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SIMPLEXES OF EXTENSIONS OF STATES OF C^*-ALGEBRAS

BY

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ABSTRACT. Let B be a C^*-subalgebra of a C^*-algebra A, F a compact face of the state space $S(B)$ of B, and $S_F(A)$ the set of all states of A whose restrictions to B lie in F. It is shown that $S_F(A)$ is a Choquet simplex if and only if (a) F is a simplex, (b) pure states in $S_F(A)$ restrict to pure states in F, and (c) the states of A which restrict to any given pure state in F form a simplex. The properties (b) and (c) are also considered in isolation.

Sets of the form $S_F(A)$ have been considered by various authors in special cases including those where B is a maximal abelian subalgebra of A, or A is a C^*-crossed product, or the Cuntz algebra $\mathcal{O}_n$.

1. Introduction. Let A be a C^*-algebra with quasi-state space $Q(A)$:

$$Q(A) = \{\phi \in A^*: \phi \geqslant 0, \|\phi\| \leqslant 1\}.$$

Let B be a C^*-subalgebra of A and F a nonempty (weak*) closed face of $Q(B)$. There are various situations in which one is interested in the structure of extensions of functionals in F. Thus one studies the nonempty closed face

$$Q_F(A) = \{\phi \in Q(A): \phi|_B \in F\}$$

of $Q(A)$. For example, B might be a maximal abelian C^*-subalgebra (masa) in A, and F consist of a single pure state; a problem of some complexity is to determine whether $Q_F(A)$ also contains only a single (pure) state [**1**, **2**, **3**]. Alternatively, A might be (the multiplier algebra of) the crossed product $G \times_\alpha A_0$ of some C^*-dynamical system (A_0, G, α), B the C^*-subalgebra $C^*(G)$ of A, and F consist of the single state ϕ_0 of B with $\phi_0(u_g) = 1$ $(g \in G)$, where u is the universal representation of G in $C^*(G)$. Then $Q_F(A)$ is isomorphic to the set $Q^G(A_0)$ of G-invariant functionals in $Q(A_0)$ [**4**]. In algebraic models of statistical mechanics, a fundamental question is therefore whether $Q_F(A)$ is a Choquet simplex [**7**, §4.3].

Here we shall be concerned with the general question of when $Q_F(A)$ is a simplex. A general criterion has been established in [**4**, **5**] for a closed face K of $Q(A)$ to be a simplex. This takes on several forms, the simplest of which is that distinct pure states in F are (unitarily) inequivalent. From this, it will be established in Theorem 4.1 that $Q_F(A)$ is a simplex if and only if the following three conditions are all satisfied:

(a) F is a simplex;

(b) any pure state in $Q_F(A)$ restricts to a pure state of B;

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(c) for any pure state ψ in F, the extensions of ψ in $Q(A)$ form a simplex.
(Here and throughout the paper, the zero functional is conventionally regarded as a "pure state", so the pure states are the extreme points of $Q(A)$.) Each of these conditions can be considered separately. For (a), this was done in [**4**, **5**]; we shall see in §2 that (b) places a strong restriction on the GNS-representation of pure states in $Q_F(A)$, and in §3 that (b) comes very close to implying (c) in the special case when $F = Q(B)$. (In this special case, (c) is the *simplex extension property* (SEP) introduced in [**6**].) The final two sections will be concerned with applications to C^*-dynamical systems. Taking $A = G \times_\alpha A_0$ and $B = G \times_\alpha B_0$ where B_0 is a G-invariant subalgebra of A_0, conditions will be obtained which ensure that the system (A_0, G, α) is abelian when it is known that (B_0, G, α) is abelian. Taking $A = G \times_\alpha A_0$ where A_0 is commutative and G is discrete, and $B = A_0$, it will be shown that A_0 has the (SEP) in $G \times_\alpha A_0$ if and only if the stabiliser in G of each (nonzero) pure state of A_0 is abelian. These results include as special cases some known properties of extensions of pure states of masas in $\mathcal{O}_n$ and $\mathcal{K} \otimes \mathcal{O}_n$ [**9**, **11**].

