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METRICALLY COMPLETE REGULAR RINGS

BY

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ABSTRACT. This paper is concerned with the structure of those (von Neumann) regular rings R which are complete with respect to the weakest metric derived from the pseudo-rank functions on R, known as the N^*-metric. It is proved that this class of regular rings includes all regular rings with bounded index of nilpotence, and all $\aleph_0$-continuous regular rings. The major tool of the investigation is the partially ordered Grothendieck group $K_0(R)$, which is proved to be an archimedean norm-complete interpolation group. Such a group has a precise representation as affine continuous functions on a Choquet simplex, from earlier work of the author and D. E. Handelman, and additional aspects of its structure are derived here. These results are then translated into ring-theoretic results about the structure of R. For instance, it is proved that the simple homomorphic images of R are right and left self-injective rings, and R is a subdirect product of these simple self-injective rings. Also, the isomorphism classes of the finitely generated projective R-modules are determined by the isomorphism classes modulo the maximal two-sided ideals of R. As another example of the results derived, it is proved that if all simple artinian homomorphic images of R are $n \times n$ matrix rings (for some fixed positive integer n), then R is an $n \times n$ matrix ring.

All rings in this paper are associative with 1, and all modules are unital right modules. For the overall theory of regular rings, we refer the reader to [**2**]; for the general development of K_0 of regular rings as partially ordered abelian groups, and the theory of partially ordered abelian groups via their state spaces, we refer the reader to [**2**, **4**]. In particular, these references should be consulted for more detail on definitions and concepts which are just sketched here.

I. N^*-completeness. Completeness of a regular ring with respect to a rank function, or with respect to a family of pseudo-rank functions, implies that the ring is right and left self-injective [**2**, Theorems 19.7 and 20.8], hence a considerable amount of structure theory is available for such rings [**2**, Chapters 9–12]. The purpose of this paper is to investigate a much broader class of regular rings, namely those which are complete with respect to the (pseudo-) metric obtained from the supremum N^* of all pseudo-rank functions on the ring. In particular, all regular rings complete with respect to a family of pseudo-rank functions are N^*-complete, but also, as we prove later in this section, all regular rings with bounded index of nilpotence and all $\aleph_0$-continuous regular rings are N^*-complete.

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In addition to these results, the present section introduces the definition and a few basic properties of N^*-completeness. §II is devoted to proving that K_0 of any N^*-complete regular ring is an archimedean norm-complete interpolation group, while the third section develops a number of structural properties of such groups. In §IV, we apply these results to the structure theory of N^*-complete regular rings.

DEFINITION. Recall that a *pseudo-rank function* on a regular ring R is a map $P\colon R \to [0, 1]$ such that (a) $P(1) = 1$; (b) $P(xy) \leqslant P(x)$, $P(y)$ for all $x, y \in R$; (c) $P(e+f) = P(e) + P(f)$ for all orthogonal idempotents $e, f \in R$. We use $\mathbf{P}(R)$ to denote the set of all pseudo-rank functions on R. Considered as a subset of the linear topological space $\mathbf{R}^R$ (which is given the product topology), $\mathbf{P}(R)$ is a compact convex set [**2**, Proposition 16.17]. In fact, $\mathbf{P}(R)$ is a Choquet simplex [**2**, Theorem 17.5].

DEFINITION. Let R be a regular ring. For each $x \in R$, we define

$$N^*(x) = \sup\{P(x) \mid P \in \mathbf{P}(R)\},$$

with the proviso that $N^*(x) = 0$ in case $\mathbf{P}(R)$ is empty. (This definition is formally different from the definition of N^* in [**2**, p. 272], but the two definitions are equivalent, as follows from [**2**, Proposition 18.10].) Thus $N^*(x)$ is a real number, and $0 \leqslant N^*(x) \leqslant 1$. Whenever the ring R needs to be emphasized, we shall write $N_R^*(x)$ in place of $N^*(x)$. In case R is unit-regular, N^* may be computed as in the following proposition. We first recall two pieces of notation.

DEFINITION. Given modules A and B, we write $A \lesssim B$ to mean that A is isomorphic to a submodule of B. Given a module A and a positive integer n, we write nA to denote the direct sum of n copies of A.

PROPOSITION 1.1. *If R is a nonzero unit-regular ring, then*

$$N^*(x) = \inf\{k/n \mid k, n \in \mathbf{N} \text{ and } n(xR) \lesssim kR_R\}$$

for all $x \in R$.

PROOF. [**2**, Proposition 18.10]. □

LEMMA 1.2. *Let R be a regular ring, and let $x, y, y_1, \ldots, y_n \in R$.*

(a) *If $t(xR) \lesssim s_1(y_1R) \oplus \cdots \oplus s_n(y_nR)$ for some positive integers $t, s_1, \ldots, s_n$, then*

$$N^*(x) \leqslant (s_1/t)N^*(y_1) + \cdots + (s_n/t)N^*(y_n).$$

(b) *If $t(xR) \cong s(yR)$ for some positive integers s, t, then $N^*(x) = (s/t)N^*(y)$.*

(c) *$N^*(xy) \leqslant N^*(x)$ and $N^*(xy) \leqslant N^*(y)$.*

(d) *$N^*(x+y) \leqslant N^*(x) + N^*(y)$.*

PROOF. These results are clear if $\mathbf{P}(R)$ is empty, so assume it is nonempty.

(a) For any $P \in \mathbf{P}(R)$, we have

$$tP(x) \leqslant s_1P(y_1) + \cdots + s_nP(y_n)$$

by [**2**, Proposition 16.1], whence

$$\begin{aligned} P(x) &\leqslant (s_1/t)P(y_1) + \cdots + (s_n/t)P(y_n) \\ &\leqslant (s_1/t)N^*(y_1) + \cdots + (s_n/t)N^*(y_n). \end{aligned}$$

Consequently, $N^*(x) \leqslant (s_1/t)N^*(y_1) + \cdots + (s_n/t)N^*(y_n)$.

(b) This follows directly from (a).

(c) For all $P \in \mathbf{P}(R)$, we have $P(xy) \leqslant P(x) \leqslant N^*(x)$, hence $N^*(xy) \leqslant N^*(x)$. Likewise, $N^*(xy) \leqslant N^*(y)$.

(d) Since $(x+y)R \leqslant xR + yR \lesssim xR \oplus yR$, we may apply (a). $\square$

DEFINITION. Let R be a regular ring. In view of Lemma 1.2, we see that the rule $\delta(x, y) = N^*(x - y)$ defines a pseudo-metric δ on R. By way of abbreviation, we shall refer to δ as *the N^*-metric on R*, even though δ is not always a metric. Note that δ is a metric if and only if $\ker(\mathbf{P}(R)) = 0$. For all $x, y, z, w \in R$, we see that

$$N^*((x+y)-(z+w)) = N^*((x-z)+(y-w)) \leqslant N^*(x-z) + N^*(y-w),$$
$$N^*(xy - zw) = N^*((x-z)y + z(y-w)) \leqslant N^*(x-z) + N^*(y-w).$$

Thus addition and multiplication in R are uniformly continuous with respect to N^*. For all $x, y \in R$, we also have

$$N^*(x) \leqslant N^*(x-y) + N^*(y),$$
$$N^*(y) \leqslant N^*(y-x) + N^*(x) = N^*(x-y) + N^*(y),$$

whence $|N^*(x) - N^*(y)| \leqslant N^*(x-y)$. Thus the map $N^*: R \to [0, 1]$ is uniformly continuous with respect to the N^*-metric.

DEFINITION. A regular ring R is called *N^*-complete* provided $\ker(\mathbf{P}(R)) = 0$ (so that the N^*-metric on R is actually a metric) and R is complete in the N^*-metric. For example, if R is a simple artinian ring, then there is a unique rank function P on R, which takes on only the values $0, 1/k, 2/k, \ldots, 1$, for some $k \in \mathbf{N}$ [**2**, Corollary 16.6]. As a result, $N^* = P$, and $N^*(x) \geqslant 1/k$ for all nonzero $x \in R$, so that the N^*-metric on R is discrete. Therefore R is N^*-complete. More generally, we have the following result.

THEOREM 1.3. *A regular ring R has bounded index of nilpotence if and only if* $\ker(\mathbf{P}(R)) = 0$ *and the N^*-metric on R is discrete, in which case R is N^*-complete.*

PROOF. First assume that $\ker(\mathbf{P}(R)) = 0$ and the N^*-metric on R is discrete. Then there exists $n \in \mathbf{N}$ such that $N^*(x) \geqslant 1/n$ for any nonzero $x \in R$, and we claim that the index of nilpotence of R is at most n. If not, R must contain a direct sum of $n+1$ nonzero pairwise isomorphic right ideals [**2**, Theorem 7.2]. Consequently, there is a nonzero element $y \in R$ satisfying $(n+1)(yR) \lesssim R_R$. But then $N^*(y) > 0$ (because $\ker(\mathbf{P}(R)) = 0$) and $N^*(y) \leqslant 1/(n+1)$ (by Lemma 1.2), contradicting our discreteness assumption. Therefore the index of nilpotence of R is at most n, as claimed.

Conversely, assume that R has bounded index of nilpotence. We first note that any maximal two-sided ideal M of R is the kernel of a pseudo-rank function on R. Namely, R/M is a simple artinian ring [**2**, Theorem 7.9], so there exists a unique rank function on R/M, which pulls back to a pseudo-rank function on R with kernel M.

As all primitive factor rings of R are artinian [**2**, Corollary 7.10], the intersection of the maximal two-sided ideals of R is zero. Since each maximal two-sided ideal is the kernel of a pseudo-rank function, we obtain $\ker(\mathbf{P}(R)) = 0$.

Let n be the index of nilpotence of R. We claim that $N^*(x) \geqslant 1/n$ for any nonzero element $x \in R$.

Choose a maximal two-sided ideal M of R such that $x \notin M$. By [**2**, Theorem 7.9], $R/M \cong M_k(D)$ for some positive integer $k \leqslant n$ and some division ring D. Then R/M has a unique rank function Q, which takes on only the values $0, 1/k, 2/k, \ldots, 1$, and $Q(x + M) \geqslant 1/k \geqslant 1/n$. Pulling Q back to a pseudo-rank function P on R, we obtain $P(x) \geqslant 1/n$, and so $N^*(x) \geqslant 1/n$, as claimed.

Therefore the N^*-metric on R is discrete, and, consequently, R is N^*-complete. $\square$

For a deeper class of examples, we now proceed to prove that every $\aleph_0$-continuous regular ring is N^*-complete. In particular, it will follow that every regular, right and left self-injective ring is N^*-complete. Recall that a regular ring R is defined to be $\aleph_0$-continuous provided the lattice of principal right ideals of R is an $\aleph_0$-continuous geometry. Equivalently, R is $\aleph_0$-continuous if and only if every countably generated right (left) ideal of R is essential in a principal right (left) ideal [**2**, Corollary 14.4].

We begin with a lemma generalizing [**2**, Corollary 14.27(a)] to finitely generated projective modules. This lemma is also implicit in [**8**, Proposition 2.1].

LEMMA 1.4. *Let R be an $\aleph_0$-continuous regular ring, let A and B be finitely generated projective right R-modules, and let $A_1 \leqslant A_2 \leqslant \ldots$ be an ascending sequence of finitely generated submodules of A. If $\bigcup A_n$ is essential in A, and each $A_n \lesssim B$, then $A \lesssim B$.*

PROOF. Let S be the maximal right $\aleph_0$-quotient ring of R [**2**, pp. 177, 178], so that S is an $\aleph_0$-continuous, regular, right and left $\aleph_0$-injective overring of R [**2**, Theorems 14.12 and 14.17]. Now $A \otimes_R S$ and $B \otimes_R S$ are finitely generated projective right S-modules, $A \otimes_R S$ has an ascending sequence of finitely generated submodules which may be labelled $A_1 \otimes_R S \leqslant A_2 \otimes_R S \leqslant \ldots$, the submodule $\bigcup (A_n \otimes_R S)$ is essential in $A \otimes_R S$, and each $A_n \otimes_R S \lesssim B \otimes_R S$. Moreover, if $A \otimes_R S \lesssim B \otimes_R S$, then [**2**, Proposition 14.28] shows that $A \lesssim B$.

Thus there is no loss of generality in assuming that R is right and left $\aleph_0$-injective. Consequently, all matrix rings over R are $\aleph_0$-continuous [**2**, Proposition 14.19]. Using the standard Morita-equivalences, we may transfer our problem to the category of right modules over any matrix ring $M_k(R)$. By choosing k large enough we may arrange for the new modules corresponding to A and B to be cyclic, hence isomorphic to right ideals of $M_k(R)$.

Therefore we may now assume, without loss of generality, that A and B are principal right ideals of R. Since $\bigcup A_n$ is essential in A, we see that A is the supremum, in the lattice of principal right ideals of R, of the family $\{A_n\}$. Consequently, [**2**, Corollary 14.27] shows that $A \lesssim B$. $\square$

LEMMA 1.5. *Let R be an $\aleph_0$-continuous regular ring, and let $x, x_1, x_2, \ldots$ be elements of R. If the right ideal $\sum x_n R$ is essential in xR, then $N^*(x) \leqslant \sum N^*(x_n)$.*

PROOF. Choose elements $y_1, y_2, \ldots \in R$ such that $y_n R = x_1 R + \cdots + x_n R$ for all n, and note from Lemma 1.2 that $N^*(y_n) \leqslant N^*(x_1) + \cdots + N^*(x_n)$. Thus it suffices to show that $N^*(x) \leqslant \sup\{N^*(y_n)\}$. If not,

$$N^*(x) > s/t > \sup\{N^*(y_n)\}$$

for some $s, t \in \mathbf{N}$.

For each n, we have $P(y_n) \leqslant N^*(y_n) < s/t$ and so $tP(y_n) < sP(1)$, for all $P \in \mathbf{P}(R)$. As a result, [**2**, Theorem 18.28] implies that $t(y_nR) \lesssim sR_R$. Inside the projective module $t(xR)$, we have finitely generated submodules $t(y_1R) \leqslant t(y_2R) \leqslant \ldots$, and $\bigcup(t(y_nR))$ is an essential submodule of $t(xR)$. Consequently, Lemma 1.4 says that $t(xR) \lesssim sR_R$. But then $N^*(x) \leqslant s/t$ (by Lemma 1.2), which is false.

Therefore $N^*(x) \leqslant \sup\{N^*(y_n)\}$. $\square$

The key to the upcoming completeness argument is the following lemma, which is a modification of a corresponding argument of von Neumann's [**10**, Lemma 17.3, p. 228]. Another completeness argument using this method occurs in [**2**, Lemma 21.6], and we can adapt the proof of that lemma with only minor changes.

LEMMA 1.6. *If R is an $\aleph_0$-continuous regular ring and e is an idempotent in R, then $eR(1-e)$ is complete in the N^*-metric.*

PROOF. We shall need the fact that R is unit-regular [**2**, Theorem 14.24].

Let $\{x_1, x_2, \ldots\}$ be a sequence in $eR(1-e)$ which is Cauchy in the N^*-metric. By passing to a subsequence, we may assume that $N^*(x_i - x_j) < 1/2^{k+1}$ whenever $i, j \geqslant k$. Set $A_n = (1 - e + x_n)R$ for all n, and note that $A_n + eR = R$. For each $k = 1, 2, \ldots$, there exists a principal right ideal B_k in R such that

$$\sum_{n=k}^{\infty} A_n \leqslant_e B_k,$$

and since $A_k \leqslant B_k$, we see that $B_k + eR = R$. Note that $B_1 \geqslant B_2 \geqslant \ldots$, and set

$$C = \bigcap_{k=1}^{\infty} B_k.$$

Inasmuch as the lattice of principal right ideals of R is $\aleph_0$-continuous, we obtain

$$C + eR = \left(\bigcap_{k=1}^{\infty} B_k\right) + eR = \bigcap_{k=1}^{\infty} (B_k + eR) = R.$$

Consequently, there exists an idempotent $f \in R$ such that $fR = eR$ and $(1-f)R \leqslant C$.

Since $fR = eR$, we have $f = ef$ and $e = fe$, hence the element $x = e - f$ lies in $eR(1-e)$. We shall show that $x_n \to x$ in the N^*-metric.

