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LEVEL SETS OF DERIVATIVES

BY

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ABSTRACT. The main result of the paper is the characterization of those triples S, G and E of subsets of the reals for which there exists an everywhere differentiable real-valued function f of one real variable such that $E = \{x;\ f'(x) > 0\}$, $G = \{x;\ f'(x) = +\infty\}$ and S is the set of those points of E at which f is discontinuous. This description is formulated with the help of a certain density-type property of subsets of the reals (called property (Z)) introduced in the paper. The main result leads to a complete description of the structure of the sets $\{x;\ f'(x) > 0\}$ and $\{x;\ f'(x) = 0\}$ for three most important classes of functions f: finitely differentiable functions, continuous differentiable functions and everywhere differentiable functions. (A complete description of the structure of these sets for the class of Lipschitz, everywhere differentiable functions was given by Zahorski in his fundamental paper **[22]**.) The connection of these results with Zahorski's classes M_2, M_3 and M_4 is discussed.

1. Introduction. This paper can be considered as a continuation of Zahorski's research **[22]**, where a characterization of the sets $\{x \in R;\ f'(x) > a\}$ for Lipschitz, everywhere differentiable functions was given. A number of problems raised by Zahorski's work were solved e.g. in **[1, 7–11, 13, 14, 16, 17, 18]**. Among other results it was shown by Lipinski that the classes M_2 and M_3 of Zahorski do not characterize the level sets of derivatives. There is also a number of papers concerning the construction of functions with derivative $+\infty$ on a given set (see e.g. **[1, 5, 11, 21]**). A general theorem of this type for differentiable, continuous functions was proved in **[22]**. As for the characterization of the level sets and the construction of discontinuous, differentiable functions very little appears to be known (see **[1, 7, 9, 11, 17]**).

Our aim is to solve the following problem: Let S, G and E be subsets of the reals. Find necessary and sufficient conditions for the existence of a differentiable function f such that $E = \{x;\ f'(x) > 0\}$, $G = \{x;\ f'(x) = +\infty\}$ and $S = \{x \in E;\ f$ is discontinuous at $x\}$. The solution of this problem enables us to construct (even discontinuous) differentiable functions with given properties as well as to characterize the level sets (and zero sets) of derivatives.

The paper is divided into five sections. In §1 we introduce some notation, remind the reader of the definitions of the classes M_0–M_5 (their properties and connections with derivatives can be found in **[22]**; more recent results as well as other information about derivatives are in the survey **[3]**) and prove two generally known lemmas

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(see e.g. the proofs of relations between descriptive and constructive definitions of Denjoy integrals in [**19**]) in the formulation convenient for later use. The main new notion, property (Z), is defined in §2, where also some equivalent conditions are given.

Necessary conditions on the sets S, G and E are found in §3. The first two lemmas of this section are slight modifications of a part of Zahorski's proof of Theorem 5 in [**22**]. The third lemma is the usual extension lemma for differentiable functions (cf. [**4**, **19**]). The main results of this section are Theorems 3.4 and 3.5 giving necessary conditions on S, G and E for the case of approximate and ordinary derivatives respectively. The sufficiency of these conditions is proved in Theorem 4.3 of §4. We give two applications of this result in Remarks 4.4 and 4.5. In the last section we define classes M^*, M_2^* and M_3^*, and show that these classes characterize the level sets of corresponding types of derivatives. This result implies also the characterization of zero sets of derivatives. Remark 5.8, Examples 5.10 (which is a simple modification of an example from [7]) and 5.12 discuss some connections among the defined classes.

We will use the following notation. R denotes the set of all real numbers with its usual topology; $\bar{R} = R \cup \{-\infty, +\infty\}$. We will use only the Euclidean metric in R. The distance between two sets A and $B \subset R$ is denoted by $d(A, B)$. The closure of a set $E \subset R$ is denoted by $\bar{E}$; its interior by $\operatorname{Int} E$. A perfect set is a nonempty compact subset of R without isolated points. A portion of a set A is an arbitrary set of the form $I \cap A \neq \varnothing$ where I is an open interval in R (or the empty set if A is empty). The interval (a, b) is denoted also by (b, a) (similarly for other types of intervals).

Lebesgue measure, integral, etc. will be called simply measure, integral, etc. The outer measure of a set $E \subset R$ will be denoted by $|E|$.

A real-valued function of a real variable (defined on some subset of R) will be simply called a function. A function f is called a jump function in $S \subset R$ if there exists a sequence $\{s_i\}$, $s_i \in S$, and real numbers a_i and b_i such that $\sum |a_i| + \sum |b_i| < +\infty$ and $f(x) = \sum_{s_i \leq x} a_i + \sum_{s_i < x} b_i$. The value of a derivative (or approximate derivative) may be finite or infinite (i.e. belongs to $\bar{R}$).

DEFINITION 1.1. Define the following classes of F_σ subsets $E \subset R$.

$E \in M_0$ if $E \cap (x, x+h) \neq \varnothing$ for every $x \in E$ and $h \neq 0$.

$E \in M_1$ if $\operatorname{card}(E \cap (x, x+h)) > \aleph_0$ for every $x \in E$ and $h \neq 0$.

$E \in M_2$ if $|E \cap (x, x+h)| > 0$ for every $x \in E$ and $h \neq 0$.

$E \in M_3$ if for every $x \in E$ and $c > 0$ there exists $\varepsilon > 0$ such that

$$|E \cap (x+h, x+h+k)| > 0$$

for every $h, k \neq 0$, $0 < h/k < c$, and $|h+k| < \varepsilon$.

$E \in M_4$ if there are closed sets F_n and numbers $\eta_n > 0$ such that $E = \bigcup_{n=1}^{\infty} F_n$ and, for every $x \in F_n$ and $c > 0$, there exists $\varepsilon > 0$ with the following property: if $h, k \neq 0$, $0 < h/k < c$ and $|h+k| < \varepsilon$, then

$$|E \cap (x+h, x+h+k)| > \eta_n |k|.$$

$E \in M_5$ if every $x \in E$ is a point of density of E; i.e. if

$$\lim_{h \to 0} \frac{|E \cap (x, x+h)|}{|h|} = 1.$$

LEMMA 1.2. *Let E be a measurable set. Then there exists a set $F \in M_5$ such that $F \subset E$ and $|E - F| = 0$.*

PROOF. Let E_1 be the set of all points of E which are points of density of E. According to the Lebesgue density theorem $|E - E_1| = 0$. Consider any F_σ subset of E_1 with $|E_1 - F| = 0$.

LEMMA 1.3. *Let $\mathfrak{R}$ be a system of open subsets of R. Suppose that*

(α) $\varnothing \in \mathfrak{R}$,

(β) *if $H \in \mathfrak{R}$, then there is a set $A \subset H$ dense in H such that $(-\infty, a) \cap H \in \mathfrak{R}$ and $(a, +\infty) \cap H \in \mathfrak{R}$ for any $a \in A$,*

(γ) *if $H \subset R$ and if there is a locally finite cover $\{I_n\}$ of H consisting of bounded open intervals such that $\bar{I}_n \subset H$ and $I_n \in \mathfrak{R}$, then $H \in \mathfrak{R}$, and*

(δ) *if $H \in \mathfrak{R}$ and $H \neq R$, then $H' - H \neq \varnothing$ for some $H' \in \mathfrak{R}$.*

Then $R \in \mathfrak{R}$.

PROOF. Let G be the union of all elements of $\mathfrak{R}$. Since the system $\mathfrak{R}$ is an open cover of the separable metric space G, it has a locally finite refinement $\{J_n\}$. Moreover, we may suppose that each J_n is an open, bounded interval and that, for each n, there is an $H \in \mathfrak{R}$ such that $\bar{J}_n \subset H$. Using condition (β) twice we find that for every $\varepsilon > 0$ there is an interval $I \in \mathfrak{R}$ such that $I \subset J_n$ and $|J_n - I| < \varepsilon$. This enables us to define (by induction) a sequence of open intervals $\{I_n\}$ such that $I_n \in \mathfrak{R}$, $I_n \subset J_n$, and $I_n \supset J_n - \bigcup_{k<n} I_k - \bigcup_{k>n} J_k$. The last inclusion shows $\bigcup_n I_n = G$, which, according to (γ), implies $G \in \mathfrak{R}$. Now, the assumption $G \neq R$ contradicts (δ). Therefore $G = R$.

2. Property (Z).

DEFINITION 2.1. A set $P \subset R$ is said to have property (Z) with respect to $E \subset R$ if, for every open set $H \subset R - E$ which intersects each component of $R - \overline{(P \cap E)}$ in a connected set, the set $(P \cap E) \cup (R - (\bar{H} \cup \overline{(P \cap E)}))$ belongs to the class M_4.

REMARK 2.2. It might be helpful to make the following observations.

(1) A set P has property (Z) w.r.t. E if and only if $P \cap E$ does.

(2) If P has property (Z) w.r.t. E and $E \subset E_1 \subset E \cup (R - P)$ then P has property (Z) w.r.t. E_1.

(3) If $E \subset P$ then P has property (Z) w.r.t. E if and only if E belongs to the class M_4. (Consider $H = R - \overline{(P \cap E)}$.)

(4) If P has property (Z) w.r.t. E then $P \cap E$ is an F_σ set.

PROPOSITION 2.3. (1) *Suppose that P has property* (Z) *w.r.t. E. If $Q \subset P$ can be written as $Q = P \cap F \cap G$ with F closed and G open then Q has property* (Z) *w.r.t. E.*

(2) *Suppose that P and E are subsets of R and that each $x \in P$ has a neighborhood U such that $P \cap U$ has property* (Z) *w.r.t. E. Then P has property* (Z) *w.r.t. E.*

(3) *Suppose that P and E are subsets of R and that $G \supset P$ is an open set. Then P has property* (Z) *w.r.t. $E \cap G$ if and only if it has property* (Z) *w.r.t. E.*

PROOF. (1) Let $H \subset R - E$ be an open set which intersects each component of $R - (\overline{Q \cap E})$ in a connected set. Then H also intersects each component of $R - (\overline{P \cap E})$ in a connected set. Therefore,

$$(P \cap E) \cup \left(R - \left(\overline{H} \cup \left(\overline{P \cap E}\right)\right)\right) \in M_4.$$

Since $R - (\overline{H} \cup (\overline{Q \cap E})) \supset R - (\overline{H} \cup (\overline{P \cap E}))$, it suffices to prove that the set $R - (\overline{H} \cup (\overline{Q \cap E}))$ contains almost all points of $(P - Q) \cap E \cap G$. This follows from the fact that $\overline{H} \subset H \cup S \cup (\overline{Q \cap E})$, where S is a countable set. Hence,

$$\begin{aligned} R - \left(\overline{H} \cup \left(\overline{Q \cap E}\right)\right) &\supset R - \left(H \cup S \cup \left(\overline{Q \cap E}\right)\right) \\ &= (R - H) \cap (R - S) \cap \left(R - \left(\overline{Q \cap E}\right)\right). \end{aligned}$$

Since $R - H \supset (P - Q) \cap E \cap G$, and since $R - (\overline{Q \cap E}) \supset (P - Q) \cap E \cap G$, the set $R - (\overline{H} \cup (\overline{Q \cap E}))$ contains $((P - Q) \cap E \cap G) - S$.