Throughout, A will be a C^*-algebra, and B will be a C^*-subalgebra of A. For a state ϕ of A with restriction ψ to B, $(\mathcal{H}_\phi, \pi_\phi, \xi_\phi)$ will be the associated cyclic representation of A, and $([\pi_\phi(B)\xi_\phi], \pi_\phi|_B, \xi_\phi)$ will be identified with $(\mathcal{H}_\psi, \pi_\psi, \xi_\psi)$. If ϕ lies in a closed face K of $Q(A)$, then p_ϕ^K will denote the orthogonal projection of $\mathcal{H}_\phi$ onto the closed subspace $\mathcal{H}_\phi^K$ of all vectors η for which the vector functional ω_ϕ^η: $a \to \langle \pi_\phi(a)\eta, \eta\rangle$ lies in the cone generated by K, and the dimension of $\mathcal{H}_\phi^K$ will be called the *K-multiplicity* of ϕ. Conventionally, the K-multiplicity of the zero functional is taken to be 1. It was shown in [**4**, **5**] that K is a simplex if and only if each pure state in K has K-multiplicity 1, or, in other words, no two pure states in K are equivalent. This fact will be used repeatedly without further reference.

For a pure state ψ of B, let $Q_\psi(A) = Q_F(A)$, where $F = \{\psi\}$, so

$$Q_\psi(A) = \{\phi \in Q(A) : \phi|_B = \psi\}.$$

Thus B has the (SEP) in A if and only if $Q_\psi(A)$ is a simplex for each pure state ψ of B, or equivalently no two distinct equivalent pure states of A have the same pure restriction to B.

In some respects, little would be lost in the following if it was assumed that A and B have a common unit, and that F is a closed (hence compact) face of the state space $S(A)$ of A. For one may otherwise adjoin a common unit obtaining C^*-algebras $\tilde{A}$ and $\tilde{B}$ and identify $Q(A)$ with $S(\tilde{A})$; nearly all the properties considered here are preserved under the passage between (A, B) and $(\tilde{A}, \tilde{B})$. However, at certain points it will be necessary to consider ideals and crossed products, so no assumption about the presence of a unit is appropriate.

2. Restriction property for faces. The first property of faces which we shall study is described in the following definition.

DEFINITION 2.1. A face K of $Q(A)$ has the *restriction property* (RP) to B if the restriction to B of each pure state in K is a pure state of B.

LEMMA 2.2. *Let K be a face of $Q(A)$ with the (RP) to B, and ϕ and ϕ' be equivalent pure states in K. Then the restrictions of ϕ and ϕ' to B are equivalent pure states.*

PROOF. Suppose that the restrictions ψ and ψ' of ϕ and ϕ' are inequivalent. By assumption there is a unit vector η in $\mathcal{H}_\phi^K$ such that $\phi' = \omega_\phi^\eta$, and the representation π_ϕ of B has inequivalent irreducible restrictions to $\mathcal{H}_\psi$ and $\mathcal{H}_{\psi'}$ $(= [\pi_\phi(B)\eta])$. Hence $\mathcal{H}_\psi$ and $\mathcal{H}_{\psi'}$ are orthogonal. If $\eta' = 2^{-1/2}(\xi_\phi + \eta) \in \mathcal{H}_\phi^K$, $\omega_\phi^{\eta'}$ is a pure state of A lying in K, so its restriction to B is pure. But $\omega_\phi^{\eta'}|_B = \frac{1}{2}(\psi + \psi')$, so $\psi = \psi'$. This is a contradiction.

THEOREM 2.3. *Let F be a closed face of $Q(B)$, $K = Q_F(A)$, ϕ a (nonzero) pure state in K, and $\psi = \phi|_B$, and suppose that K has the (RP) to B, so that ψ is a pure state of B.*

(i) *If the F-multiplicity of ψ is* 1, *then*

$$p_\phi^K \pi_\phi(b) p_\phi^K = \psi(b) p_\phi^K \qquad (b \in B).$$

(ii) *If the F-multiplicity of ψ is greater than* 1, *then $\mathcal{H}_\phi^K = \mathcal{H}_\psi^F$.*

PROOF. (i) Suppose that the F-multiplicity of ψ is 1, so that there are no other pure states in F equivalent to ψ. For any unit vector η in $\mathcal{H}_\phi^K$, the restriction ψ' of ω_ϕ^η to B is a pure state in F, and by Lemma 2.2, ψ' is equivalent to ψ, so $\psi' = \psi$. Thus

$$\langle \pi_\phi(b)\eta, \eta \rangle = \psi(b)\|\eta\|^2 \qquad (b \in B, \eta \in \mathcal{H}_\phi^K),$$

so $p_\phi^K \pi_\phi(b) p_\phi^K = \psi(b) p_\phi^K$.