For each $k = 1, 2, \ldots$, there exists an element $d_k \in R$ such that

$$\sum_{j=k+1}^{\infty} (x_j - x_{j-1})R \leqslant_e d_kR,$$

and Lemma 1.5 shows that

$$N^*(d_k) \leqslant \sum_{j=k+1}^{\infty} N^*(x_j - x_{j-1}) < \sum_{j=k+1}^{\infty} 1/2^j = 1/2^k.$$

For all $n \geqslant k$, we have

$$A_n = (1 - e + x_n)R = \left(1 - e + x_k + \sum_{j=k+1}^{n} (x_j - x_{j-1})\right) R$$
$$\leqslant (1 - e + x_k)R + \sum_{j=k+1}^{n} (x_j - x_{j-1})R \leqslant A_k + d_k R.$$

Consequently, $\sum_{n=k}^{\infty} A_n \leqslant A_k + d_k R$, whence $B_k \leqslant A_k + d_k R$.

Each $B_k = A_k \oplus u_k R$ for some $u_k \in R$. Then

$$A_k \oplus u_k R = B_k \leqslant A_k + d_k R \lesssim A_k \oplus d_k R$$

and so $u_k R \lesssim d_k R$ (because R is unit-regular). As a result, $N^*(u_k) \leqslant N^*(d_k) < 1/2^k$ (Lemma 1.2).

The idempotent $1 - f$ lies in the right ideal

$$C \leqslant B_k = A_k + u_k R = (1 - e + x_k)R + u_k R,$$

hence $1 - f = (1 - e + x_k)r + u_k s$ for some $r, s \in R$. Since $x_k \in eR(1 - e)$, we see that $1 - e + x_k$ is idempotent, whence

$$(1 - e + x_k)(1 - f) = (1 - e + x_k)r + (1 - e + x_k)u_k s$$
$$= 1 - f + (x_k - e)u_k s.$$

In addition, since $fR = eR$, we have $R(1 - f) = R(1 - e)$, and so $1 - e + x_k$ lies in $R(1 - f)$. Thus

$$1 - e + x_k = (1 - e + x_k)(1 - f) = 1 - f + (x_k - e)u_k s,$$

and consequently

$$x_k - x = x_k - e + f = (1 - e + x_k) - (1 - f) = (x_k - e)u_k s,$$

hence $N^*(x_k - x) \leqslant N^*(u_k) < 1/2^k$.

Therefore $x_k \to x$ in the N^*-metric. $\square$

We shall apply Lemma 1.6 in a situation where R is a matrix ring and e is a corner idempotent. For this purpose, and for later use, we need the following information concerning N^* in matrix rings.

LEMMA 1.7. *Let R be a regular ring, let $n \in \mathbf{N}$, and set $T = M_n(R)$. Let $\varphi: R \to T$ be the natural map, and let $\{e_{ij} \mid i, j = 1, \ldots, n\}$ be the standard matrix units in T.*

(a) $N_T^* \varphi = N_R^*$.

(b) $N_T^*(\varphi(x)e_{ij}) = N_R^*(x)/n$ *for all* $x \in R$ *and all* i, j.

(c) $N_R^*(y_{ij}) \leqslant nN_T^*(y)$ *for all* $y \in T$ *and all* i, j.

(d) $N_T^*(y) \leqslant \sum_{i,j=1}^{n} N_R^*(y_{ij})/n$ *for all* $y \in T$.

PROOF. (a) According to [2, Corollary 16.10], the rule $P \mapsto P\varphi$ defines a bijection of $\mathbf{P}(T)$ onto $\mathbf{P}(R)$, hence

$$N_R^*(x) = \sup\{P\varphi(x) \mid P \in \mathbf{P}(T)\} = N_T^*\varphi(x)$$

for any $x \in R$.

(b) For each $k = 1,\ldots,n$, we note that left multiplication by e_{ki} defines an isomorphism of $\varphi(x)e_{ij}T$ onto $\varphi(x)e_{kk}T$. Consequently,

$$\varphi(x)T = \bigoplus_{k=1}^{n} \varphi(x)e_{kk}T \cong n\big(\varphi(x)e_{ij}T\big),$$

hence $N_T^*\varphi(x) = nN_T^*(\varphi(x)e_{ij})$, by Lemma 1.2.

(c) Since $\varphi(y_{ij})e_{ij} = e_{ii}ye_{jj}$, it follows from (b) that

$$N_R^*(y_{ij}) = nN_T^*\big(\varphi(y_{ij})e_{ij}\big) = nN_T^*(e_{ii}ye_{jj}) \leqslant nN_T^*(y).$$

(d) Using (b) again, we conclude that

$$\begin{aligned} N_T^*(y) = N_T^*\left(\sum_{i,j=1}^{n} \varphi(y_{ij})e_{ij}\right) &\leqslant \sum_{i,j=1}^{n} N_T^*\big(\varphi(y_{ij})e_{ij}\big) \\ &= \sum_{i,j=1}^{n} N_R^*(y_{ij})/n. \quad \square \end{aligned}$$

THEOREM 1.8. *Every $\aleph_0$-continuous regular ring is N^*-complete.*

PROOF. For any $\aleph_0$-continuous regular ring R, [**4**, Proposition II.11.4] shows that $\ker(\mathbf{P}(R)) = 0$.

Assume for the moment that R is right and left $\aleph_0$-injective. Then the ring $T = M_2(R)$ is $\aleph_0$-continuous, by [**2**, Proposition 14.19]. Setting $e = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we obtain from Lemma 1.6 that $eT(1-e)$ is complete in the N_T^*-metric. There is a group isomorphism $x \mapsto \left(\begin{smallmatrix} 0 & 0 \\ x & 0 \end{smallmatrix}\right)$ of R onto $eT(1-e)$, and Lemma 1.7 shows that

$$N_R^*(x) = 2N_T^*\left(\begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}\right)$$

for all $x \in R$. Therefore in this case R is N^*-complete.

In general, let S denote the maximal right $\aleph_0$-quotient ring of R. Then S is an $\aleph_0$-continuous, regular, right and left $\aleph_0$-injective overring of R, and R contains all the idempotents of S [**2**, Theorems 14.12 and 14.17]. By the case above, S is N^*-complete.

Now R and S are unit-regular rings [**2**, Theorem 14.24], and we may assume they are nonzero. Using Proposition 1.1 and [**2**, Proposition 14.28], we compute that for any $x \in R$,

$$\begin{aligned} N_R^*(x) &= \inf\{k/n \mid k, n \in \mathbf{N} \text{ and } n(xR) \lesssim kR_R\} \\ &= \inf\{k/n \mid k, n \in \mathbf{N} \text{ and } n(xS) \lesssim kS_S\} = N_S^*(x). \end{aligned}$$

Consequently, the N^*-completeness of R will follow from the N^*-completeness of S provided R is closed in S in the N_S^*-metric.

We claim that $N_S^*(x) = 1$ for any $x \in S - R$. By [**2**, Proposition 3.15], S has a two-sided ideal N such that $N \subseteq R$ and the ring S/N is abelian. Then $x \notin N$, hence S has a primitive ideal M such that $N \subseteq M$ but $x \notin M$. As S/N is abelian, S/M is a division ring, hence S/M has a unique rank function Q and $Q(x + M) = 1$. Pulling Q back to a pseudo-rank function P on S, we obtain $P(x) = 1$, so that $N_S^*(x) \geqslant 1$. Thus $N_S^*(x) = 1$, as claimed.

As a result, $N_S^*(x-y)=1$ for all $x \in S-R$ and all $y \in R$, whence R is closed in S in the N_S^*-metric, as desired. Therefore R is N^*-complete. $\square$

COROLLARY 1.9. *Every regular, right and left self-injective ring is N^*-complete.* $\square$

We conclude this section by proving that N^*-completeness carries over to matrix rings and certain factor rings.

THEOREM 1.10. *If R is an N^*-complete regular ring, then any matrix ring $M_n(R)$ is N^*-complete.*

PROOF. Set $T=M_n(R)$, let $\varphi\colon R \to T$ be the natural map, and let $\{e_{ij} \mid i, j = 1,\ldots,n\}$ be the standard matrix units in T. Given any $y \in \ker(\mathbf{P}(T))$, we have $N_T^*(y)=0$, whence Lemma 1.7 shows that $N_R^*(y_{ij})=0$ for all i, j. Then all $y_{ij}=0$, so that $y=0$. Thus $\ker(\mathbf{P}(T))=0$.

Now consider any sequence $\{y^{(1)}, y^{(2)},\ldots\}$ in T that is Cauchy with respect to N_T^*. Fixing i and j for a while, we have

$$N_R^*\left(y_{ij}^{(k)}-y_{ij}^{(l)}\right) \leqslant nN_T^*\left(y^{(k)}-y^{(l)}\right)$$

for all k, l (Lemma 1.7), hence the sequence $\{y_{ij}^{(1)}, y_{ij}^{(2)},\ldots\}$ in R is Cauchy with respect to N_R^*. Consequently, there exists $y_{ij} \in R$ such that $y_{ij}^{(k)} \to y_{ij}$ in the N_R^*-metric. Having gotten such elements y_{ij} for each i, j, we obtain a matrix $y \in T$ with entries y_{ij}. Inasmuch as

$$N_T^*\left(y^{(k)}-y\right) \leqslant \sum_{i,j=1}^{n} N_R^*\left(y_{ij}^{(k)}-y_{ij}\right)/n$$

for all k (Lemma 1.7 again), we conclude that $y^{(k)} \to y$ in the N_T^*-metric.

Therefore T is N^*-complete. $\square$

LEMMA 1.11. *Let J be a two-sided ideal in a regular ring R, let A and B be finitely generated projective right R-modules, and let $n \in \mathbf{N}$. If $n(A/AJ) \lesssim B/BJ$, then there exists a decomposition $A = A' \oplus A''$ such that $nA' \lesssim B$ and $A''=A''J$.*

PROOF. As $(nA)/(nA)J \lesssim B/BJ$, we may apply [**2**, Proposition 2.20] to obtain a decomposition $nA = C \oplus D$ such that $C \lesssim B$ and $D = DJ$. Then, by [**2**, Theorem 2.8], there exist decompositions $C = C_1 \oplus \cdots \oplus C_n$ and $D = D_1 \oplus \cdots \oplus D_n$ such that $C_i \oplus D_i \cong A$ for all i. Since $D = DJ$, each $D_i = D_iJ$, hence $C_i/C_iJ \cong A/AJ$. Thus the modules C_i/C_iJ are pairwise isomorphic, so by [**2**, Proposition 2.19] there exist decompositions $C_i = E_i \oplus F_i$ for each i such that the E_i are pairwise isomorphic and each $F_i = F_iJ$.

Now $A \cong C_1 \oplus D_1 = E_1 \oplus F_1 \oplus D_1$, so there is a decomposition $A = A' \oplus A''$ with $A' \cong E_1$ and $A'' \cong F_1 \oplus D_1$. Then

$$nA' \cong nE_1 \cong E_1 \oplus \cdots \oplus E_n \leqslant C_1 \oplus \cdots \oplus C_n = C \lesssim B.$$

Since $D_1 = D_1J$ and $F_1 = F_1J$, we also have $A''=A''J$. $\square$

For the moment, we now restrict to unit-regular rings. This restriction will be removed when we prove that all N^*-complete regular rings are unit-regular (Theorem 2.3).

LEMMA 1.12. *Let J be a two-sided ideal in a unit-regular ring R. Then*

$$N^*_{R/J}(x+J)=\inf\{N^*_R(y)\mid y\in x+J\}$$

for all $x\in R$.

PROOF. This is clear if $J=R$, so assume $J\neq R$.

Given any $y\in x+J$, we see that

$$\begin{aligned}N^*_{R/J}(x+J)&=N^*_{R/J}(y+J)=\sup\{Q(y+J)\mid Q\in\mathbf{P}(R/J)\}\\&=\sup\{P(y)\mid P\in\mathbf{P}(R)\text{ and }J\subseteq\ker(P)\}\\&\leqslant\sup\{P(y)\mid P\in\mathbf{P}(R)\}=N^*_R(y).\end{aligned}$$

Thus $N^*_{R/J}(x+J)\leqslant\inf\{N^*_R(y)\mid y\in x+J\}$.

Given any real number $\alpha>N^*_{R/J}(x+J)$, Proposition 1.1 shows that there exist $k,n\in\mathbf{N}$ for which $k/n<\alpha$ and

$$n((x+J)(R/J))\lesssim k(R/J),$$

that is, $n(xR/xJ)\lesssim(kR)/(kR)J$. According to Lemma 1.11, there is a decomposition $xR=A'\oplus A''$ such that $nA'\lesssim kR_R$ and $A''=A''J$. Then $x=y+z$ for some $y\in A'$ and $z\in A''$. Note that $z\in J$, so that $y\in x+J$. Since $n(yR)\leqslant nA'\lesssim kR_R$, we conclude from Proposition 1.1 that $N^*_R(y)\leqslant k/n<\alpha$. Therefore

$$\inf\{N^*_R(y)\mid y\in x+J\}\leqslant N^*_{R/J}(x+J).\quad\square$$

THEOREM 1.13. *Let R be an N^*-complete unit-regular ring, and let J be a two-sided ideal of R. Then the following conditions are equivalent.*

(a) *R/J is N^*-complete.*

(b) *J is N^*-closed in R.*

(c) *$J=\ker(X)$ for some $X\subseteq\mathbf{P}(R)$.*

PROOF. (a) $\Rightarrow$ (c): By definition, $\ker(\mathbf{P}(R/J))=0$, hence if

$$X=\{P\in\mathbf{P}(R)\mid J\subseteq\ker(P)\},$$

then $J=\ker(X)$.

(c) $\Rightarrow$ (b): Given $x\in R-J$, we must have $P(x)>0$ for some $P\in X$. Then

$$N^*(y-x)\geqslant P(y-x)\geqslant P(x)-P(y)=P(x)$$

for all $y\in J$, hence x is not in the N^*-closure of J. Thus J is N^*-closed.

(b) $\Rightarrow$ (a): Given a nonzero coset $x+J$ in R/J, we have $x\notin J$, hence there must exist a positive real number ε such that $N^*_R(y-x)\geqslant\varepsilon$ for all $y\in J$. Then $N^*_R(z)\geqslant\varepsilon$ for all $z\in x+J$, whence $N^*_{R/J}(x+J)\geqslant\varepsilon$, by Lemma 1.12. Consequently, $\ker(\mathbf{P}(R/J))=0$.

Now consider a sequence $\{a_1,a_2,\dots\}$ in R/J that is Cauchy with respect to $N^*_{R/J}$. There is no loss of generality in assuming that $N^*_{R/J}(a_{n+1}-a_n)<1/2^n$ for all n. Choose $x_1,x_2,\dots$ in R such that each $a_n=x_n+J$.

Set $y_1=x_1$. Then $N^*_{R/J}((x_2-y_1)+J)<1/2$, so by Lemma 1.12 there exists z in $(x_2-y_1)+J$ satisfying $N^*_R(z)<1/2$. Set $y_2=y_1+z$, so that $y_2+J=x_2+J=a_2$ and $N^*_R(y_2-y_1)<1/2$. Continuing in this manner, we obtain elements $y_1,y_2,\dots$ in R such that $y_n+J=a_n$ and $N^*_R(y_{n+1}-y_n)<1/2^n$ for all n.

Thus $\{y_1, y_2, \dots\}$ is Cauchy with respect to N_R^*, hence there exists $y \in R$ such that $y_n \to y$ in the N_R^*-metric. Setting $a = y + J$, we conclude from Lemma 1.12 that $a_n \to a$ in the $N_{R/J}^*$-metric.

Therefore R/J is N^*-complete. $\square$

COROLLARY 1.14. *Let R be an N^*-complete unit-regular ring, and let M be a maximal two-sided ideal of R. Then R/M is N^*-complete.*

PROOF. As R/M is a simple unit-regular ring, [2, Corollary 18.5] shows that there is a rank function on R/M. Then there exists $P \in \mathbf{P}(R)$ such that $\ker(P) = M$, hence R/M is N^*-complete by Theorem 1.13. $\square$

II. K_0. A good deal of information about a regular ring R, particularly ideal theory and decomposition properties of the principal right ideals, is stored in the Grothendieck group $K_0(R)$. We study this group for an N^*-complete regular ring R in this section, proving that $K_0(R)$ is an archimedean, norm-complete, partially ordered abelian group with the interpolation property. In the following section, we develop a structure theory for such groups, which can then be applied, via K_0, to the structure theory of N^*-complete regular rings.

DEFINITION. Recall that the Grothendieck group K_0 of a ring R is an abelian group with generators $[A]$ corresponding to the finitely generated projective right R-modules A and with relations $[A] + [B] = [C]$ whenever $A \oplus B \cong C$. All elements of $K_0(R)$ are of the form $[A] - [B]$, for suitable A and B. We set

$$K_0(R)^+ = \{[A] \mid A \text{ is a finitely generated projective right } R\text{-module}\},$$

and we define a relation $\leqslant$ on $K_0(R)$ so that $x \leqslant y$ if and only if $y - x$ lies in $K_0(R)^+$. This relation is a translation-invariant pre-order on $K_0(R)$, so that $K_0(R)$ becomes a pre-ordered abelian group. The element $[R]$ is an *order-unit* in $K_0(R)$, meaning that for any $x \in K_0(R)$ there exists $n \in \mathbf{N}$ such that $x \leqslant n[R]$.

