if n is even, then the unique involution* $\tau \in A$ *is such that* $C_{\mathfrak{B}}(\tau) \cong {}^3D_4(\sqrt{q})$ *and* $\mathfrak{I}(\mathfrak{A} - \mathfrak{B}) = \tau^{\mathfrak{B}}$.

PROOF. Clearly $X \cong \mathfrak{B}$ and (b) follows from the methods above or [**14**, Theorem and (2A)(iii)]. Also (a) follows from results of R. Steinberg, [**34**, Theorem 36]. Thus (c) holds. Suppose that $n = 2n_1$ for some integer n_1 and τ is the unique involution of A. Then $C_{\mathfrak{B}}(\tau) \cong {}^3D_4(\sqrt{q})$ and τ is induced from the field automorphism of k_2 of order 2. Let $\mathfrak{D} = \mathfrak{B}\langle \tau \rangle$, so that $\mathfrak{I}(\mathfrak{A} - \mathfrak{B}) = \mathfrak{I}(\mathfrak{D} - \mathfrak{B})$.

To conclude the proof, we shall apply Lemma 4.2 as follows. Let $\mathfrak{G} = D_4$, let π denote the adjoint representation of $\mathfrak{G}$ and let $\bar{G}$ denote the Chevalley group obtained from $(\mathfrak{G}, \pi, K)$. Let $q_0 = p^{n_1}$ and $\sigma_0 = \lambda^{n_1} \in \mathrm{Aut}(K)$ so that $\sigma_0(u) = u^{q_0}$ for any $u \in K$ and denote the σ_0-induced endomorphism of $\bar{G}$ also by σ_0. Then $\sigma_0^2 = \sigma$ and $C_K(\sigma_0) = k_0$ has order q_0. Also let ρ denote the element of $\mathrm{Aut}(\bar{G})$ induced by the graph automorphism of order 3 (cf. [**35**, §11; **11**, Proposition 12.2.3]). Then $\rho^3 = 1$, $\rho\sigma_0 = \sigma_0\rho$ as endomorphisms of $\bar{G}$ (cf. [**11**, p. 225]), $(\rho^2\sigma_0)^2 = \rho\sigma$, $(\rho\sigma)^3 = \sigma^3$, $C_{\bar{G}}(\rho\sigma, \sigma_0\sigma) = C_{\bar{G}}(\rho^2\sigma_0) < C_{\bar{G}}(\rho\sigma) < C_{\bar{G}}(\sigma^3)$ and $(\sigma_0\sigma)^2 = \sigma^3$. Also $C_{\bar{G}}(\rho\sigma) \cong {}^3D_4(q)$ by [**34**, §11, Theorem 35 and Corollary] and Lemma 4.2 now implies that $\mathfrak{I}(\mathfrak{D} - \mathfrak{B}) = \tau^{\mathfrak{B}}$. The proof of this lemma is now complete.

LEMMA 4.22. *Let* $X = G_2(q)$, *let* $\mathfrak{A} = \mathrm{Aut}(X)$ *and let* $\mathfrak{B} = \mathrm{Inn}(X)$. *Then the following four conditions hold.*

(a) $X \cong \mathfrak{B} = \mathfrak{A}'$ *and* $\mathfrak{A}/\mathfrak{B}$ *is cyclic*;

(b) $\mathfrak{B}$ *has one conjugacy class of involutions and if* $t \in \mathfrak{I}(\mathfrak{B})$, *then* $C_{\mathfrak{B}}(t)$ *has precisely two* 2*-components* J_1 *and* J_2 *and by appropriate indexing we may assume that* $Z(J_1) = Z(J_2) = \langle t \rangle$, $J_1 \cong J_2 \cong \mathrm{SL}(2, q)$, $|C_{\mathfrak{B}}(t)/(J_1 * J_2)| = 2$ *and* $C_{\mathfrak{B}}(J_1 * J_2) = \langle t \rangle$;

(c) *if* $p \neq 3$, *then* $\mathfrak{A} = \mathfrak{B}A$ *where* $\mathfrak{B} \cap \mathfrak{A} = 1$ *and A is the subgroup of* $\mathfrak{A}$ *induced by* $\mathrm{Aut}(k)$, *so that* $A \cong \mathrm{Aut}(k) \cong Z_n$, $O^{2'}(\mathfrak{A}) = \mathfrak{B}$ *when n is odd and, when n is even, the unique involution* $\tau \in A$ *is such that* $C_{\mathfrak{B}}(\tau) \cong G_2(\sqrt{q})$ *and* $\mathfrak{I}(\mathfrak{A} - \mathfrak{B}) = \tau^{\mathfrak{B}}$; *and*

(d) *if* $p = 3$, *then* $\mathfrak{A} = \mathfrak{B}A$ *where* $\mathfrak{B} \cap A = 1$, $A \cong Z_{2n}$ *and the unique involution* $\tau \in A$ *is such that* $\mathfrak{I}(\mathfrak{A} - \mathfrak{B}) = \tau^{\mathfrak{B}}$, $C_{\mathfrak{B}}(\tau) \cong G_2(\sqrt{q})$ *if n is even and* $C_{\mathfrak{B}}(\tau) \cong {}^2G_2(q)$ *if n is odd where* ${}^2G_2(q)$ *is simple if* $q > 3$ *and* ${}^2G_2(3) \cong \mathrm{Aut}(\mathrm{PSL}(2, 8))$.

PROOF. Clearly $X \cong \mathfrak{B}$ and (b) follows from [**30**, §9 and Proposition 9(ii)] or [**14**, §2]. Suppose that $p \neq 3$. Then Lemma 4.2 and the proof of [**11**, Theorem 12.5.1] imply (c). Suppose that $p = 3$. Then [**11**, Proposition 12.4.1, Theorem 12.5.1 and p. 225] imply that $\mathfrak{A} = \mathfrak{B}A$ where $A \cap \mathfrak{B} = 1$ and $A = \langle g \rangle$ where g^2 is the automorphism of X induced by $\lambda \in \mathrm{Aut}(K)$. Since $\mathrm{Aut}(k) = \langle \lambda \rangle \cong Z_n$, we have (d) when n is even. When n is odd, then Lemma 4.2, [**11**, p. 225 and Lemma 14.1.1; **31**, Theorem 7.8 and (8.4); **11**, Proposition 12.4.1 and p. 268, Note] and the fact that a Sylow 3-subgroup of ${}^2G_2(3)$ is nonabelian by [**31**, (5.5)] yield (d). Clearly (a) holds and we are done.

LEMMA 4.23. *Suppose that* $q = 3^n$ *where* $n = 2m + 1$ *for some integer* $m \geqslant 1$. *Let* $X = {}^2G_2(q)$, *let* $\mathfrak{A} = \mathrm{Aut}(X)$ *and let* $\mathfrak{B} = \mathrm{Inn}(X)$. *Then the following two conditions hold.*

(a) $\mathfrak{A} = \mathfrak{B}A$ *where* $A \cap \mathfrak{B} = 1$, $X \cong \mathfrak{B} = \mathfrak{A}' = O^{2'}(\mathfrak{A})$ *and* A *is the subgroup of* $\mathfrak{A}$ *induced by* $\mathrm{Aut}(k)$, *so that* $A \cong \mathrm{Aut}(k) \cong Z_n$; *and*

(b) $\mathfrak{B}$ *has one conjugacy class of involutions and if* $t \in \mathfrak{I}(\mathfrak{B})$, *then* $C_{\mathfrak{B}}(t) = \langle t \rangle \times J$ *where* $J \cong \mathrm{PSL}(2, q)$.

PROOF. Clearly [**31**, Theorem 9.1] implies (a) and (b) is well known (cf. [**16**, §16.6] or [**10**, Appendix 1]).

LEMMA 4.24. *Let* $X = E_7(q)$, *let* $\mathfrak{A} = \mathrm{Aut}(X)$ *and let* $\mathfrak{B} = \mathrm{Inn}(X)$. *Then the following four conditions hold.*

(a) $\mathfrak{A}' = \mathfrak{B}$, $\mathfrak{A}/\mathfrak{B} \cong Z_2 \times Z_n$ *and there is a subgroup* $\mathfrak{C}$ *of* $\mathfrak{A}$ *such that* $\mathfrak{B} < \mathfrak{C}$, $|\mathfrak{C}/\mathfrak{B}| = 2$, $\mathfrak{C} = \langle \mathfrak{I}(\mathfrak{C}) \rangle$ *and* $\mathfrak{A} = \mathfrak{C}A$ *where* $\mathfrak{C} \cap A = 1$ *and* A *is the subgroup of* $\mathfrak{A}$ *induced by* $\mathrm{Aut}(k)$, *so that* $A \cong \mathrm{Aut}(k) \cong Z_n$;

(b) *if* n *is odd, then* $O^{2'}(\mathfrak{A}) = \mathfrak{C}$;

(c) *if* n *is even, then the unique involution* $\tau \in A$ *is such that* $C_{\mathfrak{B}}(\tau) \cong E_n(\sqrt{q})$ *and* $\mathfrak{I}(\mathfrak{A} - \mathfrak{C}) = \tau^{\mathfrak{C}}$; *and*

(d) $\mathfrak{C}$ *contains five conjugacy classes of involutions which may be represented by* $\{t_i \mid 1 \leqslant i \leqslant 5\}$ *such that*

(1) $t_1 \in \mathfrak{B}$, $C_{\mathfrak{B}}(t_1)$ *contains precisely two* 2-*components* J_1 *and* J_2 *which may be indexed so that* $Z(J_1) = Z(J_2) = \langle t_1 \rangle$, $J_1 \cong \mathrm{SL}(2, q)$, J_2 *is a proper quotient of* $\mathrm{Spin}(12, q, 1)$, $|C_{\mathfrak{C}}(t_1)/(J_1 * J_2)| = 2$ *and* $C_{\mathfrak{C}}(J_1 * J_2) = \langle t_1 \rangle$;

(2) $t_2 \in \mathfrak{B}$, $C_{\mathfrak{B}}(t_2)$ *contains exactly one* 2-*component* J *such that* J *is a quotient of* $\mathrm{SL}(8, q)$ *if* $q \equiv 1 \pmod 4$ *and of* $\mathrm{SU}(8, q)$ *if* $q \equiv -1 \pmod 4$ *and* $C_{\mathfrak{C}}(t_2) \cap C_{\mathfrak{C}}(J)$ *is cyclic*;

(3) $t_3 \in \mathfrak{B}$, $C_{\mathfrak{B}}(t_3)$ *contains exactly one* 2-*component* J *such that* J *is a quotient of* $\mathrm{Cov}(E_6(q))$ *if* $q \equiv 1 \pmod 4$ *and of* $\mathrm{Cov}({}^2E_6(q))$ *if* $q \equiv -1 \pmod 4$ *and* $C_{\mathfrak{C}}(t_3) \cap C_{\mathfrak{C}}(J)$ *is cyclic*;

(4) $t_4 \in \mathfrak{C} - \mathfrak{B}$, $C_{\mathfrak{B}}(t_4)$ *contains exactly one* 2-*component* J *such that* J *is a quotient of* $\mathrm{SU}(8, q)$ *if* $q \equiv 1 \pmod 4$ *and of* $\mathrm{SL}(8, q)$ *if* $q \equiv -1 \pmod 4$ *and* $C_{\mathfrak{C}}(t_4) \cap C_{\mathfrak{C}}(J)$ *is cyclic*; *and*

(5) $t_5 \in \mathfrak{C} - \mathfrak{B}$, $C_{\mathfrak{B}}(t_5)$ *contains exactly one* 2-*component* J *such that* J *is a quotient of* $\mathrm{Cov}({}^2E_6(q))$ *if* $q \equiv 1 \pmod 4$ *and of* $\mathrm{Cov}(E_6(q))$ *if* $q \equiv -1 \pmod 4$ *and* $C_{\mathfrak{C}}(t_5) \cap C_{\mathfrak{C}}(J)$ *is cyclic.*

PROOF. Let $\mathfrak{G} = E_7$, let π denote the adjoint representation of $\mathfrak{G}$ and let $\overline{G}$ denote the linear algebraic group obtained from the triple $(\mathfrak{G}, \pi, K)$. Note that $G = O^{p'}(\overline{G}_\sigma) \cong X$, $\overline{G}_\sigma = G\overline{H}_\sigma$, $\overline{G}_\sigma/G \cong \overline{H}_\sigma/(G \cap \overline{H}_\sigma) \cong Z_2$ by [**35**, Corollary 12.6(b)], $\overline{H}_\sigma \cong \mathrm{Hom}(\Gamma_{\mathrm{ad}}, k^\times)$ and $G \cap \overline{H}_\sigma$ corresponds by [**11**, Theorem 7.1.1] to Im(Res) where Res: $\mathrm{Hom}(\Gamma_{\mathrm{sc}}, k^\times) \to \mathrm{Hom}(\Gamma_{\mathrm{ad}}, k^\times)$ denotes restriction to Γ_{ad}. Letting λ^* denote the restriction of λ to $\overline{G}_\sigma$, it follows that λ^* induces an automorphism of G of order n, $C_{\overline{G}_\sigma\langle\lambda^*\rangle}(G) = 1$ and $\mathfrak{A} = \mathrm{Aut}(E_7(q)) \cong \overline{G}_\sigma\langle\lambda^*\rangle$ (cf. [**11**, §12.5]). Let $B = \{\alpha_1, \ldots, \alpha_7\}$, $\{\overline{\omega}_1, \ldots, \overline{\omega}_7\}$, etc. be as in [**8**, Planche VI]. Also let $w \in \overline{N}_\lambda$ be such that the w induced automorphism $\tilde{w}$ of Φ satisfies $\tilde{w}$: $\alpha_1 \leftrightarrow \alpha_6$, $\alpha_3 \leftrightarrow \alpha_5$, $\alpha_7 \leftrightarrow -\alpha_0$ and fixes α_2 and α_4; thus $w^2 \in \overline{H}_\lambda$. Here $\Gamma_{\mathrm{ad}} = \Sigma_{i=1}^7 Z\alpha_i$, $\Gamma_{\mathrm{ad}}^\perp = \Gamma_{\mathrm{sc}} = \Sigma_{i=1}^7 Z\overline{\omega}_i$, $|\Gamma_{\mathrm{sc}}/\Gamma_\pi| = 2$, $\varepsilon_i = \overline{\omega}_i$ for all $1 \leqslant i \leqslant 7$ and $\alpha_0 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$. Also by [30, Proposition 3], $Z_2 \cong \Omega_\pi = \langle T_{(q-1)\varepsilon_7}\tilde{w}\rangle$. Then it follows that $\overline{G}$ has three conjugacy classes of involutions represented by $t_i = h(\chi_{\lambda_i})$ for $i = 1, 2, 3$ where

$$\lambda_1 = \frac{q-1}{2}\overline{\omega}_1, \ \lambda_2 = \frac{q-1}{2}\overline{\omega}_2, \ \lambda_3 = \frac{q-1}{2}\tilde{\omega}_7,$$
$$t_1 = h_{\alpha_3}(-1)h_{\alpha_5}(-1)h_{\alpha_7}(-1) = h_{\alpha_0}(-1),$$
$$t_2 = h_{\alpha_2}(\nu^3)h_{\alpha_5}(\nu)h_{\alpha_6}(-1)h_{\alpha_7}(\nu^3),$$
$$t_3 = h_{\alpha_1}(-1)h_{\alpha_2}(\nu^3)h_{\alpha_4}(-1)h_{\alpha_5}(\nu)h_{\alpha_7}(\nu^3)$$

for some element $\nu \in K^\times$ of order 4. Also $t_i \in \overline{G}_\sigma = G$ for $i = 1, 2, 3$ and $C_{\overline{G}}(t_1) = (\overline{L}_1 * \overline{L}_2)\overline{H}$ where $\overline{L}_1 = \langle \overline{\mathfrak{X}}_{\pm\alpha_0}\rangle \cong \mathrm{SL}(2, K)$, $\overline{L}_2 = \langle \overline{\mathfrak{X}}_{\pm\alpha_i} \mid 2 \leqslant i \leqslant 7\rangle$ is a proper quotient of Spin(12, K) and $Z(\overline{L}_1) = Z(\overline{L}_2) = \langle t_1\rangle$, $C_{\overline{G}}(t_2) = (\overline{L}_3\overline{H})\langle w\rangle$ where $\overline{L} = \langle \overline{\mathfrak{X}}_{\pm\alpha_i} \mid i \in \{0, 1, 3, 4, 5, 6, 7\}\rangle$ is a quotient of SL(8, K) and $C_{\overline{G}}(t_3) = (\overline{L}_4\overline{H})\langle w\rangle$ where $\overline{L}_4 = \langle \overline{\mathfrak{X}}_{\pm\alpha_i} \mid 1 \leqslant i \leqslant 6\rangle$ is a quotient of Cov($E_6(K)$).

Next we apply Lemma 4.2. Let $g \in \overline{G}$ be such that $g\sigma(g)^{-1} = w$. Then $G = \overline{G}_\sigma$ has five conjugacy classes of involutions represented by $\{t_1, t_2, t_2^g, t_3, t_3^g\}$. Also if β denotes the inner automorphism of $\overline{G}$ induced by w^{-1}, then $C_G(t_i^g) = ((C_{\overline{G}}(t_i))_{\beta\sigma})^g$ for $i = 2, 3$. Now Lemmas 4.2 and 4.9 yield (a)–(c).

As in [30, §8], we have $C_G(t_1) = (J_1 * J_2)\overline{H}_\sigma$ where $J_1 = \langle \mathfrak{X}_{\pm\alpha_0}\rangle \cong \mathrm{SL}(2q)$, $J_2 = \langle \mathfrak{X}_{\pm\alpha_i} \mid 2 \leqslant i \leqslant 7\rangle$ is a proper quotient of Spin(12, q, 1) and $Z(J_1) = Z(J_2) = \langle t_1\rangle$. Clearly $|C_G(t_1)/(J_1 * J_2)| = 2$ and

$$C_G(J_1 * J_2) = C_G(t_1) \cap C_G(J_1 * J_2) = C_{\overline{H}_\sigma}(J_1 * J_2)$$
$$\leqslant C_{\overline{H}_\sigma}\big(\langle \mathfrak{X}_{-\alpha_0}\rangle * \langle \mathfrak{X}_{\alpha_i} \mid 2 \leqslant i \leqslant 7\rangle\big) = \langle t_1\rangle;$$

thus part (1) of (d) holds.

Similarly $C_G(t_2) = (J_3\overline{H}_\sigma)\langle w\rangle$ where $J_3 = \langle \mathfrak{X}_{\pm\alpha_i} \mid i \in \{0, 1, 3, 4, 5, 6, 7\}\rangle$ is a quotient of SL(8, q). Also, it is clear that $C_G(t_2) \cap C_G(J_3) = C_{\overline{H}_\sigma}(J_3)$. Since $\langle \overline{H}, J_3\rangle = \overline{L}_3\overline{H}$, we have $\langle t_2\rangle \leqslant C_{\overline{H}_\sigma}(J_3) \leqslant C_{\overline{H}}(\overline{L}_3) = \langle t_2\rangle$. Thus $C_G(t_2) \cap C_G(J_3) = \langle t_2\rangle$.