(ii) Suppose that the F-multiplicity of ψ is greater than 1, and let ξ' be a unit vector in $\mathcal{H}_\psi^F$ orthogonal to ξ_ϕ, and η any unit vector in $\mathcal{H}_\phi^K$. The restriction ψ' of ω_ϕ^η to B is a pure state equivalent to ψ (Lemma 2.2). Thus there is a unitary operator u of $\mathcal{H}_\psi$ onto $\mathcal{H}_{\psi'}$ $(= [\pi_\phi(B)\eta])$ such that

$$u\pi_\phi(b)\xi = \pi_\phi(b)u\xi \qquad (b \in B, \xi \in \mathcal{H}_\psi).$$

Now

$$\langle \pi_\phi(b)u\xi', u\xi' \rangle = \langle u\pi_\phi(b)\xi', u\xi' \rangle = \langle \pi_\phi(b)\xi', \xi' \rangle$$

so $\omega_\phi^{u\xi'}|_B = \omega_\phi^{\xi'}|_B = \omega_\psi^{\xi'} \in F$, and therefore $u\xi' \in \mathcal{H}_\phi^K$. Let $\eta' = \xi_\phi - u\xi'$, and $\eta'' = \|\eta'\|^{-1}\eta'$ (note that $\eta' \neq 0$). Since $\eta'' \in \mathcal{H}_\phi^K$ and K has the (RP) to B, the restriction ψ'' of $\omega_\phi^{\eta''}$ to B is a pure state in F.

Since π_ψ is irreducible, Kadison's Transitivity Theorem shows that there is some b_0 in B with

$$\pi_\phi(b_0)\xi_\phi = \xi_\phi, \qquad \pi_\phi(b_0)\xi' = 0.$$

Then

$$\pi_\phi(b_0)\eta' = \pi_\phi(b_0)\xi_\phi - u\pi_\phi(b_0)\xi' = \xi_\phi.$$

Since the restriction to B of π_ϕ is irreducible on $\mathcal{H}_{\psi''} = [\pi_\phi(B)\eta']$,

$$\eta' \in \left[\pi_\phi(B)\xi_\phi\right] = \mathcal{H}_\psi.$$

Hence $u\xi' = \xi_\phi - \eta' \in \mathcal{H}_\psi$. Since the restriction to B of π_ϕ is irreducible on $\mathcal{H}_{\psi'}$, which contains $u\xi'$,

$$\eta \in \left[\pi_\phi(B)u\xi'\right] \subseteq \left[\pi_\phi(B)\mathcal{H}_\psi\right] = \mathcal{H}_\psi.$$

But $\omega_\psi^\eta = \omega_\phi^\eta|_B \in F$, so $\eta \in \mathcal{H}_\psi^F$. Thus $\mathcal{H}_\phi^K \subseteq \mathcal{H}_\psi^F$. The reverse inclusion is immediately verified.

There are various extreme cases of Theorem 2.3. §3 will be devoted to the case when $F = Q(B)$. In the opposite extreme when $F = \{\psi\}$ for a pure state ψ of B, $Q_F(A)$ automatically has the (RP) to B, and Theorem 2.3(i) is applicable. A slightly less special case occurs when F is generated by two equivalent pure states, and this leads to the following result.

COROLLARY 2.4. *Let ψ be a pure state of B, and suppose that $Q_\psi(A)$ is not a simplex. Let ψ' be a pure state of B equivalent, but not equal, to ψ. Then there is a pure state ϕ of A and a real number $\lambda > 0$ such that $\phi|_B$ is not pure, and $\phi(b) \leq \lambda(\psi(b) + \psi'(b))$ $(b \in B)$.*

PROOF. Let F be the smallest face of $Q(B)$ containing ψ and ψ', so that

$$K = Q_F(A) = \{\phi \in Q(A) : \phi|_B \leq \lambda(\psi + \psi') \text{ for some } \lambda > 0\}.$$

Suppose that the conclusion of the corollary is false, so that K has the (RP) to B. Since ψ and ψ' are distinct equivalent pure states in F, the F-multiplicity of ψ is greater than 1.

Since $Q_\psi(A)$ is not a simplex, there are distinct equivalent pure states ϕ and ϕ' in $Q_\psi(A)$. Let η be a unit vector in $\mathcal{H}_\phi$ with $\omega_\phi^\eta = \phi'$. Then $\omega_\phi^\eta|_B = \psi \in F$, so $\eta \in \mathcal{H}_\phi^K = \mathcal{H}_\psi^F$ (Theorem 2.3(ii)). Now $\omega_\psi^\eta = \psi$, and π_ψ is irreducible, so η is a scalar multiple of $\xi_\psi = \xi_\phi$. But this contradicts the fact that ϕ and ϕ' are distinct.

3. The restriction property for algebras.

DEFINITION 3.1. A C^*-subalgebra B of A has the *restriction property* (RP) in A if the restriction to B of each pure state of A is pure on B.