For a unit-regular ring R, the relations between $K_0(R)$ and the finitely generated projective right R-modules are much cleaner than in general. Namely, for any finitely generated projective right R-modules A, B, C, D we have

$$[A] - [B] = [C] - [D] \quad \text{if and only if } A \oplus D \cong B \oplus C,$$
$$[A] - [B] \leqslant [C] - [D] \quad \text{if and only if } A \oplus D \lesssim B \oplus C$$

[2, Proposition 15.2]. In addition, the relation $\leqslant$ on $K_0(R)$ is actually a partial order, so that $K_0(R)$ is a partially ordered abelian group in this case. Thus, in order to deal effectively with K_0 of N^*-complete regular rings, we first prove that such rings are unit-regular. Two lemmas will be helpful in doing this.

LEMMA 2.1. *Let R be an N^*-complete regular ring, and let A, B, C be finitely generated projective right R-modules. Let $\{A_1, A_2, \dots\}$ and $\{B_1, B_2, \dots\}$ be independent sequences of finitely generated submodules of A and B, such that $A_k \cong B_k$ for all k. For each k, let A_k^* be a submodule of A such that*

$$A = A_1 \oplus \cdots \oplus A_k \oplus A_k^*,$$

and assume that $2^k t_k A_k^* \lesssim t_k C$ for some $t_k \in \mathbf{N}$. Then $A \lesssim B$.

PROOF. As all matrix rings over R are N^*-complete (Theorem 1.10), we may use the standard Morita-equivalences to transfer our problem to the category of right modules over a suitable matrix ring $M_n(R)$, with n chosen large enough so that the modules corresponding to A, B, C are cyclic. Thus there is no loss of generality in assuming that A, B, C are actually principal right ideals of R.

Choose idempotents $e, f \in R$ such that $eR = A$ and $fR = B$. Applying [**2**, Proposition 2.13] to the ascending sequence

$$A_1 \leqslant A_1 \oplus A_2 \leqslant A_1 \oplus A_2 \oplus A_3 \leqslant \ldots$$

of finitely generated submodules of A, we obtain orthogonal idempotents $e_1, e_2, \ldots$ in eRe such that

$$e_1R \oplus \cdots \oplus e_kR = A_1 \oplus \cdots \oplus A_k$$

for all k. Similarly, there exist orthogonal idempotents $f_1, f_2, \ldots$ in fRf such that

$$f_1R \oplus \cdots \oplus f_kR = B_1 \oplus \cdots \oplus B_k$$

for all k. Note that each

$$\begin{aligned} e_kR &\cong (e_1R \oplus \cdots \oplus e_kR)/(e_1R \oplus \cdots \oplus e_{k-1}R) \\ &= (A_1 \oplus \cdots \oplus A_k)/(A_1 \oplus \cdots \oplus A_{k-1}) \cong A_k \end{aligned}$$

and similarly $f_kR \cong B_k$, so that $e_kR \cong f_kR$. Thus there exist elements $x_k \in e_kRf_k$ and $y_k \in f_kRe_k$ such that $x_ky_k = e_k$ and $y_kx_k = f_k$.

For each k, we have $(e - e_1 - \cdots - e_k)R \cong A_k^*$, whence

$$2^kt_k((e - e_1 - \cdots - e_k)R) \lesssim t_kC \lesssim t_kR_R,$$

and consequently $N^*(e - e_1 - \cdots - e_k) \leqslant 1/2^k$, by Lemma 1.2. Thus $\Sigma e_k \to e$ in the N^*-metric. As

$$x_{k+1} = e_{k+1}x_{k+1} = (e - e_1 - \cdots - e_k)e_{k+1}x_{k+1}$$

for each k, we also obtain $N^*(x_{k+1}) \leqslant 1/2^k$, and similarly, $N^*(y_{k+1}) \leqslant 1/2^k$.

Now the partial sums of the series Σx_k and Σy_k are Cauchy with respect to N^*, hence there exist $x, y \in R$ such that $\Sigma x_k \to x$ and $\Sigma y_k \to y$ in the N^*-metric. As each

$$x_k = e_kx_kf_k = ee_kx_kf_kf = ex_kf,$$

we obtain $x = exf$, and likewise $y = fye$. Since $x_iy_j = x_if_if_jy_j = 0$ whenever $i \neq j$, we have

$$(x_1 + \cdots + x_k)(y_1 + \cdots + y_k) = x_1y_1 + \cdots + x_ky_k = e_1 + \cdots + e_k$$

for all k, and consequently $xy = e$. Therefore $eR \lesssim fR$, that is, $A \lesssim B$. $\square$

LEMMA 2.2. *Let A, B, C be finitely generated projective right modules over a regular ring, such that $A \oplus C \cong B \oplus C$. Then there exist decompositions*

$$A = A' \oplus A''; \qquad B = B' \oplus B''; \qquad C = C' \oplus C''$$

such that $A' \cong B'$ and $A'' \cong C'$, while also $A'' \oplus C'' \cong B'' \oplus C''$.

PROOF. According to [2, Theorem 2.8], there exist decompositions $A = A' \oplus A''$ and $C = D \oplus E$ such that $A' \oplus D \cong B$ and $A'' \oplus E \cong C$. Then we obtain decompositions $B = B' \oplus B''$ and $C = C' \oplus C''$ such that $B' \cong A'$ and $B'' \cong D$, while also $C' \cong A''$ and $C'' \cong E$. Finally,

$$A'' \oplus C'' \cong A'' \oplus E \cong C = D \oplus E \cong B'' \oplus C''. \quad \square$$

THEOREM 2.3. *Every N^*-complete regular ring is unit-regular.*

PROOF. Given an N^*-complete regular ring R, we have $\ker(\mathbf{P}(R)) = 0$ by definition, hence all matrix rings over R are directly finite [2, Proposition 16.11]. To prove that R is unit-regular, it suffices to show that if A, B, C are any finitely generated projective right R-modules satisfying $A \oplus C \cong B \oplus C$, then $A \cong B$.

Inducting on Lemma 2.2, we obtain submodules

$$A_1', A_1'', A_2', A_2'', \ldots \leq A; \quad B_1', B_1'', B_2', B_2'', \ldots \leq B; \quad C_1', C_1'', C_2', C_2'', \ldots \leq C$$

such that

$$A = A_1' \oplus A_1''; \qquad B = B_1' \oplus B_1''; \qquad C = C_1' \oplus C_1''$$

while also

$$A_i'' = A_{i+1}' \oplus A_{i+1}''; \quad B_i'' = B_{i+1}' \oplus B_{i+1}''; \quad C_i'' = C_{i+1}' \oplus C_{i+1}'';$$
$$A_i' \cong B_i'; \quad A_i'' \cong C_i'; \quad A_i'' \oplus C_i'' \cong B_i'' \oplus C_i''$$

for all i. Note that $A_1'' \geq A_2'' \geq \ldots$, and that $C_1', C_2', \ldots$ are independent submodules of C. Consequently, we obtain

$$iA_i'' \lesssim A_1'' \oplus A_2'' \oplus \cdots \oplus A_i'' \cong C_1' \oplus C_2' \oplus \cdots \oplus C_i' \leq C$$

for all i.

Now set $A_1 = A_1' \oplus A_2'$ and $B_1 = B_1' \oplus B_2'$, while $A_1^* = A_2''$. Set

$$A_k = \bigoplus_{i=2^{k-1}+1}^{2^k} A_i'; \quad A_k^* = A_{2^k}''; \quad B_k = \bigoplus_{i=2^{k-1}+1}^{2^k} B_i'$$

for all $k = 2, 3, \ldots$. Thus $\{A_1, A_2, \ldots\}$ and $\{B_1, B_2, \ldots\}$ are independent sequences of finitely generated submodules of A and B, with $A_k \cong B_k$ for all k. Also, since

$$A = A_1' \oplus A_1'' = A_1' \oplus A_2' \oplus A_2'' = \cdots = A_1' \oplus A_2' \oplus \cdots \oplus A_i' \oplus A_i''$$

for all i, we have $A = A_1 \oplus A_2 \oplus \cdots \oplus A_k \oplus A_k^*$ for all k. Inasmuch as $2^k A_k^* \lesssim C$ for all k, we conclude from Lemma 2.1 that $A \lesssim B$.

By symmetry, $B \lesssim A$. As all matrix rings over R are directly finite, it follows that $A \cong B$ [2, Proposition 5.4]. $\square$

DEFINITION. Let G be a partially ordered abelian group. Then G is said to be an *interpolation group* if given any x_1, x_2, y_1, y_2 in G satisfying $x_i \leq y_j$ for all i, j, there exists $z \in G$ such that $x_i \leq z \leq y_j$ for all i, j. Equivalently, G is an interpolation group if and only if either of the following forms of the *Riesz decomposition property* holds.

(a) If $x, y_1, \ldots, y_n \in G^+$ and $x \leq y_1 + \cdots + y_n$, then there exist $x_1, \ldots, x_n$ in G^+ such that $x = x_1 + \cdots + x_n$ and each $x_i \leq y_i$.

(b) If $x_1,\ldots,x_n$, $y_1,\ldots,y_k \in G^+$ and $x_1 + \cdots + x_n = y_1 + \cdots + y_k$, then there exist $z_{ij} \in G^+$ (for $i = 1,\ldots,n$ and $j = 1,\ldots,k$) such that $x_i = z_{i1} + \cdots + z_{ik}$ for all i and $y_j = z_{1j} + \cdots + z_{nj}$ for all j.

For any unit-regular ring R, the partially ordered abelian group $K_0(R)$ is an interpolation group [**4**, Proposition II.10.3]. In particular, this holds for any N^*-complete regular ring, because of Theorem 2.3; however, we postpone recording this fact until Theorem 2.11.

DEFINITION. Let G be a partially ordered abelian group, and let n be a positive integer. We say that G is *n-unperforated* if for all $x \in G$, we have $nx \geqslant 0$ only when $x \geqslant 0$. If G is n-unperforated for all $n \in \mathbf{N}$, then G is said to be *unperforated*. The group G is *archimedean* provided that whenever $x, y \in G$ and $nx \leqslant y$ for all $n \in \mathbf{N}$, then $x \leqslant 0$. It is easily checked that all archimedean directed abelian groups are unperforated [**4**, Lemma I.5.2]. Our next goal is to prove that K_0 of any N^*-complete regular ring R is archimedean; however, since our proof requires $K_0(R)$ to be 2-unperforated, we show that first.

LEMMA 2.4. *Let A and B be finitely generated projective right modules over a unit-regular ring R, and let $n \in \mathbf{N}$. If $A \lesssim nB$, then there exists a decomposition $A = A' \oplus A''$ such that $A' \lesssim B$ and $nA'' \lesssim (n-1)A$.*

PROOF. Since $A \lesssim nB$, there is a decomposition $A = A_1 \oplus \cdots \oplus A_n$ with each $A_i \lesssim B$ [**2**, Corollary 2.9]. Each A_i is isomorphic to a submodule $B_i \leqslant B$, and we define $B' = B_1 + \cdots + B_n$. Then B' is a finitely generated submodule of B, and

$$B' \lesssim B_1 \oplus \cdots \oplus B_n \cong A_1 \oplus \cdots \oplus A_n = A,$$

hence we obtain a decomposition $A = A' \oplus A''$ with $A' \cong B' \leqslant B$. For each $j = 1,\ldots,n$, we have $A_j \cong B_j \leqslant B' \cong A'$ and so

$$A' \oplus A'' = A = A_1 \oplus \cdots \oplus A_n \lesssim A' \oplus \left(\bigoplus_{i \neq j} A_i \right),$$

whence $A'' \lesssim \oplus_{i \neq j} A_i$. Therefore

$$nA'' \lesssim \bigoplus_{j=1}^{n} \bigoplus_{i \neq j} A_i \cong \bigoplus_{i=1}^{n} (n-1)A_i \cong (n-1)A. \quad \square$$

LEMMA 2.5. *Let A and B be finitely generated projective right modules over a unit-regular ring R, and let $n \in \mathbf{N}$. If $nA \lesssim nB$, then there exist decompositions $A = A' \oplus A''$ and $B = B' \oplus B''$ such that $A' \cong B'$ and $nA'' \lesssim nB''$, while also $2tA'' \lesssim tA$ for some $t \in \mathbf{N}$.*

PROOF. By Lemma 2.4, there exist decompositions $A = A_1 \oplus A_1^*$ and $B = B_1 \oplus B_1^*$ such that $A_1 \cong B_1$ and $nA_1^* \lesssim (n-1)A$. Since

$$nA_1 \oplus nA_1^* \cong nA \lesssim nB = nB_1 \oplus nB_1^* \cong nA_1 \oplus nB_1^*,$$

we also have $nA_1^* \lesssim nB_1^*$. Thus we may continue by induction, obtaining submodules

$$A_1, A_1^*, A_2, A_2^*, \ldots \leqslant A; \qquad B_1, B_1^*, B_2, B_2^*, \ldots \leqslant B$$

such that

$$A_i^* = A_{i+1} \oplus A_{i+1}^*; \qquad B_i^* = B_{i+1} \oplus B_{i+1}^*;$$

$$A_{i+1} \cong B_{i+1}; \qquad nA_{i+1}^* \lesssim (n-1)A_i^*; \qquad nA_{i+1}^* \lesssim nB_{i+1}^*$$

for all i. In addition,

$$n^i A_i^* \lesssim n^{i-1}(n-1)A_{i-1}^* \lesssim \cdots \lesssim n(n-1)^{i-1}A_1^* \lesssim (n-1)^i A$$

for all i.

Choose $i \in \mathbf{N}$ such that $((n-1)/n)^i \leqslant 1/2$, and set $t = (n-1)^i$, so that $2t \leqslant n^i$. Setting $A' = A_1 \oplus \cdots \oplus A_i$ and $A'' = A_i^*$, we obtain $A = A' \oplus A''$ and

$$2tA'' \lesssim n^i A_i^* \lesssim (n-1)^i A = tA.$$

Setting $B' = B_1 \oplus \cdots \oplus B_i$ and $B'' = B_i^*$, we obtain $B = B' \oplus B''$, while also $A' \cong B'$ and $nA'' \lesssim nB''$. $\square$

THEOREM 2.6. *Let R be an N^*-complete regular ring, let A and B be finitely generated projective right R-modules, and let $n \in \mathbf{N}$. If $nA \lesssim nB$, then $A \lesssim B$.*

PROOF. Inducting on Lemma 2.5, we obtain submodules

$$A_1, A_1^*, A_2, A_2^*, \ldots \leqslant A; \qquad B_1, B_1^*, B_2, B_2^*, \ldots \leqslant B$$

such that $A = A_1 \oplus A_1^*$ and $B = B_1 \oplus B_1^*$, while also

$$A_k^* = A_{k+1} \oplus A_{k+1}^*; \qquad B_k^* = B_{k+1} \oplus B_{k+1}^*;$$

$$A_k \cong B_k; \qquad nA_k^* \lesssim nB_k^*; \qquad 2^k t_k A_k^* \lesssim t_k A$$

(for some $t_k \in \mathbf{N}$) for all k. Applying Lemma 2.1, we conclude that $A \lesssim B$. $\square$

As an N^*-complete regular ring R is unit-regular, it follows immediately from Theorem 2.6 that $K_0(R)$ is unperforated. This result will be subsumed by the stronger result that $K_0(R)$ is archimedean (Theorem 2.11).