(2) From (1) we deduce that there is a sequence of open intervals $\{I_n\}$ such that $P = \bigcup_n (P \cap I_n)$ and each set $P \cap I_n$ has property (Z) w.r.t. E. Whenever $H \subset R - E$ is an open set which intersects each component of $R - (\overline{P \cap E})$ in a connected set, then the set

$$\begin{aligned} &(P \cap E) \cup \left(R - \left(\overline{H} \cup \left(\overline{P \cap E}\right)\right)\right) \\ &\quad = \bigcup_n \left\{I_n \cap \left[(P \cap I_n \cap E) \cup \left(R - \left(\left(\overline{H \cap I_n}\right) \cup \left(\overline{P \cap I_n \cap E}\right)\right)\right)\right]\right\} \\ &\quad\quad \cup \left\{R - \left(\overline{H} \cup \left(\overline{P \cap E}\right)\right)\right\} \end{aligned}$$

is a countable union of M_4 sets, and hence an M_4 set.

(3) Because of 2.2(2) and 2.3(2) it suffices to show that, whenever P has property (Z) w.r.t. E and I is an open interval, then $P \cap I$ has property (Z) w.r.t. $E \cap I$. If $H \subset R - (E \cap I)$ intersects each component of $R - (\overline{P \cap I \cap E})$ in a connected set then $H \cap I$ intersects each component of $R - (\overline{P \cap E})$ in a connected set. Hence, the set

$$\begin{aligned} &(P \cap I \cap E) \cup \left(R - \left(\overline{H} \cup \left(\overline{P \cap I \cap E}\right)\right)\right) \\ &\quad = \left\{I \cap \left[(P \cap E) \cup \left(R - \left(\left(\overline{H \cap I}\right) \cup \left(\overline{P \cap E}\right)\right)\right)\right]\right\} \\ &\quad\quad \cup \left\{R - \left(\overline{H} \cup \left(\overline{P \cap I \cap E}\right)\right)\right\} \end{aligned}$$

belongs to the class M_4.

THEOREM 2.4. *A set P has property* (Z) *with respect to E if and only if the following three conditions hold.*

(α) *If $x \in P \cap E$ and $h \neq 0$, then $E \cap (x, x + h) \neq \varnothing$.*

(β) *If $x \in P \cap E$ is not an isolated point of $P \cap E$ from the right (resp. left), then for any $c > 0$ there is an $\varepsilon > 0$ enjoying the following property*:

$E \cap (x + h, x + h + k) \neq \varnothing$ for any $h, k > 0$ (resp. $h, k < 0$) for which $h/k < c$ and $|h + k| < \varepsilon$.

(γ) *There exist a sequence of closed sets $\{F_n\}$, a sequence of positive numbers $\{\eta_n\}$, and $q \in \langle \frac{1}{2}, 1)$ such that $P \cap E = \bigcup_n F_n$ and the following assertion holds.*

(+) *If* $x \in F_n$ *and* $c > 0$, *then there is an* $\varepsilon > 0$ *such that the inequality*

$$|P \cap E \cap (x+h, x+h+k)| + |(x+h, x+h+k) - \big(H_q \cup (\overline{P \cap E})\big)| > \eta_n |k|$$

holds if $0 < h/k < c$, $|h+k| < \varepsilon$, $x + h \in \overline{P \cap E}$, *and* $x + h + k \in \overline{P \cap E}$, *where* H_q *is the union of all those intervals* J *contiguous to* $\overline{E}$ *for which there exists some bounded interval* I *contiguous to* $\overline{P \cap E}$ *such that* $J \subset I$ *and* $|J| > q|I|$.

PROOF. First suppose that P has property (Z) with respect to E. Suppose that (α) does not hold. Then there is $x \in P \cap E$ and $h \neq 0$ such that $E \cap (x, x+h) = \varnothing$. Putting $H = (x, x+h)$ in the definition of property (Z) we obtain a contradiction. Now suppose that (β) does not hold. Then there are $x \in P \cap E$ not right isolated say from $P \cap E$ and $c > 0$ such that for every $\varepsilon > 0$ there exist $h, k > 0$ such that $h/k < c$, $|h+k| < \varepsilon$ and $(x+h, x+h+k) \cap E = \varnothing$. From this we deduce the existence of decreasing sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim x_n = \lim y_n = x$, $y_{n+1} < x_n < y_n$, $(x_n - x)/(y_n - x_n) < c$, $(x_n, y_n) \cap E = \varnothing$ and $\underline{(y_{n+1}, x_n)} \cap P \cap E \neq \varnothing$. Put $H = \bigcup_{n=1}^{\infty}(x_n, y_n)$. Then the set $(P \cap E) \cup (R - (\overline{H} \cup (\overline{P \cap E})))$ is not of the type M_3 (consider the point x). Therefore, it is not an M_4 set. A similar proof gives a contradiction in the case of points not left isolated from $P \cap E$. The condition (γ) for any $q \in \langle \frac{1}{2}, 1)$ follows directly from the definition of property (Z).

Let the conditions (α), (β) and (γ) hold. First prove the following statement.

(++) The assertion (+) holds if η_n is replaced by $(1-q)\eta_n$ and H_q is replaced by any open set $H \subset R - E$ which intersects each component of $R - (\overline{P \cap E})$ in a connected set.

If I is a bounded interval contiguous to $\overline{P \cap E}$, then either $H_q \cap I \neq \varnothing$ and thus $|H \cap I| \leq |H_q \cap I|$ or $H_q \cap I = \varnothing$ and thus $|H \cap I| \leq q|I|$. Therefore, for $u, v \in \overline{P \cap E}$ we have (the summation is over the intervals contiguous to $\overline{P \cap E}$)

$$\begin{aligned}
&|P \cap E \cap (u,v)| + |(u,v) - ((\overline{P \cap E}) \cup H)| \\
&\quad = |P \cap E \cap (u,v)| + \sum_{I \subset (u,v)} |I - H| \\
&\quad \geq |P \cap E \cap (u,v)| + \sum_{\substack{I \subset (u,v) \\ H_q \cap I \neq \varnothing}} |I - H_q| + \sum_{\substack{I \subset (u,v) \\ H_q \cap I = \varnothing}} (1-q)|I| \\
&\quad \geq (1-q)\Big(|P \cap E \cap (u,v)| + \sum_{I \subset (u,v)} |I - H_q|\Big) \\
&\quad = (1-q)\big(|P \cap E \cap (u,v)| + |(u,v) - ((\overline{P \cap E}) \cup H_q)|\big).
\end{aligned}$$

Let $H \subset R - E$ be an open set intersecting each component of $R - (\overline{P \cap E})$ in a connected set. Let $x \in F_n$ and $C > 0$. If x is an isolated point of $P \cap E$ from the right (resp. left), then condition (α) implies the existence of $h > 0$ (resp. $h < 0$) such that $(x, x+h) \cap (H \cup (\overline{P \cap E})) = \varnothing$. Thus

$$(x, x+h) \subset (P \cap E) \cup \big(R - (\overline{H} \cup (\overline{P \cap E}))\big).$$

If x is not an isolated point of the set $P \cap E$ from the right (resp. left), choose $\varepsilon > 0$ such that $E \cap (x + h, x + h + k) \neq \varnothing$ for any $h, k > 0$ (resp. $h, k < 0$), $h/k < 9(C + 1)$ and $|h + k| < \varepsilon$ (condition (β) for $c = 9(C + 1)$) and that

$$|P \cap E \cap (x+h, x+h+k)| \\ + |(x+h, x+h+k) - (H \cup (\overline{P \cap E}))| > (1-q)\eta_n |k|$$

for any h, k such that $0 < h/k < 3(C + 1)$, $|h + k| < \varepsilon$, $x + h \in \overline{P \cap E}$ and $x + h + k \in \overline{P \cap E}$ (condition (+ +) for $c = 3(C + 1)$).

Let $h, k > 0$ (resp. $h, k < 0$), $h/k < C$ and $|h + k| < \varepsilon$. First note that each interval $I \subset (x + h, x + h + k)$ with $|I| \geqslant \frac{1}{9}|k|$ intersects E since $d(x, I)/|I| < 9(C + 1)$. Thus, if there exists an open interval $J = (a, b) \subset (x + h, x + h + k) - (\overline{P \cap E})$ with $|J| \geqslant \frac{1}{3}|k|$, then $(\frac{2}{3}a + \frac{1}{3}b, \frac{1}{3}a + \frac{2}{3}b) \cap E \neq \varnothing$. Therefore, $H \cap J$ is a subset of at most one of the intervals $(a, \frac{1}{3}a + \frac{2}{3}b)$ and $(\frac{2}{3}a + \frac{1}{3}b, b)$. Therefore,

$$|(x+h, x+h+k) - (\overline{H} \cup (\overline{P \cap E}))| \geqslant |J - H| \geqslant \tfrac{1}{3}|J| > \tfrac{1}{9}|k| .$$

If there is no such interval J, put $L = (m, M)$, where

$$m = \min \overline{P \cap E} \cap \langle x+h, x+h+k \rangle,$$
$$M = \max \overline{P \cap E} \cap \langle x+h, x+h+k \rangle.$$

Then $|L| > \frac{1}{3}|k|$ and $d(x, L)/|L| < 3(C + 1)$, and therefore,

$$|(x+h, x+h+k) \cap P \cap E| + |(x+h, x+h+k) - (\overline{H} \cup (\overline{P \cap E}))| \\ \geqslant |L \cap P \cap E| + |L - (\overline{H} \cup (\overline{P \cap E}))| > (1-q)\eta_n |L| > \tfrac{1}{3}(1-q)\eta_n |k| .$$

Since the set $R - (\overline{H} \cup (\overline{P \cap E}))$ is open, the set $(P \cap E) \cup (R - (\overline{H} \cup (\overline{P \cap E})))$ is of the type M_4.

In the following text we will often need a different property of a pair P, E of subsets of R, namely, that there is a portion of P possessing property (Z) with respect to E. The following proposition says that in proving such a property one may restrict his attention to special perfect sets only.

PROPOSITION 2.5. *Let E and G be subsets of R. Suppose that for each $x \in E - G$ and each $c > 0$ there is an $\varepsilon > 0$ such that $E \cap (x + h, x + h + k) \neq \varnothing$ for every $h, k \neq 0$, $0 < h/k < c$ and $|h + k| < \varepsilon$. Then the following conditions are equivalent.*

(i) *Every perfect subset P of $R - G$ such that $\overline{P \cap E} = \overline{P - E} = P$ has a portion possessing property* (Z) *with respect to E.*

(ii) *Every subset of $R - G$ which is at the same time of type F_σ and G_δ has a portion possessing property (Z) with respect to E.*

PROOF. We only need to prove (i) $\Rightarrow$ (ii). Let $P \subset R - G$ be an F_σ and a G_δ set.

(a) If $P = \varnothing$, then obviously P has property (Z) w.r.t. E.

(b) If $P \not\subset \overline{P \cap E}$, then there is a portion of P which does not intersect E. This portion has property (Z) w.r.t. E.

(c) If $P \not\subset \overline{P - E}$, then there is a portion Q of P such that $Q \subset E - G$. In this case put $Q_n = \{x \in Q;\ E \cap (x + h, x + h + k) \neq \varnothing$ for any $h, k \neq 0$, $0 < h/k \leqslant 1$, and $|h + k| < \frac{1}{n}\}$. The sets Q_n are closed in Q and $\bigcup_n Q_n = Q$. Since Q is of type

G_δ, there is an open interval I and a natural number m such that $Q_m \supset I \cap Q \neq \varnothing$. Moreover, we can assume that $|I| < \frac{1}{m}$ and that the set $P_0 = I \cap Q$ is closed in I (because Q is of type F_σ as well as G_δ). If $J = (a, b)$ is a bounded interval contiguous to $\overline{P_0}$, then a or $b \in Q_m$ and thus, for any interval $J' \subset J$ with $J' \cap E = \varnothing$ we have $|J'| \leqslant \frac{1}{2}|J|$. Hence, the set H_q from condition (γ) of Theorem 2.2 is empty. This implies that P_0 has property (Z) w.r.t. E.