Note that $C_G(t_2^g) \cong C_{\overline{G}}(t_2)_{\beta\sigma} = (J_3^0(\overline{H}_{\beta\sigma}))\langle w\rangle$ where $J_3^0 = O^{p'}((\overline{L}_3)_{\beta\sigma})$ is a quotient of SU(8, q) by [35, §11.6]. Also, it is easy to see that $\langle \overline{H}, J_3^0\rangle = \overline{L}_3\overline{H}$. Since $C_{\overline{G}}(t_2)_{\beta\sigma} \cap C_{\overline{G}}(J_3^0) \leqslant \overline{H}_{\beta\sigma}$ and $C_{\overline{H}}(\overline{L}_3) = \langle t_3\rangle$, we conclude from Lemma 4.9 that (2) and (4) of (d) hold.

Also, as above, $C_G(t_3) = (J_4\overline{H}_\sigma)\langle w\rangle$ where $J_4 = \langle \overline{\mathfrak{X}}_{\pm\alpha_i} \mid 1 \leqslant i \leqslant 6\rangle$ is a quotient of Cov($E_6(q)$) and $C_G(t_3) \cap C_G(J_4) = C_{\overline{H}_\sigma}(J_4)$. Here $\langle \overline{H}, J_4\rangle = \overline{L}_4\overline{H}$ and

$$C_{\overline{H}}(\overline{L}_4) \cong \{\chi \in \mathrm{Hom}(\Gamma_\pi, K^\times) \mid \chi(\alpha_i) = 1 \text{ for all } 1 \leqslant i \leqslant 6\} \cong K^\times,$$

so that $C_G(t_3) \cap C_G(J_4)$ is cyclic.

Finally, as above, $C_G(t_3^g) \cong C_{\overline{G}}(t_3)_{\beta\sigma} = (J_4^0(\overline{H}_{\beta\sigma}))\langle w\rangle$ where $J_4^0 = O^{p'}((\overline{L}_4)_{\beta\sigma})$ is a quotient of Cov($^2E_6(q)$) and $\langle \overline{H}, J_4^0\rangle = \overline{L}_4\overline{H}$. Since $C_{\overline{G}}(t_3)_{\beta\sigma} \cap C_{\overline{G}}(J_4^0) \leqslant \overline{H}_{\beta\sigma}$, we conclude that (3) and (5) of (d) hold and we are done.

LEMMA 4.25. *Let* $X = E_6(q)$, *let* $\mathfrak{A} = \mathrm{Aut}(X)$ *and let* $\mathfrak{B} = \mathrm{Inn}(X)$. *Then the following five conditions hold.*

(a) $\mathfrak{A} = \mathfrak{A}'(A \times \langle \tau^* \rangle)$ *where* $\mathfrak{A}' \cap (A \times \langle \tau^* \rangle) = 1$, A *is the subgroup of* $\mathfrak{A}$ *induced by* $\mathrm{Aut}(k)$, *so that* $A \cong \mathrm{Aut}(k) \cong Z_n$, *and where* τ^* *is the automorphism of* X *of order* 2 *induced by the graph automorphism of order* 2 *of the root system of type* E_6, $\mathfrak{B} = O^{2'}(\mathfrak{A}')$, $|\mathfrak{A}'/\mathfrak{B}| = (3, q-1)$, τ^* *inverts* $\mathfrak{A}'/\mathfrak{B}$ *and* $A \times \langle \tau^* \rangle \cong Z_n \times Z_2$;

(b) $\mathfrak{A}'$ *and* $\mathfrak{B}$ *have precisely two conjugacy classes of involutions which may be represented by involutions* t *and* v *such that* $t \in C_{\mathfrak{B}}(t)' \cong \mathrm{Spin}(10, q, 1)$ *and* $C_{\mathfrak{A}'}(C_{\mathfrak{B}}(t)')$ *is cyclic and such that* $C_{\mathfrak{B}}(v)$ *possesses precisely two* 2-*components* J_1 *and* J_2 *which, by appropriate indexing may be assumed to satisfy*: $J_1 \cong \mathrm{SL}(2, q)$, J_2 *is a quotient of* $\mathrm{SL}(6, q)$, $Z(J_1) = Z(J_2) = \langle v \rangle$ *and* $C_{\mathfrak{A}'}(J_1 * J_2) = \langle v \rangle$;

(c) $\mathfrak{A}'\tau^*$ *is* $\mathfrak{A}$-*invariant and* $\mathfrak{I}(\mathfrak{A}'\tau^*)$ *decomposes into two* $\mathfrak{A}'$-*orbits* (*under conjugacy by elements of* $\mathfrak{A}'$), *these two* $\mathfrak{A}'$-*orbits may be represented by* τ^* *and* $h\tau^*$ *for some involution* $h \in C_{\mathfrak{A}'}(\langle \tau^* \rangle \times A)$ *where* $C_{\mathfrak{A}'}(\tau^*) \cong F_4(q)$ *and* $C_{\mathfrak{A}'}(h\tau^*)'$ *is a quotient of* $\mathrm{Sp}(8, q)$;

(d) *if n is odd, then* $O^{2'}(\mathfrak{A}) = \mathfrak{A}'\langle \tau \rangle$; *and*

(e) *if n is even and* φ *denotes the unique involution of* A, *then* $\mathfrak{I}(\mathfrak{A} - \mathfrak{A}'\langle \tau^* \rangle) = \varphi^{\mathfrak{A}'} \cup (\tau^*\varphi)^{\mathfrak{A}'}$, $C_{\mathfrak{B}}(\varphi)' \cong E_6(\sqrt{q})$ *and* $C_{\mathfrak{B}}(\tau^*\varphi)' \cong {}^2E_6(\sqrt{q})$.

PROOF. Let $\mathfrak{G} = E_6$, let π denote the adjoint representation of $\mathfrak{G}$ and let $\overline{G}$ denote the linear algebraic group obtained from the triple $(\mathfrak{G}, \pi, K)$. Let τ denote the automorphism of $\overline{G}$ induced by the graph automorphism $\tilde{\tau}$ of Φ (of order 2) such that $\tau(x_\alpha(u)) = x_{\tilde{\tau}(\alpha)}(u)$ for all $u \in K$ and all $\alpha \in \Phi$. Clearly $\tau\lambda = \lambda\tau$ as endomorphisms of $\overline{G}$, both τ and λ leave invariant $\overline{G}_\sigma$, $G = O^{p'}(\overline{G}_\sigma) \cong X$, $\overline{G}_\sigma = G\overline{H}_\sigma$, $\overline{G}_\sigma/G \cong \overline{H}_\sigma/(G \cap \overline{H}_\sigma)$ and $G = (\overline{G}_\sigma)'$. Letting λ^* and τ^* denote the restrictions of λ and τ to $\overline{G}_\sigma$, respectively, it follows that λ^* and τ^* induce commuting automorphisms of G of orders n and 2, respectively; $C_{\overline{G}_\sigma(\langle \lambda^* \rangle \times \langle \tau^* \rangle)}(G) = 1$ and $\mathfrak{A} = \mathrm{Aut}(E_6(q)) \cong \overline{G}_\sigma(\langle \lambda^* \rangle \times \langle \tau^* \rangle)$ (cf. [**11**, §12.5]). Let $B = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$, $\{\overline{\omega}_1, \overline{\omega}_2, \overline{\omega}_3, \overline{\omega}_4, \overline{\omega}_5, \overline{\omega}_6\}$, etc., be as in [**8**, Planche V]. Then $\Gamma_{\mathrm{ad}} = \sum_{i=1}^6 Z\alpha_i$, $\Gamma_{\mathrm{ad}}^\perp = \Gamma_{\mathrm{sc}} = \sum_{i=1}^6 Z\overline{\omega}_i$, $|\Gamma_{\mathrm{sc}}/\Gamma_{\mathrm{ad}}| = 3$, $\varepsilon_i = \overline{\omega}_i$ for all $1 \leq i \leq 6$, $\alpha_0 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ and it follows that $\overline{G}$ has two conjugacy classes of involutions represented by $t_i = h(\chi_{\lambda_i})$ for $i = 1, 2$ where $\lambda_1 = \frac{q-1}{2}(\overline{\omega}_1 + \overline{\omega}_6)$, $t_1 = h_{\alpha_3}(-1)h_{\alpha_5}(-1)$, $\lambda_2 = \frac{q-1}{2}\overline{\omega}_2$ and $t_2 = h_{\alpha_1}(-1)h_{\alpha_4}(-1)h_{\alpha_6}(-1) = h_{\alpha_0}(-1)$. Also $t_i \in \overline{G}_\sigma$ for $i = 1, 2$, $C_{\overline{G}}(t_1) = \overline{L}_1\overline{H}$ where $\overline{L}_1 = \langle \overline{\mathfrak{X}}_{\pm\alpha_i} \mid i \in \{0, 2, 3, 4, 5\} \rangle$, $t_1 \in \overline{L}_1$ and $\overline{L}_1 \cong \mathrm{Spin}(10, K)$ and $C_{\overline{G}}(t_2) = (\overline{L}_2 * \overline{L}_3)\overline{H}$ where $\overline{L}_2 = \langle \overline{\mathfrak{X}}_{\pm\alpha_0} \rangle \cong \mathrm{SL}(2, K)$, $\overline{L}_3 = \langle \overline{\mathfrak{X}}_{\pm\alpha_i} \mid i \in \{1, 3, 4, 5, 6\} \rangle$, is a quotient of $\mathrm{SL}(6, K)$ and $Z(\overline{L}_2) = \langle t_2 \rangle = Z(\overline{L}_3)$.

As in [**30**, §8], we have $C_{\overline{G}_\sigma}(t_1) = J_1(\overline{H}_\sigma)$ where $t_1 \in J_1 = \langle \overline{\mathfrak{X}}_{\pm\alpha_i} \mid i \in \{0, 2, 3, 4, 5\} \rangle = (\overline{L}_1)_\sigma \cong \mathrm{Spin}(10, q, 1)$ by [**35**, Theorem 12.4]. Clearly $C_{\overline{G}_\sigma}(J_1) = C_{(\overline{H}_\sigma)}(J_1)$ and $\langle \overline{H}, J_1 \rangle = \overline{L}_1\overline{H} = C_{\overline{G}}(t_1)$. Since

$$\begin{aligned} C_{\overline{H}}(\overline{L}_1) &\cong \{\chi \in \mathrm{Hom}(\Gamma_\pi, K^\times) \mid \chi(\alpha_i) = 1 \text{ for all } i \in \{0, 2, 3, 4, 5\}\} \\ &= \{\chi \in \mathrm{Hom}(\Gamma_\pi, K^\times) \mid \chi(\alpha_i) = 1 \text{ for all } 2 \leq i \leq 5 \text{ and } \chi(\alpha_6) = \chi(\alpha_1)^{-1}\} \\ &\cong K^\times, \end{aligned}$$

it follows that $C_{\overline{G}_\sigma}(J_1)$ is cyclic.

Similarly, we have $C_{\overline{G}_\sigma}(t_2) = (J_2 * J_3)(\overline{H}_\sigma)$ where $J_2 = \langle \mathfrak{X}_{\pm\alpha_0}\rangle \cong \mathrm{SL}(2, q)$, $Z(J_2) = \langle t_2\rangle$, $J_3 = \langle \mathfrak{X}_{\pm\alpha_i} \mid i \in \{1,3,4,5,6\}\rangle$ is a quotient of $\mathrm{SL}(6, q)$ and $Z(J_3) = \langle t_2\rangle$. Clearly $C_{\overline{G}_\sigma}(J_2 * J_3) = C_{\overline{H}_\sigma}(J_2 * J_3)$ and $\langle \overline{H}, J_2, J_3\rangle = (\overline{L}_2 * \overline{L}_3)\overline{H} = C_{\overline{G}}(t_2)$. As above, it follows that $C_{\overline{H}}(\overline{L}_2 * \overline{L}_3) = \langle t_2\rangle = C_{\overline{G}_\sigma}(J_2 * J_3)$.

Let π^* denote the simply connected representation of $\mathfrak{G}$ and let G^* denote the simply connected linear algebraic group obtained from the triple $(\mathfrak{G}, \pi^*, K)$. Also let $\Delta\colon \overline{G}^* \to \overline{G}$ be the universal covering of $\overline{G}$ and extend λ, τ and $\sigma = \lambda^n$ to endomorphisms of $\overline{G}^*$ in the obvious way so that λ, τ and σ are compatible with Δ. Clearly (4.1) and (4.2) imply that $\mathrm{Ker}(\Delta) \cong \mathrm{Hom}(\Gamma_{sc}/\Gamma_\pi, K^\times)$ where $|\Gamma_{sc}/\Gamma_\pi| = 3$. Thus, if $p = 3$, then $\mathrm{Ker}(\Delta) = 1$ and if $p \neq 3$ and ν is an element of order 3 in $K^\times$, then $\mathrm{Ker}(\Delta) = \langle f\rangle \cong Z_3$ where $f = h_{\alpha_1}(\nu)h_{\alpha_3}(\nu^{-1})h_{\alpha_5}(\nu)h_{\alpha_6}(\nu^{-1})$. Note that τ inverts $\mathrm{Ker}(\Delta)$ and $\mathrm{Ker}(\Delta) \leqslant (\overline{G}^*)_\sigma$ if and only if $3 \mid (q-1)$. Now the proof of [**35**, Corollary 12.6(b)] implies that $\overline{G}_\sigma/G \cong Z_{(3,q-1)}$ and τ^* inverts $\overline{G}_\sigma/G$. Thus $(\overline{G}_\sigma(\langle\lambda^*\rangle \times \langle\tau^*\rangle))' = \overline{G}_\sigma$ and both (a) and (b) hold.

For (c), observe that $\overline{G}\langle\tau\rangle$ is a linear algebraic group and that $\overline{G}$ is a normal closed subgroup of $\overline{G}\langle\tau\rangle$ by [**6**, Chapter I, 1.11]. Since $\overline{G}$ is a simple group (cf. [**7**, §3.2(3)]), it follows that $(\overline{G}\langle\tau\rangle)^0 = \overline{G}$ by [**6**, Chapter I, (1.2), Proposition]. Also σ induces an endomorphism of $\overline{G}\langle\tau\rangle$ with $(\overline{G}\langle\tau\rangle)_\sigma = \overline{G}_\sigma\langle\tau\rangle$ finite. On the other hand, every involution of $\overline{G}\tau$ is conjugate via $\overline{G}$ to an element of $N_{\overline{G}\langle\tau\rangle}(\overline{B}) \cap N_{\overline{G}\langle\tau\rangle}(\overline{H}) = \overline{H}\langle\tau\rangle$ by [**35**, Theorem 7.5; **36**, §2.12, Corollary 2 and §2.8, Theorem 2(c)]. Note that $\tilde{\tau}\colon \alpha_1 \leftrightarrow \alpha_6$, $\alpha_3 \leftrightarrow \alpha_5$ and fixes α_2 and α_4. For each $1 \leqslant i \leqslant 6$, let $\overline{H}_i = \{h(\chi) \mid \chi(\alpha_j) = 1 \text{ for all } j \neq i \text{ with } 1 \leqslant j \leqslant 6\}$, so that $\overline{H}_i$ is a subgroup of $\overline{H}$. Then $\overline{H} = \oplus_{i=1}^6 \overline{H}_i$, $\overline{H}_1^\tau = \overline{H}_6$, $\overline{H}_3^\tau = \overline{H}_5$ and $[\tau, \overline{H}_2 \times \overline{H}_4] = 1$. For $j \in \{2,4\}$, let $h_j = h(\chi) \in \overline{H}_j$ be such that $\chi(\alpha_j) = -1$, so that h_j is the unique involution of $\overline{H}_j$. Also $h_2 = h_{\alpha_1}(-1)h_{\alpha_4}(-1)h_{\alpha_6}(-1)$ and $h_4 = h_{\alpha_2}(-1)$. Then Lemma 2.5 implies that every involution of $\overline{H}\tau$ is conjugate via $\overline{H}$ to an involution in $\langle h_2, h_4\rangle\tau$. Clearly $\langle h_2, h_4, \overline{\mathfrak{X}}_{\pm\alpha_2}, \overline{\mathfrak{X}}_{\pm\alpha_4}\rangle \leqslant C_{\overline{G}}(\tau)$ and $\langle h_2, h_4\rangle \leqslant \overline{H}_\lambda \leqslant \overline{H}_\sigma$. For $j \in \{2,4\}$, let ω_{α_j} be as defined in [**34**, p. 30, (R5)]. Then

$$\langle \omega_{\alpha_2}, \omega_{\alpha_4}\rangle \leqslant C_{\overline{G}}(\tau) \cap \overline{N}_\lambda \leqslant C_{\overline{G}}(\tau) \cap \overline{N}_\sigma \quad\text{and}\quad (h_2\tau)^{\omega_{\alpha_2}} = h_2h_4\tau = (h_4\tau)^{\omega_{\alpha_4}h_{\alpha_1}(-1)}.$$

Consequently every involution of $\overline{G}\tau$ is conjugate via $\overline{G}$ to an element of $\{\tau, h_2\tau\}$ where $\{\tau, h_2\tau\} \subseteq (\overline{G}\langle\tau\rangle)_\sigma = \overline{G}_\sigma\langle\tau\rangle$. Also, [**35**, (9.4) and (9.8)] imply that $C_{\overline{G}}(\tau)$ and $C_{\overline{G}}(h_2\tau)$ are connected and reductive linear algebraic groups. Hence Lemma 4.32 implies that $\mathfrak{I}(\overline{G}_\sigma\tau^*) = (\tau^*)^{\overline{G}_\sigma} \cup (h_2\tau^*)^{\overline{G}_\sigma}$.

The methods of [**11**, Chapter 13] can be used to determine the structure of $C_{\overline{G}}(\tau)$ and $C_{\overline{G}}(h_2\tau)$. For this process, let $\mathbb{S}$ denote the set of orbits of $\tilde{\tau}$ on Φ and for each orbit $\mathcal{O} \in \mathbb{S}$, let $\alpha_{\mathcal{O}} = (\sum_{\alpha\in\mathcal{O}}\alpha)/|\mathcal{O}|$ denote the average of the vectors in $\mathcal{O}$.