LEMMA 2.7. *Let R be an N^*-complete regular ring, let A, B, C be finitely generated projective right R-modules, and let $n \in \mathbf{N}$. If $2^n A \lesssim 2^n B \oplus C$, then there exists a decomposition $A = A' \oplus A''$ such that $A' \lesssim B$ and $2^n A'' \lesssim C$.*

PROOF. The proof of [**2**, Lemma 14.31] may be used, substituting Theorem 2.6 for [**2**, Theorem 14.30]. $\square$

THEOREM 2.8. *Let R be an N^*-complete regular ring, and let A, B, C be finitely generated projective right R-modules. If $2^n A \lesssim 2^n B \oplus C$ for all $n \in \mathbf{N}$, then $A \lesssim B$.*

PROOF. Since $2A \lesssim 2B \oplus C$, Lemma 2.7 provides us with decompositions $A = A_1 \oplus A_1^*$ and $B = B_1 \oplus B_1^*$ such that $A_1 \cong B_1$ and $2A_1^* \lesssim C$. For all $n \in \mathbf{N}$,

$$2^n A_1 \oplus 2^n A_1^* \cong 2^n A \lesssim 2^n B \oplus C \cong 2^n B_1 \oplus 2^n B_1^* \oplus C \cong 2^n A_1 \oplus 2^n B_1^* \oplus C,$$

hence $2^n A_1^* \lesssim 2^n B_1^* \oplus C$. Thus we may continue by induction, obtaining submodules

$$A_1, A_1^*, A_2, A_2^*, \ldots \leqslant A; \qquad B_1, B_1^*, B_2, B_2^*, \ldots \leqslant B$$

such that

$$A_k^* = A_{k+1} \oplus A_{k+1}^*; \qquad B_k^* = B_{k+1} \oplus B_{k+1}^*;$$
$$A_k \cong B_k; \qquad 2^k A_k^* \lesssim C; \qquad 2^n A_k^* \lesssim 2^n B_k^* \oplus C$$

for all k, n. Applying Lemma 2.1, we conclude that $A \lesssim B$. $\square$

It follows directly from Theorem 2.8 that K_0 of any N^*-complete regular ring is archimedean. We postpone recording this result until Theorem 2.11.

DEFINITION. Let (G, u) be a partially ordered abelian group with order-unit. For any $x \in G$, we define

$$\|x\|_u = \inf\{k/n \mid k, n \in \mathbf{N} \text{ and } -ku \leq nx \leq ku\},$$

and we note that $\|x\|_u$ is a nonnegative real number. When there is no danger of confusion as to the order-unit u, we just write $\|x\|$ instead of $\|x\|_u$. The function $\|\cdot\|$ behaves like a seminorm on G, for $\|mx\| = |m| \cdot \|x\|$ and $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in G$ and $m \in \mathbf{Z}$ [4, Lemma I.6.1]. In particular, it follows that the rule $\delta(x, y) = \|x - y\|$ defines a pseudo-metric δ on G. If δ is actually a metric, and G is complete in this metric, then we say that (G, u) is *norm-complete*.

It is tempting to expect norm-complete partially ordered abelian groups with order-unit, particularly those that are interpolation groups, to be archimedean, but this is not the case in general. For instance, make the group $G = \mathbf{R}^2$ into a partially ordered abelian group with positive cone

$$G^+ = \{(0,0)\} \cup \{(a, b) \in G \mid a > 0 \text{ and } b > 0\}.$$

It is clear that G is an interpolation group, and that the element $u = (1, 1)$ is an order-unit in G. Observing that

$$\|(a, b)\|_u = \max\{|a|, |b|\}$$

for all $(a, b) \in G$, we see that (G, u) is norm-complete. However, $n(-1, 0) \leq u$ for all $n \in \mathbf{N}$ while $(-1, 0) \nleq 0$, so that G is not archimedean.

LEMMA 2.9. *Let R be an N^*-complete regular ring, let $v \in K_0(R)$, and let $n \in \mathbf{N}$. If $\|v\| < 1/2^n$, then there exist $x, y \in R$ such that $v = [xR] - [yR]$, while also $N^*(x) \leq 1/2^n$ and $N^*(y) \leq 1/2^n$.*

PROOF. Write $v = [A] - [B]$ for some finitely generated projective right R-modules A and B. Since $\|v\| < 1/2^n$, there exist $s, t \in \mathbf{N}$ such that $-s[R] \leq tv \leq s[R]$ and $s/t < 1/2^n$. Since R is unit-regular (Theorem 2.3), it follows that

$$tA \lesssim sR_R \oplus tB \qquad \text{and} \qquad tB \lesssim sR_R \oplus tA.$$

Now $2^n tA \lesssim 2^n tB \oplus tR_R$, because $2^n s < t$. Then $2^n A \lesssim 2^n B \oplus R_R$ by Theorem 2.6, and similarly $2^n B \lesssim 2^n A \oplus R_R$.

In view of Lemma 2.7, there exist decompositions $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ such that $A_1 \cong B_1$ and $2^n A_2 \lesssim R_R$. Then

$$2^n B_1 \oplus 2^n B_2 \cong 2^n B \lesssim 2^n A \oplus R_R \cong 2^n A_1 \oplus 2^n A_2 \oplus R_R \cong 2^n B_1 \oplus 2^n A_2 \oplus R_R,$$

whence $2^n B_2 \lesssim 2^n A_2 \oplus R_R$. Applying Lemma 2.7 a second time, we obtain decompositions $B_2 = B_3 \oplus B_4$ and $A_2 = A_3 \oplus A_4$ such that $B_3 \cong A_3$ and $2^n B_4 \lesssim R_R$.

Now $B_4 \lesssim 2^n B_4 \lesssim R_R$, so $B_4 \cong yR$ for some $y \in R$. Then as $2^n(yR) \lesssim R_R$, we obtain $N^*(y) \leqslant 1/2^n$ from Lemma 1.2. Similarly, since $2^n A_4 \lesssim 2^n A_2 \lesssim R_R$, we have $A_4 \cong xR$ for some $x \in R$ satisfying $N^*(x) \leqslant 1/2^n$. Finally,

$$A = A_1 \oplus A_2 = A_1 \oplus A_3 \oplus A_4 \cong A_1 \oplus A_3 \oplus xR,$$
$$B = B_1 \oplus B_2 = B_1 \oplus B_3 \oplus B_4 \cong A_1 \oplus A_3 \oplus yR,$$

hence we conclude that $v = [A] - [B] = [xR] - [yR]$. □

LEMMA 2.10. *Let R be an N^*-complete regular ring, let $x_1, x_2, \ldots \in R$, and assume that $N^*(x_n) < 1/2^n$ for all n. Then there exists $x \in R$ such that*

$$\|[x_1R] + \cdots + [x_nR] - [xR]\| \to 0$$

as $n \to \infty$.

PROOF. We may clearly assume that R is nonzero.

For each n, Proposition 1.1 provides us with positive integers s_n and t_n such that $s_n/t_n < 1/2^n$ and $t_n(x_nR) \lesssim s_nR_R$. Then since $2^n s_n < t_n$, we obtain $2^n s_n(x_nR) \lesssim s_nR_R$, and so $2^n(x_nR) \lesssim R_R$, using Theorem 2.6. Consequently,

$$2^n(x_1R \oplus \cdots \oplus x_nR) \lesssim 2^{n-1}R_R \oplus 2^{n-2}R_R \oplus \cdots \oplus 2R_R \oplus R_R \lesssim 2^nR_R,$$

whence $x_1R \oplus \cdots \oplus x_nR \lesssim R_R$ (Theorem 2.6 again).

As this holds for all n, we obtain $\oplus x_nR \lesssim R_R$, by [**2**, Proposition 4.8]. Thus R contains an ascending sequence $A_1 \leqslant A_2 \leqslant \ldots$ of principal right ideals such that each

$$A_n \cong x_1R \oplus \cdots \oplus x_nR.$$

By [**2**, Proposition 2.13], there exist orthogonal idempotents $e_1, e_2, \ldots$ in R such that

$$e_1R \oplus \cdots \oplus e_nR = A_n$$

for all n. Note that $e_nR \cong A_n/A_{n-1} \cong x_nR$, when $N^*(e_n) = N^*(x_n) < 1/2^n$ (Lemma 1.2).

Now the partial sums of the series $\Sigma\, e_n$ are Cauchy with respect to N^*, hence there exists $e \in R$ such that $\Sigma\, e_n \to e$ in the N^*-metric. Note that e is an idempotent, and $e_ne = ee_n = e_n$ for all n. For each n, we compute that

$$N^*(e - e_1 - \cdots - e_n) = N^*\left(\sum_{k=n+1}^{\infty} e_k\right) \leqslant \sum_{k=n+1}^{\infty} N^*(e_k) < \sum_{k=n+1}^{\infty} 1/2^k = 1/2^n,$$

whence $2^n((e - e_1 - \cdots - e_n)R) \lesssim R_R$, using Proposition 1.1 and Theorem 2.6 again. Inasmuch as

$$eR = e_1R \oplus \cdots \oplus e_nR \oplus (e - e_1 - \cdots - e_n)R,$$

it follows that $2^n(eR) \lesssim 2^n(e_1R \oplus \cdots \oplus e_nR) \oplus R_R$. Consequently,

$$2^n([e_1R] + \cdots + [e_nR]) \leqslant 2^n[eR] \leqslant 2^n([e_1R] + \cdots + [e_nR]) + [R]$$

in $K_0(R)$. Thus

$$0 \leqslant 2^n([eR] - [e_1R] - \cdots - [e_nR]) \leqslant [R],$$

from which we conclude that

$$\|[eR] - [e_1R] - \cdots - [e_nR]\| \leqslant 1/2^n.$$

Since each $[e_nR] = [x_nR]$, this proves that

$$\|[x_1R] + \cdots + [x_nR] - [eR]\| \to 0,$$

as desired. □

THEOREM 2.11. *If R is an N^*-complete regular ring, then $(K_0(R), [R])$ is an archimedean norm-complete interpolation group with order-unit.*

PROOF. Since R is unit-regular (Theorem 2.3), it follows that $K_0(R)$ is a partially ordered (rather than just pre-ordered) abelian group [**2**, Proposition 15.2], and that $K_0(R)$ is an interpolation group [**4**, Proposition II.10.3]. Given $x, y \in K_0(R)$ such that $nx \leqslant y$ for all $n \in \mathbf{N}$, choose finitely generated projective right R-modules A, B, C, D such that $x = [A] - [B]$ and $y = [C] - [D]$. Then

$$[nA] - [nB] = nx \leqslant y \leqslant [C]$$

and so $nA \lesssim nB \oplus C$, for all $n \in \mathbf{N}$. By Theorem 2.8, $A \lesssim B$, whence $x \leqslant 0$. Therefore $K_0(R)$ is archimedean. In particular, it now follows from [**4**, Proposition I.6.2] that the pseudo-metric on $K_0(R)$ induced by $\|\cdot\|$ is actually a metric.

Finally, consider a Cauchy sequence $\{v_1, v_2, \ldots\}$ in $K_0(R)$. By passing to a subsequence, we may assume that $\|v_{n+1} - v_n\| < 1/2^{n+1}$ for all n. Using Lemma 2.9, we obtain elements $x_n, y_n \in R$ such that

$$v_{n+1} - v_n = [x_nR] - [y_nR],$$

while also $N^*(x_n) \leqslant 1/2^{n+1}$ and $N^*(y_n) \leqslant 1/2^{n+1}$. According to Lemma 2.10, there exist elements $x, y \in R$ such that

$$\|[x_1R] + \cdots + [x_nR] - [xR]\| \to 0,$$
$$\|[y_1R] + \cdots + [y_nR] - [yR]\| \to 0.$$

Since $v_{n+1} - v_1 = [x_1R] + \cdots + [x_nR] - [y_1R] - \cdots - [y_nR]$ for all n, we conclude that

$$\|(v_{n+1} - v_1) - ([xR] - [yR])\| \to 0,$$

and consequently $v_{n+1} \to v_1 + [xR] - [yR]$. Therefore $(K_0(R), [R])$ is norm-complete. □

III. Archimedean norm-complete interpolation groups. Given an archimedean norm-complete interpolation group (G, u) with order-unit, we study the state space $S(G, u)$, and the relationship between G and the space of affine continuous real-valued functions on $S(G, u)$. In particular, we investigate extreme points and closed faces of $S(G, u)$, and relate them to ideals of G. These results will be applied, via K_0, to the ideal theory of N^*-complete regular rings.

DEFINITION. Let (G, u) be a partially ordered abelian group with order-unit. A *state on* (G, u) is any positive homomorphism s: $G \to \mathbf{R}$ such that $s(u) = 1$. The *state space of* (G, u), denoted $S(G, u)$, is the set of all states on (G, u). The state space is regarded as a subset of the linear topological space $\mathbf{R}^G$ (which is given the product topology), and as such is a compact convex set [**2**, Proposition 17.11]. If G is an interpolation group, then $S(G, u)$ is a Choquet simplex [**4**, Theorem I.2.5].

DEFINITION. The *extreme boundary* of a convex set S, denoted $\partial_e S$, is the set of all *extreme points* of S, that is, points $s \in S$ such that the only convex combinations $s = \alpha s' + (1-\alpha)s''$ with $0 \leqslant \alpha \leqslant 1$ and $s', s'' \in S$ are those for which $\alpha = 0$, or $\alpha = 1$, or $s' = s'' = s$. Now suppose that $S = S(G, u)$ for some partially ordered abelian group (G, u) with order-unit. An extreme state s in $\partial_e S$ is said to be *discrete* if $s(G)$ is a cyclic subgroup of $\mathbf{R}$. Note that if s is discrete, then $s(G) = (1/m)\mathbf{Z}$ for some $m \in \mathbf{N}$, because $1 = s(u) \in s(G)$. On the other hand, if s is not discrete, then $s(G)$ is dense in $\mathbf{R}$.

DEFINITION. We use $\mathrm{Aff}(S)$ to denote the partially ordered real Banach space of all affine continuous real-valued functions on S (with the pointwise ordering and the supremum norm). Evaluation at elements of G provides a map $\varphi\colon G \to \mathrm{Aff}(S)$, so that $\varphi(x)(s) = s(x)$ for all $x \in G$ and $s \in S$. Note that φ is a positive homomorphism, and that $\varphi(u)$ is the constant function 1. We refer to φ as *the natural map from G to* $\mathrm{Aff}(S)$. The map φ is also norm-preserving; namely,

$$\|\varphi(x)\| = \sup\{|s(x)| : s \in S\} = \|x\|$$

for all $x \in G$ [**4**, Lemma I.6.1].

THEOREM 3.1. *Let (G, u) be an archimedean norm-complete interpolation group with order-unit, and set $S = S(G, u)$. For all discrete $s \in \partial_e S$, set $A_s = s(G)$; for all other $s \in \partial_e S$, set $A_s = \mathbf{R}$. Set*

$$A = \{p \in \mathrm{Aff}(S) \mid p(s) \in A_s \textit{ for all } s \in \partial_e S\}.$$

Then the natural map from G to $\mathrm{Aff}(S)$ *provides an isomorphism of (G, u) onto $(A, 1)$ (as partially ordered abelian groups with order-unit).*

PROOF. [**3**, Theorem 5.1]. □

COROLLARY 3.2. *Let (G, u) be an archimedean norm-complete interpolation group with order-unit. Then G is lattice-ordered if and only if $\partial_e S(G, u)$ is compact.*

PROOF. [**3**, Corollary 5.4]. □

In order to apply Theorem 3.1 effectively, we must be able to identify $\partial_e S$ easily. Thus we develop criteria for deciding when states are extreme. For topological considerations, we also develop similar results for compact sets of extreme states, and for closed faces of S. We begin with a result relating the archimedean property to norm properties.

PROPOSITION 3.3. *Let (G, u) be an interpolation group with order-unit. Then G is archimedean if and only if G is 2-unperforated and G^+ is norm-closed in G.*

PROOF. If G is archimedean, then G is unperforated by [**4**, Lemma I.5.2]. Now consider elements $x_1, x_2, \ldots$ in G^+ and $x \in G$ such that $x_n \to x$ in norm. We may assume that $\|x_n - x\| < 1/n$ for all n. According to [**4**, Proposition I.6.2], $n(x_n - x) \leqslant u$, and consequently $n(-x) \leqslant u$. Since this holds for all $n \in \mathbf{N}$, the archimedean property implies that $-x \leqslant 0$, so that $x \in G^+$. Thus G^+ is norm-closed in G.

Conversely, assume that G is 2-unperforated and that G^+ is norm-closed in G. Given $a, b \in G$ such that $na \leqslant b$ for all $n \in \mathbf{N}$, we must show that $a \leqslant 0$. Write $a = x - y$ for some $x, y \in G^+$, and choose $z \in G^+$ with $b \leqslant z$. Then $nx \leqslant ny + z$ for all $n \in \mathbf{N}$, and we must show that $x \leqslant y$.

For all $n \in \mathbf{N}$, we have $2^n x \leqslant 2^n y + z$, hence [**4**, Lemma I.5.7] says that $x = v_n + w_n$ for some $v_n, w_n \in G^+$ such that $v_n \leqslant y$ and $2^n w_n \leqslant z$. Then $\|w_n\| \leqslant \|z\|/2^n$, so that $w_n \to 0$, and consequently $v_n \to x$. Thus $y - v_n \to y - x$. Since each $y - v_n$ is in G^+, we conclude that $y - x$ is in G^+, as desired. Therefore G is archimedean. $\square$

In particular, if (G, u) is an archimedean interpolation group with order-unit, and we have norm-convergent sequences $x_n \to x$ and $y_n \to y$ in G with $x_n \leqslant y_n$ for all n, then $x \leqslant y$.