(d) In the last case when $(\overline{P \cap E}) \cap (\overline{P - E}) \supset P \neq \varnothing$, we have $\overline{P \cap E} = \overline{P - E}$ and we choose an open interval I such that $I \cap P \neq \varnothing$ and the set $I \cap P$ is closed in I. Next we find a bounded open interval $J \subset I$ such that $\bar{J} \subset I$ and $J \cap P \neq \varnothing$. Then the set $\overline{J \cap P}$ has the properties required in (i). Thus there exists a portion Q of $\overline{J \cap P}$ having property (Z) w.r.t. E. The set $J \cap Q$ is the required portion of P.

Corollary 2.6. *Let E be an M_3 set and let $P \subset E$ be a set of type F_σ as well as G_δ. Then P has a portion having property* (Z) *with respect to E.*

Proof. Put $G = R - E$ in the preceding proposition.

Remark 2.7. Incidentally, under the assumptions of Proposition 2.5 one can add also the following condition equivalent to (i) and (ii).

(iii) Every F_σ subset of $R - G$ can be written as a countable union of closed sets, each of which has property (Z) with respect to E.

Since (iii) $\Rightarrow$ (i) is obvious, let us sketch the proof of (ii) $\Rightarrow$ (iii). Let $F \subset R - G$ be a closed set. Let $\mathfrak{R}$ be the system of all open subsets $H \subset R$ such that $F \cap H$ can be written as a countable union of closed sets with property (Z) w.r.t. E. Since $\mathfrak{R}$ has all the properties required in Lemma 1.3, we obtain $R \in \mathfrak{R}$. (Let us note that this is a usual method of finding connections among the notions defined with the help of portions and countable unions of closed sets (cf. **[19]**). E.g. with this method one can easily prove an equivalent definition of M_4 in terms of portions.)

3. Necessary conditions.

Lemma 3.1. *Let f be a measurable function defined on a neighborhood of a point $x \in R$. Suppose that $f'_{\text{ap}}(x) = A \in (0, +\infty)$. Then for every $c > 0$ there exists $\varepsilon > 0$ such that for any $h, k \neq 0$, $0 < h/k < c$ and $|h + k| < \varepsilon$ there are h' and k' with $h' \in (h, h + \frac{1}{3}k)$, $h' + k' \in (h + \frac{2}{3}k, h + k)$, and $(f(x + h' + k') - f(x + h'))/k' > \frac{1}{2}A$.*

Proof. Let $c > 0$. Put $\eta(t) = (f(x + t) - f(x))/t - A$ for $t \neq 0$ and choose $\varepsilon > 0$ such that

$$\left|\left\{t \in (x, x + u);\, |\eta(t)| \geqslant \frac{A}{6(2c + 1)}\right\}\right| < \frac{|u|}{3(c + 1)} \quad \text{for } 0 < |u| < \varepsilon.$$

Then

$$\left|\left\{t \in (x + h, x + h + k);\, |\eta(t)| \geqslant \frac{A}{6(2c + 1)}\right\}\right| < \frac{1}{3(c + 1)}|h + k| < \frac{1}{3}|k|$$

for any $h, k \neq 0$, $0 < h/k < c$ and $|h + k| < \varepsilon$. Thus, there are h' and k' such that $x + h' \in (x + h, x + h + \frac{1}{3}k)$, $x + h' + k' \in (x + h + \frac{2}{3}k, x + h + k)$, $|\eta(h')| < A/6(2c + 1)$ and $|\eta(h' + k')| < A/6(2c + 1)$. Since $|h'/k'| < 3c + 1$, we obtain

$$\left|\frac{f(x + h' + k') - f(x + h')}{k'} - A\right| = \left|\frac{h' + k'}{k'}\eta(h' + k') - \frac{h'}{k'}\eta(h')\right| < \frac{1}{2}A.$$

LEMMA 3.2. *Let f be a function defined on a neighborhood of $x \in R$. Suppose that $f'_+(x) = A \in (0, +\infty)$. Then for every $c > 0$ there exists $\varepsilon > 0$ such that*

$$(f(x + h + k) - f(x + h))/k > \tfrac{1}{2}A$$

for any $h, k > 0$, $0 < h/k < c$ and $|h + k| < \varepsilon$. (A similar inequality holds in the case of the derivative from the left.)

PROOF. Similar to that of the preceding lemma.

LEMMA 3.3. *Let f be a measurable function defined on a neighborhood of a perfect set P and let f possess a bounded, approximate derivative on P. Suppose that $K \in R$ such that $|f'_{\mathrm{ap}}(x)| < K$ for all $x \in P$. Let us define the following function,*

$$g(x) = f(x) \quad \text{for } x \in P,$$

g is linear on the closure of each bounded interval contiguous to P.

Then there is an open interval I such that

(a) *$I \cap P \neq \varnothing$,*

(b) *the functions f and g are both defined on I and*

$$|\{t \in (x, x + h); |f(t) - f(x)| > K|t - x|\}| \leq \tfrac{1}{6}|h|$$

for every $x \in P \cap I$ and $x + h \in I$,

(c) *$|g(x) - g(y)| \leq K|x - y|$ for all x and $y \in I$,*

(d) *g'_+ and g'_- exist on I, and*

(e) *if $x \in I \cap P$ is not an isolated point of P from the right (resp. left), then $g'_+(x) = f'_{\mathrm{ap}}(x)$ (resp. $g'_-(x) = f'_{\mathrm{ap}}(x)$).*

PROOF. Let P_n be the set of those $x \in P$ for which the function f is defined on $(x - \frac{1}{n}, x + \frac{1}{n})$ and

$$|\{t \in (x, x + h); |f(t) - f(x)| > K|t - x|\}| \leq \tfrac{1}{6}|h|$$

for every h with $0 < |h| < \frac{1}{n}$. Let us prove that the sets P_n are closed. Suppose, to the contrary, that there exists $x \in \overline{P_n}$ such that

$$|\{t \in (x, x + h); |f(t) - f(x)| > K|t - x|\}| > \tfrac{1}{6}|h|$$

for some h with $0 < |h| < \frac{1}{n}$. Since f is measurable, there is a compact set Q such that f/Q is continuous, $|Q| > \frac{1}{6}|h|$ and

$$Q \subset \{t \in (x, x + h); |(f(t) - f(x))/(t - x)| > K\}$$

(see **[15]**). Thus, there is an $\eta > 0$, $\eta < 2K$, such that

(1) $|Q| > \frac{1}{6}|h| + \eta$, and

(2) $|(f(t) - f(x))/(t - x)| > K + \eta$ for every $t \in Q$.

Choose $y \in P_n$ such that
(3) $|\{t \in (x, y); |f(t) - f(x)| > K|t - x|\}| \leq \frac{1}{6}|y - x|$,
(4) $|x - y| < \eta|t - x|/2K$ for every $t \in Q$,
(5) $|x - y| < 6\eta$, and
(6) $|x - y| < \frac{1}{n} - |h|$.
The inequalities (3), (6) and $y \in P_n$ imply the existence of $t \in (x, y)$ such that $|f(t) - f(x)| \leq K|t - x|$, and $|f(t) - f(y)| \leq K|t - y|$. Hence, $|f(x) - f(y)| \leq K|x - y|$.

Using (2) and (4) one obtains, for any $t \in Q$, the following inequalities.

$$\begin{aligned} |f(t) - f(y)| &\geq |f(t) - f(x)| - |f(x) - f(y)| \\ &> (K + \eta)|t - x| - K|x - y| \\ &\geq K(|t - y| - |x - y|) + \eta|t - x| - K|x - y| \\ &= K|t - y| + \eta|t - x| - 2K|x - y| > K|t - y|. \end{aligned}$$

Further from (4) it follows that $Q \subset (y, x + h)$ (since $\eta/2K < 1$ and $Q \subset (x, x + h)$) and (5) implies that $|Q| > \frac{1}{6}|h| + \eta > \frac{1}{6}|x + h - y|$. Then (6) implies $y \notin P_n$ contrary to the choice of y.

Thus, the sets P_n are closed. Since P is their union, there is an open interval I with endpoints in P and a natural number m such that $\varnothing \neq I \cap P$, $\bar{I} \cap P \subset P_m$, $|I| < \frac{1}{m}$ and the functions f and g are both defined on $\bar{I}$. Now (a) is obvious and (b) holds even for $x \in P \cap \bar{I}$, $x + h \in \bar{I}$.

If x and $y \in \bar{I} \cap P$, then there is a $t \in (x, y)$ such that $|f(x) - f(t)| \leq K|t - x|$ and $|f(t) - f(y)| \leq K|t - y|$. Hence, $|f(x) - f(y)| \leq K|x - y|$. Now (c) follows from the definition of the function g.

Let $x \in P \cap I$, $0 < \varepsilon < \frac{1}{2}$. Choose $\delta > 0$ such that

$$\left|\{t \in (x, x + h); |f(t) - f(x) - f'_{\text{ap}}(x)(t - x)| > \varepsilon|t - x|\}\right| < \varepsilon|h|$$

for every h, $0 < |h| < \delta$. If $y \in I \cap P$ with $0 < |y - x| < \delta$, put

$$J = (2\varepsilon x + (1 - 2\varepsilon)y, y).$$

Then $|J| = 2\varepsilon|y - x|$,

$$\left|\{t \in J; |f(t) - f(x) - f'_{\text{ap}}(x)(t - x)| > \varepsilon|t - x|\}\right| < \varepsilon|y - x|,$$

and

$$\left|\{t \in J; |f(t) - f(y)| > K|t - y|\}\right| \leq \tfrac{1}{3}\varepsilon|x - y|.$$

(The last inequality we obtain using (b) at the point y.) Hence, there is $t \in J$ such that $|f(t) - f(x) - f'_{\text{ap}}(x)(t - x)| \leq \varepsilon|t - x|$ and $|f(t) - f(y)| \leq K|t - y|$. Since $|f'_{\text{ap}}(x)| < K$ and $|y - t| \leq |J| = 2\varepsilon|y - x|$, we have

$$\begin{aligned} &|f(y) - f(x) - f'_{\text{ap}}(x)(y - x)| \\ &\quad \leq |f(y) - f(t)| + |f(t) - f(x) - f'_{\text{ap}}(x)(t - x)| + |f'_{\text{ap}}(x)||y - t| \\ &\quad \leq 2K|y - t| + \varepsilon|t - x| \leq (4K + 1)\varepsilon|y - x|. \end{aligned}$$

This inequality and the definition of the function g imply (d) and (e).

THEOREM 3.4. *Let f be a function defined on an open interval I which possesses an approximate derivative on I. Let $\alpha \in R$, $E = \{x \in I;\ f'_{ap}(x) > \alpha\}$, $G = \{x \in I;$ $f'_{ap}(x) = +\infty\}$ and let S be the set of all those $x \in I$ for which $\lim_{h \to 0_+} f(x-h)$ exists and is $< f(x)$, or $\lim_{h\to 0_+} f(x+h)$ exists and is $> f(x)$. Then*

(i) *S is a countable set, G is a G_δ set of measure zero, E is an F_σ set and $S \subset G \subset E$,*

(ii) *if $x \in E - S$ and $h \neq 0$ then $|E \cap (x, x+h)| > 0$ or $S \cap (x, x+h) \neq \varnothing$,*

(iii) *if $x \in E - G$ and $c > 0$, then there is a number $\varepsilon > 0$ such that*

$$|E \cap (x+h, x+h+k)| > 0 \quad \text{or} \quad S \cap (x+h, x+h+k) \neq \varnothing,$$

for any $h, k \neq 0$ with $0 < h/k < c$ and $|h+k| < \varepsilon$, and

(iv) *any perfect subset of $R - G$ has a portion having property* (Z) *with respect to E.*

PROOF. Without loss of generality we can assume $\alpha = 0$.