Then $\{\alpha_{\mathcal{O}} \mid \mathcal{O} \in \mathbb{S}\}$ is the corresponding root system for $C_{\overline{G}}(\tau)$ and $\{\alpha_2, \alpha_4, (\alpha_1 + \alpha_6)/2, (\alpha_3 + \alpha_5)/2\}$ is a base for this root system. Since each element of $(\overline{N}/\overline{H})_\tau$ can be represented by an element of $\overline{N}_\tau$, as follows from [**35**, §8.2(5)], we conclude that $C_{\overline{G}}(\tau) = \langle\langle \overline{\mathfrak{X}}_\alpha \mid \alpha \in \Phi \text{ and } \tilde{\tau}(\alpha) = \alpha\rangle, \langle x_\alpha(u)x_{\tau(\alpha)}(u) \mid u \in K\rangle\rangle \cong F_4(K)$ and $(C_{\overline{G}}(\tau))_\sigma = C_{\overline{G}_\sigma}(\tau^*) \cong F_4(k)$.

Note that $h_2 = h(\chi)$ for the element $\chi \in \mathrm{Hom}(\Gamma_{\mathrm{ad}}, K^\times)$ such that $\chi(\alpha_i) = 1$ for all $1 \leqslant i \leqslant 6$ with $i \neq 2$ and $\chi(\alpha_2) = -1$. Let $\mathbb{S}_1 = \{\mathcal{O} \in \mathbb{S} \mid |\mathcal{O}| = 1 \text{ and } \chi(\alpha) = 1 \text{ for } \alpha \in \mathcal{O}\}$, let $\mathbb{S}_2 = \{\mathcal{O} \in \mathbb{S} \mid |\mathcal{O}| = 2 \text{ and } \chi(\alpha) = 1 \text{ for } \alpha \in \mathcal{O}\}$ and let $\mathbb{S}_3 = \{\mathcal{O} \in \mathbb{S} \mid |\mathcal{O}| = 2 \text{ and } \chi(\alpha) = -1 \text{ for } \alpha \in \mathcal{O}\}$. Then $\{\alpha_{\mathcal{O}} \mid \mathcal{O} \in \mathbb{S}_1 \cup \mathbb{S}_2 \cup \mathbb{S}_3\}$ is the corresponding root system for $C_{\bar{G}}(h_2\tau)$, is of type C_4 and has base

$$\left\{\alpha_4, \frac{\alpha_1 + \alpha_6}{2}, \frac{\alpha_3 + \alpha_5}{2}, \frac{2\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5}{2}\right\}.$$

Then [**35**, (9.2), (9.3), (9.8) and the proof of (8.2)] imply that $C_{\bar{G}}(h_2\tau) = \langle \bar{H}_{h_2\tau}, \bar{L}\rangle$ where $\bar{L} = \langle\langle \bar{\mathfrak{X}}_\alpha \mid \{\alpha\} \in \mathbb{S}_1\rangle, \langle x_\alpha(u)x_{\tau(\alpha)}(u) \mid u \in K \text{ and } \{\alpha, \tilde{\tau}(\alpha)\} \in \mathbb{S}_2\rangle, \langle x_\alpha(u)x_{\tilde{\tau}(\alpha)}(-u) \mid u \in K \text{ and } \{\alpha, \tilde{\tau}(\alpha)\} \in \mathbb{S}_3\rangle\rangle$. Hence $((C_{\bar{G}}(h_2\tau))_\sigma) = C_{\bar{G}_\sigma}(h_2\tau)$ and $C_{\bar{G}_\sigma}(h_2\tau)'$ is a quotient of $\mathrm{Sp}(4, k)$. This completes the proof of (c), and (d) is immediate. Finally, suppose that n is even and let φ be as in (e). Then φ corresponds to $(\lambda^*)^{n/2} = (\lambda^{n/2})|_{\bar{G}_\sigma}$. Since $(\lambda^{n/2})^2 = \lambda^n = \sigma$ and $(\lambda^{n/2}\tau)^2 = \sigma$, Lemma 4.2 implies that $\mathfrak{I}(\bar{G}_\sigma(\lambda^*)^{n/2}) = ((\lambda^*)^{n/2})^{G_\sigma}$ and $\mathfrak{I}(\bar{G}_\sigma(\lambda^*)^{n/2}\tau^*) = ((\lambda^*)^{n/2}\tau^*)^{G_\sigma}$. Clearly $C_{\bar{G}_\sigma}((\lambda^*)^{n/2}) = C_{\bar{G}}(\lambda^{n/2})$ and $C_{\bar{G}_\sigma}(\lambda^{*n/2}\tau^*) = C_{\bar{G}}(\lambda^{n/2}\tau)$. Thus (e) holds and we are done.

LEMMA 4.26. *Let $X = {}^2E_6(q)$, let $\mathfrak{A} = \mathrm{Aut}(X)$ and let $\mathfrak{B} = \mathrm{Inn}(X)$. Then the following three conditions hold.*

(a) *$\mathfrak{A} = \mathfrak{A}'A$ where $\mathfrak{A}' \cap A = 1$ and A is the subgroup of $\mathfrak{A}$ induced by $\mathrm{Aut}(k_1)$, so that $A \cong \mathrm{Aut}(k_1) \cong Z_{2n}$, $\mathfrak{B} \leqslant \mathfrak{A}'$ and $|\mathfrak{A}'/\mathfrak{B}| = (3, q+1)$;*

(b) *$\mathfrak{A}'$ and $\mathfrak{B}$ have precisely two conjugacy classes of involutions which may be represented by involutions t and v such that $t \in C_{\mathfrak{B}}(t)' \cong \mathrm{Spin}(10, q, -1)$ and $C_{\mathfrak{A}'}(C_{\mathfrak{B}}(t)')$ is cyclic and such that $C_{\mathfrak{B}}(v)$ possesses precisely two 2-components J_1 and J_2 which by appropriate indexing may be assumed to satisfy: $J_1 \cong \mathrm{SL}(2, q)$, J_2 is a quotient of $\mathrm{SU}(6, q)$, $Z(J_1) = Z(J_2) = \langle v\rangle$ and $C_{\mathfrak{A}'}(J_1 * J_2) = \langle v\rangle$; and*

(c) *if φ denotes the unique involution of A, then $\mathfrak{A}'\langle\varphi\rangle = \langle \mathfrak{I}(\mathfrak{A})\rangle$ and $\mathfrak{I}(\mathfrak{A}'\varphi)$ decomposes into two $\mathfrak{A}'$-orbits (under conjugacy by elements of $\mathfrak{A}'$), these two $\mathfrak{A}'$-orbits may be represented by φ and $h\varphi$ for some involution $h \in C_{\mathfrak{A}'}(A)$ where $C_{\mathfrak{A}'}(\varphi) \cong F_4(q)$ and $C_{\mathfrak{A}'}(h\varphi)'$ is a quotient of $\mathrm{Sp}(8, q)$.*

PROOF. We use the notation of the previous lemma except that we consider $\bar{G}_{\sigma\tau}$ and set $G = O^{p'}(\bar{G}_{\sigma\tau})$. Then $(\bar{G}_{\sigma\tau})' = G \cong X$, λ and τ leave $\bar{G}_{\sigma\tau}$ invariant, $\bar{G}_{\sigma\tau} = GH_{\sigma\tau} \leqslant \bar{G}_{\sigma^2}$ and $\bar{G}_{\sigma\tau}/G \cong \bar{H}_{\sigma\tau}/(G \cap \bar{H}_{\sigma\tau})$. Letting λ^* and τ^* denote the restrictions of λ and τ to $\bar{G}_{\sigma\tau}$, we have $(\lambda^*)^n = \tau^* = \sigma|_{\bar{G}_{\sigma\tau}}$, $|\lambda^*| = 2n$, $C_{\bar{G}_{\sigma\tau}\langle\lambda^*\rangle}(G) = 1$ and $\mathfrak{A} = \mathrm{Aut}({}^2E_6(q)) \cong \bar{G}_{\sigma\tau}\langle\lambda^*\rangle$ (cf. [**34**, Theorem 36]). Note that [**35**, Corollary 12.6(b)] and the proof of Lemma 4.25 readily imply that $|\bar{G}_{\sigma\tau}/G| = (3, q+1)$ and that τ inverts $\bar{G}_{\sigma\tau}/G$. Clearly $\langle t_1, t_2\rangle \leqslant G$. Then Lemma 4.4 implies that $\bar{G}_{\sigma\tau}$ has two conjugacy classes of involutions represented by t_1 and t_2. Also, as above, $C_{\bar{G}_{\sigma\tau}}(t_1) = J_1(\bar{H}_{\sigma\tau})$ where $t_1 \in J_1 = \langle\langle x_\alpha(u) \mid u \in k, \alpha \in \{\pm\alpha_2, \pm\alpha_0, \pm\alpha_4\}\rangle, \langle x_{\alpha_3}(u)x_{\alpha_5}(\sigma(u)), x_{-\alpha_3}(u)x_{-\alpha_5}(\sigma(u)) \mid u \in k_1\rangle\rangle = (\bar{L}_1)_{\sigma\tau} \cong \mathrm{Spin}(10, q, -1)$. Clearly $C_{\bar{G}_{\sigma\tau}}(J_1) = C_{\bar{H}_{\sigma\tau}}(J_1)$ and $\langle \bar{H}, J_1\rangle = \bar{L}_1\bar{H} = C_{\bar{G}}(t_1)$, so that $C_{\bar{G}_{\sigma\tau}}(J_1)$ is cyclic. Similarly, we have $C_{\bar{G}_{\sigma\tau}}(t_2) = (J_2 * J_3)(\bar{H}_{\sigma\tau})$ where $J_2 = \langle \mathfrak{X}_{\pm\alpha_0}\rangle \cong \mathrm{SL}(2, q)$, $Z(J_2) = \langle t_2\rangle$, $J_3 = \langle\langle \mathfrak{X}_{\pm\alpha_4}\rangle, \langle x_\alpha(u)x_{\tau(\alpha)}(\sigma(u)) \mid u \in k_1 \text{ and } \alpha \in \{\pm\alpha_1, \pm\alpha_3\}\rangle\rangle$ is a quotient of $\mathrm{SU}(6, q)$ and $Z(J_3) = \langle t_2\rangle$. Since $\langle \bar{H}, J_2, J_3\rangle = (\bar{L}_2 * \bar{L}_3)\bar{H}$, we conclude, as above, that $C_{\bar{G}_{\sigma\tau}}(J_2 * J_3) = \langle t_2\rangle$. Thus both (a) and (b) hold.

For (c), as in the preceding lemma, we have $\mathfrak{I}(\overline{G}_{\sigma\tau}\tau^*) = (\tau^*)^{\overline{G}_{\sigma\tau}} \cup (h_2\tau^*)^{\overline{G}_{\sigma\tau}}$, $C_{\overline{G}_{\sigma\tau}}(\tau^*) = C_{\overline{G}}(\tau)_{\sigma\tau} = C_{\overline{G}}(\tau)_\sigma \cong F_4(k)$ and $C_{\overline{G}_{\sigma\tau}}(h_2\tau^*) = C_{\overline{G}}(h_2\tau)_{\sigma\tau}$. However $h_2 \in C_{\overline{G}}(h_2\tau)_{\sigma\tau}$ and $C_{\overline{G}}(h_2\tau)$ is connected. Thus $C_{\overline{G}}(h_2\tau)_{\sigma\tau} = C_{\overline{G}}(h_2\tau)_{\beta\sigma\tau} = C_{\overline{G}}(h_2\tau)_{\beta\tau\sigma} = C_{\overline{G}}((h_2)\tau)_\sigma$ where β denotes the inner automorphism of $C_{\overline{G}}(h_2\tau)$ induced by conjugation by h_2. Now (c) follows from Lemma 4.25 and our proof is complete.

The information about Aut(PSL$(2, q)$) that we require appears in [**14**, §1]. For the remaining cases we present

LEMMA 4.27. *Let* $X = \mathrm{PSL}(m, q)$ *for some integer* $m \geqslant 3$, *let* $\mathfrak{A} = \mathrm{Aut}(X)$ *and* $\mathfrak{B} = \mathrm{Inn}(X)$. *Then the following four conditions hold.*

(a) $\mathfrak{A} = \mathfrak{C}(A \times \langle\tau^*\rangle)$ *where* $\mathfrak{C} \cap (A \times \langle\tau^*\rangle) = 1$, *$A$ is the subgroup of $\mathfrak{A}$ induced by* Aut(k), *so that* $A \cong \mathrm{Aut}(k) \cong Z_n$, *$\tau^*$ is the graph automorphism of order 2 induced by the graph automorphism of order 2 of the root system of type* A_{m-1}, $\mathfrak{B} = \mathfrak{C}'$, $\mathrm{PGL}(m, q) \cong \mathfrak{C} \trianglelefteq \mathfrak{A}$ *and τ^* inverts* $\mathfrak{C}/\mathfrak{B} \cong Z_{(m,q-1)}$;

(b) *if m is odd then* $\mathfrak{I}(\mathfrak{C}\tau^*) = (\tau^*)^{\mathfrak{C}}$ *and if m is even, then* $\mathfrak{I}(\mathfrak{C}\tau^*)$ *decomposes into three orbits under conjugation by* $\mathfrak{C}$;

(c) *if n is odd, then* $O^{2'}(\mathfrak{A}) = \mathfrak{C}\langle\tau^*\rangle$; *and*

(d) *if n is even and φ denotes the unique involution of A, then* $\mathfrak{I}(\mathfrak{A} - \mathfrak{C}\langle\tau^*\rangle) = \varphi^{\mathfrak{C}} \cup (\tau^*\varphi)^{\mathfrak{C}}$, $C_{\mathfrak{B}}(\varphi) \cong \mathrm{PGL}(m, \sqrt{q})$ *and* $C_{\mathfrak{B}}(\tau^*\varphi) \cong \mathrm{PU}(m, \sqrt{q})$.

PROOF. Let $\mathfrak{G} = A_{m-1}$, let π denote the adjoint representation of $\mathfrak{G}$ and let $\overline{G}$ denote the linear algebraic group obtained from the triple $(\mathfrak{G}, \pi, K)$. Let τ denote the automorphism of $\overline{G}$ induced by the graph automorphism $\tilde{\tau}$ of Φ or order 2 such that $\tau(x_\alpha(u)) = x_{\tilde{\tau}(\alpha)}(u)$ for all $u \in K$ and all $\alpha \in \Phi$. Clearly $\tau\lambda = \lambda\tau$ as endomorphisms of $\overline{G}$, both τ and λ leave invariant $\overline{G}_\sigma \cong \mathrm{PGL}(m, q)$, $(\overline{G}_\sigma)' = G \cong \mathrm{PSL}(m, q)$ and $\overline{G}_\sigma/G \cong Z_{(m,q-1)}$. Letting λ^* and τ^* denote the restrictions of λ and τ to $\overline{G}_\sigma$, respectively, it follows that λ^* and τ^* induce commuting automorphisms of $\overline{G}_\sigma$ of orders n and 2, respectively,

$$C_{\overline{G}_\sigma(\langle\lambda^*\rangle\times\langle\tau^*\rangle)}(G) = 1 = \overline{G}_\sigma \cap (\langle\lambda^*\rangle \times \langle\tau^*\rangle)$$

and

$$\mathfrak{A} = \mathrm{Aut}(\mathrm{PSL}(m, q)) \cong \overline{G}_\sigma(\langle\lambda^*\rangle \times \langle\tau^*\rangle)$$

(cf. [**11**, §12.5]). Let $B = \{\alpha_1, \ldots, \alpha_{m-1}\}$, $\{\bar{\omega}_1, \ldots, \bar{\omega}_{m-1}\}$, etc., be as in [**8**, Planche I]. Thus, we may assume that $\tilde{\tau}$: $\alpha_i \leftrightarrow \alpha_{m-i}$ for all $1 \leqslant i \leqslant (m-1)/2$ and that $\tilde{\tau}$ fixes $\alpha_{m/2}$ if $m-1$ is odd. Then, from [**8**, Planche I], we conclude that $\tilde{\tau}$ inverts $\Gamma_{\mathrm{sc}}/\Gamma_{\mathrm{ad}}$. Note that $\overline{G}_\sigma = G(\overline{H}_\sigma)$, $\overline{G}_\sigma/G \cong \overline{H}_\sigma/(G \cap \overline{H}_\sigma) \cong Z_{(m,q-1)}$, $\Gamma_{\mathrm{sc}} = Z\bar{\omega}_1 + \Gamma_{\mathrm{ad}}$ and $\tilde{\tau}(\bar{\omega}_1) = -\bar{\omega}_1 + r$ for some $r \in \Gamma_{\mathrm{ad}}$. As in Lemma 4.25, it is easy to see that τ^* inverts $\overline{G}_\sigma/G$. This may also be demonstrated as follows. If $\chi \in \mathrm{Hom}(\Gamma_\pi, k^\times)$, then $\chi\chi^{\tilde{\tau}}$ extends to an element of $\mathrm{Hom}(\Gamma_{\mathrm{sc}}, k^\times)$ by defining

$$(\chi\chi^{\tilde{\tau}})(zw_1 + s) = \chi(r)^z\chi(s)\chi(\tilde{\tau}(s)) \quad \text{for all } z \in Z \text{ and } s \in \Gamma_{\mathrm{ad}}.$$

Now [**11**, Theorem 7.1.1] implies that τ^* inverts $\overline{H}_\sigma/(G \cap \overline{H}_\sigma)$ and $\overline{G}_\sigma/G$. Thus (a) and (c) hold. Also (d) follows from Lemma 4.2 and [**35**, §11.6]. For any $1 \leqslant i \leqslant m-1$, let $\overline{H}_i = \{h(\chi) \mid \chi(\alpha_j) = 1 \text{ for all } j \neq i \text{ with } 1 \leqslant j \leqslant m-1\}$, so that $\overline{H}_i$ is a subgroup of $\overline{H}$ and $\overline{H}_i \cong K^\times$. Clearly $\overline{H} = \oplus_{i=1}^{m-1} \overline{H}_i$ and Lemma 2.5 implies that $\mathfrak{I}(\overline{H}\tau) = \tau^{\overline{H}}$ if $m-1$ is even and $\mathfrak{I}(\overline{H}\tau) = \tau^{\overline{H}} \cup (h\tau)^{\overline{H}}$ where h is the unique involution of $\overline{H}_{m/2}$ if $m-1$ is odd. As in Lemma 4.25, we observe that $\overline{G}\langle\tau\rangle$ is a linear algebraic group and $(\overline{G}\langle\tau\rangle)^0 = \overline{G}$. Also σ induces an endomorphism of $\overline{G}\langle\tau\rangle$ with $(\overline{G}\langle\tau\rangle)_\sigma = \overline{G}_\sigma\langle\tau\rangle$, and $C_{\overline{G}}(\tau)$ is a connected linear algebraic group by [**35**, 8.3(b)]. Thus, by Lemma 4.4, we may assume that $m-1$ is odd. Since $|C_{\overline{G}}(h\tau)/(C_{\overline{G}}(h\tau))^0| = 2$ (cf. [**10**, §4.3]), (b) follows from Lemma 4.4 and we are done.