THEOREM 3.4. *Let (G, u) be an interpolation group with order-unit, and let $s \in S(G, u)$. Then s is an extreme point of $S(G, u)$ if and only if*

$$\min\{s(x), s(y)\} = \sup\{s(z) \mid z \in G^+ ; z \leqslant x; z \leqslant y\}$$

for all $x, y \in G^+$.

PROOF. [**3**, Theorem 3.1]. $\square$

COROLLARY 3.5. *Let (G, u) be an interpolation group with order-unit, and let X be a compact subset of $\partial_e S(G, u)$. Given $x, y \in G^+$ and a positive real number ε, there exists $z \in G^+$ such that $z \leqslant x$ and $z \leqslant y$, while also*

$$s(z) > \min\{s(x), s(y)\} - \varepsilon$$

for all $s \in X$.

PROOF. Set $W = \{w \in G^+ \mid w \leqslant x \text{ and } w \leqslant y\}$, and note, because of the interpolation property, that W is upward directed. For each $w \in W$, set

$$V(w) = \{s \in S(G, u) \mid s(w) > \min\{s(x), s(y)\} - \varepsilon\},$$

which is an open subset of $S(G, u)$. In view of Theorem 3.4, we see that these $V(w)$'s cover $\partial_e S(G, u)$, and so cover X. As X is compact, it follows that

$$X \subseteq V(w_1) \cup \cdots \cup V(w_n)$$

for some $w_1, \ldots, w_n \in W$. Since W is upward directed, there exists $z \in W$ such that all $w_i \leqslant z$, and z has the desired properties. $\square$

COROLLARY 3.6. *Let (G, u) be an interpolation group with order-unit, let $a, b \in G^+$, and let $m \in \mathbf{N}$. Let X be a compact subset of $\partial_e S(G, u)$, and assume that $s(ma) \leqslant s(b)$ for all $s \in X$. Given any positive real number ε, there exists $c \in G^+$ such that $c \leqslant a$ and $mc \leqslant b$, while also $s(c) > s(a) - \varepsilon$ for all $s \in X$.*

PROOF. By Corollary 3.5, there exists $x \in G^+$ such that $x \leqslant ma$ and $x \leqslant b$, while also $s(x) > s(ma) - \varepsilon$ for all $s \in X$. Then, using Riesz decomposition, $x = x_1 + \cdots + x_m$ for some $x_i \in G^+$ satisfying $x_i \leqslant a$. For each i, note that $x - x_i \leqslant (m-1)a$, whence $x - (m-1)a \leqslant x_i$. Since $0 \leqslant x_i$ for each i as well, interpolation provides an element $c \in G$ such that

$$x - (m-1)a \leqslant c \leqslant x_i \quad \text{and} \quad 0 \leqslant c \leqslant x_i$$

for all i. Thus $c \in G^+$ and $c \leqslant x_1 \leqslant a$, while

$$mc \leqslant x_1 + \cdots + x_m = x \leqslant b.$$

As $x - (m-1)a \leqslant c$, we have $a - c \leqslant ma - x$, whence

$$s(a) - s(c) \leqslant s(ma) - s(x) < \varepsilon$$

for all $s \in X$. Therefore $s(c) > s(a) - \varepsilon$ for all $s \in X$. □

THEOREM 3.7. *Let (G, u) be an archimedean norm-complete interpolation group with order-unit, and let X be a compact subset of $\partial_e S(G, u)$. Given any $x, y \in G$, there exist $z, w \in G$ such that $z \leqslant x \leqslant w$ and $z \leqslant y \leqslant w$, while also*

$$s(z) = \min\{s(x), s(y)\} \qquad \text{and} \qquad s(w) = \max\{s(x), s(y)\}$$

for all $s \in X$. If $x, y \in G^+$, then such z, w can be found in G^+.

PROOF. First assume that $x, y \in G^+$. The rule $p(s) = \min\{s(x), s(y)\}$ defines a continuous map p of $S(G, u)$ into $\mathbf{R}^+$. We construct elements $z_1 \leqslant z_2 \leqslant \ldots$ in G^+ such that each $z_n \leqslant x$ and $z_n \leqslant y$, while also

$$s(z_n) > p(s) - (1/2^n) \qquad \text{and} \qquad \|z_{n+1} - z_n\| \leqslant 1/2^n$$

for all n and all $s \in X$. To begin, we obtain z_1 directly from Corollary 3.5.

Now assume that $z_1, \ldots, z_n$ have been constructed, for some n. According to Corollary 3.5, there exists $a \in G^+$ such that $a \leqslant x$ and $a \leqslant y$, while also

$$s(a) > p(s) - (1/2^{n+2})$$

for all $s \in X$. Since $z_n \leqslant x$ and $z_n \leqslant y$ as well, there is some $b \in G^+$ satisfying $a \leqslant b \leqslant x$ and $z_n \leqslant b \leqslant y$. Note that

$$s(b) \geqslant s(a) > p(s) - (1/2^{n+2})$$

for all $s \in X$.

The element $b - z_n$ lies in G^+, and for all $s \in X$ we have

$$s(z_n) > \min\{s(x), s(y)\} - (1/2^n) \geqslant s(b) - (1/2^n),$$

whence $s(2^n(b - z_n)) < 1 = s(u)$. By Corollary 3.6, there exists $c \in G^+$ such that $c \leqslant b - z_n$ and $2^n c \leqslant u$, while also

$$s(c) > s(b - z_n) - (1/2^{n+2})$$

for all $s \in X$. Set $z_{n+1} = z_n + c$, noting that $z_n \leqslant z_{n+1} \leqslant b \leqslant x$ and $z_{n+1} \leqslant b \leqslant y$. For all $s \in X$, we have

$$s(z_{n+1}) = s(z_n) + s(c) > s(b) - (1/2^{n+2}) > p(s) - (1/2^{n+1}).$$

Since $0 \leqslant 2^n(z_{n+1} - z_n) = 2^n c \leqslant u$, we also have $\|z_{n+1} - z_n\| \leqslant 1/2^n$, which completes the induction step.

Having constructed a Cauchy sequence $\{z_1, z_2, \ldots\}$ in G, we must have $z_n \to z$ for some $z \in G$. Since $0 \leqslant z_n \leqslant x$ and $0 \leqslant z_n \leqslant y$ for all n, we obtain $0 \leqslant z \leqslant x$ and $0 \leqslant z \leqslant y$. On the other hand, $z_k \geqslant z_n$ whenever $k \geqslant n$, hence $z \geqslant z_n$ for all n. Consequently, for any $s \in X$,

$$\min\{s(x), s(y)\} \geqslant s(z) \geqslant s(z_n) > \min\{s(x), s(y)\} - (1/2^n),$$

and thus $s(z) = \min\{s(x), s(y)\}$.

Now consider arbitrary elements $x, y \in G$, and choose $a \in G^+$ such that $x + a \geqslant 0$ and $y + a \geqslant 0$. By the above, there exists $b \in G^+$ such that $b \leqslant x + a$ and $b \leqslant y + a$, while also

$$s(b) = \min\{s(x + a), s(y + a)\}$$

for all $s \in X$. Setting $z = b - a$, we obtain $z \leqslant x$ and $z \leqslant y$, while

$$s(z) = \min\{s(x), s(y)\}$$

for all $s \in X$.

Finally, use the result above to obtain $c \in G$ such that $c \leqslant -x$ and $c \leqslant -y$, while also

$$s(c) = \min\{s(-x), s(-y)\}$$

for all $s \in X$. Set $w = -c$. $\square$

COROLLARY 3.8. *If (G, u) is an archimedean norm-complete interpolation group with order-unit, and X is a compact subset of $\partial_e S(G, u)$, then* $\ker(X)$ *is an ideal of G, and $G/\ker(X)$ is a lattice-ordered abelian group.*

PROOF. Set $K = \ker(X)$. Clearly K is a convex subgroup of G. Given $x \in K$, we have $s(x) = s(0)$ for all $s \in X$. According to Theorem 3.7, there exists $z \in G$ such that $z \leqslant x$ and $z \leqslant 0$, while also $s(z) = s(x) = s(0)$ for all $s \in X$. Thus $x - z$ and $-z$ are elements of $G^+ \cap K$ satisfying $(x - z) - (-z) = x$, which proves that K is a directed subgroup of G. Therefore K is an ideal of G.

Now consider any elements $x + K$ and $y + K$ in G/K. According to Theorem 3.7, there exists $z \in G$ such that $z \leqslant x$ and $z \leqslant y$, while also $s(z) = \min\{s(x), s(y)\}$ for all $s \in X$. Then $z + K \leqslant x + K$ and $z + K \leqslant y + K$, and we claim that $z + K$ is the infimum of $x + K$ and $y + K$ in G/K.

Given $a + K$ in G/K satisfying $a + K \leqslant x + K$ and $a + K \leqslant y + K$, we have $a \leqslant x + k'$ and $a \leqslant y + k''$ for some $k', k'' \in K$. Since K is directed, we may choose $k \in K$ such that $k' \leqslant k$ and $k'' \leqslant k$, whence $a \leqslant x + k$ and $a \leqslant y + k$. Now

$$a - k \leqslant x; \qquad a - k \leqslant y; \qquad z \leqslant x; \qquad z \leqslant y.$$

Interpolating, we obtain $b \in G$ such that $a - k \leqslant b \leqslant x$ and $z \leqslant b \leqslant y$. In particular,

$$s(z) \leqslant s(b) \leqslant \min\{s(x), s(y)\} = s(z)$$

for all $s \in X$, whence $b - z \in K$. Thus

$$a + K = (a - k) + K \leqslant b + K = z + K,$$

proving that $z + K$ is indeed the infimum of $x + K$ and $y + K$.

Therefore G/K is lattice-ordered. $\square$

We thank the referee for pointing out that the hypothesis of norm-completeness in Theorem 3.7 and Corollary 3.8 is essential, as the following example shows. (This is a simplified version of the referee's example.)

EXAMPLE 3.9. There exists an archimedean interpolation group (G, u) with order-unit possessing an extreme state s such that $\ker(s) = \{0\}$ but G is not lattice-ordered.

PROOF. Let G be the $\mathbf{Q}$-subspace of $\mathbf{R}^2$ spanned by the vectors $u = (1, 1)$ and $v = (\pi, -\pi)$. Note that since u and v are $\mathbf{R}$-linearly independent, G is dense in $\mathbf{R}^2$ in the usual Euclidean topology. Give $\mathbf{R}^2$ the direct product ordering, and give G the relative ordering inherited from $\mathbf{R}^2$, so that $G^+ = G \cap (\mathbf{R}^2)^+$. Then $\mathbf{R}^2$ and G are archimedean partially ordered abelian groups, and u is an order-unit in each of them.

To check interpolation, consider elements x_1, x_2, y_1, y_2 in G satisfying $x_i \leqslant y_j$ for all i, j. Then each

$$x_i = (a_i + b_i\pi, a_i - b_i\pi) \quad \text{and} \quad y_j = (c_j + d_j\pi, c_j - d_j\pi)$$

for suitable $a_i, b_i, c_j, d_j \in \mathbf{Q}$. First assume that $a_i = c_j$ for some i, j, say $a_1 = c_1$. Since

$$a_1 + b_1\pi \leqslant c_1 + d_1\pi \quad \text{and} \quad a_1 - b_1\pi \leqslant c_1 - d_1\pi,$$

it follows that $b_1 = d_1$, whence $x_1 = y_1$. In this case, $x_i \leqslant x_1 \leqslant y_j$ for all i, j. Now assume that $a_i \neq c_j$ for all i, j. Then

$$a_i + b_i\pi \neq c_j + d_j\pi \quad \text{and} \quad a_i - b_i\pi \neq c_j - d_j\pi$$

for all i, j. Consequently, the set W consisting of those (α, β) in $\mathbf{R}^2$ satisfying

$$a_i + b_i\pi < \alpha < c_j + d_j\pi \quad \text{and} \quad a_i - b_i\pi < \beta < c_j - d_j\pi$$

for all i, j is a nonempty open subset of $\mathbf{R}^2$. Since G is dense in $\mathbf{R}^2$, there exists an element z in $G \cap W$, and $x_i \leqslant z \leqslant y_j$ for all i, j. Therefore G is an interpolation group.

Define $t\colon \mathbf{R}^2 \to \mathbf{R}$ by the rule $t(\alpha, \beta) = \alpha$, and note that t is an extreme point of $S(\mathbf{R}^2, u)$. Let s denote the restriction of t to G. As $G^+ = G \cap (\mathbf{R}^2)^+$, all states in $S(G, u)$ extend to states in $S(\mathbf{R}^2, u)$, by [**2**, Proposition 18.1]. Thus the restriction map $S(\mathbf{R}^2, u) \to S(G, u)$ is an affine homeomorphism, from which we see that s is an extreme point of $S(G, u)$. It is clear that $\ker(s) = \{0\}$.

Consider any $x \in G$ satisfying $x \leqslant u$ and $x \leqslant v$. Then $x = (a + b\pi, a - b\pi)$ for some $a, b \in \mathbf{Q}$, and from the relations $x \leqslant u$ and $x \leqslant v$ we obtain $a \leqslant \alpha$, where

$$\alpha = \min\{1 - b\pi, 1 + b\pi, (1 - b)\pi, (b - 1)\pi\}.$$

Note that α must be irrational. Thus $a < \alpha$, and we may choose $c \in \mathbf{Q}$ such that $a < c < \alpha$. Setting $y = (c + b\pi, c - b\pi)$, we conclude that $x < y$ while also $y \leqslant u$ and $y \leqslant v$. Therefore the set $\{u, v\}$ has no infimum in G, proving that G is not lattice-ordered. $\square$

In case the set X in Corollary 3.8 is a singleton, we can precisely identify the quotient partially ordered abelian group $G/\ker(X)$, as follows.

THEOREM 3.10. *Let (G, u) be an archimedean norm-complete interpolation group with order-unit, and let $s \in \partial_e S(G, u)$. Then $\ker(s)$ is an ideal of G, and s induces an isomorphism of $G/\ker(s)$ onto $s(G)$ as partially ordered abelian groups. Moreover, either $s(G) = \mathbf{R}$ or $s(G) = (1/m)\mathbf{Z}$ for some $m \in \mathbf{N}$.*

PROOF. The subgroup $K = \ker(s)$ is an ideal of G by Corollary 3.8. Obviously the induced map $\bar{s}\colon G/K \to s(G)$ is a group isomorphism, and a positive map as well. Given an element $x + K$ in G/K with $\bar{s}(x + K) \geqslant 0$, we have $s(x) \geqslant 0 = s(0)$. Then

Theorem 3.7 provides us with an element $z \in G$ such that $z \leqslant x$ and $z \leqslant 0$, while also $s(z) = 0$. Now $z \in K$ and $x - z \geqslant 0$, hence

$$x + K = (x - z) + K \geqslant 0$$

as well. Thus for all $a \in G/K$, we have $a \geqslant 0$ if and only if $\bar{s}(a) \geqslant 0$. Therefore $\bar{s}$ is an isomorphism of partially ordered abelian groups.

If s is discrete, then $s(G) = (1/m)\mathbf{Z}$ for some $m \in \mathbf{N}$. Now assume that s is not discrete, so that $s(G)$ is dense in $\mathbf{R}$.

Given $\alpha \in \mathbf{R}$, we construct elements $x_1, x_2, \ldots$ in G such that

$$\alpha - (1/2^n) < s(x_n) < \alpha \qquad \text{and} \qquad \|x_{n+1} - x_n\| \leqslant 1/2^n$$

for all n. To begin, we obtain x_1 from the density of $s(G)$ in $\mathbf{R}$.

Now assume that $x_1, \ldots, x_n$ have been constructed, for some n. Choose an element $a \in G$ such that

$$\alpha - (1/2^{n+2}) < s(a) < \alpha.$$

By Theorem 3.7, there exists $b \in G$ such that $b \geqslant x_n$ and $b \geqslant a$, while also $s(b) = \max\{s(x_n), s(a)\}$. Note that

$$\alpha - (1/2^{n+2}) < s(b) < \alpha.$$

Since $s(b) < \alpha$ and $s(x_n) > \alpha - (1/2^n)$, we find that

$$s(2^n b) < 2^n \alpha < s(2^n x_n) + 1,$$

whence $s(2^n(b - x_n)) < 1 = s(u)$. As $b - x_n \in G^+$, Corollary 3.6 provides us with an element $c \in G^+$ such that $c \leqslant b - x_n$ and $2^n c \leqslant u$, while also

$$s(c) > s(b - x_n) - (1/2^{n+2}).$$

Set $x_{n+1} = x_n + c$, so that $x_n \leqslant x_{n+1} \leqslant b$. Thus $s(x_{n+1}) \leqslant s(b) < \alpha$, and

$$s(x_{n+1}) = s(x_n) + s(c) > s(b) - (1/2^{n+2}) > \alpha - (1/2^{n+1}).$$

In addition, $\|x_{n+1} - x_n\| = \|c\| \leqslant 1/2^n$, which completes the induction step.