(i) The set S is countable (see e.g. **[15]**), G is of type G_δ and E is of type F_σ since f'_{ap} is a function of the first Baire class on I (see **[18]**) and $|G| = 0$ according to the Denjoy-Young-Saks theorem for approximate derivatives (see **[19]**).

(ii) Suppose, to the contrary, that there are $x \in E - S$ and $h \neq 0$ such that $|E \cap (x, x+h)| = 0$ and $S \cap (x, x+h) = \varnothing$. According to **[18]** the function f is nonincreasing on $(x, x+h)$. Since $x \notin S$, we have $f'_{ap}(x) \leqslant 0$ which contradicts $x \in E$.

(iii) Let $x \in E - G$ and $c > 0$. Choose $\varepsilon > 0$ such that for any $h, k \neq 0$, $0 < h/k < c$ and $|h+k| < \varepsilon$ there are h' and k' such that $h' \in (h, h + \frac{1}{3}k)$, $h' + k' \in (h + \frac{2}{3}k, h+k)$ and $(f(x+h'+k') - f(x+h'))/k' > 0$ (see Lemma 3.1). If $|E \cap (x+h, x+h+k)| = 0$ and $S \cap (x+h, x+h+k) = \varnothing$ for some such h and k, then f is nonincreasing on $(x+h, x+h+k)$ (see **[18]**). Hence, $(f(x+h'+k') - f(x+h'))/k' \leqslant 0$, which is a contradiction.

(iv) Let $P \subset R - G$ be a perfect set. We may assume that each portion of P meets E. If x is a point of continuity of $f'_{ap}|_P$ then it follows that $0 \leqslant f'_{ap}(x) < +\infty$ and hence there is an open interval J and $K > 0$ such that $x \in J$ and $|f'_{ap}(y)| < K$ for $y \in \overline{J \cap P}$. We apply Lemma 3.3 to the function f and the set $\overline{J \cap P}$. Let g be the function defined in 3.3 and I an interval with the properties (a)—(e). Using Theorem 2.4 we prove that the set $Q = I \cap J \cap P$ has property (Z) w.r.t. E. The conditions (α) and (β) follow directly from (iii). Let us prove (γ) with $q = \frac{1}{2}$. (We denote H_q by H.)

Let (a, b) (where $a < b$) be a bounded interval contiguous to $\overline{Q}$. If $\varnothing \neq H \cap (a, b) = (c, d)$ $(c < d)$, then $d - c > \frac{1}{2}(b-a)$ and the function f is nonincreasing on (c, d) (see **[18]**). If $c = a$, then f is nonincreasing on $\langle c, d)$ and put $c' = c$. If $c > a$, then 3.3(b) implies

$$\left|\{t \in (a, 2c-a);\ |f(t) - f(a)| > K|t-a|\}\right| \leqslant \tfrac{1}{3}(c-a).$$

Hence, there is $c' \in (c, 2c-a)$ such that f is nonincreasing on $\langle c', d)$ and $f(c') - f(a) \leqslant K(c'-a)$. Thus, in both cases we have found $c' \in \langle c, 2c-a\rangle$ such that f is nonincreasing on $\langle c', d)$ and $f(c') - f(a) \leqslant K(c'-a)$. Similarly, we find $d' \in \langle 2d-b, d\rangle$ such that f is nonincreasing on $(c, d'\rangle$ and $f(b) - f(d') \leqslant K(b-d')$. Moreover, we have $c' \leqslant 2c - a < 2d - b \leqslant d'$ and f is nonincreasing on $\langle c', d'\rangle$.

Hence,

$$\begin{aligned} f(b)-f(a) &\leq K(b-d')+K(c'-a)=K((b-a)-(d'-c')) \\ &\leq K((b-a)-((2d-b)-(2c-a))) \\ &= 2K((b-a)-(d-c))=2K\,|(a,b)-H|\,. \end{aligned}$$

The inequality $f(b)-f(a)\leq 2K\,|(a,b)-H|$ holds for any bounded interval (a,b) contiguous to $\bar{Q}$. (For $(a,b)\cap H=\varnothing$ it follows directly from 3.3(c).)

Since g is a Lipschitz function, it is an indefinite integral of its derivative. Thus, for $y, z\in Q$, $y<z$, we have (the summation is over intervals (a,b), $a<b$, contiguous to $\bar{Q}$)

$$\begin{aligned} f(z)-f(y)=g(z)-g(y) &= \int_{Q\cap(y,z)} g' + \sum_{(a,b)\subset(y,z)} (g(b)-g(a)) \\ &= \int_{Q\cap(y,z)} f'_{\mathrm{ap}} + \sum_{(a,b)\subset(y,z)} (f(b)-f(a)) \\ &\leq 2K(|Q\cap E\cap(y,z)|+|(y,z)-(H\cup Q)|). \end{aligned}$$

If $x\in Q\cap E$ and $c>0$, then, according to 3.2 and 3.3(e), there is an $\varepsilon>0$ such that for every $h,k\neq 0$, $0<h/k<c$, $|h+k|<\varepsilon$ for which $x+h$, $x+h+k\in Q$ we have

$$\frac{f(x+h+k)-f(x+h)}{k}=\frac{g(x+h+k)-g(x+h)}{k}>\frac{1}{2}f'_{\mathrm{ap}}(x).$$

Thus,

$$\begin{aligned} &|Q\cap E\cap(x+h,x+h+k)| \\ &\qquad +|(x+h,x+h+k)-(H\cup Q)|>\frac{1}{4K}f'_{\mathrm{ap}}(x)\,|k|\,. \end{aligned}$$

From this inequality we see that to prove 2.4(γ) it suffices to choose closed sets $F_{n,m}$ such that $\{x\in Q\cap E;\ f'_{\mathrm{ap}}(x)>\frac{1}{n}\}=\bigcup_{m=1}^{\infty}F_{n,m}$ and to put $\eta_{n,m}=1/4Kn$.

THEOREM 3.5. *Let f be a function defined on an open interval I which possesses a derivative on I. Let $\alpha\in R$, $E=\{x\in I;\ f'(x)>\alpha\}$, $G=\{x\in I;\ f'(x)=+\infty\}$ and let S be the set of all those $x\in E$ at which f is discontinuous. Then the sets S, G and E fulfill the conditions* (i)–(iv) *of Theorem* 3.4.

PROOF. The countability of S is well known (see e.g. **[15]**). Now the statement follows from 3.4 since any triple $\tilde{S}$, G, E fulfills the conditions (i)–(iv) of 3.4 if S, G, E does, $S\subset\tilde{S}\subset G$ and $\tilde{S}$ is a countable set.

REMARK 3.6. The condition 3.4(iii) can be formulated a bit simpler. Instead of (iii) one need only conclude $E\cap(x+h,x+h+k)\neq\varnothing$. Condition (iii) then follows from (ii). Note also that for $x\in E-G$ (ii) follows from (iii). On the other hand (iv) does not follow from (i)–(iii) as may be easily seen from 4.3 and 5.12. Let us also remark that (i), (ii) and (iv) do not imply (iii)—consider

$$S=G=\varnothing,\qquad E=\{0\}\cup\bigcup_{n=1}^{\infty}(-2^{-2n},-2^{-2n+1})\cup(2^{-2n},2^{-2n+1}).$$

4. Sufficient conditions.

LEMMA 4.1. *Let $P \neq \varnothing$ be a compact subset of R and let $A \subset R$ be a bounded set. Then there is a compact set $T \subset R$ containing P such that $T - P \subset A$,*

(i) *$T - P$ is an isolated set in $R - P$, and*

(ii) *if J is any interval contiguous to T, then there exists an open, connected set $H \subset J$, $H \cap A = \varnothing$ such that $|K| \leq d^2(K, P)$ for each component K of the set $J - H$.*

PROOF. Put $U_x = \{t \in R;\ |t - x| < \frac{1}{4}\min(d(x, P), d^2(x, P))\}$ for $x \in R - P$. We can find a locally finite covering $\{V_i\}$ of $R - P$ consisting of bounded open intervals such that, for any i, there is an $x \in R - P$ with $\bigcup_{V_j \cap V_i \neq \varnothing} V_j \subset U_x$. For each i with $A \cap V_i \neq \varnothing$ choose $x_i \in A \cap V_i$. We prove that the set $T = P \cup \{x_i\}$ is the required set. Since (i) is obvious we need only verify (ii). First note that $|U_x| \leq \frac{1}{2} d^2(x, P)$ and that, for any $t \in U_x$, $d(t, P) \geq d(x, P) - |x - t| \geq 0$. Thus $d^2(t, P) \geq d^2(x, P) - 2d(x, P)|x - t| \geq \frac{1}{2} d^2(x, P)$. Hence, $|U_x| \leq d^2(t, P)$ for any $t \in U_x$.

For $u \in R - P$ let K_u be the union of all V_i containing u. Since there is $x \in R - P$ such that $K_u \subset U_x$, we have $|K_u| \leq d^2(K_u, P)$. For $u \in P \cup \{+\infty, -\infty\}$ put $K_u = \varnothing$. Let $J = (u, v)$ be an interval contiguous to T. If $K_u \cap K_v \neq \varnothing$, then there are V_i and V_j such that $V_i \cap V_j \neq \varnothing$ and $V_i \cup V_j \supset J$. Since $V_i \cup V_j \subset U_x$ for some x, we have $|J| \leq d^2(J, P)$ and it is sufficient to put $H = \varnothing$. If $K_u \cap K_v = \varnothing$, put $H = J - (\bar{K}_u \cup \bar{K}_v)$. The assumption $H \cap A \neq \varnothing$ would imply that $V_i \cap H \cap A \neq \varnothing$ for some i. Hence, $V_i \cap T \neq \varnothing$ and consequently, $u \in V_i$ or $v \in V_i$. This would imply $V_i \subset K_u \cup K_v$, which is a contradiction.

LEMMA 4.2. *Let S, G and E be sets fulfilling the conditions* (i)–(iv) *of Theorem* 3.4 *and let P be a compact subset of R. Suppose that*

(A) *$P \subset E$, or*

(B) *$P \cap G = \varnothing$ and P has property* (Z) *with respect to E.*

Then there exists a nondecreasing function f defined and differentiable everywhere on R such that

(a) *$E \cap P = \{x \in P; f'(x) > 0\}$,*

(b) *$G \cap P = \{x \in P; f'(x) = +\infty\}$,*

(c) *f is continuous at every point of $R - S$ and at every point of $S \cap P$ it is discontinuous from the right as well as from the left,*

(d) *$\{x \in R; f'(x) > 0\} \subset E$, $\{x \in R; f'(x) = +\infty\} \subset G$, and*

(e) *f can be written as a sum of a nondecreasing absolutely continuous function and a nondecreasing jump function in S.*

PROOF. It suffices to assume $P = \overline{P \cap E} \neq \varnothing$. (If $P \cap E = \varnothing$ we may take $f = 0$; if $P \neq \overline{P \cap E}$ then (B) holds, $\overline{P \cap E} \cap G = \varnothing$, the set $\overline{P \cap E}$ has the property (Z) with respect to E according to Proposition 2.3 and any function f fulfilling (a)–(e) with P replaced by $\overline{P \cap E}$ has the required properties.) Let $F \subset E$ be an M_5 set for which $|E - F| = 0$ and let L be a bounded, open interval containing P. Let T be the compact set constructed in the preceding lemma with $A = (F \cup S) \cap L$.