Remark 4.28. Let $G = \mathrm{GL}(m, k)$ for some integer $m \geqslant 3$, the group of nonsingular $m \times m$ matrices over k. Let $H = \mathrm{SL}(m,k) = \{x \in G \mid \det(x) = 1\}$ and let τ denote the transpose-inverse automorphism of G. Then $G = H' = G^{(\infty)}$ and [**29**, Proposition 8.9] implies that $C_G(\tau) \cong O((V/k, f))$ and $C_H(\tau) \cong SO((V/k, f))$ where $(V/k, f)$ is a nonsingular orthogonal vector space of dimension m with $D((V/k, f)) = (k^\times)^2$. Also if $c \in k^\times - (k^\times)^2$ and h denotes the inner automorphism of G induced by the $m \times m$ diagonal matrix with c in position m and 1 in the remaining (diagonal) positions, then $\tau h = h^{-1}\tau$ and $(h\tau)^2 = \tau^2 = I_G$. Also, by [**29**, Proposition 8.9], $C_G(h\tau) \cong O((V/k, f))$ and $C_H(h\tau) \cong SO((V/k, f))$ where $(V/k, f)$ is a nonsingular orthogonal vector space of dimension m with $D((V/k, f)) = c(k^\times)^2$. Suppose that m is even and let g denote the inner automorphism of G induced by the matrix A on [**11**, p. 3]. Then $\tau g = g\tau$, $g^2 = I_G \neq g$ and $C_H(\tau g) = \mathrm{Sp}(V/k)$ where V/k is a nonsingular symplectic vector space of dimension m by [**29**, Proposition 9.13]. Suppose that n is even and let $\sigma_0 = \lambda^{n/2}$, so that $\sigma_0^2 = \sigma$. Clearly λ and σ_0 induce automorphisms, in the natural way, of G which we shall also denote by λ and σ_0. Then $\lambda^{n/2} = \sigma_0 \neq I_G = \sigma_0^2$ and $\tau\sigma_0 = \sigma_0\tau$, etc. Letting $C_K(\sigma_0) = C_k(\sigma_0) = k_0$, we have $C_G(\sigma_0) = \mathrm{GL}(m, k_0)$ and $C_H(\sigma_0) = \mathrm{SL}(m, k_0)$. Also, by [**29**, Proposition 8.8], we have $C_G(\sigma_0\tau) = U(V/k)$ and $C_H(\sigma_0\tau) = \mathrm{SU}(V/k)$ where V/k is a nonsingular unitary vector space of dimension m.

Lemma 4.29. *Assume that n is even and let $X = \mathrm{PSU}(m, \sqrt{q})$ for some integer $m \geqslant 3$, let $\mathfrak{A} = \mathrm{Aut}(X)$ and let $\mathfrak{B} = \mathrm{Inn}(X)$. Then the following two conditions hold.*

(a) *$\mathfrak{A} = \mathfrak{C}A$ where $\mathfrak{C} \trianglelefteq \mathfrak{A}$, $\mathfrak{C} \cap A = 1$, A is the subgroup of $\mathfrak{A}$ induced by* $\mathrm{Aut}(k)$, *so that $A \cong \mathrm{Aut}(k) \cong Z_n$, $\mathfrak{B} = \mathfrak{C}'$, $\mathrm{PU}(m, \sqrt{q}) \cong \mathfrak{C} \trianglelefteq \mathfrak{A} \cong P\Gamma U(m, \sqrt{q})$, and $\mathfrak{C}/\mathfrak{B} \cong Z_{(m, \sqrt{q}+1)}$; and*

(b) *if φ denotes the unique involution in A, then $\mathfrak{I}(\mathfrak{C}\varphi)$ decomposes under conjugation by $\mathfrak{C}$ into one orbit if m is odd and into three orbits if m is even.*

Proof. Utilize the notation of Lemma 4.27 and set $\sigma_0 = \lambda^{n/2}$ and $G = O^{p'}(\overline{G}_{\sigma_0\tau})$. Clearly $\sigma_0^2 = \sigma$, $\tau\sigma_0 = \sigma_0\tau$, $(\tau\sigma_0)^2 = \sigma$,

$$\overline{G}_{\sigma_0\tau} \cong \mathrm{PU}\big(m, \sqrt{q}\big),\ G = \overline{G}'_{\sigma_0\tau} \cong \mathrm{PSU}\big(m, \sqrt{q}\big)$$

and

$$\overline{G}_{\sigma_0\tau} \leqslant \overline{G}_\sigma \cong \mathrm{PGL}(m, q).$$

Let λ^*, τ^*, σ_0^* denote the restrictions of λ, τ and σ_0 to $\overline{G}_{\sigma_0\tau}$, respectively. Then $\tau^* = \sigma_0^* = (\lambda^*)^{n/2}$, $(\lambda^*)^n = I_{G^*}$, $C_{\overline{G}_{\sigma_0\tau}\langle\lambda^*\rangle}(G) = 1$ and $\mathfrak{A} = \mathrm{Aut}(\mathrm{PSU}(m, \sqrt{q})) \cong \overline{G}_{\sigma_0\tau}\langle\lambda^*\rangle$ (cf. [**34**, Theorem 36]). Thus (a) holds. Clearly $\sigma_0\tau$ induces an endomorphism of the linear algebraic group $\overline{G}\langle\tau\rangle$, $(\overline{G}\langle\tau\rangle)_{\sigma_0\tau} = \overline{G}_{\sigma_0\tau}\langle\tau\rangle$ and (b) follows as in Lemma 4.27.

REMARK 4.30. Suppose that n is even and let $\sigma_0 = \lambda^{n/2}$ and $k_0 = C_k(\sigma_0)$. Let $(V/k, f)$ be a nonsingular unitary vector space of dimension m. Then [**29**, Proposition 8.8] implies that V/k has a basis $B = \{v_1, \ldots, v_m\}$ such that $f(v_i, v_j) = \delta_{ij}$ for all $1 \leqslant i, j \leqslant m$. Then $W = \sum_{i=1}^m k_0 v_i$ is a vector space over k_0 of dimension m and $(W/k_0, f|_{W/k_0})$ is a nonsingular orthogonal vector space with $D(W/k_0) = (k_0^\times)^2$ by [**29**, Proposition 8.9]. If α_B: $U(V/k) \to \mathrm{GL}(m, k)$ denotes the monomorphism defined via the basis B, then $\mathrm{Im}(\alpha_B)$ is invariant under σ_0, $\mathrm{Im}(\alpha_B)_{\sigma_0} = O(W/k_0)$ and $\alpha_B(\mathrm{SU}(V/k))_{\sigma_0} = SO(W/k_0)$. Next let N: $k^\times \to k_0^\times$ denote the norm map, let $c \in k^\times$ be such that $N(c) \notin (k_0^\times)^2$, let $v_i' = v_i$ for all $1 \leqslant i \leqslant m-1$ and $v_m' = cv_m$, let $B' = \{v_1', \ldots, v_m'\}$ and let $W' = \sum_{i=1}^m k_0 v_i'$. Then B' is a basis of V/k such that $f(v_i', v_j') = \delta_{ij}$ for all $1 \leqslant i, j \leqslant m$ with $i \neq m$ or $j \neq m$ and $f(v_m', v_m') = N(c)$, W' is a vector space over k_0 of dimension m and $(W'/k_0, f|_{W'/k_0})$ is a nonsingular orthongonal vector space with $D(W/k_0) = N(c)(k_0^\times)^2$ by [**29**, Proposition 8.9]. If $\alpha_{B'}$: $U(V/k) \to \mathrm{GL}(m, k)$ deotes the monomorphism defined via the basis B then $\mathrm{Im}(\alpha_{B'})$ is invariant under σ_0, $\mathrm{Im}(\alpha_{B'})_{\sigma_0} = O(W'/k_0)$ and $\alpha_{B'}(\mathrm{SU}(V/k))_{\sigma_0} = SO(W'/k_0)$. Next, suppose that m is also even. By [**29**, Proposition 9.14], V/k has a basis $B^* = \{w_1^{(i)}, w_2^{(i)} \mid 1 \leqslant i \leqslant \frac{m}{2}\}$ such that $f(w_1^{(i)}, w_2^{(i)}) = f(w_2^{(i)}, w_1^{(i)}) = 1$ for all $1 \leqslant i \leqslant \frac{m}{2}$ and $f(w_r^{(i)}, w_s^{(j)}) = 0$ if $i \neq j$ or $r = s$ with $1 \leqslant i, j \leqslant \frac{m}{2}$ and $1 \leqslant r, s \leqslant 2$. Let $d \in k^\times$ be such that $\sigma_0(d) = -d$ and set $f^* = df$. Then $W^* = \sum_{i=1}^{m/2} k_0 w_1^{(i)} + \sum_{i=1}^{m/2} k_0 w_2^{(i)}$ is a vector space over k_0 of dimension m and $(W^*/k_0, f^*|_{W^*/k_0})$ is a nonsingular symplectic vector space by [**29**, Proposition 9.13]. If α_{B^*}: $U(V/k) \to \mathrm{GL}(n, k)$ denotes the monomorphism defined via the bases B^*, then $\mathrm{Im}(\alpha_{B^*})$ is invariant under σ_0 and $\mathrm{Im}(\alpha_{B^*})_{\sigma_0} = \mathrm{Sp}(W^*/k_0) = \alpha_{B^*}(\mathrm{SU}(V/k))_{\sigma_0}$.

Clearly, the following two results are easy consequences of the methods utilized in this section.

LEMMA 4.31. *let* $X = \mathrm{PSp}(2m, q)$ *for some integer* $m \geqslant 2$, *let* $\mathfrak{A} = \mathrm{Aut}(X)$ *and let* $\mathfrak{B} = \mathrm{Inn}(X)$. *Then the following three conditions hold.*

(a) $\mathfrak{A}' = \mathfrak{B}$, $\mathfrak{A} = \mathfrak{C}A$ *where* $\mathfrak{B} \trianglelefteq \mathfrak{C} \trianglelefteq \mathfrak{A}$, $\mathfrak{C} \cong (2m, k)$, $|\mathfrak{C}/\mathfrak{B}| = 2$, $A \cap \mathfrak{C} = 1$ *and A is the subgroup of $\mathfrak{A}$ induced by* $\mathrm{Aut}(k)$, *so that* $A \cong \mathrm{Aut}(k) \cong Z_n$;

(b) *if n is odd, then* $O^{2'}(\mathfrak{A}) = \mathfrak{C}$;

(c) *if n is even, then the unique involution* $\tau \in A$ *is such that* $C_{\mathfrak{B}}(\tau) \cong \mathrm{PGSp}(2m, \sqrt{q})$ *and* $\mathfrak{I}(\mathfrak{A} - \mathfrak{C}) = \tau^{\mathfrak{C}}$.

LEMMA 4.32. *Let* $X = P\Omega(m, q)$ *for some odd integer* $m \geqslant 7$, *let* $\mathfrak{A} = \mathrm{Aut}(X)$ *and let* $\mathfrak{B} = \mathrm{Inn}(X)$. *Then the following three conditions hold.*

(a) $\mathfrak{A}' = \mathfrak{B}$, $\mathfrak{A} = \mathfrak{C}\mathfrak{A}$ *where* $\mathfrak{B} \trianglelefteq \mathfrak{C} \trianglelefteq \mathfrak{A}$, $\mathfrak{C} \cong PGO(m, k) \cong SO(m, q)$, $|\mathfrak{C}/\mathfrak{B}| = 2$, $A \cap \mathfrak{C} = 1$ *and A is the subgroup of $\mathfrak{A}$ induced by* $\mathrm{Aut}(k)$, *so that* $A \cong \mathrm{Aut}(k) \cong Z_n$;

(b) *if n is odd, then* $O^{2'}(\mathfrak{A}) = \mathfrak{C}$; *and*

(c) *if n is even, then the unique involution $\tau \in A$ is such that* $C_{\mathfrak{B}}(\tau) \cong SO(m, \sqrt{q})$ *and* $\mathfrak{I}(\mathfrak{A} - \mathfrak{C}) = \tau^{\mathfrak{C}}$.

LEMMA 4.33. *Let* $X = P\Omega(2m, q, 1)$ *for some integer* $m \geqslant 4$, *let* $\mathfrak{A} = \mathrm{Aut}(X)$ *and let* $\mathfrak{B} = \mathrm{Inn}(X)$. *Then the following seven conditions hold.*

(a) *There is a normal subgroup* $\mathfrak{C}$ *of* $\mathfrak{A}$ *such that* $\mathfrak{C}' = \mathfrak{B}$, $\mathfrak{C}/\mathfrak{B} \cong E_4$ *if m is even and* $\mathfrak{C}/\mathfrak{B} \cong Z_{(4, q-1)}$ *if m is odd*;

(b) $\mathfrak{A}$ *contains a subgroup A induced by* $\mathrm{Aut}(k)$, *so that* $A \cong \mathrm{Aut}(k) \cong Z_n$;

(c) $C_{\mathfrak{A}}(A)$ *contains a subgroup B such that* $B \cap A = 1$, $|B| = 2$ *if* $m \neq 4$ *and* $B \cong \Sigma_3$ *if* $m = 4$ *and* $\mathfrak{A} = \mathfrak{C}(B \times A)$ *with* $\mathfrak{C} \cap (B \times A) = 1$;

(d) $\mathfrak{C}B/\mathfrak{B} \cong \Sigma_4$ *if* $m = 4$, $\mathfrak{C}B/\mathfrak{B} \cong D_8$ *if m is even or m is odd and* $q \equiv 1 \pmod 4$ *and* $\mathfrak{C}B/\mathfrak{B} \cong E_4$ *if m is odd and* $q \equiv -1 \pmod 4$;

(e) *if* $\tau \in \mathfrak{I}(B)$, *then* $\mathfrak{C}\langle\tau\rangle \cong PGO(2m, q, 1)$;

(f) *if n is odd, then* $O^{2'}(\mathfrak{A}) = \mathfrak{C}B$; *and*

(g) *if n is even and φ denotes the unique involution of A and if* $\tau \in \mathfrak{I}(B)$, *then*

$$\mathfrak{I}(\mathfrak{C}\varphi) = \varphi^{\mathfrak{C}}, \quad \mathfrak{I}(\mathfrak{C}\tau\varphi) = (\tau\varphi)^{\mathfrak{C}}, \quad C_{\mathfrak{C}}(\varphi)' = P\Omega\big(2m, \sqrt{q}, 1\big)$$

and $C_{\mathfrak{C}}(\tau\varphi)' \cong P\Omega(2m, \sqrt{q}, -1)$.

PROOF. Let $\mathfrak{G} = D_m$, let π denote the linear algebraic group obtained from the triple $(\mathfrak{G}, \pi, K)$. Let B denote the group of automorphisms of $\overline{G}$ induced by the group of graph automorphisms of Φ, as in Lemma 4.27. Thus $B \cong \Sigma_3$ if $m = 4$ and $|B| = 2$ otherwise and $[B, \lambda] = 1$ as endomorphisms of $\overline{G}$. Also B and λ leave invariant $\overline{G}_\sigma$ and $G = O^{p'}(\overline{G}_\sigma) = (\overline{G}_\sigma)' \cong X$. Let $\lambda^* = \lambda|_G$ and for $b \in B$, let $b^* = b|_G$. Then, as endomorphisms of G, $|\lambda^*| = n$, λ^* commutes with $B^* = \langle b^* \mid b \in B\rangle$, $B^* \cong B$, $C_{\overline{G}_\sigma(\langle\lambda^*\rangle\times B^*)}(G) = 1 = \overline{G}_\sigma \cap (\langle\lambda^*\rangle \times B^*)$ and $\mathfrak{A} \cong \overline{G}_\sigma(\langle\lambda^*\rangle \times B^*)$ by [**11**, Theorem 12.5.1]. Also from [**8**, Planche IV], we have $\Gamma_{\mathrm{sc}}/\Gamma_{\mathrm{ad}} \cong E_4$ if m is even and $\Gamma_{\mathrm{sc}}/\Gamma_{\mathrm{ad}} \cong Z_4$ if m is odd. Note that $\overline{G}_\sigma = G(\overline{H}_\sigma)$ and $\overline{G}_\sigma/G \cong \overline{H}_\sigma/(G \cap \overline{H}_\sigma) \cong \mathrm{Hom}(\Gamma_{\mathrm{sc}}/\Gamma_{\mathrm{ad}}, k^\times)$ by [**11**, Theorem 7.1.1]. Applying [**35**, Corollary 12.6(b)] and the proofs of Lemmas 4.13, 4.15 and 4.17 (as in Lemma 4.25), we conclude that (a)–(d) and (f) hold. Note that $P\Gamma O(2m, q, 1)$ is isomorphic to a subgroup of $\mathfrak{A}$ and if $|\mathfrak{A}| \neq |P\Gamma O(2m, q, 1)|$, then $m = 4$ and $|\mathfrak{A}| = 3|P\Gamma O(2m, q, 1)|$. Thus we may assume that n is even and $q \equiv 1 \pmod 4$. Let $\varphi = (\lambda^*)^{n/2}$ and let $\tau \in \mathfrak{I}(B^*)$. Then (g) holds by Lemma 4.4 and [**35**, §11.6]. Note that $\langle \mathfrak{I}(P\Omega(2m, q, 1))\rangle = P\Omega(2m, q, 1)$ and that $\langle \mathfrak{I}(PO(2m, q, 1))\rangle = PO(2m, q, 1)$ and $\langle \mathfrak{I}(PGO(2m, q, 1))\rangle = PGO(2m, q, 1)$ from §3D. Thus there is a monomorphism $\delta\colon PGO(2m, q, 1) \to \mathfrak{C}\langle\tau, \varphi\rangle = \mathfrak{C} \cup \mathfrak{C}\tau \cup \mathfrak{C}\varphi \cup \mathfrak{C}\tau\varphi$. Then §3D and (g) imply that $\delta(PGO(2m, q, 1)) \leqslant \mathfrak{C}\langle\tau\rangle$. Since $|PGO(2m, q, 1)| = |\mathfrak{C}\langle\tau\rangle|$, we have $\delta(PGO(2m, q, 1)) = \mathfrak{C}\langle\tau\rangle$ and we are done.

LEMMA 4.34. *Assume that n is even and let* $X = P\Omega(2m, \sqrt{q}, -1)$ *for some integer* $m \geqslant 4$, *let* $\mathfrak{A} = \mathrm{Aut}(X)$ *and let* $\mathfrak{B} = \mathrm{Inn}(X)$. *Then the following five conditions hold.*

(a) *There is a normal subgroup* $\mathfrak{C}$ *of* $\mathfrak{A}$ *such that* $\mathfrak{C}' = \mathfrak{B}$, $|\mathfrak{C}/\mathfrak{B}| = 2$ *if m is even, and* $\mathfrak{C}/\mathfrak{B} \cong Z_{(4, \sqrt{q}+1)}$ *if m is odd*;

(b) $\mathfrak{A}$ *contains a subgroup A induced by* $\mathrm{Aut}(k)$, *so that* $A \cong \mathrm{Aut}(k) \cong Z_n$;

(c) $\mathfrak{A} = \mathfrak{C}A$ *and* $\mathfrak{C} \cap A = 1$;

(d) $\mathfrak{A} \cong P\Gamma O(2m, \sqrt{q}, -1)$; *and*

(e) *if* φ *denotes the unique involution of* A, *then* $\mathfrak{C}\langle\varphi\rangle \cong PGO(2m, \sqrt{q}, -1)$ *and* φ *inverts* $\mathfrak{C}/\mathfrak{B}$.