Thus B has the (RP) in A if and only if $Q(A)$ has the (RP) to B. Any (closed two-sided) ideal has the (RP) in A; an abelian C^*-subalgebra has the (RP) in A if and only if it is contained in the centre of A; a C^*-subalgebra B has the (RP) in A if A coincides with the C^*-subalgebra $(B : A)$ generated by operators of the form zb, where $b \in B$ and z is a central multiplier of A.

The next two results follow from Theorem 2.3 and Corollary 2.4 on taking $F = Q(B)$, so that $K = Q(A)$, $\mathcal{H}_\psi^F = \mathcal{H}_\psi$, $\mathcal{H}_\phi^K = \mathcal{H}_\phi$, and the F-multiplicity of ψ is 1 if and only if ψ is multiplicative. The proof of Theorem 2.3 can be made very short in this case.

PROPOSITION 3.2. *Suppose B has the (RP) in A, and let ψ be a pure state of B and ϕ a pure state of A extending ψ.*

(i) *If ψ is multiplicative, then $\pi_\phi(b) = \psi(b)\mathbf{1}$ $(b \in B)$.*

(ii) *If ψ is not multiplicative, then $\mathcal{H}_\phi = \mathcal{H}_\psi$.*

COROLLARY 3.3. *Suppose B has the (RP) in A, and ψ is a nonmultiplicative pure state of B. Then $Q_\psi(A)$ is a simplex.*

It is to be expected that the (RP) will be related to properties of restrictions of irreducible representations. The precise extent of this connection will now be discussed.

DEFINITION 3.4. A C^*-subalgebra B of A has the *irreducible representation property* (IRP) in A if, for each irreducible representation π of A, $\pi(B)$ is either irreducible or zero.

It is immediate from Definition 3.4 that a C^*-subalgebra which is rich (in the sense of [**10**, §11.1.1]) has the (IRP); any ideal has the (IRP) in A; if $A = (B : A)$, then B has the (IRP) in A. If Ω is a locally compact Hausdorff space, $\{A_\omega : \omega \in \Omega\}$ is a family of C^*-algebras, Γ_1 and Γ_2 are continuous vector fields over this family with $\Gamma_2 \subseteq \Gamma_1$, and A and B are the corresponding C^*-algebras, then B has the (IRP) in A [**10**, Théorème 10.4.3].

PROPOSITION 3.5. *Let J_0 be the ideal in A generated by the commutators $\{ab - ba: a \in A, b \in B\}$. The following are equivalent*:

(i) *B has the (RP) in A*;

(ii) *$B \cap J_0$ has the (IRP) in J_0*;

(iii) *there is an ideal J in A such that $(B + J)/J$ is contained in the centre of A/J, and $B \cap J$ has the (IRP) in J.*

PROOF. (i) $\Rightarrow$ (ii) Let $(\mathcal{H}, \pi)$ be an irreducible representation of J_0, $(\mathcal{H}, \tilde{\pi})$ its unique extension to A, ξ a unit vector in $\mathcal{H}'$ and ϕ the vector state: $\phi(a) = \langle \pi(a)\xi, \xi\rangle$. Since π does not vanish on J_0, it follows from Proposition 3.2 that ϕ is not multiplicative on B, and hence that $\mathcal{H} = [\tilde{\pi}(B)\xi]$. Thus $(\mathcal{H}, \tilde{\pi}|_B)$ is the GNS-representation of the pure state $\phi|_B$, and is therefore irreducible. Hence $\pi(B \cap J_0)$ is zero or irreducible.

(ii) $\Rightarrow$ (iii) This is immediate on taking $J = J_0$.

(iii) $\Rightarrow$ (i) Let ϕ be a pure state of A. If ϕ vanishes on J, then ϕ induces a pure state $\tilde{\phi}$ of A/J, which is therefore multiplicative on $(B + J)/J$. Hence ϕ is multiplicative and therefore pure on B.

If ϕ does not vanish on J, then $\pi_\phi(J)$ is irreducible on $\mathcal{H}_\phi$. By assumption, $\pi_\phi(B \cap J)$ is either zero or irreducible. If $\pi_\phi(B \cap J)$ is irreducible, then $\phi|_B$ is the unique extension of the pure state $\phi|_{B \cap J}$ to a state of B, so ϕ is pure on B.

If ϕ vanishes on $B \cap J$, then ϕ induces a pure state of $B/(B \cap J)$ which is isomorphic to $(B + J)/J$, so by the Hahn-Banach theorem, there is a pure state ϕ' of A coinciding with ϕ on B and vanishing on J. As in the first part of this proof, $\phi'|_B$ is pure, so $\phi|_B$ is pure.