First define an auxiliary number α and function g, by alteration of which we obtain the function f. If (A) holds, put $\alpha = \frac{1}{2}$ and, since $G \cap P$ is a G_δ set of measure zero, choose a nondecreasing absolutely continuous function g differentiable on R such that $G \cap P = \{x \in R; g'(x) = +\infty\}$ and $g'(x) > 0$ for all $x \in R$ (see [**21** or **22**, Theorem 7]). If (A) fails and (B) holds, put $\alpha = 0$ and first choose, for every interval I contiguous to T for which there is a bounded interval (a, b) contiguous to P with $\frac{1}{2}(a+b) \in I$, an open connected set $H_I \subset I$ such that $H_I \cap A = \varnothing$ and $|K| \leq d^2(K, P)$ for any component K of the set $I - H_I$. If $\frac{1}{2}(a+b) \notin I$, let $H_I = \varnothing$. Let H be the union of all H_I. The set $(P \cap E) \cup (R - (\overline{H} \cup \overline{(P \cap E)}))$ is of the type M_4. (This follows from condition (B), since by 4.1(ii) $H \subset R - E$.) Thus there exists a function g possessing a bounded, nonnegative derivative on R such that

$$\{x \in R; g'(x) > 0\} = (P \cap E) \cup \left(R - \left(\overline{H} \cup \left(\overline{P \cap E}\right)\right)\right)$$

(see [**22**, Theorem 8]).

The next step will be done for both cases (A) and (B) together. Let $t \in T$. If there is $t' \in T$, $t' > t$ such that the interval $I = (t, t')$ is contiguous to T, find that interval (a, b) contiguous to P which contains I, and put

$$A^+(t) = \alpha(g(t') - g(t)) \quad \text{if } t' \leq \tfrac{1}{2}(a+b),$$
$$A^+(t) = \tfrac{1}{2}(g(t') - g(t)) \quad \text{if } \tfrac{1}{2}(a+b) \in I \text{ and (A) holds or } H_I = \varnothing,$$
$$A^+(t) = g(u) - g(t) \quad \text{if } \tfrac{1}{2}(a+b) \in I, \text{(A) fails, and } H_I = (u, v) \neq \varnothing,$$
$$A^+(t) = (1-\alpha)(g(t') - g(t)) \quad \text{if } \tfrac{1}{2}(a+b) \leq t.$$

If there is no such t', put $A^+(t) = 0$.

Similarly we define $A^-(t)$. If there is $t' \in T$, $t' < t$ such that the interval (t', t) is contiguous to T, find that interval (a, b) contiguous to P which contains $I = (t', t)$ and put

$$A^-(t) = \alpha(g(t) - g(t')) \quad \text{if } t' \geq \tfrac{1}{2}(a+b),$$
$$A^-(t) = \tfrac{1}{2}(g(t) - g(t')) \quad \text{if } \tfrac{1}{2}(a+b) \in I \text{ and (A) holds or } H_I = \varnothing,$$
$$A^-(t) = g(t) - g(v) \quad \text{if } \tfrac{1}{2}(a+b) \in I, \text{(A) fails, and } H_I = (u, v) \neq \varnothing,$$
$$A^-(t) = (1-\alpha)(g(t) - g(t')) \quad \text{if } t \leq \tfrac{1}{2}(a+b).$$

If there is no such t', put $A^-(t) = 0$.

Let us prove that $A^+(t) = A^-(t) = 0$ for any $t \in P - S$. If $J = (t, t')$ where $t \in P$, $t' \in T$, and $J \cap T = \varnothing$, then choose an open connected set $H' \subset J$ such that $H' \cap A = \varnothing$ and $|K| \leq d^2(K, P)$ for any component K of $J - H'$. Since $t \in P$, it follows that $H' = (t, u)$ where $u \in (t, t'\rangle$. If (A) holds, then 3.4(ii) implies that $t \in S$. If (A) fails, choose $v \in \overline{R}$ such that (t, v) is the interval contiguous to P containing J. If $\frac{1}{2}(t+v) \in J$, then arguing as with H' above $H_J = (t, u')$ where $u' \in (t, t'\rangle$. Hence $A^+(t) = 0$. If $\frac{1}{2}(t+v) \notin J$, then $A^+(t) = 0$ because $\alpha = 0$. That $A^-(t) = 0$ is proved in a similar fashion.

If $t \in T \cap S$, put $w^+(t) = A^+(t)$ and $w^-(t) = A^-(t)$. If $t \in T - S$, put $w^+(t) = w^-(t) = 0$. If $t \in T \cap S$, put $\varphi_t^+(x) = \varphi_t^-(x) = 0$ for $x \in R$. If $A^+(t) = 0$

(resp. $A^-(t) = 0$), put $\varphi_t^+(x) = 0$ (resp. $\varphi_t^-(x) = 0$) for $x \in R$. If $A^+(t) > 0$ (resp. $A^-(t) > 0$) and $t \in T - S$ let $I = (t, t')$, where $t < t'$ (resp. $t > t'$), be an interval contiguous to T. Since $t \notin P \cup S$, we have $t \in F$. Hence t is a point of accumulation of $F \cap I$. Choose $u \in (t, \frac{1}{2}(t + t'))$ (resp. $(\frac{1}{2}(t + t'), t)$) such that $|u - t| < d(t, P)$ and $(t, u) \cap H_I = \varnothing$. Since $F \cap (t, u)$ is a nonempty set of type M_5, we can find a bounded, nonnegative function φ_t^+ (resp. φ_t^-) approximately continuous on R such that $\{x; \varphi_t^+(x) > 0\} \subset F \cap (t, u)$ and $\int_{-\infty}^{+\infty} \varphi_t^+(x)\, d(x) = A^+(t)$ (resp. $\{x; \varphi_t^-(x) > 0\} \subset F \cap (t, u)$ and $\int_{-\infty}^{+\infty} \varphi_t^-(x)\, dx = A^-(t)$) (see [22, Lemma 11]).

Put $m = \inf T$ and $M = \sup T$. Then

$$
\begin{aligned}
&\sum_{t \in T} \left(w^+(t) + w^-(t) + \int_{-\infty}^{+\infty} \varphi_t^+(x)\, dx + \int_{-\infty}^{+\infty} \varphi_t^-(x)\, dx \right) \\
&\qquad = \sum_{t \in T} (A^+(t) + A^-(t)) \\
&\qquad = \sum_{\substack{t, t' \in T, t < t' \\ (t, t') \cap T = \varnothing}} (g(t') - g(t)) \leq g(M) - g(m).
\end{aligned} \tag{1}
$$

Further put $v(x) = \sum_{t \leq x, t \in T} w^-(t) + \sum_{t < x, t \in T} w^+(t)$, $\varphi(x) = g'(x)$ for $x \in P$, and $\varphi(x) = \sum_{t \in T} (\varphi_t^+(x) + \varphi_t^-(x))$ for $x \notin P$. The function φ is nonnegative and $\int_{-\infty}^{+\infty} \varphi < +\infty$ according to (1). Choose $x_0 \in P$ and choose the indefinite integral g_1 of φ such that $g_1(x_0) + v(x_0) = g(x_0)$. Let $f_1 = g_1 + v$. For $x, y \in T$, $x < y$, we have

$$
\begin{aligned}
f_1(y) - f_1(x) &= \int_{\langle x, y \rangle \cap P} g'(u)\, du \\
&\quad + \sum_{(t, t') \subset (x, y)} \left(\int_t^{t'} (\varphi_t^+(u) + \varphi_{t'}^-(u))\, du + w^+(t) + w^-(t') \right) \\
&= \int_{\langle x, y \rangle \cap P} g'(u)\, du + \sum_{(t, t') \subset (x, y)} (g(t') - g(t)) = g(y) - g(x).
\end{aligned}
$$

(The summation is over all intervals (t, t') contiguous to T, $t < t'$.) Hence, $f_1 = g$ on T.

Since the family of supports of the functions φ_t^+ and φ_t^- is locally finite in $R - P$ and since φ_t^+ and φ_t^- are bounded and approximately continuous, we have $g_1' = \varphi$ on $R - P$. Using the fact that the set $(T \cap S) - P$ is isolated in $R - P$ we find that $f_1' = \varphi$ on $R - (P \cup (T \cap S))$.

Let $x \in P \cap (E - G)$ and let $c > 1$. According to 3.4(iii) there is an $\varepsilon \in (0, 1)$ such that $E \cap (x + h, x + h + k) \neq \varnothing$ for every $h, k \neq 0$, $0 < h/k < c$ and $|h + k| < \varepsilon$. Moreover, 3.4(ii) implies that x is a bilateral limit point of A and hence of T. So there is a $\delta \in (0, \varepsilon)$ such that each interval contiguous to T which intersects $(x - \delta, x + \delta)$ is a subset of $(x - \varepsilon, x + \varepsilon) \cap L$. Let $|h| < \delta$, $x + h \in (t, t')$, where (t, t') is an interval contiguous to T. Choose the notation so that $t \in (x, x + h)$. Let $H' \subset (t, t')$ be the set from 4.1(ii) and note that the measures of the components considered there equal $d(t, H')$ and $d(t', H')$ (these we put equal to $|t - t'|$, if

$H' = \varnothing$). Further, $d(x, H') \geqslant c|H'|$ since $H' \cap E = \varnothing$ by 4.1(ii). Hence,

$$|H'| \leqslant c^{-1}d(x, H') = c^{-1}(|t - x| + d(t, H')) \leqslant c^{-1}(|t - x| + (t - x)^2).$$

Using that f_1 is nondecreasing and agrees with g on T it can be shown that

$$\frac{t-x}{h}\,\frac{g(t)-g(x)}{t-x} \leqslant \frac{f_1(x+h)-f_1(x)}{h} \leqslant \frac{g(t')-g(x)}{t'-x}\,\frac{t'-x}{h}.$$

Since

$$\frac{t-x}{h} \geqslant \frac{t-x}{t'-x} = 1 - \frac{t'-x}{t'-x}$$

and since

$$\frac{t'-x}{h} \leqslant \frac{t'-x}{t-x} = 1 + \left(\frac{t'-t}{t'-x}\right)\left(1 - \frac{t'-t}{t'-x}\right)^{-1},$$

to prove that $f_1'(x) = g'(x)$ it suffices to show that $(t' - t)/(t' - x)$ can be made arbitrarily small by taking c large enough. This holds since

$$\begin{aligned}|t' - t| &\leqslant d(t', H') + |H'| + d(t, H')\\ &\leqslant |H'| + 2(t' - x)^2\\ &\leqslant c^{-1}(|t - x| + (t - x)^2) + 2(t' - x)^2.\end{aligned}$$

Hence $f_1'(x) = g'(x) = \varphi(x)$.

Let us consider a point $x \in P \cap (G - S)$. Such a point exists only when condition (A) holds. Thus, $\alpha = \frac{1}{2}$ and $g'(x) = +\infty$. Let $h \neq 0$, $x + h \in (t, t')$, where (t, t') is a bounded interval contiguous to T. Choose the notation so that $t \in (x, x + h)$. If $|x + h - t| \leqslant |t - x|$, then

$$|h| \leqslant |x + h - t| + |t - x| \leqslant 2|t - x|.$$

Hence,

$$\frac{f_1(x+h)-f_1(x)}{h} \geqslant \frac{g(t)-g(x)}{h} \geqslant \frac{1}{2}\,\frac{g(t)-g(x)}{t-x}.$$

If $|x + h - t| > |t - x|$, then $|f_1(x + h) - f_1(x)| \geqslant \frac{1}{2}|g(t') - g(t)|$. This is clear if $t \in S$, if $t \notin S$ it follows from the fact that the support of the corresponding function φ_t^+ (or φ_t^-) is a subset of $\langle t, u\rangle$ where $|u - t| < d(t, P) \leqslant |t - x|$. Hence,

$$\begin{aligned}|f_1(x + h) - f_1(x)| &= |f_1(x + h) - f_1(t)| + |g(t) - g(x)|\\ &\geqslant \tfrac{1}{2}|g(t') - g(t)| + |g(t) - g(x)|\\ &\geqslant \tfrac{1}{2}|g(t') - g(x)|.\end{aligned}$$

Thus

$$\frac{f_1(x+h)-f_1(x)}{h} \geqslant \frac{1}{2}\,\frac{g(t')-g(x)}{h} \geqslant \frac{1}{2}\,\frac{g(t')-g(x)}{t'-x}.$$

Hence, $f_1'(x) = +\infty = \varphi(x)$.