PROOF. Utilize the notation of Lemma 4.33 but let $\tau \in \mathfrak{I}(B)$ and set $\sigma_0 = \lambda^{n/2}$ and $G = O^{p'}(\overline{G}_{\sigma_0\tau})$. Clearly $\sigma_0^2 = \sigma$, $\tau\lambda = \lambda\tau$ and $(\tau\sigma_0)^2 = \sigma$. Thus λ and τ leave $\overline{G}_{\sigma_0\tau}$ invariant, $G = (\overline{G}_{\sigma_0\tau})' \cong X$, $\sigma_0|_{\overline{G}_{\sigma_0\tau}} = \tau|_{\overline{G}_{\sigma_0\tau}}$, $\overline{G}_{\sigma_0\tau} = G\overline{H}_{\sigma_0\tau}$ and $\overline{G}_{\sigma_0\tau}/G \cong \overline{H}_{\sigma_0\tau}/(G \cap \overline{H}_{\sigma_0\tau})$. Applying [35, Corollary 12.6(b)] and the proofs of Lemmas 4.13, 4.15 and 4.17, we conclude that $|\overline{G}_{\sigma_0\tau}/G| = 2$ if m is even and $\overline{G}_{\sigma_0\tau}/G \cong Z_{(4,\sqrt{q}+1)}$ if m is odd. Also τ inverts $\overline{G}_{\sigma_0\tau}/G$ in all cases. Let $\lambda^* = \lambda|_{\overline{G}_{\sigma_0\tau}}$. Then

$$(\lambda^*)^{n/2} = \sigma_0|_{\overline{G}_{\sigma_0\tau}} = \tau|_{\overline{G}_{\sigma_0\tau}} \neq I_{\overline{G}_{\sigma_0\tau}}, \qquad C_{\overline{G}_{\sigma_0\tau}\langle\lambda^*\rangle}(G) = 1$$

and

$$\mathfrak{A} \cong \overline{G}_{\sigma_0\tau}\langle\lambda^*\rangle$$

by [34, Theorem 36]. Thus (a)–(c) hold. Since $P\Gamma O(2m, \sqrt{q}, -1)$ is isomorphic to a subgroup of $\mathfrak{A}$ and $|P\Gamma O(2m, \sqrt{q}, -1)| = |\mathfrak{A}|$, we have (d). Since

$$\langle \mathfrak{I}\big(PGO\big(2m, \sqrt{q}, -1\big)\big)\rangle = PGO\big(2m, \sqrt{q}, -1\big)$$

by §3D, $|PGO(2m, \sqrt{q}, -1)| = 2|\overline{G}_{\sigma_0\tau}|$ and $(\overline{G}_{\sigma_0\tau}\langle\lambda^*\rangle)/\overline{G}_{\sigma_0\tau} \cong Z_n$, we have (e) also and we are done.

Finally, we note that §§3A–3D and Lemmas and Remarks 4.27–4.34 give a complete survey of the conjugacy classes of involutions in the automorphism groups of the classical linear groups over finite fields of odd order that extends [12].

5. Additional preliminary results. In this section, we utilize our previous work to derive further results that are required in our proofs of Theorems 1–3 (as presented in §§6–8, respectively).

Throughout this section, p will denote an odd prime integer and $q = p^n$ for some positive integer n.

LEMMA 5.1. *Let* X *be a simple Chevalley group over a finite field of order* q. *Let* $z \in \mathfrak{I}(X)$ *and set* $H = C_G(z)$. *Then the following seven conditions hold*:

(a) *every* 2-*component* H *is a Chevalley group over a finite field of order* q, q^2, q^3 *or* q^4 *and* $E(H) = L_{2'}(H)$;

(b) $H/E(H)$ *is solvable*;

(c) *if* $q = 3$, *then* $O(H) = 1$;

(d) *if* H *possesses a solvable* 2-*component, then* $q = 3$;

(e) H *does not contain a* 2-*component* J *with* $z \in J$ *and* $J \cong \mathrm{Spin}(7, p^r)$ *for any positive integer* r;

(f) *if* $q \neq 3$, *then* $S(H) = C_H(E(H))$ *and* $S(H)$ *is cyclic or dihedral*; *and*

(g) *if* $q = 3$ *and* $E_s(H)$ *denotes the product of all solvable* 2-*components of* H, *then* $C_H(E(H)E_s(H))$ *is cyclic or dihedral.*

PROOF. Clearly [9, Lemma (c)] yields (c). Applying Lemmas 4.19–4.26, it follows that we may assume that X is a classical linear group. Then Lemmas 3.1–3.2, [25, Propositions 2–5] and §§3A–3D yield the result.

LEMMA 5.2. *Let K be a 2-quasisimple group such that $K/Z^*(K)$ is a simple Chevalley group over a finite field of order q. Then exactly one of the following three conditions holds.*

(a) *There exists an involution $t \in K$ such that $C_K(t)$ possesses an intrinsic 2-component J with $J/O(J) \cong \mathrm{SL}(2, q)$;*

(b) $K/O(K) \cong \mathrm{PSL}(2, q)$; *or*

(c) $p = 3$, *n is odd*, $n \geqslant 3$ *and* $K/O(K) \cong {}^2G_2(q)$.

PROOF. Let K be a counterexample of minimal order to this lemma. Then $O(K) = 1$ by Lemma 2.19 and hence $Z^*(K) = Z(K) = O_2(K)$. Let $L = \mathrm{Cov}(K)$. Then $L = \mathrm{Cov}(K/Z^*(K))$ by Corollary 2.7.1 and there is an epimorphism $\pi\colon L \to K$ such that $O(L) \leqslant \mathrm{Ker}(\pi) \leqslant Z(L)$. Now Lemmas 2.14, 2.27, 4.9, 4.10, 4.12–4.26 and §§3A–3D force the conclusion of this lemma.

LEMMA 5.3. *Let K be a 2-quasisimple group such that $K/Z^*(K)$ is a simple Chevalley group over a finite field of order q. Then exactly one of the following five conditions holds.*

(a) $K/O(K) \cong \mathrm{PSL}(2, q)$;

(b) $p = 3$, *n is odd*, $n \geqslant 3$ *and* $K/O(K) \cong {}^2G_2(q)$;

(c) $q = 3$ *and $K/O(K)$ is isomorphic to* PSL(3, 3), PSU(3, 3), PSL(4, 3), PSU(4, 3), PSp(4, 3), $G_2(3)$, $P\Omega(7, 3)$, $P\Omega(8, 3, 1)$ *or to* $P\Omega(8, 3, -1)$;

(d) $q = 3$, $|Z^*(K)|_2 = 2$ *and* $K/Z^*(K) \cong \mathrm{PSU}(4, 3)$; *or*

(e) *K contains an involution t such that $C_K(t)$ possesses a perfect intrinsic 2-component of $\mathfrak{M}(p)$-type.*

PROOF. Let K be a counterexample of minimal order to this lemma. Then, as in the previous lemma, we have $O(K) = 1$. Moreover, we must have $q = 3$ by the previous lemma. Suppose that K is simple. Then Lemmas 4.19–4.26, Lemma 2.14 and §§3A–3D yield a contradiction. Thus $S(K) = Z(K) = O_2(K) \neq 1$. Since, $K = C_K(j)$ for any involution $j \in Z(K)$, we conclude that K is a proper quotient of $\mathrm{SL}(2m, 3)$ or of $\mathrm{SU}(2m, 3)$ for some integer $m \geqslant 2$. However Lemma 2.14 and §§3A and 3C yield a contradiction in this case also and we are done.

LEMMA 5.4. *Let K be a 2-quasisimple group such that $K/Z^*(K)$ is isomorphic to a simple Chevalley group over a finite field of order q. Assume that t is an involution in $Z^*(K)$ and that $K/Z^*(K)$ is not isomorphic to* PSL(2, q). *Then there is an involution $z \in K - Z^*(K)$ such that $C_K(z)$ possesses a 2-component J with $z \in Z(J)$ and $J/O(J) \cong \mathrm{SL}(2, q)$ and at least one other 2-component L of $\mathfrak{M}(p)$-type with $L/Z^*(L)$ isomorphic to a Chevalley group over a finite field of order q. Moreover, if K is of $\mathfrak{M}(p)$-type, then L may be chosen to satisfy $Z(L) \cap \{t, tz\} \neq \varnothing$ also.*

PROOF. As above, we may assume that $O(K) = 1$. Then §§3A–3C and Lemmas 2.14, 2.27, 4.9, 4.10 and 4.12–4.18 imply the desired conclusions.

Our next result is clearly a consequence of §§3A and 3C.

LEMMA 5.5. *Let $X = \mathrm{PSL}(2m, q)$ or $X = \mathrm{PSU}(2m, q)$ for some positive integer $m \geq 2$. Let $z \in \mathfrak{I}(X)$, let $H = C_X(z)$ and let L be a 2-component of H. Then the following two conditions hold.*

(a) *If L is a 2-component of H, then either L is isomorphic to $\mathrm{SL}(j, q)$ or to $\mathrm{SU}(j, q)$ for some integer j with $2 \leq j \leq 2m - 1$ or $L/Z(L) \cong \mathrm{PSL}(m, q^2)$; and*

(b) *if L is a perfect 2-component of H such that $|Z^*(L)|_2 \neq |\mathfrak{M}(L/Z^*(L))|_2$, then $L/Z(L) \cong \mathrm{PSL}(m, q^2)$ and L is the unique 2-component of H.*

LEMMA 5.6. *Let X be a simple Chevalley group over a finite field of characteristic p such that $|\mathfrak{M}(X)|$ is even. Assume that X contains an involution z such that $H = C_X(z)$ contains distinct solvable or perfect 2-components J and L such that $z \in L$ and $J/O(J) \cong \mathrm{PSL}(2, q)$. Then $O(J) = 1$, $|\mathfrak{M}(X)| = 2$ and exactly one of the following six conditions holds.*

(a) *$X \cong P\Omega(7, q)$, and $L \cong \mathrm{SL}(2, q)$;*

(b) *$X \cong P\Omega(2m + 1, q)$ for some integer $m \geq 4$, $L \cong \Omega(2m - 2, q, 1)$ if m is odd or if m is even and $q \equiv 1 \pmod 4$ and $L \cong \Omega(2m - 2, q, -1)$ if m is even and $q \equiv -1 \pmod 4$;*

(c) *n is even, $X \cong P\Omega(8, r, -1)$, $r^2 = q$ and $L \cong \mathrm{SL}(2, r)$;*

(d) *n is even, $X \cong P\Omega(2m, r, -1)$ for some even integer $m \geq 6$, $r^2 = q$ and $L \cong \Omega(2m - 4, r, 1)$;*

(e) *n is even, $X \cong P\Omega(2m, r, 1)$ for some odd integer $m \geq 5$, $r^2 = q$, $r \equiv -1 \pmod 4$ and $L \cong \Omega(2m - 4, r, -1)$; or*

(f) *n is even, $X \cong P\Omega(2m, r, -1)$ for some odd integer $m \geq 5$, $r^2 = q$, $r \equiv 1 \pmod 4$ and $L \cong \Omega(2m - 4, r, 1)$.*

PROOF. Applying Lemmas 2.27 and 4.24 and §§3A–3C, we conclude that $X \cong P\Omega(m, p^s)$ for positive integers m and s with $m \geq 7$. Since $\Omega(3, q) \cong \mathrm{PSL}(2, q)$ and $\Omega(4, q, -1) \cong \mathrm{PSL}(2, q^2)$, it is easy to see that this result follows from §3D.

LEMMA 5.7. *Let G be a group such that $O^{2'}(G)$ is 2-quasisimple and $O^{2'}(G)/O(O^{2'}(G))$ is a Chevalley group over a finite field of order q. Assume that G contains an involution z such that $H = C_G(z)$ possesses a solvable 2-component J such that $O^{2'}(J)$ is not contained in $Z^*(O^{2'}(G))$. Then $q = 3$, $z \notin S(G)$, $O(H) \leq O(G)$ and $O^{2'}(G) \cap H$ contains a solvable 2-component J_1 such that $J_1 \trianglelefteq \trianglelefteq H$, $J = O(H)J_1$ and $O^{2'}(J) = O^{2'}(J_1)$.*

PROOF. Let G be a counterexample of minimal order to this result and set $M = O^{2'}(G)$.

Suppose that $z \in Z^*(M)$. Then $K = C_M(z)^{(\infty)}$ is a perfect 2-component of H such that $M = O(M)K = (O(G)K)^{(\infty)}$ by Lemma 2.15. Since $[J, K] \leq O(K) \leq O(M)$ by Lemma 2.11, we have $[J, M] \leq O(M)$ and hence $J \leq S(G) = O(G)Z^*(M)$ by Lemma 2.13. Since $O^{2'}(S(G)) = O^{2'}(Z^*(M))$, we have a contradiction. Thus $z \notin S(G)$ since $Z^*(M) = S(G) \cap M$.

Next observe that $q = 3$ implies $O(H) \leq O(G)$. To see this, set $\overline{G} = G/S(G)$. Then $\bar{z} \in \mathfrak{I}(\overline{G})$, $F^*(\overline{G}) = \overline{M} = O^{2'}(\overline{G})$ and $O(C_{\overline{G}}(\bar{z})) = 1$ by Lemma 2.13 and [**9**, Lemma (c)]. Since $\overline{O(H)} \leqslant O(C_{\bar{G}}(\bar{z}))$ by [**18**, Proposition 3.11], we have $O(H) \leqslant O^2(S(G)) = O(G)$ and our assertion is proved.

Suppose that $O(M) \neq 1$. Set $\bar{G} = G/O(M)$. Then $O^{2'}(\bar{G}) = \bar{M} \cong M/O(M)$, $Z^*(\bar{M}) = \overline{Z^*(M)}$, $\bar{z} \in \mathfrak{I}(\bar{G})$, $\bar{H} = C_{\bar{G}}(\bar{z})$, $O(\bar{H}) = \overline{O(H)}$ and $\bar{J}$ is a solvable 2-component of $\bar{H}$ by Lemmas 2.2, 2.12, 2.16 and 2.17. Since $|\bar{G}|<|G|$, we have $q = 3$, $O(\bar{H}) \leqslant O(\bar{G}) = \overline{O(G)}$ and there is a solvable 2-component J_1 of $C_M(z)$ such that $\bar{J} = O(\bar{H})\bar{J}_1$ by Lemma 2.18. Hence $O(M)J = O(M)O(H)J_1$. As $C_{O(M)}(z) \leqslant O(H)$, we have $J = O(H)J_1$. Also, since $J_1 \trianglelefteq \trianglelefteq J$, we have $O^{2'}(J) = O^{2'}(J_1)$. Thus $O(M) = 1$, $S(G) = O(G) \times Z(M) = C_G(M)$ and M is quasisimple.

Suppose that $O(G) \neq 1$. Set $\bar{G} = G/O(G)$. Then, as above, we conclude that $q = 3$, $\bar{H} = C_{\bar{G}}(\bar{z})$, $O(H) \leqslant O(G)$ and $(O(G) \times M) \cap H$ possess a solvable 2-component $\mathfrak{J}$ such that $\bar{J} = O(\bar{H})\bar{\mathfrak{J}}$. As $O(G) \leqslant H$ since $z \in M$, we have $O(H) = O(G)$ and $J = \mathfrak{J}$. Thus $J = O(G) \times (M \cap J)$. Set $J_1 = M \cap J$. Then $J_1 = M \cap J \trianglelefteq \trianglelefteq H, J_1 \trianglelefteq \trianglelefteq M \cap H, J_1 \trianglelefteq J, O(M \cap H) \leqslant O(H) \cap M = 1, J_1$ is a solvable 2-component of $M \cap H$ and $O^{2'}(J) = O^{2'}(J_1)$. Consequently $O(G) = 1$ and $S(G) = Z(M) = O_2(M)$.

Suppose that $S(G) \neq 1$. Set $\bar{G} = G/S(M)$. Then $O(C_{\bar{G}}(\bar{z}))\bar{J}$ is a solvable 2-component of $C_{\bar{G}}(\bar{z})$ by Lemma 2.14 since $O^{2'}(J) \nleqslant S(G)$. Thus $q = 3$, $O(H) = 1$, $O(C_{\bar{G}}(\bar{z})) \leqslant O(\bar{G}) = 1$ and $\bar{J} \leqslant \bar{M} = O^{2'}(\bar{G})$. Thus J is a solvable 2-component of $M \cap H$. This contradiction implies that $S(G) = 1$, $F^*(G) = M$ and M is simple.

Let $R = O^{2'}(J)$. Then $R \trianglelefteq \trianglelefteq H \cap M$, $H \cap M \trianglelefteq H$, $E(H) = L_{2'}(H) = E(H \cap M)$ and $[R, E(H)] = [R, E(H \cap M)] \leqslant [J, E(H)] = 1$ by Lemmas 2.11 and 5.1. Let $T \in \mathrm{Syl}_2(R)$, so that either $T \cong Q_8$ or $T \cong E_4$ and $z \notin T$. Thus $q = 3$ by Lemma 5.1(f) and $O(H) = 1$ by [**9**, Lemma (c)]. Let $E_s(H \cap M)$ denote the product of all solvable 2-components of $H \cap M$. Suppose that J is not a solvable 2-component of $H \cap M$. Then

$$[R, E(H \cap M)E_s(H \cap M)] \leqslant [J, E(H \cap M)E_s(H \cap M)] = 1$$

by Lemma 2.11 and Lemma 5.1(g) yields a contradiction since $\langle z, R\rangle \leqslant C_{H\cap M}(E(H \cap M)E_s(H \cap M))$. Thus $J \leqslant H \cap M$ which is also a contradiction and the proof is complete.