Now there exists $x \in G$ such that $x_n \to x$ in norm. Since

$$|s(x_n) - s(x)| = |s(x_n - x)| \leqslant \|x_n - x\|$$

for all n [**4**, Lemma I.6.1], it follows that $s(x_n) \to s(x)$, and thus $s(x) = \alpha$. Therefore $s(G) = \mathbf{R}$, as desired. □

DEFINITION. A *face* of a convex set S is a convex subset $F \subseteq S$ (possibly empty) such that whenever $s = \alpha s' + (1 - \alpha)s''$ is a positive convex combination with $s \in F$ and $s', s'' \in S$, then $s', s'' \in F$.

LEMMA 3.11. *Let (G, u) be an interpolation group with order-unit, and let X be either a compact subset of $\partial_e S(G, u)$ or a closed face of $S(G, u)$. Let $t \in \partial_e S(G, u)$ such that $t \notin X$. Then there exists $x \in G^+$ such that $t(x) > 1$ but $s(x) < 1$ for all $s \in X$.*

PROOF. Set $A = \{a \in G^+ \mid t(a) > 1\}$, and note that $2u \in A$. Also, A is downward directed, by Theorem 3.4. For all $a \in A$, set

$$W(a) = \{s \in S(G, u) \mid s(a) < 1\},$$

which is an open subset of $S(G, u)$. We claim that these $W(a)$'s cover X.

Thus consider any $s \in X$. Since $\{t\}$ is a face of $S(G, u)$, and either $\{s\}$ or X is a face of $S(G, u)$, we see that s and t lie in disjoint faces of $S(G, u)$. By [**3**, Lemma 2.8], $2u = a + b$ for some $a, b \in G^+$ such that $s(a) + t(b) < 1$. Then $t(a) = 2 - t(b) > 1$, whence $a \in A$ and $s \in W(a)$. Therefore the $W(a)$'s do cover X, as claimed.

By compactness, $X \subseteq W(a_1) \cup \cdots \cup W(a_n)$ for some elements $a_i \in A$. As A is downward directed, there exists $x \in A$ such that each $a_i \geqslant x$, and x has the desired properties. □

THEOREM 3.12. *Let (G, u) be an archimedean norm-complete interpolation group with order-unit, and let X be a compact subset of $\partial_e S(G, u)$. Then*

$$X = \{s \in \partial_e S(G, u) \mid G^+ \cap \ker(X) \subseteq \ker(s)\}.$$

PROOF. Consider $t \in \partial_e S(G, u)$ such that $t \notin X$. By Lemma 3.11, there exists $x \in G^+$ such that $t(x) > 1$ but $s(x) < 1$ for all $s \in X$. Applying Theorem 3.7 to the elements $x, u \in G^+$ and the compact subset $X \cup \{t\}$ of $\partial_e S(G, u)$, we obtain $y \in G^+$ such that $y \geqslant x$ and $y \geqslant u$, while $t(y) > 1$ and $s(y) = 1$ for all $s \in X$. Consequently, $y - u$ is an element of $G^+ \cap \ker(X)$ which does not lie in $\ker(t)$. □

We now turn to closed faces of state spaces. The results above, concerning compact sets of extreme states, carry over fairly directly, with similar proofs.

THEOREM 3.13. *Let (G, u) be an interpolation group with order-unit, let $x, y \in G^+$, and let F be a closed face of $S(G, u)$. Assume that $s(x) \leqslant s(y)$ for all $s \in F$. Given any positive real number ε, there exists $z \in G^+$ such that $z \leqslant x$ and $z \leqslant y$, while also $s(z) > s(x) - \varepsilon$ for all $s \in F$.*

PROOF. [**3**, Theorem 3.4]. □

COROLLARY 3.14. *Let (G, u) be an interpolation group with order-unit, and let F be a closed face of $S(G, u)$. Let $a, b \in G^+$ and $m \in \mathbf{N}$, and assume that $s(ma) \leqslant s(b)$ for all $s \in F$. Given any positive real number ε, there exists $c \in G^+$ such that $c \leqslant a$ and $mc \leqslant b$, while also $s(c) > s(a) - \varepsilon$ for all $s \in F$.*

PROOF. As Corollary 3.6, using Theorem 3.13 in place of Corollary 3.5. □

THEOREM 3.15. *Let (G, u) be an archimedean norm-complete interpolation group with order-unit, let $x, y \in G$, and let F be a closed face of $S(G, u)$. If $s(x) \leqslant s(y)$ for all $s \in F$, then there exist $z, w \in G$ such that $z \leqslant x \leqslant w$ and $z \leqslant y \leqslant w$, while also $s(z) = s(x)$ and $s(w) = s(y)$ for all $s \in F$. If $x, y \in G^+$, then such z, w can be found in G^+.*

PROOF. As Theorem 3.7, using Theorem 3.13 and Corollary 3.14 in place of Corollaries 3.5 and 3.6. □

COROLLARY 3.16. *If (G, u) is an archimedean norm-complete interpolation group with order-unit, and F is a closed face of $S(G, u)$, then $\ker(F)$ is an ideal of G.*

PROOF. As Corollary 3.8, using Theorem 3.15 in place of Theorem 3.7. □

THEOREM 3.17. *Let (G, u) be an archimedean norm-complete interpolation group with order-unit, and let F be a closed face of $S(G, u)$. Then*

$$F = \{s \in S(G, u) \mid G^+ \cap \ker(F) \subseteq \ker(s)\}.$$

PROOF. Set $F' = \{s \in S(G, u) \mid G^+ \cap \ker(F) \subseteq \ker(s)\}$. Clearly F' is a closed convex subset of $S(G, u)$, and we claim that F' is a face of $S(G, u)$ as well. Thus consider any positive convex combination $s = \alpha s' + (1 - \alpha)s''$ in $S(G, u)$ with $s \in F'$. For any $x \in G^+ \cap \ker(F)$, we have

$$\alpha s'(x) + (1 - \alpha)s''(x) = 0; \qquad s'(x) \geqslant 0; \qquad s''(x) \geqslant 0$$

and so $s'(x) = s''(x) = 0$. Therefore $s', s'' \in F'$, proving that F' is indeed a face of $S(G, u)$.

Being a compact convex set, F' equals the closure of the convex hull of its extreme boundary $\partial_e F'$, by the Krein-Milman Theorem. Thus if $F' \nsubseteq F$, there must be some $t \in \partial_e F'$ which does not lie in F. As F' is a face of $S(G, u)$, we see that actually t is an extreme point of $S(G, u)$.

By Lemma 3.11, there exists $x \in G^+$ such that $t(x) > 1$ but $s(x) < 1$ for all $s \in F$. According to Theorem 3.15, there exists $w \in G^+$ such that $x \leqslant w$ and $u \leqslant w$, while also $s(w) = 1$ for all $s \in F$. Thus $w - u$ is an element of $G^+ \cap \ker(F)$. On the other hand, $t(w) \geqslant t(x) > 1$ and so $w - u$ is not in $\ker(t)$, which contradicts the fact that $t \in F'$.

Therefore $F' \subseteq F$. The reverse inclusion is automatic, hence $F = F'$, as desired. □

IV. ***N*-complete regular rings.*** We apply the results of §III, via K_0, to the structure of N^*-complete regular rings R. The thrust of most of these results is that R behaves much like a ring of sections of a sheaf of simple self-injective rings. Namely, for every maximal two-sided ideal M of R, the factor ring R/M is right and left self-injective, and many properties of R and its projective modules are determined by what happens modulo these maximal two-sided ideals. For instance, R is an $n \times n$ matrix ring (for some fixed $n \in \mathbf{N}$) if and only if each R/M is an $n \times n$ matrix ring. For another example, given finitely generated projective right R-modules A and B, we have $A \cong B$ if and only if $A/AM \cong B/BM$ for all M. Many of our results generalize parallel results for $\aleph_0$-continuous regular rings in [**4**, **7**, **8**]. At the end of the section, we indicate how our results relate to these papers.

All the results of this section depend on Theorems 2.3 and 2.11: That any N^*-complete regular ring R is unit-regular, and that for such rings $(K_0(R), [R])$ is an archimedean norm-complete interpolation group with order-unit. We shall use these results repeatedly without further reference to them. One other basic result is needed, to identify the state space of $(K_0(R), [R])$.

PROPOSITION 4.1. *For any regular ring R, there is a natural affine homeomorphism θ: $S(K_0(R), [R]) \to \mathbf{P}(R)$ such that $\theta(s)(x) = s([xR])$ for all s in $S(K_0(R), [R])$ and all $x \in R$.*

PROOF. [**2**, Proposition 17.12]. □

THEOREM 4.2. *Let R be an N^*-complete regular ring.*

(a) *If X is a compact subset of $\partial_e\mathbf{P}(R)$, then*

$$X = \{P \in \partial_e\mathbf{P}(R) \mid \ker(X) \subseteq \ker(P)\}.$$

(b) *If F is a closed face of $\mathbf{P}(R)$, then*

$$F = \{P \in \mathbf{P}(R) \mid \ker(F) \subseteq \ker(P)\}.$$

PROOF. Set $S = S(K_0(R),[R])$, and let $\theta: S \to \mathbf{P}(R)$ be the affine homeomorphism given in Proposition 4.1.

(a) Set $Y = \theta^{-1}(X)$, which is a compact subset of $\partial_e S$. Let $Q \in \partial_e\mathbf{P}(R)$ such that $\ker(X) \subseteq \ker(Q)$, and set $t = \theta^{-1}(Q)$. We shall show that $K_0(R)^+ \cap \ker(Y)$ is contained in $\ker(t)$.

Given $a \in K_0(R)^+ \cap \ker(Y)$, we have $a = [A]$ for some finitely generated projective right R-module A. Then

$$A \cong x_1R \oplus \cdots \oplus x_nR \qquad \text{and} \qquad [A] = [x_1R] + \cdots + [x_nR]$$

for some elements $x_i \in R$. For each i, we have $0 \leqslant [x_iR] \leqslant [A]$, whence $s([x_iR]) = 0$ for all $s \in Y$, and so $P(x_i) = 0$ for all $P \in X$. Then each x_i lies in $\ker(X)$, hence $x_i \in \ker(Q)$, and so $t([x_iR]) = 0$. Consequently,

$$t(a) = t([x_1R]) + \cdots + t([x_nR]) = 0.$$

Therefore $K_0(R)^+ \cap \ker(Y) \subseteq \ker(t)$, as claimed.

Now Theorem 3.12 shows that $t \in Y$, and therefore $Q \in X$.

(b) As (a), using Theorem 3.17 in place of Theorem 3.12. □

COROLLARY 4.3. *If M is a maximal two-sided ideal in an N^*-complete regular ring R, then R/M is a simple, unit-regular, right and left self-injective ring. There is a unique rank function on R/M, and R/M is complete in the rank-metric.*

PROOF. According to Corollary 1.14, R/M is N^*-complete, hence there is no loss of generality in assuming that $M = 0$.

Since R is a nonzero unit-regular ring, [**2**, Corollary 18.5] shows that $\mathbf{P}(R)$ is nonempty. By the Krein-Milman Theorem, there exists at least one P in $\partial_e\mathbf{P}(R)$. Note that $\ker(P) = 0$, because R is simple. Inasmuch as $\{P\}$ is a closed face of $\mathbf{P}(R)$, we now conclude from Theorem 4.2(b) that $\mathbf{P}(R) = \{P\}$. Thus P is the unique rank function on R.

Now $N^* = P$, hence the N^*-metric on R coincides with the P-metric. Therefore R is complete in the P-metric. According to [**2**, Theorem 19.7], R is thus right and left self-injective. □

COROLLARY 4.4. *Let R be an N^*-complete regular ring, and let $P \in \mathbf{P}(R)$. Then the following conditions are equivalent.*

(a) *P is an extreme point of $\mathbf{P}(R)$.*

(b) $\ker(P)$ *is a maximal two-sided ideal of R.*

(c) *There is a unique pseudo-rank function on $R/\ker(P)$.*

PROOF. Let $\bar{P}$ denote the rank function on $R/\ker(P)$ induced by P.

(a) $\Rightarrow$ (c): Since $\{P\}$ is a closed face of $\mathbf{P}(R)$, Theorem 4.2(b) says that P is the only pseudo-rank function on R whose kernel contains $\ker(P)$. Thus $\bar{P}$ is the only pseudo-rank function on $R/\ker(P)$.

(c) $\Rightarrow$ (b): Since $R/\ker(P)$ is a unit-regular ring possessing a unique rank function, [**2**, Corollary 18.6] shows that $R/\ker(P)$ is a simple ring.

(b) $\Rightarrow$ (a): By Corollary 4.3, $\bar{P}$ is the only rank function on $R/\ker(P)$, hence P is the only pseudo-rank function on R whose kernel contains $\ker(P)$. Given a positive convex combination $P = \alpha P_1 + (1-\alpha)P_2$ in $\mathbf{P}(R)$, we see that $\ker(P) \subseteq \ker(P_1)$ because $\alpha > 0$, whence $P_1 = P$, and similarly $P_2 = P$. Therefore P is an extreme point of $\mathbf{P}(R)$. $\square$

With the help of Corollaries 4.3 and 4.4, we can show that in any N^*-complete regular ring R, there is a natural bijection between $\partial_e\mathbf{P}(R)$ and the set of maximal two-sided ideals of R. In fact, this bijection is continuous with respect to the usual topology on the maximal ideal space, as follows.

DEFINITION. For any ring R, we use $\operatorname{MaxSpec}(R)$ to denote the family of all maximal two-sided ideals of R, equipped with the usual hull-kernel topology.

THEOREM 4.5. *Let R be an N^*-complete regular ring.*

(a) *There is a continuous bijection θ: $\partial_e\mathbf{P}(R) \to \operatorname{MaxSpec}(R)$ given by the rule $\theta(P) = \ker(P)$.*

(b) *θ maps compact subsets of $\partial_e\mathbf{P}(R)$ onto closed subsets of* $\operatorname{MaxSpec}(R)$.

(c) *θ is a homeomorphism if and only if $\partial_e\mathbf{P}(R)$ is compact, if and only if* $\operatorname{MaxSpec}(R)$ *is Hausdorff.*

PROOF. (a) It is clear from Corollaries 4.3 and 4.4 that θ defines a bijection of $\partial_e\mathbf{P}(R)$ onto $\operatorname{MaxSpec}(R)$. If X is a closed subset of $\operatorname{MaxSpec}(R)$, then

$$X = \{M \in \operatorname{MaxSpec}(R) \mid Y \subseteq M\}$$

for some $Y \subseteq R$. Consequently,

$$\theta^{-1}(X) = \{P \in \partial_e\mathbf{P}(R) \mid P(y) = 0 \text{ for all } y \in Y\},$$

which is closed in $\partial_e\mathbf{P}(R)$. Therefore θ is continuous.

(b) If X is a compact subset of $\partial_e\mathbf{P}(R)$, then

$$X = \{P \in \partial_e\mathbf{P}(R) \mid \ker(X) \subseteq \theta(P)\},$$

by Theorem 4.2(a). As a result,

$$\theta(X) = \{M \in \operatorname{MaxSpec}(R) \mid \ker(X) \subseteq M\},$$

which is closed in $\operatorname{MaxSpec}(R)$.

(c) Note that $\partial_e\mathbf{P}(R)$ is Hausdorff while $\operatorname{MaxSpec}(R)$ is compact. Thus if θ is a homeomorphism, then $\partial_e\mathbf{P}(R)$ must be compact and $\operatorname{MaxSpec}(R)$ must be Hausdorff. On the other hand, if $\partial_e\mathbf{P}(R)$ is compact, then we see from (b) that θ is a closed map, whence θ is a homeomorphism.

Now assume that $\operatorname{MaxSpec}(R)$ is Hausdorff. We shall show that $\partial_e\mathbf{P}(R)$ is compact, by showing that $\partial_e\mathbf{P}(R)$ is closed in $\mathbf{P}(R)$.