COROLLARY 3.6. *If B has no multiplicative states, or if A is simple and B is nontrivial, then the (RP) and the (IRP) are equivalent.*

If A is type I and B has the (IRP) in A, then there is a composition series of ideals $(I_\rho)_{0 \leq \rho \leq \rho_0}$ for A such that for each ordinal $\rho < \rho_0$, either $(B \cap I_{\rho+1} + I_\rho)/I_\rho$ is contained in the centre of $I_{\rho+1}/I_\rho$, or $((B \cap I_{\rho+1} + I_\rho)/I_\rho: I_{\rho+1}/I_\rho) = I_{\rho+1}/I_\rho$. However, the converse of this is not valid. For example, let $\mathcal{K}$ be the C^*-algebra of compact operators on some Hilbert space $\mathcal{H}$, B a (type I) C^*-algebra of operators on $\mathcal{H}$ with $B \cap \mathcal{K} = \{0\}$, $A = B + \mathcal{K}$, $I_0 = (0)$, $I_1 = \mathcal{K}$, $I_2 = A$. The identity representation is irreducible on A but not on B.

4. $Q_F(A)$ as a simplex. It has already been seen in Corollaries 2.4 and 3.3. that the (RP) is related to whether extension faces $Q_\psi(A)$ are simplexes. The following result strengthens this point.

THEOREM 4.1. *Let F be a closed face of $Q(B)$. Then $Q_F(A)$ is a simplex if and only if the following three conditions are all satisfied*:

(a) *F is a simplex*;

(b) *$Q_F(A)$ has the (RP) to B*;

(c) *for each pure state ψ of B in F, $Q_\psi(A)$ is a simplex.*

PROOF. Suppose first that $Q_F(A)$ is a simplex. For any pure state ψ in F, $Q_\psi(A)$ is a face of $Q_F(A)$ and is therefore a simplex.

Let ψ and ψ' be equivalent pure states in F, and ϕ be any pure state in $Q_\psi(A)$. There is a vector η in $\mathcal{H}_\psi$ ($\subseteq \mathcal{H}_\phi$) such that $\psi' = \omega_\psi^\eta = \omega_\phi^\eta|_B$. Thus ω_ϕ^η belongs to the simplex $Q_F(A)$, so $\phi = \omega_\phi^\eta$ and $\psi = \psi'$. Hence F is a simplex.

Let ϕ be a pure state in $Q_F(A)$, $\psi = \phi|_B$, and suppose that $\psi = \frac{1}{2}(\psi_1 + \psi_2)$ for some ψ_1 and ψ_2 in F. Since $\psi_j \leq 2\psi$ ($j = 1, 2$), there are vectors η_j in $\mathcal{H}_\psi$ ($\subseteq \mathcal{H}_\phi$) such that $\psi_j = \omega_\psi^{\eta_j}$. Let $\phi_j = \omega_\phi^{\eta_j}$, so that $\phi_j|_B = \psi_j$. Then ϕ_1 and ϕ_2 are equivalent pure states of A belonging to the simplex $Q_F(A)$. Hence $\phi_1 = \phi_2$, so $\psi_1 = \psi_2$. Thus ψ is pure.

Conversely, suppose that conditions (a)–(c) are satisfied. Let ϕ and ϕ' be equivalent pure states in $Q_F(A)$. It follows from (b) and Lemma 2.2 that $\phi|_B$ and $\phi'|_B$ are equivalent pure states of B. It now follows from (a) that $\phi|_B = \phi'|_B$, so ϕ and ϕ' are equivalent pure states in $Q_\psi(A)$ for some pure state ψ in F. Finally, condition (c) shows that $\phi = \phi'$, so $Q_F(A)$ is a simplex.

COROLLARY 4.2. *Suppose B has the (RP) in A, and F is a closed face of $Q(B)$ containing no multiplicative functionals (in particular, $0 \notin F$). Then $Q_F(A)$ is a simplex if and only if F is a simplex.*

PROOF. This is immediate from Theorem 4.1 and Corollary 3.3.

5. C^*-dynamical systems. Let $G \times_\alpha A$ be the C^*-crossed product of a C^*-dynamical system (A, G, α); let u: $G \to M(G \times_\alpha A)$ be the universal representation of G in the multiplier algebra of $G \times_\alpha A$, and regard A as embedded as a C^*-subalgebra of $M(G \times_\alpha A)$. Let

$$F_G(A) = \{\phi \in Q(G \times_\alpha A) \colon \phi(u_g) = \|\phi\| (g \in G)\}.$$

Then $F_G(A)$ is a closed face of $Q(G \times_\alpha A)$, and the restriction map is an affine homeomorphism of $F_G(A)$ onto the set $Q^G(A)$ of G-invariant functionals in $Q(A)$ [**4**, Theorem 4.2]. It will be convenient to refer to the extreme points of $Q^G(A)$ (including 0) as "G-ergodic states". The system (A, G, α) is said to be *abelian* if $Q^G(A)$ is a simplex. (See [**4**, **5**] for equivalent definitions.)

