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A FAKE TOPOLOGICAL HILBERT SPACE

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ABSTRACT. We give an example of a topologically complete separable metric AR space X which is *not* homeomorphic to the Hilbert space l^2, but which has the following properties:

(i) X imbeds as a convex subset of l^2;
(ii) every compact subset of X is a Z-set;
(iii) $X \times X \approx l^2$;
(iv) X is homogeneous;
(v) $X \approx X \setminus G$ for every countable subset G.

1. Introduction. Torunczyk [**15**] has recently obtained the following topological characterization of the separable Hilbert space l^2.

1.1. THEOREM. *A topologically complete separable metric* AR *space X is homeomorphic to l^2 if and only if every map $f\colon \oplus_1^\infty Q_i \to X$ of the countable free union of Hilbert cubes into X is strongly approximable by maps $g\colon \oplus_1^\infty Q_i \to X$ for which the collection $\{g(Q_i)\}$ is discrete.*

This extremely useful characterization has now become the standard method for recognizing topological Hilbert spaces, in situations ranging from hyperspaces to infinite products to topological groups (see [**7**, **15**, **9**]). The above approximation property, referred to as the *strong discrete approximation property*, can be stated in various equivalent ways:

1.2. For each map $f\colon \oplus_1^\infty Q_i \to X$ and each open cover $\mathcal{U}$ of X, there exists a map $g\colon \oplus_1^\infty Q_i \to X$ such that f and g are $\mathcal{U}$-close and $\{g(Q_i)\}$ is discrete.

1.3. With respect to some admissible metric d on X, for each map $f\colon \oplus_1^\infty Q_i \to X$ and each map $\varepsilon\colon X \to (0, \infty)$, there exists a map $g\colon \oplus_1^\infty Q_i \to X$ such that $d(f(y), g(y)) < \varepsilon(f(y))$ for each y and $\{g(Q_i)\}$ is discrete.

1.4. With respect to *every* admissible metric d on X, for each map $f\colon \oplus_1^\infty Q_i \to X$ and each $\varepsilon > 0$, there exists a map $g\colon \oplus_1^\infty Q_i \to X$ such that $d(f(y), g(y)) < \varepsilon$ for each y and $\{g(Q_i)\}$ is discrete.

The equivalence of (1.2) through (1.4) is well known and easily demonstrated (to show that (1.4) $\Rightarrow$ (1.2) use Theorem 9.4 in [**10**, p. 196]).

The example we describe in §§3–6 shows that the strong discrete approximation property cannot be relaxed by considering only positive constants $\varepsilon > 0$ and a fixed

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metric d on X. Specifically, our example is a topologically complete separable metric AR space (X, d) with the following properties.

(1) For each map f: $\oplus_1^\infty Q_i \to X$ and $\varepsilon > 0$, there exists a map g: $\oplus_1^\infty Q_i \to X$ such that $d(f(q), g(q)) < \varepsilon$ for each q and $\{g(Q_i)\}$ is discrete (we call this the *weak discrete approximation property* for (X, d));

(2) every compact subset of X is a Z-set (which implies that X is nowhere locally-compact);

(3) X imbeds as a convex subset of l^2;

(4) $X \times X \approx l^2$;

(5) X is strongly locally homogeneous (which implies that X is homogeneous and countable dense homogeneous);

(6) every countable subset of X is strongly negligible (in particular, $X \setminus \{\text{countable set}\} \approx X$);

(7) no Cantor set is negligible in X.

Since in l^2 every compact set is negligible **[12]**, property (7) shows that $X \not\approx l^2$. Thus X is a counterexample for the problems $TC3$ and $ANR5$ of **[11]**. Liem **[14]** has previously shown that the condition $X \times X \approx l^2$ does not imply $X \approx l^2$. His example, however, is not homogeneous (there exists $p \in X$ such that $\{p\}$ is not a Z-set, while $X \setminus \{p\}$ is an l^2-manifold).

In §7 we state another criterion by which our example may be seen to be nonhomeomorphic to l^2, and which leads to the related construction of a counterexample to the Capset Characterization Theorem of **[13]**.