If not, then some $P \in \mathbf{P}(R)$ lies in the closure of $\partial_e \mathbf{P}(R)$ but not in $\partial_e \mathbf{P}(R)$. Set $K = \ker(P)$, and note from Corollary 4.4 that there exist more than one pseudo-rank functions on R/K. Because of the Krein-Milman Theorem, there must exist at least two extreme points in $\mathbf{P}(R/K)$. As R/K is N^*-complete (Theorem 1.13), it follows from (a) that R/K has at least two maximal two-sided ideals. Thus there exist distinct M_1 and M_2 in MaxSpec(R) which contain K.

Since MaxSpec(R) is Hausdorff, there exist disjoint open sets V_1 and V_2 in MaxSpec(R) such that each $M_i \in V_i$. There are subsets $X_i \subseteq R$ such that each

$$V_i = \{M \in \operatorname{MaxSpec}(R) \mid X_i \nsubseteq M\}.$$

For each i, choose $x_i \in X_i$ such that $x_i \notin M_i$. Note that if M is in MaxSpec(R) and $x_i \notin M$, then $M \in V_i$. Thus for any M in MaxSpec(R), we must have either $x_1 \in M$ or $x_2 \in M$.

Inasmuch as $x_i \notin M_i$, we have $x_i \notin K$. Set

$$W = \{Q \in \mathbf{P}(R) \mid Q(x_1) > 0 \text{ and } Q(x_2) > 0\},$$

which is an open subset of $\mathbf{P}(R)$. Note that $P \in W$, because each $x_i \notin \ker(P)$. Since P lies in the closure of $\partial_e \mathbf{P}(R)$, there exists Q in $W \cap \partial_e \mathbf{P}(R)$. But then $\ker(Q)$ is a maximal two-sided ideal of R which contains neither x_i, a contradiction.

Thus $\partial_e \mathbf{P}(R)$ is indeed closed in $\mathbf{P}(R)$, and so is compact. Therefore θ is a homeomorphism in this case also. □

COROLLARY 4.6. *If R is an N^*-complete regular ring, then the intersection of the maximal two-sided ideals of R is zero. Consequently, R is a subdirect product of simple, unit-regular, right and left self-injective rings.*

PROOF. If an element $x \in R$ lies in all maximal two-sided ideals, then by Theorem 4.5, $P(x) = 0$ for all P in $\partial_e \mathbf{P}(R)$. As the convex hull of $\partial_e \mathbf{P}(R)$ is dense in $\mathbf{P}(R)$ (the Krein-Milman Theorem), it follows that $P(x) = 0$ for all P in $\mathbf{P}(R)$. Then $N^*(x) = 0$, whence $x = 0$.

The subdirect product statement now follows from Corollary 4.3. □

COROLLARY 4.7. *Let R be an N^*-complete regular ring, and let J be a two-sided ideal of R. Then the following conditions are equivalent.*

(a) *R/J is N^*-complete.*

(b) *J is N^*-closed in R.*

(c) *$J = \ker(X)$ for some $X \subseteq \mathbf{P}(R)$.*

(d) *$J = \bigcap Y$ for some $Y \subseteq$ MaxSpec(R).*

PROOF. As R is unit-regular, properties (a), (b), and (c) are equivalent by Theorem 1.13. We obtain (a) $\Rightarrow$ (d) from Corollary 4.6, while it follows from Theorem 4.5(a) that (d) $\Rightarrow$ (c). □

COROLLARY 4.8. *Let R be an N^*-complete regular ring. Then $K_0(R)$ is a lattice-ordered abelian group if and only if $\partial_e \mathbf{P}(R)$ is compact, if and only if* MaxSpec(R) *is Hausdorff.*

PROOF. By Corollary 3.2 and Proposition 4.1, $K_0(R)$ is lattice-ordered if and only if $\partial_e \mathbf{P}(R)$ is compact. Theorem 4.5 shows that $\partial_e \mathbf{P}(R)$ is compact if and only if $\operatorname{MaxSpec}(R)$ is Hausdorff. $\square$

An alternate view of Theorem 4.5(a) is that for any N^*-complete regular ring R, there is another topology on $\partial_e \mathbf{P}(R)$, coarser than the usual topology, with respect to which the bijection $P \mapsto \ker(P)$ provides a homeomorphism of $\partial_e \mathbf{P}(R)$ onto $\operatorname{MaxSpec}(R)$. This topology coincides with a known topology defined on extreme boundaries of compact convex sets, as follows.

DEFINITION. Let K be any compact convex set, and let $\mathfrak{F}$ denote the family of subsets of $\partial_e K$ of the form $F \cap \partial_e K$, where F is a closed split-face of K. (See [**1**, p. 133] for the definition of a split-face.) According to [**1**, Proposition II.6.20], $\mathfrak{F}$ is closed under finite unions and arbitrary intersections. Thus $\mathfrak{F}$ is the family of closed sets for a topology on $\partial_e K$, known as the *facial topology* [**1**, p. 143]. When K is a Choquet simplex, every closed face of K is a split-face [**1**, Theorem II.6.22], hence in this case $\mathfrak{F}$ consists of all sets of the form $F \cap \partial_e K$ where F is any closed face of K. This simplification will apply in our considerations because $\mathbf{P}(R)$, for any regular ring R, is a Choquet simplex [**2**, Theorem 17.5].

THEOREM 4.9. *Let R be an N^*-complete regular ring. If $\partial_e \mathbf{P}(R)$ is given the facial topology, then the rule $P \mapsto \ker(P)$ defines a homeomorphism of $\partial_e \mathbf{P}(R)$ onto* $\operatorname{MaxSpec}(R)$.

PROOF. By Theorem 4.5(a), the rule $\theta(P) = \ker(P)$ defines a bijection θ of $\partial_e \mathbf{P}(R)$ onto $\operatorname{MaxSpec}(R)$. If X is a closed subset of $\operatorname{MaxSpec}(R)$, then

$$X = \{M \in \operatorname{MaxSpec}(R) \mid Y \subseteq M\},$$

for some $Y \subseteq R$. Set $F = \{P \in \mathbf{P}(R) \mid Y \subseteq \ker(P)\}$, and recall that F is a closed face of $\mathbf{P}(R)$ [**2**, Lemma 16.18]. Observing that

$$\theta^{-1}(X) = F \cap \partial_e \mathbf{P}(R),$$

we see that $\theta^{-1}(X)$ is closed in $\partial_e \mathbf{P}(R)$ in the facial topology. Thus θ is continuous with respect to the facial topology.

If F is any closed face of $\mathbf{P}(R)$, then Theorem 4.2(b) says that

$$F = \{P \in \mathbf{P}(R) \mid \ker(F) \subseteq \ker(P)\}.$$

As a result,

$$\theta(F \cap \partial_e \mathbf{P}(R)) = \{M \in \operatorname{MaxSpec}(R) \mid \ker(F) \subseteq M\},$$

which is closed in $\operatorname{MaxSpec}(R)$. Thus θ is a closed map with respect to the facial topology. $\square$

Part of Theorem 4.5(c) may be proved using Theorem 4.9, because of the general result that in a Choquet simplex K, the facial topology on $\partial_e K$ is Hausdorff if and only if the usual topology on $\partial_e K$ is compact [**1**, Theorem II.7.8].

We now turn to the study of finitely generated projective modules over an N^*-complete regular ring R. In particular, we show that their isomorphism classes are determined both by the values of pseudo-rank functions on R, and by their isomorphism classes modulo maximal two-sided ideals of R.

THEOREM 4.10. *Let R be an N^*-complete regular ring, let A and B be finitely generated projective right R-modules, and get*

$$A \cong x_1 R \oplus \cdots \oplus x_n R; \qquad B \cong y_1 R \oplus \cdots \oplus y_k R$$

for some elements $x_1, \ldots, x_n, y_1, \ldots, y_k \in R$.

(a) *$A \lesssim B$ if and only if $A/AM \lesssim B/BM$ for all $M \in$ MaxSpec(R), if and only if*

$$P(x_1) + \cdots + P(x_n) \leqslant P(y_1) + \cdots + P(y_k)$$

for all $P \in \partial_e \mathbf{P}(R)$.

(b) *$A \cong B$ if and only if $A/AM \cong B/BM$ for all $M \in$ MaxSpec(R), if and only if*

$$P(x_1) + \cdots + P(x_n) = P(y_1) + \cdots + P(y_k)$$

for all $P \in \partial_e \mathbf{P}(R)$.

PROOF. (a) If $A \lesssim B$, then obviously $A/AM \lesssim B/BM$ for all M in MaxSpec(R).

Now assume that $A/AM \lesssim B/BM$ for all $M \in$ MaxSpec(R). Consider any P in $\partial_e \mathbf{P}(R)$, and set $M = \ker(P)$, which is in MaxSpec(R) by Corollary 4.4. Let $\overline{P}$ denote the rank function induced on R/M by P. Using $x \mapsto \bar{x}$ for the natural map $R \to R/M$, we have

$$\bar{x}_1(R/M) \oplus \cdots \oplus \bar{x}_n(R/M) \cong A/AM \lesssim B/BM \cong \bar{y}_1(R/M) \oplus \cdots \oplus \bar{y}_k(R/M).$$

Applying [2, Proposition 16.1], we obtain

$$\overline{P}(\bar{x}_1) + \cdots + \overline{P}(\bar{x}_n) \leqslant \overline{P}(\bar{y}_1) + \cdots + \overline{P}(\bar{y}_k),$$

and consequently $\Sigma P(x_i) \leqslant \Sigma P(y_j)$.

Finally, assume that $\Sigma P(x_i) \leqslant \Sigma P(y_j)$ for all $P \in \partial_e \mathbf{P}(R)$. In view of Proposition 4.1, it follows that

$$s([A]) = s([x_1 R] + \cdots + [x_n R]) \leqslant s([y_1 R] + \cdots + [y_k R]) = s([B])$$

for all extreme states s on $(K_0(R), [R])$. Since $K_0(R)$ is archimedean, we conclude from [4, Proposition I.5.3] that $[A] \leqslant [B]$. Therefore $A \lesssim B$.

(b) This follows directly from (a) and the unit-regularity of R. □

DEFINITION. Let R be any regular ring, and set $S = S(K_0(R), [R])$. By Proposition 4.1, there is a natural affine homeomorphism θ: $S \to \mathbf{P}(R)$ such that $\theta(s)(x) = s([xR])$ for all $s \in S$ and all $x \in R$. Then θ induces an isomorphism θ^* of Aff($\mathbf{P}(R)$) onto Aff(S), as partially ordered Banach spaces. We also have the natural evaluation map φ from $K_0(R)$ to Aff(S). Composing φ with $(\theta^*)^{-1}$, we obtain a positive homomorphism

$$\psi = (\theta^*)^{-1}\varphi \colon K_0(R) \to \mathrm{Aff}(\mathbf{P}(R))$$

such that $\psi(x)(P) = \theta^{-1}(P)(x)$ for all $x \in K_0(R)$ and all $P \in \mathbf{P}(R)$. In particular, $\psi([xR])(P) = P(x)$ for all $x \in R$ and all $P \in \mathbf{P}(R)$. We refer to ψ as *the natural map from $K_0(R)$ to* Aff($\mathbf{P}(R)$). Note that $\psi([R])$ is the constant function 1.

THEOREM 4.11. *Let R be an N^*-complete regular ring. Whenever $P \in \partial_e \mathbf{P}(R)$ and $R/\ker(P)$ is isomorphic to an $m \times m$ matrix ring over a division ring, set $A_P = (1/m)\mathbf{Z}$; for all other $P \in \partial_e \mathbf{P}(R)$, set $A_P = \mathbf{R}$. Set*

$$A = \{q \in \mathrm{Aff}(\mathbf{P}(R)) \mid q(P) \in A_P \text{ for all } P \in \partial_e \mathbf{P}(R)\}.$$

Then the natural map ψ: $K_0(R) \to \mathrm{Aff}(\mathbf{P}(R))$ provides an isomorphism of $(K_0(R), [R])$ onto $(A, 1)$ (as partially ordered abelian groups with order-unit).

PROOF. Define S, θ, φ as in the definition above, so that $\psi = (\theta^*)^{-1}\varphi$. For all discrete $s \in \partial_e S$, set $B_s = s(K_0(R))$; for all other $s \in \partial_e S$, set $B_s = \mathbf{R}$. Set

$$B = \{p \in \mathrm{Aff}(S) \mid p(s) \in B_s \text{ for all } s \in \partial_e S\}.$$

According to Theorem 3.1, φ provides an isomorphism of $(K_0(R), [R])$ onto $(B, 1)$. Thus we need only show that $(\theta^*)^{-1}$ restricts to an isomorphism of $(B, 1)$ onto $(A, 1)$. To prove this, it suffices to show that $A_{\theta(s)} = B_s$ for all $s \in \partial_e S$.

Thus let $s \in \partial_e S$, and set $P = \theta(s)$ and $M = \ker(P)$, so that $P \in \partial_e \mathbf{P}(R)$ and M is a maximal two-sided ideal of R. Let $\bar{P}$ be the rank function induced by P on R/M.

If s is discrete, then s induces an isomorphism of $K_0(R)/\ker(s)$ onto $(1/m)\mathbf{Z}$ for some $m \in \mathbf{N}$ (Theorem 3.10), whence

$$\bar{P}(R/M) = P(R) = \{0, 1/m, 2/m, \ldots, 1\}.$$

Choosing $x \in R/M$ such that $\bar{P}(x) = 1/m$, we infer that $x(R/M)$ is a minimal right ideal of R/M, whence R/M is a simple artinian ring. Thus $R/M \cong M_k(D)$ for some $k \in \mathbf{N}$ and some division ring D. There is a unique rank function on $M_k(D)$, and its range of values is $\{0, 1/k, 2/k, \ldots, 1\}$ [**2**, Corollary 16.6]. Consequently, we must have $k = m$, and so

$$A_{\theta(s)} = A_P = (1/m)\mathbf{Z} = s(K_0(R)) = B_s.$$

If s is not discrete, then s induces an isomorphism of $K_0(R)/\ker(s)$ onto $\mathbf{R}$, by Theorem 3.10, whence

$$\bar{P}(R/M) = P(R) = [0, 1].$$

In this case, R/M cannot be a simple artinian ring, hence

$$A_{\theta(s)} = A_P = \mathbf{R} = B_s,$$

as desired. □

COROLLARY 4.12. *Let R be an N^*-complete regular ring. If R has no simple artinian homomorphic images, then the natural map from $K_0(R)$ to $\mathrm{Aff}(\mathbf{P}(R))$ is an isomorphism of partially ordered abelian groups.* □

THEOREM 4.13. *Let R be an N^*-complete regular ring, let B be a finitely generated projective right R-module, and let $n \in \mathbf{N}$. Assume, for each maximal two-sided ideal M of R such that R/M is artinian, that B/BM is a direct sum of n pairwise isomorphic submodules. Then B is a direct sum of n pairwise isomorphic submodules. In particular, if R has no simple artinian homomorphic images, then this holds for all $n \in \mathbf{N}$.*

PROOF. In the notation of Theorem 4.11, we wish to show that the function $\psi([B])/n$ lies in A. Thus consider any $P \in \partial_e \mathbf{P}(R)$ such that $R/\ker(P) \cong M_m(D)$ for some $m \in \mathbf{N}$ and some division ring D. The ideal $M = \ker(P)$ is a maximal two-sided ideal of R, and R/M has a unique simple right module S. Then $B/BM \cong kS$ for some $k \in \mathbf{Z}^+$, and $\psi([B])(P) = k/m$. Since B/BM is assumed to be a direct sum of n pairwise isomorphic submodules, n must divide k, hence

$$\psi([B])(P)/n = (k/n)/m \in (1/m)\mathbf{Z} = A_P.$$

As this holds for all $P \in \partial_e \mathbf{P}(R)$ such that $R/\ker(P)$ is simple artinian, we find that $\psi([B])/n$ does lie in A, as desired.

In fact, $\psi([B])/n$ must lie in A^+. Because of the isomorphism given in Theorem 4.11, it follows that there exists $x \in K_0(R)^+$ such that $nx = [B]$. Then $x = [C]$ for some finitely generated projective right R-module C, and $[nC] = [B]$. Therefore $B \cong nC$. $\square$

COROLLARY 4.14. *Let R be an N^*-complete regular ring, and let $n \in \mathbf{N}$. If every simple artinian homomorphic image of R is an $n \times n$ matrix ring, then R is an $n \times n$ matrix ring. In particular, if R has no simple artinian homomorphic images, then R is an $n \times n$ matrix ring for every $n \in \mathbf{N}$.* $\square$

As another application of the affine representation of K_0 of an N^*-complete regular ring (Theorem 4.11 and Corollary 4.12), we derive criteria for the following properties.