THEOREM 5.1. *Let B be a G-invariant C^*-subalgebra of A, and suppose that the following three conditions are all satisfied*:

(a) (B, G, α) *is abelian*;

(b) *every G-ergodic state of A restricts to a G-ergodic state of B*,

(c) *For any G-ergodic state ψ of B, the set ψ of B, the set $Q^G_\psi(A) = \{\phi \in Q^G(A)\colon \phi|_B = \psi\}$ is a simplex (possibly empty).*

Then (A, G, α) is abelian.

Conversely, if (A, G, α) is abelian, then (b) *and* (c) *hold. If G is amenable and (A, G, α) is abelian, then* (a) *also holds.*

PROOF. Let $A_1 = G \times_\alpha A$, $\Phi\colon G \times_\alpha B \to A_1$ be the canonical $*$-homomorphism acting as the identity on B and on u_G, $B_1 = \Phi(G \times_\alpha B)$, and

$$F = \{\phi|_{B_1}\colon \phi \in F_G(A)\}, \qquad F' = \{\phi \circ \Phi\colon \phi \in F_G(A)\}.$$

Then F and F' are affinely homeomorphic closed faces of $S(B_1)$ and $F_G(B)$, respectively, so (a) implies that F is a simplex. Since $Q_F(A_1) = F_G(A)$, (b) and (c) reduce to the corresponding conditions of Theorem 4.1 for the C^*-algebra A_1 and subalgebra B_1. If G is amenable, then Φ is isometric, and $F' = F_G(B)$, so (a) is equivalent to F being a simplex. Thus the results follow from Theorem 4.1.

COROLLARY 5.2. *Let B be a G-invariant C^*-subalgebra of A, and suppose that $\alpha(G)$ includes all the inner automorphisms of A implemented by unitaries in $\tilde{B}$. Then (A, G, α) is abelian if and only if, for any G-ergodic state ψ of B, the set $Q^G_\psi(A)$ is a simplex.*

PROOF. Any C^*-dynamical system containing all the inner automorphisms is abelian, so Theorem 5.1(a) is satisfied. It has been shown in [**12**, Lemma 3] that (b) is satisfied.

Recall that a C^*-subalgebra B has the *simplex extension property* (SEP) in A if $Q_\psi(A)$ is a simplex for each pure state of B [**6**]; a functional ϕ in A^* is *B-central* if $\phi(ab) = \phi(ba)$ $(a \in A,\ b \in B)$. The relationship between B-central states and extensions has been studied in [**3**] in the case when B is a masa in A. It was shown there that an abelian C^*-subalgebra B has the (SEP) in a type I C^*-algebra A if $\pi(B)$ is a masa of $\pi(A)$ for each irreducible representation π of A, and that under these circumstances the B-central functionals form a simplex. This latter fact is a special case of the following result.

COROLLARY 5.3. *Let B be an abelian C^*-subalgebra of A. Then B has the (SEP) in A if and only if the B-central functionals in $Q(A)$ form a simplex.*

PROOF. Let G be the unitary group of $\tilde{B}$ acting as inner automorphisms of A. Then $Q^G(A)$ is the set of B-central functionals in $Q(A)$, and the G-ergodic states are precisely those pure states of A which are multiplicative on B [**3**, **12**]. The result now follows immediately from Corollary 5.2.

COROLLARY 5.4. *Let (A, G, α) be an abelian C^*-dynamical system, where G is abelian, and let $(G \times_\alpha A, G, \tilde{\alpha})$ be the system defined by*

$$\tilde{\alpha}_g(x) = u_g x u_g^* \qquad (x \in G \times_\alpha A, g \in G).$$

Then $(G \times_\alpha A, G, \tilde{\alpha})$ is abelian.

PROOF. Let B be the C^*-subalgebra of $G \times_\alpha A$ generated by u_G, so B is (isomorphic to) the group C^*-algebra of G, whose pure state space is the dual group $\hat{G}$. The G-invariant functionals on $G \times_\alpha A$ are precisely those which are B-central, so (a slight extension of) Corollary 5.3 shows that it is sufficient to prove that $Q_\gamma(G \times_\alpha A)$ is a simplex for each γ in $\hat{G}$. Let $(G \times_\alpha A, \hat{G}, \hat{\alpha})$ be the dual system of (A, G, α) [**13**, §7.9]. Then $\phi \to \phi \circ \hat{\alpha}_\gamma$ is an affine homeomorphism of $Q_\gamma(G \times_\alpha A)$ onto $F_G(A)$, which, by assumption, is a simplex.