Further let us consider a point $x \in P - E$. Such a point exists only when (A) fails. Thus $\alpha = 0$ and, since $P = \overline{P \cap E}$, $g'(x) = 0$. Let $h \neq 0$, $x + h \in I$, where $I = (t, t')$

is an interval contiguous to T with $t \in (x, x+h)$. Choose an interval (a, b) contiguous to P such that $I \subset (a, b)$, $a \in \langle x, x+h)$.

If $x + h \in \langle \frac{1}{2}(t+t'), t')$, then $|h| \geqslant \frac{1}{2}|t'-x|$. Hence,

$$\frac{f_1(x+h)-f_1(x)}{h} \leqslant \frac{g(t')-g(x)}{h} \leqslant 2\frac{g(t')-g(x)}{t'-x}.$$

If $x + h \in \langle \frac{1}{2}(a+b), b)$, then $|h| \geqslant \frac{1}{2}|b-x|$. Hence,

$$\frac{f_1(x+h)-f_1(x)}{h} \leqslant \frac{g(b)-g(x)}{h} \leqslant 2\frac{g(b)-g(x)}{b-x}.$$

If $x + h \in (t, \frac{1}{2}(t+t'))$, $t' \in (a, \frac{1}{2}(a+b)\rangle$, then since $\alpha = 0$,

$$\frac{f_1(x+h)-f_1(x)}{h} = \frac{g(t)-g(x)}{h} \leqslant \frac{g(t)-g(x)}{t-x}.$$

If $\frac{1}{2}(a+b) \in (t, t')$ and $H_I = \varnothing$, then $|t'-x| = |t-t'| + |t-x| \leqslant (t-x)^2 + |t-x| \leqslant |h|(1+|h|)$. Hence,

$$\frac{f_1(x+h)-f_1(x)}{h} \leqslant \frac{g(t')-g(x)}{h} \leqslant (1+|h|)\frac{g(t')-g(x)}{t'-x}.$$

If $H_I \neq \varnothing$, let $H_I = (u, v)$ with $|u-x| < |v-x|$. First suppose $x+h \in (t, v)$. Then $|u-x| \leqslant |u-t| + |t-x| \leqslant (t-x)^2 + |t-x| \leqslant |h|(1+|h|)$. Since f_1, g are both constant on (u, v), we have

$$\frac{f_1(x+h)-f_1(x)}{h} \leqslant \frac{g(u)-g(x)}{h} \leqslant (1+|h|)\frac{g(u)-g(x)}{u-x}.$$

Finally suppose that $x+h \in \langle v, t')$. Then $|t'-x| \leqslant |t'-v| + |v-x| \leqslant (v-x)^2 + |v-x| \leqslant |h|(1+|h|)$. Hence,

$$\frac{f_1(x+h)-f_1(x)}{h} \leqslant \frac{g(t')-g(x)}{h} \leqslant (1+|h|)\frac{g(t')-g(x)}{t'-x}.$$

Therefore, $f_1'(x) = 0 = \varphi(x)$.

It follows that f_1 is almost the required function. Only one more change is needed. The set $B = (G \cap P) \cup ((S \cap T) - P)$ is of type G_δ (since the set $T - P$ is isolated in $R - P$). Let $\{F_i\}$ be a nondecreasing sequence of closed sets such that $R - B = \bigcup_{i=1}^{\infty} F_i$ and let $\{s_i\}$ be a sequence of all points of $S \cap T$. Put $a_i = \min(2^{-i}, 2^{-i}d^2(s_i, F_i))$ and $v_1(x) = \Sigma_{s_i \leqslant x} a_i + \Sigma_{s_i < x} a_i$. Then v_1 is a nondecreasing jump function in S which is discontinuous at each point of $S \cap T$ from the right as well as from the left. (Thus, $v_1'(x) = +\infty$ for $x \in S \cap T$.) Let $x \in R - B$. Then $x \in F_k$ for some k. Let $\delta > 0$ be such that $(x-\delta, x+\delta) \cap \{s_1, \ldots, s_k\} = \varnothing$ and let $|h| < \delta$. From the definition of v_1 we have $|v_1(x+h) - v_1(x)| \leqslant 2\Sigma_{i>k, s_i \in \langle x, x+h\rangle} a_i \leqslant 2\Sigma_{i=k+1}^{\infty} 2^{-i}h^2 \leqslant h^2$. Hence, $v_1'(x) = 0$. Thus, it is sufficient to put $f = f_1 + v_1$.

THEOREM 4.3. *Let S, G and E be subsets of R such that the conditions* 3.4(i)—(iv) *hold. Then there exists a nondecreasing function f possessing a derivative on R such that*

(a) *f is continuous at $x \in R$ if and only if $x \notin S$; at any point $x \in S$ it is discontinuous from the right as well as from the left,*

(b) $G = \{x \in R; f'(x) = +\infty\}$,

(c) $E = \{x \in R; f'(x) > 0\}$, *and*

(d) *f can be written as a sum of an absolutely continuous nondecreasing function and a nondecreasing jump function in S.*

PROOF. Let $\mathfrak{R}$ be the system of all open sets $H \subset R$ (called regular) for which the statement of the theorem holds with S, G and E replaced by $S \cap H$, $G \cap H$ and $E \cap H$. We will verify the assumptions of Lemma 1.3.

Obviously $\varnothing \in \mathfrak{R}$. (Consider $f = 0$.)

Let $H \in \mathfrak{R}$. Put $A = (H - E) \cup \operatorname{Int}(H \cap E)$. Then the set A is dense in H. If $a \in A$, choose a function f from the definition of the regularity of H such that $f(a) = 0$. If $a \in H - E$, then $f'(a) = 0$ and to prove the regularity of $(-\infty, a) \cap H$ (resp. $(a, +\infty) \cap H$) we put $f_1(x) = f(x)$ for $x \leqslant a$ (resp. for $x \geqslant a$), and $f_1(x) = 0$ for $x > a$ (resp. for $x < a$). If $a \in \operatorname{Int}(H \cap E)$, choose $\delta > 0$ such that $(a - \delta, a + \delta) \subset \operatorname{Int}(H \cap E)$ and find a continuously differentiable function φ on R such that $\varphi(a) = 0$ and φ' is positive on $(a, a + \delta)$, negative on $(a - \delta, a)$ and equals 0 on $(-\infty, a - \delta) \cup (a + \delta, +\infty)$. Put $f_1(x) = \varphi(x)f(x)$. Then $f_1'(x) = \varphi'(x)f(x) + \varphi(x)f'(x)$ for $x \neq a$ and

$$f_1'(a) = \lim_{x \to a} \frac{\varphi(x)}{x - a} f(x) = 0$$

since $\varphi(a) = 0$, $\varphi'(a) = 0$, and f is bounded on a neighborhood of a. Hence, f_1 is nondecreasing and the conditions 4.3(a)—(c) hold with S, G and E replaced by $(S \cap H) - \{a\}$, $(G \cap H) - \{a\}$ and $(E \cap H) - \{a\}$. Further let $f = g + v$ be the decomposition of f according to (d) where v is a jump function in $S = \{s_i\}$. Hence, $v(x) = \sum_{s_i \leqslant x} a_i + \sum_{s_i < x} b_i$. Put $v_1(x) = \sum_{s_i \leqslant x} a_i \varphi(s_i) + \sum_{s_i < x} b_i \varphi(s_i)$, and

$$h(x) = \varphi(x)v(x) - v_1(x) = \sum_{s_i \leqslant x} a_i(\varphi(x) - \varphi(s_i)) + \sum_{s_i < x} b_i(\varphi(x) - \varphi(s_i)).$$

The function h is zero on $(-\infty, a - \delta)$ and constant on $(a + \delta, +\infty)$, and for $x < y$ we have

$$h(y) - h(x) = \sum_{x < s_i \leqslant y} a_i(\varphi(y) - \varphi(s_i)) + \sum_{x \leqslant s_i < y} b_i(\varphi(y) - \varphi(s_i)) + \sum_{s_i \leqslant x} a_i(\varphi(y) - \varphi(x)) + \sum_{s_i < x} b_i(\varphi(y) - \varphi(x)).$$

Since there is a $K \in R$ such that $|\varphi'| < K$, we obtain

$$|h(y) - h(x)| \leqslant K \sum_{s_i \in R} (a_i + b_i)\,|x - y|\,.$$

Thus, h is absolutely continuous. Moreover, $f(x) = (g(x) - g(a)) + (v(x) - v(a))$. Hence, $f_1(x) = \varphi(x)(g(x) - g(a)) + h(x) - \varphi(x)v(a) + v_1(x)$ and the function $\varphi(x)(g(x) - g(a)) + h(x) - \varphi(x)v(a)$ is absolutely continuous. To finish the proof of the regularity of $(-\infty, a) \cap H$ (resp. $(a, +\infty) \cap H$) it suffices to put $f_2(x) = f_1(x)$ for $x \leqslant a$ (resp. for $x \geqslant a$), and $f_2(x) = f_1(a)$ for $x > a$ (resp. for $x < a$).

Let us prove the condition 1.3(γ). If $H = \bigcup_{n=1}^{\infty} I_n$, where $I_n \in \mathfrak{R}$ are bounded regular intervals, $\bar{I}_n \subset H$, choose functions $f_n = g_n + v_n$ from the definition of the regularity of I_n (where each g_n is absolutely continuous and each v_n is a nondecreasing jump function). Since f_n, g_n and v_n are constant on each component of $R - I_n$, we can suppose that their absolute values are not greater than

$$\min\left(2^{-n}, 2^{-n} d^2(I_n, R - H)\right).$$

Put $f = \sum_{n=1}^{\infty} f_n$, $g = \sum_{n=1}^{\infty} g_n$, and $v = \sum_{n=1}^{\infty} v_n$. Then $f = g + v$, g is nondecreasing and absolutely continuous, and v is a nondecreasing jump function in $S \cap H$. Since the covering $\{I_n\}$ of H is locally finite, the only statement we have to prove is that $f'(a) = 0$ for $a \notin H$. For such an a we have

$$|f(x) - f(a)| \leq \sum_{n=1}^{\infty} |f_n(x) - f_n(a)| \leq \sum_{n=1}^{\infty} 2^{-n}(x-a)^2 = (x-a)^2.$$

Let us prove 1.3(δ). Let $H \in \mathfrak{R}$, $H \neq R$ and $P = R - H$. Let f be the function from the definition of the regularity of H. There is a bounded open interval I intersecting P such that $I \cap P \subset E$ or $I \cap P \cap G = \varnothing$. Let $\{J_n\}$ and $\{I_n\}$ be locally finite coverings of I by open intervals such that $\varnothing \neq \bar{J}_n \subset I_n \subset \bar{I}_n \subset I$. According to Lemma 4.2 (with S, G, E and P replaced by $S \cap I_n$, $G \cap I_n$, $E \cap I_n$ and $\overline{P \cap J_n}$) we construct functions $f_n = g_n + v_n$. Moreover, we can suppose that $|f_n|$, $|g_n|$, and $|v_n| \leq 2^{-n} d^2(R - I, I_n)$. The function $f + \sum_{n=1}^{\infty} f_n$ has all the properties required in the definition of the regularity of $H \cup I$.

Hence, $R \in \mathfrak{R}$ according to 1.3.