DEFINITION. Let R be a unit-regular ring. We say that R satisfies *countable interpolation* provided that given any elements $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ in R satisfying $x_i R \lesssim y_j R$ for all i, j, there exists $z \in R$ such that $x_i R \lesssim zR \lesssim y_j R$ for all i, j. According to [4, Proposition II.12.1], this property is left-right symmetric, and is equivalent to the countable interpolation property in $K_0(R)$. The ring R is said to satisfy *general comparability* provided that given any $x, y \in R$, there exists a central idempotent $e \in R$ such that $exR \lesssim eyR$ and $(1-e)yR \lesssim (1-e)xR$.

DEFINITION. Let X be a compact Hausdorff space. The space X is said to be an *F-space* if disjoint open F_σ subsets of X always have disjoint closures. The space X is said to be *basically disconnected* if the closure of every open F_σ subset of X is open.

THEOREM 4.15. *Let R be an N^*-complete regular ring with no simple artinian homomorphic images, and assume that $\partial_e \mathbf{P}(R)$ is compact. Then R has countable interpolation if and only if $\partial_e \mathbf{P}(R)$ is an F-space, if and only if* MaxSpec(R) *is an F-space.*

PROOF. Set $X = \partial_e \mathbf{P}(R)$, and note from Theorem 4.5 that X is homeomorphic to MaxSpec(R). Combining Corollary 4.12 with [1, Proposition II.3.13], we see that

$$K_0(R) \cong \operatorname{Aff}(\mathbf{P}(R)) \cong C(X, \mathbf{R})$$

as partially ordered abelian groups. Thus R has countable interpolation if and only if $C(X, \mathbf{R})$ satisfies the countable interpolation property (as a partially ordered set). According to [9, Theorem 1.1], this happens if and only if X is an F-space. $\square$

Theorem 4.15 may fail if R is allowed to have simple artinian homomorphic images, as the following example shows.

EXAMPLE 4.16. There exists an N^*-complete regular ring R such that $\partial_e \mathbf{P}(R)$ is a compact F-space, but R does not have countable interpolation.

PROOF. Choose a field K, set $R_n = M_2(K)$ for all $n \in \mathbf{N}$, and set

$$R = \left\{x \in \prod R_n \mid x_n \in K \text{ for all but finitely many } n \in \mathbf{N}\right\}.$$

Clearly R is a regular ring whose index of nilpotence is 2. By Theorem 1.3, R is N^*-complete.

The Boolean algebra $B(R)$ of central idempotents in R is a direct product of copies of $\{0, 1\}$ and so is complete. Consequently, its maximal ideal space $BS(R)$ is compact, Hausdorff, and extremally disconnected. In particular, $BS(R)$ is a compact F-space. Observing that R satisfies general comparability, we see by [**2**, Theorem 16.28] that $\partial_e \mathbf{P}(R)$ is homeomorphic to $BS(R)$. Thus $\partial_e \mathbf{P}(R)$ is a compact F-space.

Choose $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ in R so that $\operatorname{rank}(x_{kn}) = \operatorname{rank}(y_{kn}) = 1$ for all $n = 1, \ldots, k$, whereas $x_{kn} = 0$ and $y_{kn} = 1$ for all $n > k$. Clearly $x_i R \lesssim y_j R$ for all i, j. If there exists $z \in R$ such that $x_i R \lesssim zR \lesssim y_j R$ for all i, j, then $x_{nn} R \lesssim z_n R \lesssim y_{nn} R$ for all $n \in \mathbf{N}$, whence $\operatorname{rank}(z_n) = 1$ for all n. But then $z_n \notin K$ for all n, which is impossible for an element $z \in R$. Therefore R does not satisfy countable interpolation. $\square$

THEOREM 4.17. *Let R be an N^*-complete regular ring. If $\partial_e \mathbf{P}(R)$ is a compact totally disconnected F-space, then R satisfies general comparability.*

PROOF. Set $\Delta = \partial_e \mathbf{P}(R)$. Given $x, y \in R$, set

$$X = \{P \in \Delta \mid P(x) < P(y)\}; \qquad Y = \{P \in \Delta \mid P(x) > P(y)\}.$$

Inasmuch as the rule $P \mapsto P(x) - P(y)$ defines a continuous real-valued map on Δ, we see that X and Y are disjoint open F_σ subsets of Δ. Since Δ is assumed to be an F-space, the closures of X and Y must be disjoint. Consequently, it follows from the total disconnectedness of Δ that there is a clopen set $V \subseteq \Delta$ such that $X \subseteq V$ and $Y \subseteq \Delta - V$.

Now let $\theta \colon \Delta \to \operatorname{MaxSpec}(R)$ be the homeomorphism given by Theorem 4.5, so that $\theta(V)$ is a clopen subset of $\operatorname{MaxSpec}(R)$. As the intersection of the maximal two-sided ideals of R is zero (Corollary 4.6), there must exist a central idempotent $e \in R$ such that

$$\theta(V) = \{M \in \operatorname{MaxSpec}(R) \mid e \notin M\}.$$

Given any $P \in V$, we have $\ker(P) \in \theta(V)$ and so $e \notin \ker(P)$. Then $1 - e$ lies in $\ker(P)$, hence $P(x) = P(ex)$ and $P(y) = P(ey)$, by [**2**, Lemma 16.2]. Consequently, $P(ex) < P(ey)$ for all $P \in X$, and $P(ex) = P(ey)$ for all $P \in V - X$. On the other hand, for $P \in \Delta - V$, we have $\ker(P) \notin \theta(V)$ and so $e \in \ker(P)$, whence $P(ex) = 0 = P(ey)$. Thus $P(ex) \leqslant P(ey)$ for all $P \in \Delta$, hence $exR \lesssim eyR$, by Theorem 4.10. Similarly, $(1 - e)yR \lesssim (1 - e)xR$. Therefore R satisfies general comparability. $\square$

The proof of Theorem 4.17 can be considerably shortened if R has no simple artinian homomorphic images. For in this case R has countable interpolation by Theorem 4.15, and then [**4**, Theorem II.14.7] shows that R satisfies general comparability.

COROLLARY 4.18. *Let R be an N^*-complete regular ring with no simple artinian homomorphic images. Then R satisfies general comparability if and only if $\partial_e \mathbf{P}(R)$ is a compact totally disconnected F-space, if and only if* MaxSpec(R) *is a Hausdorff totally disconnected F-space.*

PROOF. It is clear from Theorem 4.5 that $\partial_e \mathbf{P}(R)$ is a compact totally disconnected F-space if and only if MaxSpec(R) is a Hausdorff totally disconnected F-space. These conditions imply general comparability in R by Theorem 4.17.

Conversely, assume that R has general comparability. It follows from [**2**, Theorem 16.28] that $\partial_e \mathbf{P}(R)$ is compact and totally disconnected.

Now consider any disjoint open F_σ subsets X and Y in $\partial_e \mathbf{P}(R)$. Then there exists a continuous real-valued function f on $\partial_e \mathbf{P}(R)$ such that $f > 0$ on X and $f < 0$ on Y. (This is an exercise in applying Urysohn's Lemma, which we leave to the reader.) As $\partial_e \mathbf{P}(R)$ is compact, we can modify f to obtain a continuous function

$$g\colon \partial_e \mathbf{P}(R) \to [0, 1]$$

such that $g > 1/2$ on X and $g < 1/2$ on Y. By [**1**, Proposition II.3.13], g extends to an affine continuous function $g^*\colon \mathbf{P}(R) \to [0, 1]$, to which we apply Corollary 4.12. Since $0 \leqslant g^* \leqslant 1$, we obtain an element $b \in K_0(R)$ such that $0 \leqslant b \leqslant [R]$ and the map induced by b in Aff($\mathbf{P}(R)$) coincides with g^*. Thus $b = [xR]$ for some $x \in R$, and $g^*(P) = P(x)$ for all $P \in \mathbf{P}(R)$.

We use general comparability to compare the projective modules $2(xR)$ and R_R, via [**2**, Proposition 8.8]. Thus we obtain a central idempotent $e \in R$ such that

$$2(exR) \lesssim eR \qquad \text{and} \qquad (1-e)R \lesssim 2((1-e)xR).$$

Consider any $P \in X$, so that $P(x) = g(P) > 1/2$. If $e \notin \ker(P)$, then $1 - e$ is in $\ker(P)$, because $\ker(P)$ is a maximal two-sided ideal of R (Corollary 4.4). But then

$$2P(x) = 2P(ex) \leqslant P(e) = 1$$

(since $2(exR) \lesssim eR$), which contradicts the fact that $P(x) > 1/2$. Thus $e \in \ker(P)$ and so $P(e) = 0$. Similarly, for any $P \in Y$ we have $1 - e \in \ker(P)$ and so $P(e) = 1$. As the map $P \mapsto P(e)$ is a continuous map from $\mathbf{P}(R)$ to $\mathbf{R}$, we conclude that X and Y must have disjoint closures.

Therefore $\partial_e \mathbf{P}(R)$ is an F-space. □

Corollary 4.18 may fail if R is allowed to have simple artinian homomorphic images, as the following example shows.

EXAMPLE 4.19. There exists an N^*-complete regular ring R such that R satisfies general comparability, but $\partial_e \mathbf{P}(R)$ is not an F-space.

PROOF. Choose a field K, and let R be the ring of all eventually constant sequences

$$(\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha, \alpha, \alpha, \ldots)$$

of elements of K. Clearly R is a regular ring whose index of nilpotence is 1. By Theorem 1.3, R is N^*-complete. It is also clear that R satisfies general comparability.

The maximal ideals of R are easily identified, from which one sees that MaxSpec(R) is homeomorphic to the one-point compactification of $\mathbf{N}$. In particular, MaxSpec(R) is Hausdorff but is not an F-space. By Theorem 4.5, $\partial_e \mathbf{P}(R)$ is homeomorphic to MaxSpec(R), hence $\partial_e \mathbf{P}(R)$ is not an F-space. $\square$

As we mentioned in the introduction to this section, many of our results—particularly Corollaries 4.3 and 4.6, Theorems 4.10 and 4.13, and Corollary 4.14—show that an N^*-complete regular ring R behaves much like a ring of sections of a sheaf of simple self-injective rings. The obvious candidate for a topological space on which such a sheaf should live is MaxSpec(R). However, MaxSpec(R) is usually not Hausdorff, and even when it is Hausdorff it need not be disconnected, which would seem to be required for sheaf-theoretic proofs of results such as Corollary 4.14. The lack of Hausdorffness, at least, of MaxSpec(R) may be remedied by using the space $\partial_e \mathbf{P}(R)$ instead; the fact that we have to pay attention to how $\partial_e \mathbf{P}(R)$ sits inside $\mathbf{P}(R)$ is in some sense the price for overcoming the lack of compactness of $\partial_e \mathbf{P}(R)$. This feeling may be made more precise at the level of $K_0(R)$: if R were to resemble the sections of a sheaf based on $\partial_e \mathbf{P}(R)$, then $K_0(R)$ should resemble the sections of a sheaf of functions based on $\partial_e \mathbf{P}(R)$. When $\partial_e \mathbf{P}(R)$ is compact, this does happen: Combining Theorem 4.11 with [**1**, Proposition II.3.13] yields

$$K_0(R) \cong \{q \in C(\partial_e \mathbf{P}(R), \mathbf{R}) \mid q(P) \in A_P \text{ for all } P \in \partial_e \mathbf{P}(R)\}.$$

On the other hand, when $\partial_e \mathbf{P}(R)$ is not compact, continuous real-valued functions on $\partial_e \mathbf{P}(R)$ do not correspond to elements of $K_0(R)$ unless they can be made to respect the affine relations present in $\mathbf{P}(R)$.

There is one situation in which R is isomorphic to the ring of sections of a sheaf-like object, namely when R is a continuous regular ring. This idea is developed by Handelman in [**5**] under the additional assumption that R is N^*-complete; however, we now know that N^*-completeness holds for all continuous regular rings, by Theorem 1.8. Handelman's construction is based on the space $\partial_e \mathbf{P}(R)$, which in this case is a compact Hausdorff extremally disconnected space. (Of course, we could equally well use MaxSpec(R), in view of Theorem 4.5.) The stalk at a point $P \in \partial_e \mathbf{P}(R)$ is the simple self-injective ring $R/\ker(P)$. What prevents Handelman's object from being an actual sheaf is that the rank-metric topologies on the rings $R/\ker(P)$ must be respected, and these topologies are not discrete unless the rings $R/\ker(P)$ are artinian.

Another aspect of Handelman's development of this construction provides a general means for constructing N^*-complete regular rings. Namely, given any regular ring R, one can complete R with respect to the N^*-metric. Since the ring operations on R are uniformly continuous with respect to N^* (as we observed in §I), the N^*-completion $\bar{R}$ is again a ring; moreover, $\bar{R}$ is actually a regular ring [**6**, Proposition 1.4]. In addition, the restriction map $\mathbf{P}(\bar{R}) \to \mathbf{P}(R)$ is an affine homeomorphism, as stated in [**5**, Proposition 15]. (The proof of this proposition is incomplete, but Handelman has informed me that he and Walter Burgess have developed a complete proof.) In particular, this result provides a convenient means for constructing examples. Combining it with [**2**, Theorem 17.23], we find that any metrizable Choquet simplex S is affinely homeomorphic to $\mathbf{P}(R)$ for a suitable N^*-complete regular ring R. For instance, there exists an N^*-complete regular ring R such that $\mathbf{P}(R)$ is affinely homeomorphic to the Choquet simplex of all probability measures on the unit interval [0, 1], whence $\partial_e \mathbf{P}(R)$ is homeomorphic to [0, 1].

A number of the results proved here for N^*-complete regular rings were first proved for $\aleph_0$-continuous regular rings [**7**, **8**], or, somewhat more generally, for unit-regular rings satisfying countable interpolation [**4**]. (All $\aleph_0$-continuous regular rings, and their factor rings, satisfy countable interpolation by [**4**, Theorem II.12.3].) In the first case, our results are generalizations, since all $\aleph_0$-continuous regular rings are N^*-complete (Theorem 1.8). However, in the second case our results are not strict generalizations except at the level of K_0: Namely, a unit-regular ring R satisfying countable interpolation need not be N^*-complete, but as long as $\ker(\mathbf{P}(R)) = 0$, then $K_0(R)$ is an archimedean norm-complete interpolation group [**4**, Theorems II.12.7 and I.6.6]. Thus the appropriate results for unit-regular rings with countable interpolation follow from the K_0 results in §III, in the same manner as the derivations in §IV for N^*-complete regular rings. The relationship between our results and these earlier results is as follows.

Theorem 4.2 corresponds to [**4**, Corollary II.13.6 and Theorem II.13.7], while Corollary 4.3 corresponds to [**7**, Corollary 3.2]. Theorems 4.5 and 4.9 correspond to [**4**, Proposition II.14.5 and Theorem II.14.6], while Corollary 4.6 and Theorem 4.10 correspond to [**8**, Theorem 2.3]. Theorems 4.11 and 4.13, along with Corollaries 4.12 and 4.14, correspond to [**4**, Theorems II.15.1, II.15.3, and II.15.4, and Corollary II.15.2].

Note added in proof. The N^*-completion results mentioned above are included in a paper by Burgess and Handelman, *The N^*-metric completion of regular rings*, submitted for publication.

References

1. E. M. Alfsen, *Compact convex sets and boundary integrals*, Springer-Verlag, Berlin, 1971.
2. K. R. Goodearl, *Von Neumann regular rings*, Pitman, London, 1979.
3. K. R. Goodearl and D. E. Handelman, *Metric completions of partially ordered abelian groups*, Indiana Univ. Math. J. **29** (1980), 861–895.
4. K. R. Goodearl, D. E. Handelman and J. W. Lawrence, *Affine representations of Grothendieck groups and applications to Rickart C^*-algebras and $\aleph_0$-continuous regular rings*, Mem. Amer. Math. Soc. No. 234 (1980).
5. D. Handelman, *Representing rank complete continuous rings*, Canad. J. Math. **28** (1976), 1320–1331.
6. ______, *Simple regular rings with a unique rank function*, J. Algebra **42** (1976), 60–80.
7. ______, *Finite Rickart C^*-algebras and their properties*, Studies in Analysis, Advances in Math. Suppl. Studies, Vol. 4, 1979, pp. 171–196.
8. D. Handelman, D. Higgs and J. Lawrence, *Directed abelian groups, countably continuous rings, and Rickart C^*-algebras*, J. London Math. Soc. (2) **21** (1980), 193–202.
9. G. L. Seever, *Measures on F-spaces*, Trans. Amer. Math. Soc. **133** (1968), 267–280.
10. J. von Neumann, *Continuous geometry*, Princeton Univ. Press, Princeton, N. J., 1960.

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