Corollary 5.4 may fail if G is not abelian. For example, one may take G to be the alternating group on 7 letters, A to be the group C^*-algebra of G, and α to be the action by conjugation.

6. Topological dynamical systems. In this final section, let (C, G, α) be a C^*-dynamical system, where C is abelian and G is discrete. (Some of the discussion can be modified for locally compact groups.) Let Ω be the pure state space of C, so that $C \cong C_0(\Omega)$, and there is an action of G on Ω such that $\omega(\alpha_g(x)) = (g^{-1} \cdot \omega)(x)$ $(x \in C,\ \omega \in \Omega,\ g \in G)$. The alternative notation $C^*(\Omega, G)$ will be used for the C^*-crossed product $G \times_\alpha C$. Now C is a subalgebra of $C^*(\Omega, G)$, and Proposition 6.1 will identify the faces $Q_\omega(C^*(\Omega, G))$. Let P: $C^*(\Omega, G) \to C$ be the canonical projection, so that

$$\begin{aligned} P(xu_g) &= 0 \qquad (x \in C, g \in G, g \neq e), \\ P(x) &= x \qquad (x \in C). \end{aligned}$$

States ϕ of $C^*(\Omega, G)$ will be identified with normalised positive-definite functions Φ: $G \to C^*$ given by

$$\Phi(g)(x) = \phi(xu_g).$$

In particular, there are no states of $C^*(\Omega, G)$ which vanish on C.

PROPOSITION 6.1. *Let ω be a point of Ω. Then $Q_\omega(C^*(\Omega, G))$ is affinely homeomorphic to the state space of the group C^*-algebra $C^*(G_\omega)$ of the stabilizer G_ω of ω in G. Furthermore $\omega \circ P$ is pure if and only if G_ω is trivial.*

PROOF. Let Ψ: $G \to \mathbf{C}$ be a normalised positive-definite function, and define Φ: $G \to C^*$ by

$$\begin{aligned} \Phi(g) &= \Psi(g)\omega \quad (g \in G_\omega), \\ &= 0 \qquad\quad (g \in G \setminus G_\omega). \end{aligned}$$

For g_i in G and x_i in C $(i = 1, \dots, n)$,

$$\sum_{i,j=1}^{n} \Phi\left(g_i^{-1}g_j\right)\left(\alpha_{g_i^{-1}}(x_i^* x_j)\right) = \sum_{g_i^{-1}g_j \in G_\omega} \overline{(g_i \cdot \omega)(x_i)}\,(g_i \cdot \omega)(x_j)\Psi\left(g_i^{-1}g_j\right)$$

$$= \sum_{\omega' \in \Omega} \sum_{i,j=1}^{n} \overline{\lambda_i(\omega')}\,\lambda_j(\omega')\Psi\left(g_i^{-1}g_j\right) \geqslant 0,$$

where $\lambda_i(g_i \cdot \omega) = (g_i \cdot \omega)(x_i)$, $\lambda_i(\omega') = 0$ $(\omega' \neq g_i \cdot \omega)$. Thus Φ is positive-definite. Furthermore the corresponding state ϕ of $C^*(\Omega, G)$ satisfies

$$\phi(x) = \Phi(e)(x) = \omega(x) \qquad (x \in C)$$

so ϕ belongs to $Q_\omega(C^*(\Omega, G))$.

Conversely, for ϕ in $Q_\omega(C^*(\Omega, G))$, x in C and g in G,

$$\omega(x)\phi(u_g) = \phi(xu_g) = \overline{\phi\left(u_g^* x^*\right)} = \overline{\phi\left(\alpha_{g^{-1}}(x^*)u_g^*\right)} = (g \cdot \omega)(x)\phi(u_g).$$

Thus $\phi(u_g) = 0 = \phi(xu_g)$ $(g \in G \setminus G_\omega)$ and $\phi(xu_g) = \omega(x)\phi(u_g)$ $(g \in G_\omega)$, so ϕ is of the above form. It is clear that $\Psi \to \Phi$ is an affine homeomorphism.

If G_ω is trivial, then $\omega \circ P$ is the unique state extension of the pure state ω to $C^*(\Omega, G)$, and is therefore pure.