2. Definition and terminology. Let $Q = \prod_1^\infty [-1, 1]_i$ and $s = \prod_1^\infty (-1, 1)_i$. The space s is homeomorphic to l^2 **[1]**, and is called the *pseudo-interior* of Q. We sometimes write I_i for $[-1, 1]_i$. On these product spaces we use the standard metric $d((x_i), (y_i)) = \sum_1^\infty 2^{-i} |x_i - y_i|$.

Let $\mathcal{U}$ be an open cover of a space X. Maps f, g: $Y \to X$ are *$\mathcal{U}$-close* if, for each $y \in Y$, there exists $U \in \mathcal{U}$ containing both $f(y)$ and $g(y)$. For a metric d on X, we write $d(f, g) < \varepsilon$ if $d(f(y), g(y)) < \varepsilon$ for each $y \in Y$. A map h: $X \to X$ is *limited by* $\mathcal{U}$ if h is $\mathcal{U}$-close to the identity map. We say h: $X \to X$ is *supported on* $V \subset X$ if h restricts to the identity map on $X \setminus V$.

A collection $\mathcal{D}$ of closed subsets of X is *discrete* if each point of X has a neighborhood intersecting at most one member of $\mathcal{D}$.

A closed subset $A \subset X$ is a *Z-set* in X if, for each map f: $Q \to X$ and $\varepsilon > 0$, there exists a map g: $Q \to X$ with $d(f, g) < \varepsilon$ and $g(Q) \cap A = \varnothing$. In l^2 every compact set is a Z-set. A *σ-Z-set* in X is a countable union of Z-sets. We will use the fact, easily shown, that for every σ-Z-set $B \subset Q$, there exists a deformation α: $Q \times [0, 1] \to Q$ with $\alpha(q, 0) = q$ and $\alpha(q, t) \in Q \setminus B$, for all $q \in Q$ and $t > 0$.

A space X is *strongly locally homogeneous* if it has an open base $\mathcal{U}$ such that, for each $U \in \mathcal{U}$ and points $x, y \in U$, there exists a homeomorphism h: $X \to X$ with $h(x) = y$ and h supported on U. Clearly, every connected strongly locally homogeneous space is homogeneous.

A subset $K \subset X$ is *negligible* in X if $X \approx X \setminus K$. K is *strongly negligible* if there exist homeomorphisms h: $X \to X \setminus K$ limited by arbitrary open covers of X. In l^2, a subset K is strongly negligible if and only if K is a σ-Z-set **[3]**.

3. Complements of σ-Z-sets in Q. For each $i = 1, 2, \ldots$, let

$$W_i = \prod_{j \neq i} [-1 + 2^{-i}, 1 - 2^{-i}]_j \times \{1\}_i \subset Q.$$

W_i is a "shrunken endface" in the ith coordinate direction, and is a Z-set in Q. We will show that the space $Y = Q \setminus \bigcup_1^\infty W_i$ has the properties specified in §1, except for the negligibility property (6). It then follows that, for any countable dense set $D \subset Y$, the space $X = Y \setminus D$ has *all* the required properties (Theorem 5.3). The results of this section will be used in verifying the properties (1)–(4), and (7).

3.1. THEOREM. *Let $B \subset Q$ be a σ-Z-set. Then $Q \setminus B$ is an infinite-dimensional topologically complete separable metric* AR *which imbeds as a convex subset of l^2.*

PROOF. We consider the "elliptic" Hilbert cube $K = \{(x_i) \in l^2\colon \Sigma_1^\infty i^2 x_i^2 \leq 1\}$. There exists a homeomorphism $h\colon Q \to K$ such that $h(B) \subset \{(x_i) \in l^2\colon \Sigma_1^\infty i^2 x_i^2 = 1\}$ **[5]**. It is easily seen that $K \setminus h(B)$ is convex. It follows by Dugundji's Extension Theorem that $Q \setminus B \approx K \setminus h(B)$ is an AR. (An alternate proof that $Q \setminus B$ is an AR follows from the existence of a deformation $\alpha\colon Q \times [0, 1] \to Q$ with $\alpha(Q \times (0, 1]) \subset Q \setminus B$.) $Q \setminus B$ is topologically complete since B is σ-compact. The converse of this theorem is also true **[9]**.