LEMMA 5.8. *Let W be a 4-subgroup of the group G and let $W^{\#} = \{z_1, z_2, z_3\}$. Suppose that $C_G(W)$ contains solvable or perfect 2-components L_1 and L_2 such that $z_1 \in L_1$, $z_2 \in L_2$ and $L_1/O(L_1) \cong \mathrm{SL}(2, q)$. Assume that $O(C_G(z))L_1$ is not subnormal in $C_G(z_1)$ if L_1 is perfect and that $O(C_G(z_1))O^{3'}(L_1)$ is not subnormal in $C_G(z_1)$ if L_1 is solvable. Then there is a unique perfect 2-component K of $C_G(z_1)$ such that $O^{2'}(L_2) \leqslant K$. Moreover the following two conditions hold.*

(a) *$z_2 \notin Z^*(K)$, $C_G(W) \leqslant N_G(K)$, $L_1 \neq L_2$, $L_1 \leqslant K$ if $q \neq 3$ and $[K, O^{3'}(L_1)] = K$ if $q = 3$; and*

(b) *if $K/Z^*(K)$ is a simple Chevalley group over a finite field of characteristic p, then $z_1 \in Z(K)$ and exactly one of the following four conditions holds:*

(i) *$K/O(K) \cong \mathrm{Spin}(7, q)$, $C_K(z_2)$ contains unique 2-components J_1, J_2, J_3 such that $z_i \in J_i$, $J_i \trianglelefteq C_G(W)$ and $J_i/O(J_i) \cong \mathrm{SL}(2, q)$ for $i = 1, 2, 3$. Also, for $i \in \{1, 2\}$, we have $J_i = L_i$ if $q \neq 3$ and $J_i \trianglelefteq L_i = O(C_G(W))J_i$ if $q = 3$;*

(ii) $K/O(K) \cong \mathrm{Spin}(2m+1, q)$ *for some odd integer* $m \geqslant 5$; $W \leqslant L_2 \trianglelefteq C_G(W)$, $L_2/O(L_2) \cong \mathrm{Spin}(2(m-1), q, 1)$, L_2 *is a* 2-*component of* $C_K(z_2)$ *and* $C_K(z_2)$ *contains exactly one other* 2-*component* J. *Moreover*, $z_1 \in J \trianglelefteq C_G(W)$, $J = L_1$ *if* $q \neq 3$ *and* $L_1 = O(C_G(W))J$ *if* $q = 3$;

(iii) n *is even*, $K/O(K) \cong \mathrm{Spin}(8, r, -1)$ *where* $q = r^2$, $L_1 \trianglelefteq C_G(W)$, L_1 *is a* 2-*component of* $C_K(z_2)$ *and* $C_K(z_2)$ *contains precisely two other* 2-*components* J_2 *and* J_3. *Moreover* $J_i \trianglelefteq C_G(W)$ *and* $J_i/O(J_i) \cong \mathrm{SL}(2, r)$ *for* $i = 2, 3$ *and, by appropriate indexing, we may assume that* $z_2 \in J_2, z_3 \in J_3, J_2 = L_2$ *if* $r \neq 3$ *and* $L_2 = O(C_G(W))J_2$ *if* $r = 3$; *or*

(iv) n *is even*, $K/O(K) \cong \mathrm{Spin}(2m, r, -1)$ *for some even integer* $m \geqslant 6$, $q = r^2$; $\{L_1, L_2\}$ *is the set of* 2-*components of* $C_K(z_2)$, $L_2/O(L_2) \cong \mathrm{Spin}(2(m-2), r, 1)$, $W \leqslant L_2 \trianglelefteq C_G(W)$ *and* $L_1 \trianglelefteq C_G(W)$.

PROOF. Set $H = C_G(z_1)$. As $O_2(Z(L_1)) = \langle z_1 \rangle$, we have $L_1 \neq L_2$. Clearly $C_G(W) = C_H(z_2)$ and Lemma 2.26 implies that $O^{2'}(L_2)$ is contained in a unique perfect 2-component K of H. Moreover Lemma 2.26 yields $z_2 \notin Z^*(K)$, $C_G(W) \leqslant N_G(K)$, $C_K(z_2) \trianglelefteq C_G(W)$, $L_1 \leqslant K$ if $q \neq 3$ and $[K, O^{3'}(L_1)] = K$ if $q = 3$. Thus, for the remainder of this proof, we may assume that $K/Z^*(K)$ is a simple Chevalley group over a finite field of order $q_1 = p^r$ for some positive integer r. Recall that $K/Z^*(K)$ is θ-balanced when $p = 3$. Thus [**1**, Theorem 2(3)] and Lemma 2.25 imply that $O^{2'}(L_1) \leqslant K$ and $z_1 \in Z(K)$. Set $M = KL_1L_2$. Then $M^{(\infty)} = O^{2'}(M) = K$ and $C_M(z_2) = C_K(z_2)L_1L_2$. Clearly $O^{2'}(L_1)$ and $O^{2'}(L_2)$ are not contained in $Z^*(K)$. Also, if L_i is solvable for $i = 1$ or 2, then $O(L_i) = O(C_G(W)) \leqslant O(C_M(z_2)) \trianglelefteq C_M(z_2) \trianglelefteq \trianglelefteq C_G(W)$ and hence $O(C_G(W)) = O(L_i) = O(C_M(z_2))$. Consequently, L_1 and L_2 are 2-components of $C_M(z_2)$ and $q_1 = 3$ if L_1 or L_2 is solvable by Lemma 5.7. Also, Lemma 5.7 implies that $C_K(z_2)$ contains 2-components J_1 and J_2 such that $J_i \trianglelefteq \trianglelefteq C_M(z_2) \trianglelefteq \trianglelefteq C_G(W)$, $O^{2'}(L_i) = O^{2'}(J_i)$, $J_i = L_i$ if L_i is perfect and $L_i = O(C_G(W))J_i$ if L_i is solvable for $i = 1$ and 2. Since $O(K) = O(H) \cap K$ and $z_2 \notin Z^*(K)$, we may assume that $G = H = K$. Then Lemmas 2.16–2.18 and induction imply that we may assume that $O(G) = 1$ and hence that $S(G) = Z(G) = O_2(G)$. Set $\bar{G} = G/Z(G)$. Then $|\mathfrak{M}(\bar{G})|$ is even since $z_1 \in Z(G)$ and $\bar{z}_2 \in \mathfrak{I}(\bar{G})$. Clearly $L_1 \cap Z(G) = \langle z_1 \rangle$, $\bar{L}_1/O(\bar{L}_1) \cong \mathrm{PSL}(2, q)$, $\bar{z}_2 \in \bar{L}_2$ and $\bar{L}_1$ and $\bar{L}_2$ are 2-components of $C_{\bar{G}}(\bar{z}_2)$ by Lemma 2.17. Then Lemma 5.6 implies that $Z(G) = \langle z_1 \rangle$ and $\bar{G}$ satisfies one of conditions (a)–(f) of Lemma 5.6. Then Lemmas 4.10, 4.12, 4.14, 4.16, 4.17 and 4.18 combine to complete this proof.

6. A proof of Theorem 1. We begin this section with an extension of a portion of [**3**, Corollary III] that is the solvable 2-component case of Theorem 1.

LEMMA 6.1. *Let* G *be a group such that* $O^{2'}(G)$ *is* 2-*quasisimple. Suppose that* $z \in \mathfrak{I}(G)$ *and* $H = C_G(z)$ *possesses an intrinsic solvable* 2-*component* J. *Then the following three conditions hold.*

(a) $O(H) \leqslant O(G)$;

(b) *there is an intrinsic solvable* 2-*component* J_1 *of* $H \cap O^{2'}(G)$ *such that* $J_1 \trianglelefteq \trianglelefteq H$, $O^{2'}(J) = O^{2'}(J_1)$ *and* $J = O(H)J_1$; *and*

(c) *either* $G = O(G)O^{2'}(G)$ *and* $G/O(G) \cong M_{11}$ *or* $O^{2'}(G)/O(O^{2'}(G))$ *is a Chevalley group over a field of* 3 *elements.*

PROOF. Assume that G is a counterexample of minimal order to this lemma and set $M = O^{2'}(G)$. Suppose that $z \in Z^*(M) = S(G) \cap M$. Then $K = C_M(z)^{(\infty)}$ is a perfect 2-component of H and $M = O(M)K$. Thus $[J, M] \leqslant O(M)$ and $J \leqslant C_G(M/O(M)) = S(G) = O(G)Z^*(M)$ by Lemmas 2.11 and 2.3. As $J = O^2(J)$, this is impossible. Thus $z \notin S(G)$ and $S(G) \cap J \leqslant O(J)$.

Suppose that $S(G) \neq 1$ and set $\bar{G} = G/S(G)$. Then $F^*(\bar{G}) = \bar{M} \cong M/Z^*(M)$, $\bar{z} \in \mathcal{I}(\bar{J})$ and $O(C_{\bar{G}}(\bar{z}))\bar{J}$ is a solvable 2-component of $\bar{G}$ by Lemmas 2.13 and 2.14. Hence $\overline{O(H)} \leqslant O(C_{\bar{G}}(\bar{z})) = 1$ by [**18**, Proposition 3.11], and $\bar{G} = \bar{M} \cong M_{11}$ or $\bar{M}$ is a Chevalley group over a field of 3 elements. Thus $O(H) \leqslant O^2(S(G)) = O(G)$ and Lemma 5.7 implies that $\bar{G} = \bar{M} \cong M_{11}$. Then $Z^*(M) = O(M)$ since $|\mathfrak{M}(M_{11})| = 1$ (cf. [**13**, §2]) and $S(G) = O(G)$. Then $G = O(G)M$, $M/O(M) \cong M_{11}$ and $\bar{J}$ is the unique 2-component of $C_{\bar{G}}(\bar{z}) = \bar{H}$ by [**4**, Table 1]. Also we conclude that $C_M(z)$ contains a unique 2-component K from Lemma 2.18. Moreover K is solvable, $K \trianglelefteq\trianglelefteq H$, $z \in K$ and $\bar{J} = \bar{K}$. Hence $O(G)J = O(G)K$. Since $C_{O(G)}(z) = O(H)$, we have $J = O(H)K$ and $O^{2'}(J) = O^{2'}(K)$ by Lemma 2.1. Thus $S(G) = 1$ and $F^*(G) = M$ is simple. Then [**3**, Corollary III; **4**, Table 1] imply that $G = M \cong M_{11}$ or M is a Chevalley group over a finite field of odd order. Consequently Lemma 5.7 implies $G = M \cong M_{11}$. Thus $O(H) = 1$ by [**4**, Table 1] and we have a contradiction, which concludes our proof of this result.

We now commence to prove Theorem 1. Thus let G, L, z and p be as in the hypotheses of Theorem 1 and assume that G is a counterexample of minimal order to the theorem.

Thus L is perfect by Lemma 6.1, $L \leqslant O^{2'}(G) = G$ and Lemmas 2.12–2.14 imply that $O(G) = 1$. Consequently $O_2(G) = Z(G) = C_G(O^{2'}(G)) = S(G)$ and $z \notin Z(G)$ since $G \neq L$.

Suppose that $Z(G) \neq 1$ and set $\bar{G} = G/Z(G)$. Then $\bar{G}$ is simple, $|\bar{G}| < |G|$, $\bar{z} \in \mathcal{I}(\bar{G})$ and $\bar{L}$ is an intrinsic perfect 2-component of $C_{\bar{G}}(\bar{z})$ of $\mathfrak{M}(p)$-type by Lemmas 2.12 and 2.28. Then, by induction, $\bar{G} \cong M_{11}$ or $\bar{G}$ is isomorphic to a Chevalley group over a finite field of characteristic p. Since $|\mathfrak{M}(M_{11})| = 1$, we have a contradiction. Thus $Z(G) = 1$ and G is simple. Then [**3**, Corollary III] and Lemma 5.1 imply that G does not contain an involution u such that $C_G(u)$ contains an intrinsic 2-component J with $J/O(J) \cong \mathrm{SL}(2, p^k)$ for some integer $k \geqslant 1$.

Suppose that W is a 4-subgroup of G with $W^\# = \{z_1, z_2, z_3\}$ and such that $C_G(W)$ contains 2-components L_1, L_2 such that $z_1 \in L_1$, $z_2 \in L_2$, $L_1/O(L_1) \cong \mathrm{SL}(2, p^k)$ for some integer $k \geqslant 1$ and L_2 is of $\mathfrak{M}(p)$-type.

Applying Lemma 5.8, we obtain a unique perfect 2-component K of $C_G(z_1)$ such that $O^{2'}(L_2) \leqslant K$, $z_2 \notin Z^*(K)$, $C_G(W) \leqslant N_G(K)$, $L_1 \leqslant K$ if $p^k \neq 3$ and $[K, O^{3'}(L_1)] = K$ if $p^k = 3$. Set $X = KL_2$. Then $X \leqslant C_G(z_1) < G$, $O^{2'}(X) = K$, $L_2 \trianglelefteq\trianglelefteq C_X(z_2) = C_K(z_2)L_2 \trianglelefteq\trianglelefteq C_G(W)$, $O(L_2) \leqslant O(C_X(z_2)) \leqslant O(C_G(W))$ and hence L_2 is an intrinsic 2-component of $C_X(z_2)$ of $\mathfrak{M}(p)$-type. We conclude, by induction, that $K/O(K)$ is a Chevalley group over a finite field of characteristic p or $X = O(X)K$ and $X/O(X) \cong K/O(K) \cong M_{11}$. Also, when $p^k = 3$, $K/O(K)$ is always θ-balanced since M_{11} is balanced. Then [**1**, Theorem 2(3)] and Lemma 2.25 imply that $O^{2'}(L_1) \leqslant K$. Thus $z_1 \in Z(K)$, $K/Z^*(K)$ is a simple Chevalley group over a finite field of order p^n for some integer $n \geqslant 1$ and Lemma 5.8 yields a great deal of information about this situation. In particular, $L_2/O(L_2)$ is not isomorphic to $\mathrm{Spin}(7, p^r)$ for any positive integer r.

Set $H = C_G(z)$, $Q = C_H(L/O(L))$ and $\overline{H} = H/O(H)$.

First suppose that $L/O(L) \cong \mathrm{Spin}(7, q)$, where $q = p^n$ for some positive integer n. Let $1 \neq B$ be a 2-subgroup of Q. Then Lemma 2.15 implies that $J = C_L(B)^{(\infty)} = O^{2'}(C_L(B))$ is a 2-component of $C_G(B\langle z \rangle)$ with $z \in J$, $L = O(L)J$ and $J/O(J) \cong L/O(L)$. Hence J is contained in a unique 2-component K of $C_G(B)$ by Lemma 2.19. Thus J is a 2-component of $C_K(z)$ and hence $\tilde{K} = K/Z^*(K)$ is a simple Chevalley group over a finite field of characteristic p by induction since $K \leqslant C_G(B) < G$. Assume that $z \notin Z^*(K)$. Then $\tilde{z} \in \mathfrak{I}(\tilde{J})$, $\tilde{J}$ is a 2-component of $C_{\tilde{K}}(\tilde{z})$ and $\tilde{J}/O(\tilde{J}) \cong \mathrm{Spin}(7, q)$ since $Z^*(J) = O(H) \times \langle z \rangle$. Then Lemma 5.1(e) yields a contradiction. Thus $z \in Z^*(K)$, $J = C_K(z)^{(\infty)}$, $K = O(K)J = (O(C_G(B))J)^{(\infty)}$ and $K/O(K) \cong J/O(J) \cong L/O(L) \cong \mathrm{Spin}(7, q)$ by Lemma 2.15. Let $j \in \mathfrak{I}(H - L)$ be such that $C_H(j)$ contains an intrinsic 2-component J with $J/O(J) \cong \mathrm{SL}(2, p^r)$ for some positive integer r. Then Lemma 2.21 implies that $[J, L] \leqslant O(L)$ since $j \in J - L$. Hence $j \in J \leqslant Q$ and the remarks above with $W = \langle j, z \rangle$ yield a contradiction. Thus, if $j \in \mathfrak{I}(H - L)$, then $C_H(j)$ does not contain an intrinsic 2-component J with $J/O(J) \cong \mathrm{SL}(2, p^r)$ for some positive integer r. Hence, if K is a perfect 2-component of H with $K \neq L$ and $K/O(K)$ isomorphic to a Chevalley group over a finite field of order $q = p^s$ for some positive integers s, then $K/O(K) \cong \mathrm{PSL}(2, q)$ and $q \neq 3$ or $p = 3$, s is odd, $s \geqslant 3$ and $K/O(K) \cong {}^2G_2(q)$ by Lemma 5.2. Thus L char H and Q char H.

Let $S \in \mathrm{Syl}_2(H)$. Thus $S \cap Q \trianglelefteq S$, $S \cap L \trianglelefteq S$, $S \cap Q \cap L = \langle z \rangle \leqslant Z(S)$ and $[S \in Q, S \cap L] = 1$. Note that all involutions of $L - \langle z \rangle$ are conjugate in L and $z^G \cap S \neq \{z\}$ by Glauberman's Z^*-theorem [**15**, Corollary 1].

Let $\tau \in \mathfrak{I}(S - L)$ be such that $C_G(\tau)$ contains an intrinsic 2-component K with $K/O(K) \cong L/O(L)$. Suppose that $C_L(\tau)$ contains a perfect 2-component J such that $z \in J$ and $J/O(J)$ is isomorphic to one of the following groups: $\mathrm{Spin}(7, q)$, $\mathrm{SL}(4, q)$, $\mathrm{SU}(4, q)$, $\mathrm{SL}(2, q^2)$, $\mathrm{Sp}(4, q)$ or n is even and $J/O(J) \cong \mathrm{Spin}(7, \sqrt{q})$. Then $z \in Z(J)$, J is a perfect 2-component of $C_G(\tau, z)$ and J is contained in a unique perfect 2-component Y of $C_G(\tau)$ by Lemma 2.19. Since J is a perfect 2-component of $\mathfrak{M}\mathfrak{L}(p)$-type of $C_Y(z)$, we conclude by induction that $Y/O(Y)$ is a Chevalley group over a finite field of characteristic p. Clearly $Y \neq K$ by Lemma 4.10 and hence $X = C_K(z)^{(\infty)} = O^{2'}(C_K(z))$ is a perfect 2-component of $C_G(\tau, z) = C_H(\tau)$ such that $\tau \in X$ and $X/O(X) \cong K/O(K)$ by Lemma 2.15. Then X is contained in a unique perfect 2-component $\mathcal{Y}$ of H. By induction, $\mathcal{Y}/O(\mathcal{Y})$ is isomorphic to a Chevalley group over a finite field characteristic of p. Hence $\mathcal{Y} = L$, $\mathcal{Y} = O(\mathcal{Y})X$ and $\tau \in X \leqslant \mathcal{Y} = L$. Since $\tau \notin L$, we have a contradiction. Moreover as $C_{\overline{H}}(\overline{L}) = \overline{Q} \trianglelefteq \overline{H}$, we may apply Lemmas 4.10–4.11 to the quotient $\overline{H}/\overline{Q}$ to conclude that $\tau = \tau_1\tau_2$ where $\tau_1 \in \mathfrak{I}(S \cap Q)$ and $\tau_2 \in \mathfrak{I}((S \cap L) - \langle z \rangle)$. Then $C_L(\tau) = O(C_L(\tau))C_L(\langle \tau, \tau_2 \rangle)$, $O(C_L(\tau)) \leqslant O(L)$ and $C_L(\langle \tau, \tau_2 \rangle)$ contains a 2-component X such that $z \in X$ and $X/O(X) \cong \mathrm{SL}(2, q)$ by Lemma 4.10. Thus $C_G(\langle \tau, z \rangle)$ contains a 2-component J such that $z \in J$ and $J/O(J) \cong \mathrm{SL}(2, q)$. Suppose that $[K, z] \leqslant O(K)$. Then Lemma 2.15 implies that $Y = C_K(z)^{(\infty)} = O^{2'}(C_K(z))$ is a 2-component of $C_G(\langle \tau, z \rangle)$ such that $\tau \in Y$ and $Y/O(Y) \cong L/O(L)$. Applying an observation above with $W = \langle \tau, z\rangle$ yields a contradiction. Thus $[K, z] = K$ and Lemma 2.21 implies that $z \in K$. Then Lemma 4.10 implies that $C_G(\langle \tau, z\rangle)$ contains a 2-component J_1 such that $\tau \in J_1$ and $J_1/O(J_1) \cong \mathrm{SL}(2, q)$. As noted above, this implies that $\tau \in L$ and we have a contradiction.