REMARK 4.4. If E is an F_σ, as well as a G_δ, set of measure zero and if S is a countable set dense in E containing all points of E which are not points of bilateral accumulation of E (such a set always exists), then the triple S, E, E fulfills the conditions of the preceding theorem. Thus, there is a function f defined on R such that $f' = +\infty$ on E and $f' = 0$ on $R - E$. The case E countable was posed as a problem in **[22]** and solved in **[1]** and in **[17]**.

REMARK 4.5. If S is a countable set and $G \subset R$ is a G_δ set of measure zero containing S, then there is a function f possessing a derivative on R such that S is the set of the points of discontinuity of f and $G = \{x \in R;\, f'(x) = +\infty\}$. This solves the problem posed in **[11]**.

5. Level sets of derivatives.

DEFINITION 5.1. Let us define the following classes of subsets of the reals.

$E \in M^*$ if E is an F_σ set and for each closed set $P \subset R$

(i) $Q \subset E$ for some portion Q of P or

(ii) there is a portion Q of P such that

(a) Q has property (Z) with respect to E, and

(b) if $x \in Q \cap E$ and $c > 0$, then there exists $\varepsilon > 0$ such that $E \cap (x + h, x + h + k) \neq \varnothing$ for every h, $k \neq 0$, $0 < h/k < c$ and $|h + k| < \varepsilon$.

$M_2^* = M^* \cap M_2$.

$M_3^* = M^* \cap M_3$.

THEOREM 5.2. 1. *If f possesses an approximate derivative on an open interval I, then* $\{x \in I; f'_{\mathrm{ap}}(x) > \alpha\} \in M^*$ *and* $\{x \in I; f'_{\mathrm{ap}}(x) < \alpha\} \in M^*$ *for every* $\alpha \in R$.

2. *If $E \in M^*$, then there is a nondecreasing function f (which can be written as a sum of a nondecreasing, absolutely continuous function and a nondecreasing jump function) possessing a derivative on R such that* $E = \{x \in R; f'(x) > 0\}$.

3. *If $E_1, E_2 \in M^*$ are disjoint, then there is a function f possessing a derivative on R such that* $E_1 = \{x \in R; f'(x) > 0\}$ *and* $E_2 = \{x \in R; f'(x) < 0\}$.

PROOF. 1. If P is a closed set, then one of its portions is a subset of $\{x \in R; f'(x) > \alpha\}$ or there is a portion Q of P which does not intersect $\{x \in R; f'(x) = +\infty\}$. In the latter case Q has a portion having property (Z) with respect to $\{x \in R; f'(x) > \alpha\}$ according to 3.4(iv) and 2.5. The condition 5.1(b) follows directly from 3.4(iii).

2. An open set $H \subset R$ will be called regular ($H \in \mathfrak{R}$) if there are sets S_H and G_H such that the assumptions of Theorem 4.3 hold with S, G and E replaced by S_H, G_H and $E \cap H$.

Obviously $\varnothing \in \mathfrak{R}$, an open subset of a regular set is again regular (for $L \subset H$ it suffices to put $S_L = S_H \cap L$ and $G_L = G_H \cap L$), and condition (γ) of Lemma 1.3 also holds. (Consider $S_H = \bigcup_n S_{I_n}$ and $G_H = \bigcup_n G_{I_n}$.) We have to prove 1.3(δ). Let $H \in \mathfrak{R}$, $H \neq R$ and $P = R - H$. If there is an open interval I such that $\varnothing \neq Q = I \cap P \subset E$, we choose a countable set $Z \subset Q$ dense in Q containing all those points of Q which are not points of bilateral accumulation of Q. Let $F \subset Q - Z$ be a set of type M_5 such that $|Q - F| = 0$. Put $S_I = (S_H \cap I) \cup Z$ and $G_I = (G_H \cap I) \cup (Q - F)$. The verification of conditions 3.4(i) – (iii) is straightforward. We prove 3.4(iv). If $D \subset R - G_I$ is a perfect set, then either $D - \overline{Q} \neq \varnothing$ and one obtains the required portion from the definition of the regularity of H and 2.3(3) or $D \subset \overline{Q}$. In the latter case $\varnothing \neq I \cap D \subset F$ and one can use 2.6. Hence, $I \in \mathfrak{R}$ and $I - H = I \cap P \neq \varnothing$. If there is an open interval I such that the set $Q = I \cap P \neq \varnothing$ has property (ii) from Definition 5.1, we put $S_I = S_H \cap I$ and $G_I = G_H \cap I$. Conditions 3.4(i) and (ii) for $x \in G_I - S_I$ follow directly from the definition of the regularity of H. To prove 3.4(iii) (which implies 3.4(ii) for $x \in (E \cap I) - G_I$) one needs also 5.1(ii)(b) and the following observation. If $J \subset I$ is an open interval such that $J \cap E \neq \varnothing$ then $J \cap S_I \neq \varnothing$ or $|J \cap E| > 0$. If $(J \cap E) - Q \neq \varnothing$ this is obvious from the definition of the regularity of H and 3.4(ii). If $J \cap E \subset Q$, then we infer from 2.3(1), (3) and 2.2(3) first that $J \cap Q$ has property (Z) w.r.t. $J \cap E$ and then, since $J \cap E \subset J \cap Q$, that $J \cap E \in M_4$. Since $J \cap E \neq \varnothing$, this implies $|J \cap E| > 0$. Finally, let us prove 3.4(iv). If $D \subset R - G_I$ is a perfect set, then either $D - \overline{Q} \neq \varnothing$ and one can use the definition of the regularity of H or $D \subset \overline{Q}$. In the latter case $\varnothing \neq I \cap D$ and the set $I \cap D$ has property (Z) w.r.t. $E \cap I$ according to 5.1(ii)(a) and 2.3.

Hence, $R \in \mathfrak{R}$ (Lemma 1.3) and we can use Theorem 4.3.

3. It suffices to construct functions f_1 and f_2 for the sets E_1 and E_2 according to statement 2 of the theorem and to put $f = f_1 - f_2$.

THEOREM 5.3. 1. *If f is a Darboux function possessing an approximate derivative on an open interval I, then $\{x \in I; f'_{\text{ap}}(x) > \alpha\} \in M_2^*$ and $\{x \in I; f'_{\text{ap}}(x) < \alpha\} \in M_2^*$ for every $\alpha \in R$.*

2. *If $E \in M_2^*$, then there exists an absolutely continuous nondecreasing function f possessing a derivative on R such that $E = \{x \in R; f'(x) > 0\}$.*

3. *If $E_1, E_2 \in M_2^*$ are disjoint, then there is an absolutely continuous function f possessing a derivative on R such that $E_1 = \{x \in R; f'(x) > 0\}$ and $E_2 = \{x \in R; f'(x) < 0\}$.*

PROOF. 1. Use 5.2 and 3.4(ii).

2. An open set $H \subset R$ will be called regular ($H \in \mathfrak{R}$) if there is a set G_H such that the assumptions of Theorem 4.3 hold with S, G and E replaced by $\varnothing$, G_H and $E \cap H$. The proof of conditions 1.3(α)–(γ) is similar to that in 5.2(2). We prove 1.3(δ). Let $H \in \mathfrak{R}$, $H \neq R$ and $P = R - H$. If $Q = I \cap P \neq \varnothing$ is a portion of $R - H$ such that $Q \subset E$, find a set $F \subset Q$ of type M_5 such that $|Q - F| = 0$ and put $G_I = (G_H \cap I) \cup (Q - F)$. If a portion $Q = I \cap P \neq \varnothing$ of the set P has property (ii) from Definition 5.1, it suffices to put $G_I = G_H \cap I$. In view of Lemma 1.3 we can use Theorem 4.3.

3. It follows from 2 above just as in the proof of 5.2(3).

THEOREM 5.4. 1. *If f is a function possessing a finite approximate derivative on an open interval I, then $\{x \in R; f'_{\text{ap}}(x) > \alpha\} \in M_3^*$ and $\{x \in R; f'_{\text{ap}}(x) < \alpha\} \in M_3^*$ for every $\alpha \in R$.*

2. *If $E \in M_3^*$, then there exists a nondecreasing (absolutely continuous) function f possessing a finite derivative on R such that $E = \{x \in R; f'(x) > 0\}$.*

3. *If $E_1, E_2 \in M_3^*$ are disjoint, then there exists an absolutely continuous function f possessing a finite derivative such that $E_1 = \{x \in R; f'(x) > 0\}$ and $E_2 = \{x \in R; f'(x) < 0\}$.*

PROOF. 1. Use 5.2 and 3.4(iii).

2. We prove that the assumptions of Theorem 4.3 hold with S, G and E replaced by $\varnothing$, $\varnothing$ and E. Conditions 3.4(i)–(iii) are easy. We prove 3.4(iv). If P is a perfect set, then one of its portions is a subset of E and we can use 2.6, or there is a portion satisfying 5.1(ii), which clearly is the required portion.

3. This part follows directly from 2.

REMARK 5.5. Using the previous results one readily sees that the classes M^*, M_2^*, M_3^* are closed under finite unions. On the other hand, countable unions of sets of type M^* (resp. M_2^*, M_3^*) are exactly the sets of type F_σ (resp. M_2, M_3). This we easily prove considering, for a closed set $A \subset E$, the set $A \cup F$ where $F \subset E$ is a set of type M_5 such that $|E - F| = 0$.

REMARK 5.6. Since the classes M^*, M_2^* and M_3^* are closed under finite unions we have also a characterization of sets where derivatives equal zero as complements of sets of type M^*, M_2^* or M_3^* (depending on the type of the function).

REMARK 5.7. Any set which is at the same time of type F_σ and G_δ is of type M^*. Thus, if P is a closed set, there is a function f possessing a derivative on R which is constant on any interval disjoint from P and nonconstant on any interval intersecting P. A necessary and sufficient condition for the existence of such a continuous function f is that $|I \cap P| > 0$ for every open interval I intersecting P.

REMARK 5.8. Let us study the inclusions among the different classes. First define the following class: $E \in L$ if $E \subset R$ is an F_σ set and for every $x \in E$ there is an $\eta > 0$ such that, given $c > 0$, we can find $\varepsilon > 0$ possessing the following property: $|E \cap (x+h, x+h+k)| > \eta |k|$ for every $h, k \neq 0$, $0 < h/k < c$ and $|h+k| < \varepsilon$.

The following inclusions are obvious.

$$\begin{array}{ccccccccc} M^* & \supset & M_2^* & \supset & M_3^* & \supset & M_4 & \supset & M_5 \\ \cap & & \cap & & \cap & & \cap & & \\ F_\sigma & \supset & M_2 & \supset & M_3 & \supset & L & & \end{array}$$

We have no equalities among these inclusions which is easy to establish except for $L \neq M_4$ (see [8] and also 5.10) and $M_3 \neq M_3^*$ (see [7] and also 5.12).

From the definitions one obtains: If $E \in M_2$ (resp. $E \in M_3$), then $E \in M_2^*$ (resp. $E \in M_3^*$) if and only if $E \in M^*$. In other words: If $E \in M_2$ (resp. $E \in M_3$), then either there exists a continuous differentiable (resp. finitely differentiable) function f with $E = \{x; f'(x) > 0\}$, or there exists no differentiable function f with $E = \{x; f'(x) > 0\}$. Hence, the following questions arise: Is $L \cap M^* = M_4$? Is $L \subset M^*$? Examples 5.10 and 5.12 show that the answers are negative. Note that this means that the connection between L and M_4 differs from that between M_2 and M_2^* (resp. M_3 and M_3^*). In this connection the following question arises.

Is there any "natural" class of functions for which the analogs of Theorems 5.2, 5.3 and 5.4 hold with the class $L \cap M^*$?