In general, the GNS-representation of $\omega \circ P$ may be identified with the induced representation $\pi \times \lambda$ of $C^*(\Omega, G)$ on $l^2(G)$, where λ is the left regular representation of G, and

$$(\pi(x)\xi)(h) = (h \cdot \omega)(x)\xi(h) \qquad \left(x \in C, \xi \in l^2(G), h \in G\right).$$

Let ρ be the right regular representation of G on $l^2(G)$, so that $\rho_G \subseteq \lambda'_G$. For g in G_ω,

$$\left(\pi(x)\rho_g\xi\right)(h) = (h \cdot \omega)(x)\xi(hg) = (hg \cdot \omega)(x)\xi(hg) = \left(\rho_g\pi(x)\xi\right)(h)$$

so $\rho_g \in (\pi \times \lambda)(C^*(\Omega, G))'$. Thus, if $\omega \circ P$ is pure, ρ_g is a scalar, so $g = e$.

A C^*-subalgebra B of a C^*-algebra A is said to have the *extension property* (EP) in A if $Q_\psi(A)$ contains a unique functional for each pure state ψ of B.

COROLLARY 6.2. *Let G be a discrete group acting on a locally compact Hausdorff space Ω, and let $C = C_0(\Omega)$. The abelian C^*-subalgebra C has the (EP) in $C^*(\Omega, G)$ if and only if G acts freely on Ω; C has the (SEP) in $C^*(\Omega, G)$ if and only if the stabiliser of each point in Ω is abelian.*

Let $n \geqslant 2$ be a fixed integer, $\mathbf{Z}_n$ be the group of integers mod n, $\Omega_- = \bigoplus_{i=-\infty}^{-1} \mathbf{Z}_n$ (in the discrete topology), $\Omega_+ = \prod_{i=0}^{\infty} \mathbf{Z}_n$ (in the product topology), $\Omega = \Omega_- \times \Omega_+$ and $C = C_0(\Omega)$. Let $G_0 = \bigoplus_{i=-\infty}^{\infty} \mathbf{Z}_n$, and G be the semidirect product $G_0 \times_\lambda \mathbf{Z}$ of G_0 by the shift λ to the right. The discrete group G acts on Ω by

$$((r_i), m) \cdot (s_i) = (r_i + s_{i-m}) \qquad \left((r_i) \in G_0, m \in \mathbf{Z}, (s_i) \in \Omega\right).$$

For $\omega = (\omega_-, \omega_+) \in \Omega$, G_ω is either trivial or isomorphic to $\mathbf{Z}$, so by Proposition 6.1, $Q_\omega(C^*(\Omega, G))$ either contains $\omega \circ P$ only, or is isomorphic to the space of Radon probability measures on the unit circle, the latter case occurring when the sequence

$\omega_+ = (r_i)_{i\geq 0}$ eventually becomes periodic. If $\omega_- = 0$, then $\omega \circ P$ is the state $\phi_{[e_{r_i}]}$ considered in [**11**, Theorem 3.4], which result is therefore a special case of Proposition 6.1.

It was shown in [**11**, Proposition 3.3] (see also [**5**, §2.1]) that $C^*(\Omega, G)$ is isomorphic to the unique C^*-tensor product $\mathcal{K} \otimes \mathcal{O}_n$ of the algebra $\mathcal{K}$ of compact operators on a separable Hilbert space and the Cuntz algebra $\mathcal{O}_n$ generated by isometries $S_1, \ldots, S_n$ satisfying

$$S_i^* S_i = \mathbf{1} = \sum_{j=1}^{n} S_j S_j^*.$$

Let $\mathfrak{D}$ be the masa of all operators in $\mathcal{K}$ which are diagonal with respect to some basis of $\mathcal{K}$, q a minimal projection in $\mathfrak{D}$, p in C the characteristic function of the subset $\{0\} \times \Omega_+$ of Ω, and $\mathfrak{D}_n$ the masa in $\mathcal{O}_n$ generated by words of the form $S_{i_1} \cdots S_{i_k} S_{i_k}^* \cdots S_{i_1}^*$. The isomorphism may be chosen so that C corresponds to $\mathfrak{D} \otimes \mathfrak{D}_n$, and p to $q \otimes \mathbf{1}$. Thus there is an induced isomorphism between $pC^*(\Omega, G)p$ and $\mathcal{O}_n$ taking $pC = C(\Omega_+)$ onto $\mathfrak{D}_n$. If $\omega_- = 0$, the description of $Q_\omega(C^*(\Omega, G)) = Q_{\omega_+}(pC^*(\Omega, G)p)$ obtained above reduces to the result of [**9**, Proposition 3.1].

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