3.2. THEOREM. *Let $B \subset Q$ be a σ-Z-set such that for each $\varepsilon > 0$ there exists a map $\mathfrak{N}\colon Q \to B$ with $d(\mathfrak{N}, \mathrm{id}) < \varepsilon$. Then the metric space $(Q \setminus B, d)$ has the weak discrete approximation property.*

PROOF. Let a map $f\colon \oplus_1^\infty Q_i \to Q \setminus B$ and $\varepsilon > 0$ be given. Choose $\mathfrak{N}\colon Q \to B$ such that $d(\mathfrak{N}, \mathrm{id}) < \varepsilon$. Since B is a σ-Z-set, there exist maps of Q into $Q \setminus B$ arbitrarily close to the identity map. Composing the map $\mathfrak{N}$ with such maps, we obtain a sequence of maps $\{g_i\colon Q \to Q \setminus B\}$ such that

(i) $d(g_i, \mathfrak{N}) < 1/i$;

(ii) $d(g_i, \mathrm{id}) < \varepsilon$;

(iii) $g_i(Q) \cap g_j(Q) = \varnothing$ if $i \neq j$.

Define $g\colon \oplus_1^\infty Q_i \to Q \setminus B$ by $g(q) = g_i(f(q))$, for $q \in Q_i$. Then $d(f, g) < \varepsilon$, and the collection $\{g(Q_i)\} = \{g_i(f(Q_i))\}$ is discrete in $Q \setminus B$.

3.3. THEOREM. *Let (X, d) be a metric space with the weak discrete approximation property. Then every compact subset of X is a Z-set.*

PROOF. Consider a compact subset K, and let a map $f\colon Q \to X$ and $\varepsilon > 0$ be given. Define $\hat{f}\colon \oplus_1^\infty Q_i \to X$ by $\hat{f}(q) = f(q)$. By hypothesis there exists a map $g\colon \oplus_1^\infty Q_i \to X$ with $d(\hat{f}, g) < \varepsilon$ and $\{g(Q_i)\}$ a discrete collection. Since K is compact, $K \cap g(Q_i) = \varnothing$ for almost all i. Thus K is a Z-set.

A σ-Z-set $B \subset Q$ for which $Q \setminus B \approx s \approx l^2$ is called a *boundary set*. In **[8]**, various necessary and sufficient conditions are given for a σ-Z-set to be a boundary set. Here, we use the following result.

3.4. LEMMA. *Let $B \subset Q$ be a σ-Z-set for which there exists a deformation of Q through B (i.e., a deformation $\alpha\colon Q \times [0, 1] \to Q$ with $\alpha(q, 0) = q$ and $\alpha(q, t) \in B$, for all $q \in Q$ and $t > 0$). Then B is a boundary set.*

3.5. THEOREM. *Let $B \subset Q$ be a σ-Z-set such that for each $\varepsilon > 0$ there exists a map $\mathfrak{N}$: $Q \to B$ with $d(\mathfrak{N}, \mathrm{id}) < \varepsilon$. Then $(Q \setminus B) \times (Q \setminus B) \approx l^2$.*

PROOF. We construct a deformation $\{\alpha_t\}$ of $Q \times Q$ through the σ-Z-set $\tilde{B} = (B \times Q) \cup (Q \times B)$. For each $i \geq 1$, choose a map η_i: $Q \to B$ with $d(\eta_i, \mathrm{id}) < 1/i$. Set $\alpha_0 = \mathrm{id} \times \mathrm{id}$, $\alpha_{1/(2i-1)} = \eta_i \times \eta_i$, $\alpha_{1/2i} = \eta_{i+1} \times \eta_i$, and $\alpha_{1/(2i+1)} = \eta_{i+1} \times \eta_{i+1}$. Define α_t for $1/(2i+1) < t < 1/2i$ or $1/2i < t < 1/(2i-1)$ by using the straight-line homotopy in Q between η_i and η_{i+1}. It follows from 3.4 that $(Q \setminus B) \times (Q \setminus B) = (Q \times Q) \setminus \tilde{B} \approx l^2$.