We have shown that if $\tau \in \mathcal{I}(S)$ is such that $C_G(\tau)$ contains an intrinsic 2-component K such that $K/O(K) \cong L/O(L)$, then $\tau \in L$. Hence $z^G \cap S = \mathcal{I}(S \cap L)$, $z^G \cap H = \mathcal{I}(L)$, $\langle z^G \cap H\rangle = L$ and $S \in \mathrm{Syl}_2(G)$ since $Z(S \cap L) = \langle z\rangle$ and $S \cap L = \langle \mathcal{I}(S \cap L)\rangle$ by Lemma 4.10(a).

Let $z_1 \in \mathcal{I}(L - \langle z\rangle)$ and let $z_1 = z^g$ for $g \in G$. Then $z_1 \sim z_1 z$ in L, $z^G \cap H^g = \mathcal{I}(L^g)$, $\langle z^G \cap H^g\rangle = L^g$ and $z \sim z_1 z$ in L^g. Hence $N_G(\langle z, z_1\rangle)/C_G(\langle z, z_1\rangle) \cong \Sigma_3$. Note that $\overline{Q} * C_{\overline{L}}(\bar{z}_1) \trianglelefteq C_{\overline{H}}(\bar{z}_1) = \overline{C_H(z_1)}$, $Q = O(H)C_Q(z_1)$, $C_{\overline{L}}(\bar{z}_1) = \overline{C_L(z_1)}$, $LC_H(z_1) = H$, $|L : C_L(z_1)|_2 = 2 = |H : C_H(z_1)|_2$ and $\langle z, z_1\rangle$ is the center of a Sylow 2-subgroup of $C_L(z_1)$ by Lemma 4.10. Also $C_L(z_1)$ contains precisely three 2-components J_1, J_2, J_3 such that $z \in J_1$, $z_1 \in J_2$, $zz_1 \in J_3$, $O(J_i) \leqslant O(L)$ and $J_i/O(J_i) \cong \mathrm{SL}(2, q)$ for $i = 1,2,3$. Set $E = \langle z, z_1\rangle$, $M = C_G(E) = C_H(z_1)$, $\mathcal{J}_i = J_i$ if $q \neq 3$ and $\mathcal{J}_i = O(M)J_i$ if $q = 3$ for $i = 1,2,3$. Thus, as $C_L(z_1) \trianglelefteq M$, we have $J_i \trianglelefteq M$ and $\mathcal{J}_i$ is a 2-component of M for $i = 1,2,3$. Suppose that $\mathcal{K}$ is a 2-component of M such that $z_1 \in \mathcal{K}$ and $\mathcal{K}/O(\mathcal{K}) \cong \mathrm{SL}(2, q)$. Then we conclude from Lemma 2.21 that $O^{2'}(\mathcal{K}) \leqslant L$. If $q \neq 3$, then $\mathcal{K} = O^{2'}(\mathcal{K}) \leqslant L$ and $\mathcal{K} = J_2 = \mathcal{J}_2$. Suppose that $q = 3$ and set $B = L\mathcal{K}$. Then $\mathcal{K} \trianglelefteq \trianglelefteq C_B(z_1)$, $O^{2'}(B) = L$, $C_B(z_1) = C_L(z_1)\mathcal{K} \trianglelefteq \trianglelefteq M$ and $O(\mathcal{K}) \leqslant O(C_B(z_1)) \leqslant O(M) = O(\mathcal{K}) = O(\mathcal{J}_2)$. Hence $\mathcal{K} = O(M)J_2 = \mathcal{J}_2$ by Lemma 5.7. Thus $\mathcal{K} = \mathcal{J}_2$ in all cases. It follows that $\mathcal{J}_i \trianglelefteq M$ for $i = 1,2,3$ and $N_G(\langle z, z_1\rangle)$ permutes $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{J}_3$ in the obvious way. Applying [**18**, Proposition 3.11] and Lemma 4.11(f) to the group $\overline{H}/\overline{Q}$, we conclude that $O(C_{\overline{H}}(\bar{z}_1)) \trianglelefteq \overline{Q} \trianglelefteq \overline{M} = C_{\overline{H}}(\bar{z}_1)$ and hence $O(M) = M \cap O(H)$. This implies that $\overline{\mathcal{J}}_i \trianglelefteq \overline{M}$ and $\overline{\mathcal{J}}_i \cong \mathrm{SL}(2, q)$ for $i = 1,2,3$. Also $C_{\overline{M}}(\overline{\mathcal{J}}_1\overline{\mathcal{J}}_2\overline{\mathcal{J}}_3) = \overline{Q} \times \langle \bar{z}_1\rangle$ by Lemma 4.11(f). Moreover Lemma 4.10 implies that one may choose z_1 such that $z_1 \in S \cap L$, $z_1^S = z_1\langle z\rangle$, $C_S(z_1) = T \in \mathrm{Syl}_2(M)$, $Z(T \cap L) = E \lhd S$ and $|S : T| = 2$. By the Frattini argument, there is a 3-element $\pi \in N_G(T) \cap N_G(E)$ such that π acts transitively on $E^\#$ and on $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$. Thus

$$\langle S, \pi\rangle \leqslant N_G((S \cap Q) \times \langle z_1\rangle) \cap N_G(T) \cap N_G(E)$$

since $S \cap Q = T \cap Q$. Suppose that $S \cap Q \neq \langle z\rangle$. Then $(S \cap Q) \cap ((S \cap Q)^\pi) \neq 1$. Since $S \cap Q \cap E = \langle z\rangle$, we have

$$\left((S \cap Q)^\pi\right) \cap E = \langle z^\pi\rangle \quad \text{and} \quad (S \cap Q) \cap \left((S \cap Q)^\pi\right) \cap E = 1.$$

Let $\tau \in \mathcal{I}((S \cap Q) \cap ((S \cap Q)^\pi))$. Then $X = C_L(\tau)^{(\infty)}$ is a 2-component of $C_G(\tau, z)$ such that $S \cap L \leqslant X$, $Z^*(X) = O(X) \times \langle z\rangle$ and $X/O(X) \cong \mathrm{Spin}(7, q)$. Also $Y = C_{L^\pi}(\tau)^{(\infty)}$ is a 2-component of $C_G(\tau, z^\pi)$ such that $E = \langle z, z^\pi\rangle \leqslant Y$, $Z^*(Y) = O(Y) \times \langle z^\pi\rangle$ and $Y/O(Y) \cong \mathrm{Spin}(7, q)$. A previous observation implies that $\langle z, z^\pi\rangle = E \leqslant Z^*(L_{2'}(C_G(\tau)))$. Since $X \leqslant L_{2'}(C_G(\tau))$, we have $[X, z^\pi] \leqslant O(L_{2'}(C_G(\tau))) \cap X = O(X)$. Hence $z^\pi \in Z^*(X) = O(X) \times \langle z\rangle$. Since $z^\pi \neq z$, we conclude that $S \cap Q = \langle z\rangle$, $Q = O(H) \times \langle z\rangle$, $H' \leqslant O(H)L$ by Lemma 4.11, $S' \leqslant S \cap L$ and $T' \leqslant T \cap L$.

Applying [**32**, Theorem 3.4], we conclude that $S \cap L \neq S$. Then, Lemma 4.11 applied to $\overline{H}/\overline{Q} = \overline{H}/\langle \bar{z}\rangle$ yields the existence of a normal subgroup V of S such that $S \cap L \leqslant V \leqslant S$, $|V/(S \cap L)| \leqslant 2$, S/V is cyclic, $\{v \in V \mid v^2 \in \langle z \rangle\} \subseteq S \cap L$ and $S \neq V$ if and only if there is an element $x \in S - L$ such that $x^2 \in \langle z \rangle$. Since $z^G \cap S = \mathcal{I}(S \cap L)$, it follows that $\mathcal{I}(S) = \mathcal{I}(S \cap L) = z^G \cap S$ by [**20**, Corollary 2.1.2]. Note that $T \cap L \in \mathrm{Syl}_2(C_L(z_1))$, $Z(T \cap L) = E$, $\Omega_1(T \cap L) = T \cap L$ and $\Omega_1(S \cap L) = S \cap L$ by Lemma 4.10. Hence $T \cap L$ and $S \cap L$ are weakly closed in T and S with respect to G, respectively. Moreover, $S = (S \cap L)T$ since $H = LM$, $S \cap L \in \mathrm{Syl}_2(L)$ and $T \in \mathrm{Syl}_2(M)$. Also $|(S \cap L)/(T \cap L)| = 2$, $T \cap L \lhd S$, $S \cap L \lhd S \in \mathrm{Syl}_2(N_G(T) \cap N_G(E))$, $[S, T] \leqslant T'[S \cap L, T] \leqslant T \cap L$ and $(N_G(T) \cap N_G(E))/N_M(T) \cong \Sigma_3$ imply that $\langle S, \pi \rangle$ acts trivially on $T/(T \cap L)$.

As $H/(O(H)L)$ is abelian, we have $O(H)LV \trianglelefteq H$. Suppose that $Z(V) \neq \langle z \rangle$. Then $V \neq S \cap L$ and there is a subgroup $F \leqslant Z(V)$ such that $z \in F$ and $|F| = 4$. Since $\{v \in V \mid v^2 \in \langle z \rangle\} = \{v \in S \cap L \mid v^2 \in \langle z \rangle\}$, we have $F \leqslant S \cap L$. Since $Z(S \cap L) = \langle z \rangle$, this is impossible. Thus $Z(V) = \langle z \rangle$. Also $V = (S \cap L)(T \cap V)$, so that $|V/(V \cap T)| = 2$ and $V \cap T \in \mathrm{Syl}_2(M \cap (O(H)LV))$.

We shall now demonstrate the following condition.

($*$) If $\mathfrak{T} \in \{(S \cap L)^\#, V - (S \cap L), S - V\}$ and $x \in \mathfrak{T}$, then x is conjugate in G to an element $y \in \mathfrak{T}$ such that $\Omega_1(\langle y \rangle) = \langle z \rangle$.

For, if x and $\mathfrak{T}$ satisfy the hypotheses of ($*$), we may clearly assume that $\Omega_1(\langle x \rangle) = \langle j \rangle$ for some involution $j \in S - \langle z \rangle$. Then $j \in S \cap L$ and by Lemma 4.10, there is an element $g \in H$ such that $j^g = z_1$ and $x^g \in T \cap \mathfrak{T}$ since $T \in \mathrm{Syl}_2(C_H(z_1))$, $T \cap V \in \mathrm{Syl}_2((O(H)LV) \cap C_H(z_1))$ and $T \cap L \in \mathrm{Syl}_2(C_L(z_1))$. However $\langle S, \pi \rangle \leqslant N_G(T) \cap N_G(T \cap L)$ and $\langle S, \pi \rangle$ acts trivially on $T/(T \cap L)$ and $T \cap L \leqslant T \cap V \leqslant T$. Thus there is an element $h \in \langle S, \pi \rangle$ such that $j^{gh} = z$ and $x^{gh} \in \mathfrak{T}$. We have established condition ($*$) above.

Suppose that $V \neq S$. Then there is an element $x \in S - V$ such that $x^2 = z$. By [**20**, Corollary 2.1.2], x is conjugate in G to an element x_1 in V. By condition ($*$), x_1 is conjugate in G to an element $y \in V$ such that $y^2 = z$. Thus $x \sim y$ in G. Since $y^2 = z = x^2$, we have $x \sim y$ in H. However $x \notin O(H)LV \leqslant H$ and we have a contradiction. Thus $V = S$ and $|S/(S \cap L)| = 2$.

Choose an element x in $S - (S \cap L)$ of minimal order. Since $\mathcal{I}(S) = \mathcal{I}(S \cap L)$, $|x| > 2$. Also $\mathrm{ccl}_G(x^{2^i}) \cap S \leqslant S \cap L$ for all integers $i \geqslant 1$. Then [**20**, Corollary 2.1.2] and condition ($*$) imply that there are elements w and v in $\mathrm{ccl}_G(x)$ such that $w \in S - (S \cap L)$, $v \in S \cap L$ and $\Omega_1(\langle w \rangle) = \Omega_1(\langle v \rangle) = \langle z \rangle$. Thus $w \sim v$ in H. Since $w \notin O(H)L \trianglelefteq H$, we have a contradiction. We conclude that $L/O(L)$ is not isomorphic to $\mathrm{Spin}(7, q)$.

In the general case, Lemma 5.4 implies that there is an involution $u \in L - Z^*(L)$ such that $C_L(u)$ possesses 2-components J_1 and J_2 such that $u \in J_1$, $J_1/O(J_1) \cong \mathrm{SL}(2, q)$, $\{z, uz\} \cap J_2 \neq \varnothing$, J_2 is of $\mathfrak{M}(p)$-type and $J_2/Z^*(J_2)$ is isomorphic to a Chevalley group over a finite field of order q. Set $W = \langle u, z \rangle$, $M = C_G(W)$, $\mathcal{J}_i = J_i$ if J_i is perfect and $\mathcal{J}_i = O(M)J_i$ if J_i is solvable for $i = 1, 2$. Then $\mathcal{J}_1$ and $\mathcal{J}_2$ are 2-components of M, $u \in \mathcal{J}_1$, $\mathcal{J}_1/O(\mathcal{J}_1) \cong \mathrm{SL}(2, q)$, $\{z, uz\} \cap \mathcal{J}_2 \neq \varnothing$, $\mathcal{J}_2$ is of $\mathfrak{M}(p)$-type and $\mathcal{J}_2/Z^*(\mathcal{J}_2) \cong J_2/Z^*(J_2)$. From an observation above, we conclude that $C_G(u)$ contains a unique perfect 2-component K such that $M \leqslant N_G(K)$, $\langle O^{2'}(\mathcal{J}_1), O^{2'}(\mathcal{J}_2) \rangle \leqslant K$, $u \in Z(K)$, $z \in K - Z^*(K)$ and $K/Z^*(K)$ is a simple

Chevalley group over a finite field of characteristic p. Note that $\mathfrak{J}_1/O(\mathfrak{J}_1)$ and $\mathfrak{J}_2/O(\mathfrak{J}_2)$ are both isomorphic to Chevalley groups over a finite field of order q. Consequently Lemma 5.8 implies that $K/O(K) \cong \mathrm{Spin}(2m+1, q)$ for some odd integer $m \geq 3$. From the preceding discussion and Lemma 5.8, we must have $m \geq 5$, $J_2 = \mathfrak{J}_2 \leq K$, $J_2/O(J_2) \cong \mathrm{Spin}(2(m-1), q, 1)$ and $Z^*(K) = O(K) \times \langle u \rangle$. Consequently we may assume that $L/O(L) \cong \mathrm{Spin}(2r+1, q)$ for some odd integer $r \geq 5$. Thus Lemma 4.12 implies that we may assume that $J_1/O(J_1) \cong J_2/O(J_2) \cong \mathrm{SL}(2, q)$. Since $\mathrm{SL}(2, q)$ and $\mathrm{Spin}(2(m-1), q, 1)$ are not isomorphic, we have a contradiction and our proof of Theorem 1 is complete.

7. A proof of Theorem 2. We now commence to prove Theorem 2. Thus let G, W, $W^\# = \{z_1, z_2, z_3\}$, L_1, L_2, p_1 and p_2 satisfy the hypotheses of Theorem 2 and assume that G is a counterexample of minimal order to the theorem.

Assume that $O(G) \neq 1$. Set $\bar{G} = G/O(G)$. Then $\bar{z}_i \in \bar{L}_i$, $\bar{L}_i$ is a 2-component of $C_{\bar{G}}(\bar{W})$ and $\bar{L}_i$ is of $\mathfrak{M}(p_i)$-type by Lemmas 2.17 and 2.28. Hence, since $|\bar{G}| < |G|$, either (a) $O^{2'}(\bar{L}_i) = \overline{O^{2'}(L_i)} \trianglelefteq\trianglelefteq \bar{G}$ or (b) $O^{2'}(\bar{L}_i) = \overline{O^{2'}(L_i)}$ is contained in a unique perfect 2-component $\bar{K}_i$ of $\bar{G}$ such that $\bar{K}_i \cong M_{11}$ or $\bar{K}_i$ is isomorphic to a perfect Chevalley group over a finite field of characteristic p_i for $i = 1$ and 2. Let $i \in \{1, 2\}$. Thus (b) holds and Lemma 2.18 yields a contradiction. Hence $O(G) = 1$.

Suppose that L_1 and L_2 are both solvable. Thus $p_1 = p_2 = 3$. Set $M = C_G(z_1)$ and $J_1 = O(M)O^{3'}(L_1)$. Thus $O^{2'}(L_1) \leq O^{2'}(J_1)$ by Lemma 2.9. Assume, for the moment, that $J_1 \trianglelefteq\trianglelefteq M$. Suppose that $O^{2'}(J_1) \leq O_2(G)$. Then $O(M) = 1$, $E(G) = E(M)$ and $J_1 = O^{3'}(L_1) \trianglelefteq\trianglelefteq G$ by Lemma 2.23. Since $O^{2'}(J_1) = O^{2'}(L_1)$, we have $O^{2'}(L_1) \trianglelefteq\trianglelefteq G$. Suppose that $O^{2'}(J_1) \nleq O_2(G)$. Then Lemmas 2.21 and 2.22 imply that there is a unique perfect component K of G such that $O^{2'}(L_1) \leq O^{2'}(J_1) \leq K$. Clearly $J_1 \leq N_G(K)$ and $z_1 \notin Z(K)$. Then Lemma 6.1 applied to KJ_1 implies that $K \cong M_{11}$ or K is a Chevalley group over a field of 3 elements. Now assume that J_1 is not subnormal in M. Then Lemma 2.26 implies that $O^{2'}(L_2)$ is contained in a unique perfect 2-component K of M. Also $z_2 \notin Z^*(K)$, K is $C_G(W) = C_M(z_2)$-invariant and $[K, O^{3'}(L_1)] = K$. Applying Lemma 6.1 to KL_2, we conclude that $K/O(K) \cong M_{11}$ or $K/O(K)$ is a Chevalley group over a field of order 3. Then Lemma 2.25 implies that $O^{2'}(L_1) \leq K$ and $z_1 \in K$. Also Lemma 5.8(b) implies that K is of $\mathfrak{M}(3)$-type. By Lemma 2.19, K is contained in a unique component $\mathcal{K}$ of G. Then $\langle O^{2'}(L_i) \mid i = 1, 2 \rangle \leq \mathcal{K}$ and Theorem 1 eliminates this case. A similar argument applied to $C_G(z_2)$ now yields a contradiction. Thus, by symmetry, we may assume that L_1 is perfect.