LEMMA 5.9. *Let $I \subset R$ be a bounded open interval and let $k \geq 1$. Then there exists an open set $H \subset I$ such that*

(i) *$\overline{H} - H$ is a countable set,*

(ii) *if $J \subset I$ is an interval with $d(J, R - I) \leq k|J|$, then $|J \cap H| \geq \frac{1}{2}|J|$,*

(iii) *for every $c > 0$ there is $\varepsilon > 0$ such that for every interval $J \subset I$ with*

$$d(J, R-I) < c|J| \quad \text{and} \quad d(J, R-I) < \varepsilon$$

we have $|J \cap H| \geq k^{-1}|J|$, and

(iv) *for every $\varepsilon > 0$ there is an interval $J \subset I$ such that $d(J, R-I) < 3k|J|$, $|J \cap H| \leq 2k^{-1}|J|$ and $d(J, R-I) < \varepsilon$.*

PROOF. Let s_0 be the center of the interval $I = (\alpha, \beta)$, $\alpha < \beta$, and define by induction points $s_n \in I$ $(n = 1, 2, \ldots)$ such that $\beta - s_n = k(s_n - s_{n-1})$. Then $\lim_{n\to\infty} s_n = \beta$. Put $s_{-n} = 2s_0 - s_n$. Let

$$H_1 = \bigcup_{p=-\infty}^{+\infty} \left[\left(s_p, \tfrac{2}{3}s_p + \tfrac{1}{3}s_{p+1}\right) \cup \left(\tfrac{1}{3}s_p + \tfrac{2}{3}s_{p+1}, s_{p+1}\right)\right].$$

If $\bar{J} \subset I$, $d(J, R-I) < k|J|$, $J = (u, v)$ (where $u < v$), then choose the greatest p and the smallest q such that $s_p \leq u$ and $s_q \geq v$. Since $d((s_p, s_{p+1}), R-I) = k|(s_p, s_{p+1})|$, $s_{p+1} < s_q$ and therefore, $s_{p+1} \in J$. Hence, $|J \cap H_1| \geq \frac{1}{2}|J|$.

Let $\mathfrak{D}$ be the family of all open intervals which we obtain by partitioning (s_p, s_{p+1}) into $3(|p|+1)$ equal intervals (p is an arbitrary integer). If $D \in \mathfrak{D}$, $D \subset (s_p, s_{p+1})$ then

$$|D|^{-1}d(D, R-I) \geqslant \left(\frac{1}{3(|p|+1)}\,|(s_p, s_{p+1})|\right)^{-1} d\big((s_p, s_{p+1}), R-I\big)$$
$$= 3(|p|+1)k.$$

Hence, if $J \subset I$, $d(J, R-I) < c|J|$ and $d(J, R-I)$ is sufficiently small, then J is not a subset of any interval belonging to $\mathfrak{D}$. If

$$H_2 = \bigcup_{(a,b)\in\mathfrak{D}} \left[\left(a, \frac{k}{1+k}a + \frac{1}{1+k}b\right) \cup \left(\frac{1}{1+k}a + \frac{k}{1+k}b, b\right)\right],$$

then $|J \cap H_2| \geqslant k^{-1}|J|$. Moreover, for $J = (\frac{2}{3}s_p + \frac{1}{3}s_{p+1}, \frac{1}{3}s_p + \frac{2}{3}s_{p+1})$ we have

$$|J|^{-1}d(J, R-I) < \left(\tfrac{1}{3}|(s_p, s_{p+1})|\right)^{-1} d\big((s_p, s_{p+1}), R-I\big) = 3k,$$

$J \cap H_1 = \varnothing$ and $|J \cap H_2| \leqslant 2k^{-1}|J|$. It follows that the set $H = H_1 \cup H_2$ has the required properties.

EXAMPLE 5.10. Let P be a perfect set of measure zero contained in a bounded, open interval I. Then there exists a set $E \in L$ such that

(a) $P \subset E \subset I$,

(b) $E - P$ is an open set,

(c) $E \notin M_4$, and

(d) $E \in M^*$.

PROOF. Let $\{I_n\}$ be the sequence of all intervals contiguous to $P \cup (R - I)$. For each I_n choose a set H_n according to the preceding lemma (where $k = n$) and put $E = P \cup \bigcup_{n=1}^{\infty} H_n$. If $x \in P$ is not an isolated point of P from the right, choose $\varepsilon > 0$ such that $(x, x+\varepsilon) \subset I$ and $(x, x+\varepsilon) \cap I_n = \varnothing$ for $n \leqslant 3(c+1)$. Let $J \subset (x, x+\varepsilon)$, $d(x, J) < c|J|$. If there is an interval $J' \subset J$ with endpoints in P such that $|J'| \geqslant \frac{1}{3}|J|$, then $|J \cap E| \geqslant |J' \cap E| \geqslant \frac{1}{2}|J'| \geqslant \frac{1}{6}|J|$ (use 5.9(ii)). If there is no such interval, choose $J' \subset J$ such that $J' \cap P = \varnothing$ and $|J'| \geqslant \frac{1}{3}|J|$. Then $J' \subset I_n$ for some $n > 3(c+1)$ and

$$|J'|^{-1}d(J', P) \leqslant \left(\tfrac{1}{3}|J|\right)^{-1}\big(d(J, x) + |J|\big) < 3(c+1).$$

Now 5.9(ii) implies $|J \cap E| \geqslant |J' \cap E| \geqslant \frac{1}{2}|J'| \geqslant \frac{1}{6}|J|$. We can proceed similarly for those points of P that are not isolated from the left. This and 5.9(iii) imply $E \in L$.

Let us prove (c). From the assumption $E \in M_4$ we deduce the existence of a portion Q of P and of a number $\eta > 0$ such that for every $x \in Q$ and $c > 0$ there is an $\varepsilon > 0$ with the following property: $|E \cap (x+h, x+h+k)| > \eta|k|$ for every $h, k \neq \varnothing$, $0 < h/k < c$ and $|h+k| < \varepsilon$. We can find an interval I_n with $n > 2\eta^{-1}$ whose endpoints belong to Q. Using 5.9(iv) we obtain a contradiction.

To prove (d) it suffices to note that E is both an F_σ and a G_δ set.

LEMMA 5.11. *Let $I \subset R$ be a bounded, open interval. Then there exists a nowhere dense set $F \subset I$ such that $F \in L$ and $F \notin M_4$.*

PROOF. Choose a perfect subset $P \subset I$ of measure zero and construct a set E according to 5.10. Find a compact set $T \supset P$ according to Lemma 4.1 (where $A = I$). If J is a bounded interval contiguous to T, choose a nowhere dense set $F_J \subset J \cap E$ of type M_5 with $|F_J| \geq \frac{1}{2}|J \cap E|$. Let F be the union of P and of all F_J. Then F is a nowhere dense set and $F \subset I$. We have $F \notin M_4$ since $E \notin M_4$, the set $E - P$ is open and $P \subset F \subset E$.

We prove that $F \in L$. Let $x \in P$. (For $x \in F - P$ we have nothing to prove since $F - P \in M_5$.) There is $\eta > 0$ such that for every $c > 0$ we can find $\varepsilon > 0$ such that $|E \cap (x + h, x + h + k)| > \eta |k|$ whenever $h, k \neq 0, 0 < h/k < c$ and $|h + k| < \varepsilon$. Moreover, we can assume that $\varepsilon < \frac{1}{2}\eta(1 + 2c + 2c^2)^{-1}$ and $(x - \varepsilon, x + \varepsilon) \subset (\inf T, \sup T)$. Let $0 < h/k < c$ and $|h + k| < \varepsilon$. If $x + h \in T$, put $u = x + h$. If $x + h \in (t, t')$, where (t, t') is an interval contiguous to T with $t \in \langle x, x + h)$, put $u = t'$. If $x + h + k \in T$, put $v = x + h + k$. If $x + h + k \in (t, t')$, where (t, t') is an interval contiguous to T with $t \in \langle x, x + h + k)$, put $v = t$. Then $|u - (x + h)| \leq h^2$ and $|x + h + k - v| \leq (h + k)^2$ (as follows from 4.1). Therefore,

$$\begin{aligned}|u - (x + h)| + |x + h + k - v| &\leq h^2 + (h + k)^2 = 2h^2 + 2hk + k^2 \\ &\leq (2c^2 + 2c + 1)k^2 \leq \tfrac{1}{2}\eta |k| .\end{aligned}$$

This inequality implies that the intervals $(x + h, u)$ and $(v, x + h + k)$ are disjoint and that $|(u, v) \cap E| > \frac{1}{2}\eta |k|$. Hence,

$$|F \cap (x + h, x + h + k)| \geq |(u, v) \cap F| \geq \tfrac{1}{2}|(u, v) \cap E| > \tfrac{1}{4}\eta |k| .$$

EXAMPLE 5.12. There is a set $E \in L$ which is not of the type M^*.

PROOF. Let $\{I_n\}$ be the sequence of all open intervals with rational endpoints. According to 5.11 we can construct a nowhere dense set $E_1 \subset I_1$, $E_1 \in L$ and $E_1 \notin M_4$. By induction we define sets E_n as follows: Since $\bigcup_{i \leq n-1} E_i$ is a nowhere dense set, there is an open interval $I \subset I_n$ which does not intersect this set. We can assume that $|I| < 2^{-n} d^2(I, \bigcup_{i<n} E_i)$ and $\bar{I} \cap \overline{\bigcup_{i<n} E_i} = \varnothing$. Let E_n be a nowhere dense set such that $E_n \subset I$, $E_n \in L$ and $E_n \notin M_4$.

Put $E = \bigcup_{i=1}^{\infty} E_i$. Then E is a set of the type L. We prove that $E \notin M^*$. Suppose to the contrary, that $E \in M^*$. Then there is an open interval I having property (Z) w.r.t. E. Since E is dense in R, $E \cap I \in M_4$. Choose n such that $I_n \subset I$. Let $x \in E_n$. Choose $\delta > 0$ such that $(x - \delta, x + \delta) \cap \bigcup_{i=1}^{n-1} E_i = \varnothing$. If $hk > 0$ and $|h + k| < \delta$, then, for $i \neq n$, we have either $(x + h, x + h + k) \cap E_i = \varnothing$ and therefore,

$$|(x + h, x + h + k) \cap E_i| < 2^{-i}(h + k)^2$$

or $(x + h, x + h + k) \cap E_i \neq \varnothing$ and therefore, $i > n$ and $|E_i| < 2^{-i} d^2(x, E_i) \leq 2^{-i}(h + k)^2$. Hence, $|(x + h, x + h + k) \cap \bigcup_{i \neq n} E_i| \leq (h + k)^2$. Let $E \cap I = \bigcup_{k=1}^{\infty} F_k$ and $\eta_k > 0$ according to the definition of M_4. If $x \in F_k \cap E_n$ and $c > 0$, then find $\varepsilon < \frac{1}{2}\eta_k(c+1)^{-2}$, $0<\varepsilon<\delta$, such that $|(x+h, x+h+h_1) \cap E| > \eta_k|h_1|$ for all $h, h_1 \neq 0$, $0 < h/h_1 < c$ and $|h+h_1| < \varepsilon$. For such h and h_1 we have

$$\begin{aligned} |(x+h, x+h+h_1) \cap E_n| \geq |(x+h, x+h+h_1) \cap E| \\ -\Big|(x+h, x+h+h_1) \cap \bigcup_{i\neq n} E_i\Big| \\ > \eta_k|h_1| - (c+1)^2 h_1^2 > \tfrac{1}{2}\eta_k|h_1| \end{aligned}$$

since $|h_1| < \frac{1}{2}\eta_k(c+1)^{-2}$. Hence, $E_n \in M_4$, which is a contradiction.

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