3.6. LEMMA. *Let B_1 and B_2 be σ-Z-sets in Q such that $Q \setminus B_1 \approx Q \setminus B_2$. Then there exist a compact space M and monotone surjections Π_i: $M \to Q$, $i = 1, 2$, such that $\Pi_1^{-1}(B_1) = \Pi_2^{-1}(B_2)$.*

PROOF. Let h: $Q \setminus B_1 \to Q \setminus B_2$ be a homeomorphism, and let $\Gamma \subset Q \times Q$ be the graph of h. Take $M = \overline{\Gamma}$, and Π_i: $M \to Q$ the projection maps, $i = 1, 2$. Clearly, $\Pi_1^{-1}(B_1) = \overline{\Gamma} \setminus \Gamma = \Pi_2^{-1}(B_2)$, and since $Q \setminus B_i$ is dense in Q, each Π_i is onto. By symmetry, it suffices to verify that Π_1 is monotone. For $x \in Q \setminus B_1$, $\Pi_1^{-1}(x)$ is a point in Γ. Suppose that for some $x \in B_1$, $\Pi_1^{-1}(x)$ is not connected. Let $\Pi_1^{-1}(x) = F \cup G$ be a separation, and choose disjoint open sets U and V in M containing F and G, respectively. Since Π_1 is a closed map, there exists an open neighborhood W of x in Q such that $\Pi_1^{-1}(W) \subset U \cup V$, and we may assume W is connected. Then $W \setminus B_1$ is also connected (in fact, path-connected: any path in W between points of $W \setminus B_1$ may be deformed to a path in $W \setminus B_1$ via a deformation α: $Q \times [0, 1] \to Q$ such that $\alpha(Q \times (0, 1]) \subset Q \setminus B_1$). But

$$W \setminus B_1 = \big(W \cap \Pi_1(U \cap \Gamma)\big) \cup \big(W \cap \Pi_1(V \cap \Gamma)\big)$$

is a separation. Thus Π_1 and Π_2 are monotone.

4. Strong local homogeneity. Given points p, q in $Y = Q \setminus \bigcup_1^\infty W_i$ we will construct a homeomorphism h: $Q \to Q$ such that $h(p) = q$ and $h(W_i) = W_i$ for each i. Then h restricts to a homeomorphism of Y. Moreover, our constructions will show that Y is strongly locally homogeneous.

4.1. THEOREM. *Let p and q be points in s. Then there exists a homeomorphism h: $Q \to Q$ such that $h(p) = q$ and $h(W_i) = W_i$ for each i.*

PROOF. We may assume $p_i \leq q_i$ for each i. Choose a_i and b_i such that

$$-1 < a_i < p_i \leq q_i < b_i < 1.$$

Let H_i: $I_i \times [0, 1] \to I_i$ be an isotopy such that

(i) each level $H_{i,t}$ is supported on $[a_i, b_i]$;

(ii) $H_{i,0} = \mathrm{id}$;

(iii) $H_{i,1}(p_i) = q_i$.

For each i, let $B_i = \prod_1^i I_j$ and $D_i = \prod_1^i [a_j, b_j]$. There exists a map α_i: $B_i \to [0, 1]$ such that

(a) $\alpha_i(D_i) = 1$;

(b) $\alpha_i(\partial B_i) = 0$;

(c) if (s_j), $(t_j) \in B_i$ such that for each j either $s_j = t_j$ or $\{s_j, t_j\} \subset [a_j, b_j]$, then $\alpha_i((s_j)) = \alpha_i((t_j))$.
(Collapse each interval $[a_j, b_j]$ to a point, then apply a Urysohn map.)

For each i, choose an integer $\tilde{i} \geqslant i$ such that $[a_i, b_i] \subset [-1 + 2^{-\tilde{i}}, 1 - 2^{-\tilde{i}}]$. Let Π_i: $Q \to I_i$ be the projection map, with $\Pi_i(x) = x_i$. Define a homeomorphism h_i: $Q \to Q$ by the formulas

(1) $\Pi_j h_i = \Pi_j$ if $j \neq i$;

(2) $\Pi_i h_i(x) = H_i(x_i, \alpha_{\tilde{i}}(x_1, \ldots, x_{\tilde{i}}))$.

The verification that h_i is 1-1 uses conditions (i) and (c).