Set $M = C_G(z_1)$ and $J_1 = O(M)L_1$. Assume, for the moment, that $J_1 \trianglelefteq\trianglelefteq M$. Then there is a unique perfect component K_1 of G such that $L_1 \leq J_1^{(\infty)} \leq K_1$. Then, since $J_1^{(\infty)} = O(J_1^{(\infty)})L_1$ is a 2-component of $C_K(z_1)$, Theorem 1 implies that K_1 is a Chevalley group over a finite field of characteristic p_1. Suppose that $O^{2'}(L_2)$ is contained in a perfect 2-component K_2 of $M = C_G(z_1)$. Then $O^{2'}(L_2)$ is not contained in $Z^*(K_2)$, K_2 is unique, $K_2 = [L_{2'}(M), O^{2'}(L_2)]$ and $L_2 \leq N_G(K_2)$. Then Theorem 1 applied to the group K_2L_2 implies that $K_2/O(K_2) \cong M_{11}$ or $K_2/O(K_2)$ is isomorphic to a Chevalley group over a finite field of characteristic p_2. Suppose that $K_2 \leq K_1$. If $z_1 \in Z(K_1)$, then $K_1 = K_2$ and $p_1 = p_2$ which is impossible. Thus $z_1 \notin Z(K_1)$. Set $\bar{K}_1 = K_1/Z(K_1)$. Then $\bar{K}_2$ is a perfect 2-component of

$C_{\bar{K}_1}(\bar{z}_1)$, $p_1 = p_2$ and $\bar{K}_2$ is a Chevalley group over a field of characteristic p_1 by Lemma 5.1. As $\langle O^{2'}(L_1), O^{2'}(L_2)\rangle \leq K_1$, this is impossible. Hence, since $z_1 \in K_1$, K_2 is contained in a unique component X of G with $X \neq K_1$ by Lemma 2.19. Since $[K_1, X] = 1$, $X = K_2$ and again we have a contradiction. Consequently $O^{2'}(L_2)$ is not contained in a perfect 2-component of M and hence L_2 is a solvable 2-component of $C_M(z_2) = C_G(W)$ by Lemma 2.19. But then Lemmas 2.21–2.23 imply that $O(C_G(W)) \leq O(M)$, $J_2 = O(M)L_2$ is a solvable 2-component of M and $[J_2, X] = 1$ for all components X of G with $X \neq K_1$. However $O^{2'}(L_2) \leq O^{2'}(J_2)$ and $J_2 \leq N_G(K_1)$. Suppose that $O^{2'}(J_2) \leq K_1$. Then Lemma 5.7 applied to the group K_1J_1 implies that $p_2 = 3 = p_1$. Since $O^{2'}(L_2) \leq O^{2'}(J_2)$, this is impossible. Hence $O^{2'}(J_2) \nleq K_1$ and [**1**, Theorem 2(3)] implies that $[O^{2'}(J_2), E(G)] = 1$. Consequently $O^{2'}(J_2) \leq O_2(G)$ by Lemma 2.22. Then $O^{2'}(L_2) = O^{2'}(J_2) \trianglelefteq \trianglelefteq G$, which is impossible. Consequently $J_1 = O(M)L_1$ is not subnormal in M.

Now Lemma 2.26 implies that $O^{2'}(L_2)$ is contained in a unique perfect 2-component K of M. Also $z_2 \notin Z^*(K)$, K is $C_G(W) = C_M(z_2)$-invariant and $L_1 \leq K$. Thus $z_1 \in Z(K)$, $\bar{K} = K/Z^*(K)$ is a Chevalley group over a field of characteristic p_2 by Theorem 1 applied to KL_2. Since $\bar{L}_1$ is a perfect 2-component of $C_{\bar{K}}(\bar{z}_2)$, we have $p_1 = p_2$ by Lemma 5.1. Also K is contained in a unique component of $\mathcal{K}$ of G by Lemma 2.19, $L_2 \leq K$ if L_2 is perfect and $C_K(z_2)$ contains a solvable 2-component X_2 such that $L_2 = O(L_2)X_2$, $O^{2'}(L_2) = O^{2'}(X_2)$ and $O(C_K(z_2)) \leq O(K)$ if L_2 is solvable by Lemma 5.7. Moreover, K is properly contained in a unique component $\mathcal{K}$ of G. Then, Theorem 1 implies that K is not of $\mathfrak{M}(p_1)$-type. Thus $\bar{K} \cong \mathrm{PSL}(2r, q)$ or $\bar{K} \cong \mathrm{PSU}(2r, q)$ where r is an integer, $r \geq 2$ and $q = p_1^s$ for some positive integer s. Note that $\bar{L}_1$ is a perfect component of $C_{\bar{K}}(\bar{z}_2)$ with $|Z(\bar{L}_1)|_2 \neq |\mathfrak{M}(\bar{L}_1/Z(\bar{L}_1))|_2$ since $z_1 \in Z(L_1) \cap Z(K)$. Then Lemma 5.5 implies that $\bar{L}_1 = \bar{L}_2$ and $\bar{L}_1/Z(\bar{L}_1) \cong \mathrm{PSL}(r, q^2)$. Hence $L_1 = L_2$ by Lemma 2.18. Since $L_1/Z^*(L_1) \cong \bar{L}_1/Z(\bar{L}_1)$, the Sylow 2-subgroups of $Z^*(L_1)$ are cyclic. Hence $z_1 = z_2$. This contradiction completes our proof of Theorem 2.

We remark that Theorem 1 follows easily from Theorem 2. For, assume Theorem 2 and let G, z, L and p be as in the hypotheses of Theorem 1. Then $O^{2'}(G)$ is 2-quasisimple and, by Lemma 6.1, we may assume that L is perfect. Then, by Lemma 5.4, there is an involution $t \in L - Z^*(L)$ such that $C_L(t)$ contains 2-components J and K of $\mathfrak{M}(p)$-type with $t \in J$ and $Z(K) \cap \{z, tz\} \neq \varnothing$. Set $W = \langle t, z\rangle$. Then Theorem 2 yields the conclusion of Theorem 1.

8. A proof of Theorem 3. We now present a proof of Theorem 3. Thus let G, W, L, p and w satisfy the hypotheses of Theorem 3 and assume that G is a counterexample to Theorem 3.

Set $H = C_G(w)$, $\bar{H} = H/O(H)$ and $K = \langle L^{L_{2'}(H)}\rangle$ and let $t \in W - \langle w\rangle$. Thus $\bar{L}$ is a perfect 2-component of $C_{\bar{K}}(\bar{t})$ and $K = K_1K_2$ where K_1 and K_2 are distinct 2-components of H by Lemma 2.20. Also $\bar{K} = \bar{K}_1 * \bar{K}_2$, $\bar{L} = \langle \bar{k}_1\bar{k}_1^t \mid \bar{k}_1 \in \bar{K}_1\rangle$ and the mapping $\bar{k}_1 \to \bar{k}_1\bar{k}_1^t$ is a homomorphism of $\bar{K}_1$ onto $\bar{L}$ by [**17**, Lemma 2.1]. Consequently K_1 and K_2 are of $\mathfrak{M}(p)$-type by Lemma 2.28. Thus $w \notin K_1 \cup K_2$ by Theorem 1.

Suppose that $\bar{K}_1 \cap \bar{K}_2 = 1$. Then there is an involution $k_1 \in Z^*(K_1)$ such that $w = k_1 k_1^t$, $k_1 \neq k_1^t$, $[k_1, k_1^t] = 1$ and it is clear from Lemma 2.15 that Theorem 2 applies to the 4-subgroup $\langle k_1, k_1^t \rangle$ of G to yield a contradiction. Thus $\bar{K}_1 \cap \bar{K}_2 \neq 1$.

Hence there is an involution $k \in Z^*(K_1) \cap Z^*(K_2) \cap C_H(t)$. Since $\overline{kk^t} = 1$, we conclude from Lemma 2.27 that

$$L/Z^*(L) \cong K_1/Z^*(K_1) \cong K_2/Z^*(K_2) \cong P\Omega(m, p^n, \pm 1)$$

for some positive integers m, n with m even and $m \geqslant 8$, $|Z^*(L)|_2 = 2 < |\mathcal{M}(L/Z^*(L))|_2 = 4 = |Z^*(K_1)|_2 = |Z^*(K_2)|_2$ and $\langle k \rangle \in \mathrm{Syl}_2(Z^*(K_1) \cap Z^*(K_2))$ since $w \notin K_1 \cup K_2$. Moreover there is an element $\nu_1 \in Z^*(K_1)$ such that $\nu_1^2 \in \langle k \rangle$, $\langle \nu_1, k \rangle \in \mathrm{Syl}_2(Z^*(K_1))$, $\nu_1^t \neq \nu_1$, $[\nu_1, \nu_1^t] = 1$ and $w = \nu_1 \nu_1^t$. Set $\nu_2 = \nu_1^t$, $M = C_G(k)$, $\tilde{M} = M/(O(M) \times \langle k \rangle)$ and $J_i = C_{K_i}(k)^{(\infty)} = O^{2'}(C_{K_i}(k))$ for $i = 1, 2$. Then Lemmas 2.14 and 2.15 imply that J_i is a 2-component of $C_M(w)$ with $K_i = O(K_i)J_i$ and $\langle \nu_i, k \rangle \in \mathrm{Syl}_2(Z^*(J_i))$ for $i = 1, 2$, $O(\tilde{M}) = 1$, $\tilde{w} \in \mathcal{I}(\tilde{M})$ and $\tilde{J}_i$ is a 2-component of $C_{\tilde{M}}(\tilde{w})$ such that $\langle \tilde{\nu}_i \rangle \in \mathrm{Syl}_2(Z^*(\tilde{J}_i))$ and $\tilde{J}_i/Z^*(\tilde{J}_i) \cong L/Z^*(L)$ for $i = 1, 2$. Also $\tilde{w} \in \langle \tilde{\nu}_1, \tilde{\nu}_2 \rangle \cong E_4$. Set $B = \langle \nu_1, \nu_2, k \rangle$. Thus B is abelian of order 8, $w \in B$, $\tilde{B} \cong E_4$ and $[K_i, B] \leqslant O(K_i)$ and $[J_i, O(J_i)] \leqslant O(J_i)$ for $i = 1, 2$. Set $\mathcal{J}_i = C_{J_i}(B)^{(\infty)} = O^{2'}(C_{J_i}(B))$ for $i = 1, 2$. Thus $J_i = O(J_i)\mathcal{J}_i$ and $\mathcal{J}_i$ is a 2-component of $C_G(B) = C_M(B)$ such that $Z^*(\mathcal{J}_i) = O(\mathcal{J}_i) \times \langle k, \nu_i \rangle$ for $i = 1, 2$. Also $\tilde{\mathcal{J}}_i$ is a 2-component of $C_{\tilde{M}}(\tilde{B})$ such that $Z^*(\tilde{\mathcal{J}}_i) = O(\tilde{\mathcal{J}}_i) \times \langle \tilde{\nu}_i \rangle$ and $\tilde{\mathcal{J}}_i/Z^*(\tilde{\mathcal{J}}_i) \cong \mathcal{J}_i/Z^*(\mathcal{J}_i) \cong L/Z^*(L)$ for $i = 1, 2$ by Lemmas 2.12 and 2.14. If $\tilde{\mathcal{J}}_i \trianglelefteq \trianglelefteq \tilde{M}$, then $O(M)\mathcal{J}_i \trianglelefteq \trianglelefteq M$ and Theorem 1 yields a contradiction for $i = 1$ or 2. Thus Theorem 2 and Lemmas 2.17–2.18 imply that $\mathcal{J}_1$ is contained in a unique perfect 2-complete $\mathcal{K}$ of M such that $\mathcal{K}/O(\mathcal{K})$ is a Chevalley group over a finite field of characteristic p. Since $J_1 = O(J_1)\mathcal{J}_1$, we have $J_1 \leqslant \mathcal{K}$. If $[\mathcal{K}, w] \leqslant Z^*(\mathcal{K})$, then $\mathcal{K} = O(\mathcal{K})J_1$ and Theorem 1 yields a contradiction since $k \in \mathcal{K}$. Thus $[\mathcal{K}, w] = \mathcal{K}$ and $w \in B \leqslant L_{2'}(C_M(w)) \leqslant L_{2'}(M)$ so that w acts as an inner automorphism on $\hat{\mathcal{K}} = \mathcal{K}/Z^*(\mathcal{K})$. However $\hat{J}_1$ is a perfect 2-component of $C_{\hat{\mathcal{K}}}(w)$ such that $\hat{J}_1/Z^*(\hat{J}_1) \cong L/Z^*(L)$ and $\mathcal{K}$ is not of $\mathcal{M}(p)$-type by Theorem 1 since $k \in \mathcal{K}$. Consequently we have $\hat{\mathcal{K}} \cong \mathrm{PSL}(2r, q)$ or $\hat{\mathcal{K}} \cong \mathrm{PSU}(2r, q)$ where r is an integer with $r \geqslant 2$ and $q = p^s$ for some positive integer s. Then Lemma 5.5 and the fact that $\hat{J}_1/Z^*(\hat{J}_1) \cong P\Omega(m, p^n, \pm 1)$ for positive integers m, n with $m \geqslant 8$ yield a contradiction. This completes our proof of Theorem 3.

References

1. M. Aschbacher, *2-components in finite groups*, Comm. Algebra **3** (1975), 901–911.

2. ______, *On finite groups of component type*, Illinois J. Math. **19** (1975), 87–115.

3. ______, *A characterization of Chevalley groups over fields of odd order*, Ann. of Math. (2) **106** (1977), 353–398.

4. M. Aschbacher and G. M. Seitz, *On groups with a standard component of known type*, Osaka J. Math. **13** (1976), 439–482.

5. H. Bender, *On groups with Abelian Sylow 2-subgroups*, Math. Z. **17** (1970), 164–176.

6. A. Borel, *Linear algebraic groups*, Benjamin, New York, 1961.

7. ______, *Properties and linear representations of Chevalley groups*, Lecture Notes in Math., vol. 131, Springer-Verlag, Berlin, 1970.

8. N. Bourbaki, *Groupes et algèbres de Lie*, Hermann, Paris, 1968, Chapitres 4–6.

9. N. Burgoyne, *Finite groups with Chevalley-type components*, Pacific J. Math. **72** (1977), 341–350.

10. N. Burgoyne and C. Williamson, *Semi-simple classes in Chevalley type groups*, Pacific J. Math. **70** (1977), 83–100.

11. R. Carter, *Simple groups of Lie type*, Wiley, London, 1972.

12. J. A. Dieudonné, *La géométrie des groupes classiques*, Springer-Verlag, Berlin, 1971.

13. W. Feit, *The current situation in the theory of finite simple groups*, Actes Congrès Internat. Math., vol. 1, Gauthier-Villars, Paris, 1971, pp. 55–93.

14. P. Fong and W. J. Wong, *A characterization of the finite simple groups* PSp(4, q), $G_2(q)$, $D_4^2(q)$. I, Nagoya Math. J. **36** (1969), 143–184.

15. G. Glauberman, *Central elements in core-free groups*, J. Algebra **4** (1966), 403–420.

16. D. Gorenstein, *Finite groups*, Harper and Row, New York, 1968.

17. D. Gorenstein and J. H. Walter, *Centralizers of involutions in balanced groups*, J. Algebra **20** (1972), 284–319.

18. ______, *Balance and generation in finite groups*, J. Algebra **33** (1975), 224–287.

19. R. L. Griess, *Schur multipliers of the known finite simple groups*, Bull. Amer. Math. Soc. **78** (1972), 68–71.

20. M. E. Harris, *Finite groups with Sylow 2-subgroups of type* PSp(6, q), *q odd*, Comm. Algebra **2** (1974), 181–232.

21. ______, *A universal mapping problem, covering groups and automorphism groups of finite groups*, Rocky Mountain J. Math. **7** (1977), 289–295.

22. ______, *A note on 2-components of finite groups*, Arch. Math. **28** (1977), 130–132.

23. ______, *A note on solvable 2-components of finite groups*, Arch. Math. **29** (1977), 344–348.

24. ______, *Finite groups containing an intrinsic 2-component of symplectic type*, unpublished.

25. ______, *A note on the classical linear groups over finite fields*, Rev. Roumaine Math. Pures Appl. (to appear).

26. ______, PSL(2, q)-*type 2-components and the unbalanced group conjecture*, J. Algebra **68** (1981), 190–235.

27. M. E. Harris and R. Solomon, *Finite groups having an involution centralizer with a 2-component of dihedral type*. I, Illinois J. Math. **21** (1977), 575–620.

28. B. Huppert, *Endliche Gruppen*. I, Springer-Verlag, Berlin, 1967.

29. ______, *Geometric algebra*, University of Illinois at Chicago Circle Lecture Notes, 1970.

30. N. Iwahori, *Centralizers of involutions in finite Chevalley groups*, Lecture Notes in Math., vol. 131, Springer, Berlin, 1970.

31. R. Ree, *A family of simple groups associated with the simple Lie algebra of type* (G_2), Amer. J. Math. **83** (1961), 432–462.

32. R. Solomon, *Finite groups with Sylow 2-subgroups of type* 3, J. Algebra **28** (1974), 182–198.

33. T. A. Springer and R. Steinberg, *Conjugacy classes*, Lecture Notes in Math., vol. 131, Springer-Verlag, Berlin, 1970.

34. R. Steinberg, *Lectures on Chevalley groups*, Notes by J. Faulkner and R. Wilson, Yale University, 1967.

35. ______, *Endomorphisms of linear algebraic groups*, Mem. Amer. Math. Soc. No. 80 (1968).

36. ______, *Conjugacy classes in algebraic groups*, Lecture Notes in Math., vol. 366, Springer, Berlin, 1974.

37. J. G. Thompson, *Notes on the B-conjecture*, unpublished.

38. J. H. Walter, *Characterization of Chevalley groups*. I. *Finite groups*, Sapporo and Kyoto, 1974, Japan Society for the Promotion of Science, Tokyo, 1976.

39. ______, *Characterization of Chevalley groups*, preprint.

SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA 55455