Finally, define h: $Q \to Q$ by $h = \lim_{i\to\infty}(h_i \circ \cdots \circ h_1)$. Clearly, h is continuous and onto, and $h(p) = q$. Consider distinct points $x, y \in Q$. Suppose $x_i \neq y_i$, and suppose $h(x)$ and $h(y)$ agree in the first $i - 1$ coordinates. Let

$$(v_1, \ldots, v_{i-1}, x_i, \ldots, x_{\tilde{i}}) \quad \text{and} \quad (v_1, \ldots, v_{i-1}, y_i, \ldots, y_{\tilde{i}})$$

be the first $\tilde{i}$ coordinates of $(h_{i-1} \circ \cdots \circ h_1)(x)$ and $(h_{i-1} \circ \cdots \circ h_1)(y)$, respectively. If $\alpha_{\tilde{i}}(v_1, \ldots, x_{\tilde{i}}) = \alpha_{\tilde{i}}(v_1, \ldots, y_{\tilde{i}})$, then $\Pi_i h(x) \neq \Pi_i h(y)$. And otherwise, by condition (c), we must have $x_j \neq y_j$ and $\{x_j, y_j\} \not\subset [a_j, b_j]$ for some $i \leqslant j \leqslant \tilde{i}$. Then by condition (i), $\Pi_j h(x) \neq \Pi_j h(y)$. Thus h is 1-1.

For $j \leqslant \tilde{i}$, h_i restricts to the identity on W_j. For $j > \tilde{i}$, $h_i(W_j) = W_j$, since $[a_i, b_i] \subset [-1 + 2^{-j}, 1 - 2^{-j}]$. Thus $h(W_j) = W_j$ for each j.

4.2. Remark. The above proof shows that if for some n, U is a neighborhood of the product set $\Pi_1^n[a_j, b_j] \times \Pi_{n+1}^\infty I_j$, then the homeomorphism h may be constructed so as to be supported on U.

To construct a homeomorphism of Y sending a point $p \in Y \setminus s$ into s, we employ an inductive convergence procedure stated in **[2]**. For each homeomorphism g of Q and $\varepsilon > 0$, let $\mathfrak{N}(g, \varepsilon) = \inf\{d(g(x), g(y)) \colon d(x, y) \geqslant \varepsilon\}$. If $\{h_i\}$ is a sequence of homeomorphisms of Q such that $d(h_i, \mathrm{id}) < 3^{-i} \cdot \mathfrak{N}(h_j \circ \cdots \circ h_{i-1}, 2^{-j})$ for each $j < i$, it is easily verified that $h = \lim_{i\to\infty}(h_i \circ \cdots \circ h_1)$ is a homeomorphism of Q.

4.3. Theorem. *For every $p \in Y$ there exists a homeomorphism h: $Q \to Q$ such that $h(p) \in s$ and $h(W_i) = W_i$ for each i.*

Note. The construction via the inductive convergence criterion of a homeomorphism of Q sending an arbitrary point into the pseudo-interior s is well known. We need only to carry out this standard construction with a little extra control to insure that each W_i remains invariant.

Proof. Assume that homeomorphisms $h_1, \ldots, h_{i-1}$ of Q have been defined such that

(a) $\Pi_j \circ h_{i-1} \circ \cdots \circ h_1(p) \in (-1, 1)$ for each $j < i$;

(b) $h_{i-1} \circ \cdots \circ h_1(W_j) = W_j$ for each j.

For some sufficiently small $\delta > 0$, we construct a homeomorphism h_i such that

(1) $d(h_i, \mathrm{id}) < \delta$;

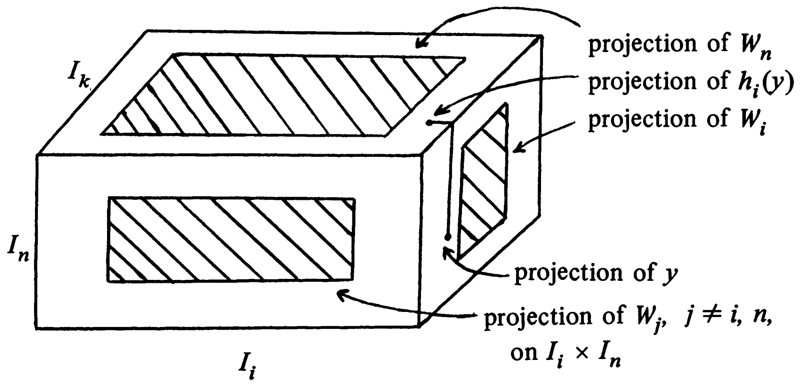
(2) $\Pi_j \circ h_i = \Pi_j$ for each $j < i$;

(3) $\Pi_i \circ h_i \circ \cdots \circ h_1(p) \in (-1, 1)$;

(4) $h_i(W_j) = W_j$ for each j.

Then $h = \lim_{i\to\infty}(h_i \circ \cdots \circ h_1)$ is the desired homeomorphism.

Let $y = h_{i-1} \circ \cdots \circ h_1(p)$. If $y_i \in (-1, 1)$, take $h_i = \mathrm{id}$. Otherwise, assume $y_i = 1$ (the case $y_i = -1$ is simpler). Since $y \notin W_i$, there exists an integer $k \neq i$ such that $y_k \notin [-1 + 2^{-i}, 1 - 2^{-i}] = \Pi_k(W_i)$. Suppose the inductive convergence procedure requires that $d(h_i, \mathrm{id}) < \delta$; choose $n > i$, $n \neq k$, such that $2^{-n} < \delta/3$. The homeomorphism h_i is constructed as the product of a homeomorphism $\tilde{h}_i$ on the 3-cube $I_i \times I_n \times I_k$ and the identity homeomorphism on the product of the remaining factors of Q. The homeomorphism $\tilde{h}_i$ will actually move only the ith and the nth coordinates.



Let $D = [1 - 2^{-(n+1)}, 1] \times [-1, 1] \subset I_i \times I_n$, and for each $0 < r < 1$ let $B_r = [-1 + r, 1 - r] \times [-1 + r, 1 - r] \subset I_i \times I_n$. There exists an isotopy H_i: $I_i \times I_n \times [0, 1] \to I_i \times I_n$ with the following properties:

(i) $H_{i,0} = \mathrm{id}$;

(ii) $H_{i,t}$ is supported on D for all t;

(iii) $H_{i,t}(B_r) = B_r$ for all r, t;

(iv) $H_{i,1}(\{1\} \times I_n) \subset (1 - 2^{-(n+1)}, 1) \times \{1\}$.

Let α: $I_k \to [0, 1]$ be a map such that $\alpha(\Pi_k(W_i)) = 0$ and $\alpha(y_k) = 1$. The homeomorphism $\tilde{h}_i$ on $I_i \times I_n \times I_k$ is then defined by $\tilde{h}_i(x_i, x_n, x_k) = (H_i(x_i, x_n, \alpha(x_k)), x_k)$. The homeomorphism $h_i = \tilde{h}_i \times \mathrm{id}$ on Q has the specified properties (1)–(4). (In particular, h_i restricts to the identity on W_i and W_n, and takes each W_j onto itself.)

4.4. Remark. The above proof shows that the homeomorphism h taking p into s may be constructed so as to be supported on a given neighborhood of p (the role played by the map α: $I_k \to [0, 1]$ is taken over by a map β: $I_{k_1} \times \cdots \times I_{k_j} \to [0, 1]$).

Combining 4.2 and 4.4, we conclude that Y is strongly locally homogeneous.

5. Countable dense homogeneity. In this section we show that the strong local homogeneity of Y implies that Y is countable dense homogeneous, and therefore the space $X = Y \setminus \{\text{countable dense subset}\}$ has the property that $X \approx X \setminus \{\text{countable subset}\}$. In fact, X is also strongly locally homogeneous, and countable subsets of X are strongly negligible.

A separable space X is *countably dense homogeneous* if, for every pair A, B of countable dense sets in X, there exists a homeomorphism of X taking A onto B. Bennett [4] showed that every countable dense homogeneous connected metric space is homogeneous, and that every strongly locally homogeneous locally compact separable metric space is countable dense homogeneous. The latter result is true more generally for topologically complete spaces. For its proof, we need a formulation of the inductive convergence procedure in a complete metric space.

5.1. LEMMA. *Let X be a complete metric space, and let $\{\mathfrak{U}_n\}$ be a sequence of open covers and $\{h_n\}$ a sequence of homeomorphisms of X satisfying the following conditions.*

(1) *$\mathfrak{U}_n$ is a barycentric refinement of $\mathfrak{U}_{n-1}$;*

(2) *$\mathfrak{U}_n$ has mesh less than 2^{-n};*

(3) *$(h_{n-1} \circ \cdots \circ h_1)^{-1}(\mathfrak{U}_n)$ has mesh less than 2^{-n};*

(4) *h_n is limited by $\mathfrak{U}_n$.*

Then $h = \lim_{n\to\infty}(h_n \circ \cdots \circ h_1)$ is a homeomorphism of X.

Note. A cover $\mathfrak{V}$ of X is a barycentric refinement of a cover $\mathfrak{U}$ if $\{\mathrm{St}(x, \mathfrak{V})\colon x \in X\}$ refines $\mathfrak{U}$. Every open cover of a paracompact space has an open barycentric refinement.

PROOF. Conditions (2) and (4) show that h is a map with a dense image. We show that h is 1-1 and closed. Suppose that, for some $\varepsilon > 0$, there exist sequences $\{x_k\}$ and $\{y_k\}$ in X such that $d(x_k, y_k) > \varepsilon$ for each k and $\lim_{k\to\infty} h(x_k) = z = \lim_{k\to\infty} h(y_k)$. Choose n such that $2^{-n} < \varepsilon/5$, and choose $U \in \mathfrak{U}_n$ such that $z \in U$. Then for some $k > n$, $\{h_k \circ \cdots \circ h_1(x_k), h_k \circ \cdots \circ h_1(y_k)\} \subset U$. By conditions (1) and (4), there exist V, $W \in \mathfrak{U}_n$ such that

$$\{h_n \circ \cdots \circ h_1(x_k), h_k \circ \cdots \circ h_1(x_k)\} \subset V$$

and

$$\{h_n \circ \cdots \circ h_1(y_k), h_k \circ \cdots \circ h_1(y_k)\} \subset W.$$

There also exist V', $W' \in \mathfrak{U}_n$ such that

$$\{h_{n-1} \circ \cdots \circ h_1(x_k), h_n \circ \cdots \circ h_1(x_k)\} \subset V'$$

and

$$\{h_{n-1} \circ \cdots \circ h_1(y_k), h_n \circ \cdots \circ h_1(y_k)\} \subset W'.$$

Thus (V', V, U, W, W') is a chain in $\mathfrak{U}_n$ with $h_{n-1} \circ \cdots \circ h_1(x_k) \in V'$ and $h_{n-1} \circ \cdots \circ h_1(y_k) \in W'$. Applying $(h_{n-1} \circ \cdots \circ h_1)^{-1}$, we obtain a 5-chain in $(h_{n-1} \circ \cdots \circ h_1)^{-1}(\mathfrak{U}_n)$ between the points x_k and y_k. By condition (3), $d(x_k, y_k) < 5 \cdot 2^{-n} < \varepsilon$, a contradiction. It follows that h is 1-1 and closed, and therefore a homeomorphism of X.

5.2. THEOREM. *Every topologically complete separable metric space which is strongly locally homogeneous is countable dense homogeneous.*

PROOF. Let countable dense subsets $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$ of such a space X be given (assume A and B are faithfully indexed). The hypothesis of strong local homogeneity implies that for each neighborhood U of a point x, and for any dense set G in X, there exists a homeomorphism of x which is supported on U and takes x into G. Using the inductive convergence procedure (with respect to some complete metric on X), we construct a sequence $\{h_n\}$ of homeomorphisms of X such that $h = \lim_{n\to\infty}(h_n \circ \cdots \circ h_1)$ is a homeomorphism and such that the following conditions (which insure that $h(A) = B$) are satisfied.

(1) $h_n \circ \cdots \circ h_1(a_i) = h_{2i} \circ \cdots \circ h_1(a_i) \in B$ for each i and each $n \geqslant 2i$;

(2) $(h_n \circ \cdots \circ h_1)^{-1}(b_i) = (h_{2i+1} \circ \cdots \circ h_1)^{-1}(b_i) \in A$ for each i and each $n \geqslant 2i + 1$.

Assume $h_1,\dots,h_{2i-1}$ have been defined. If $h_{2i-1} \circ \cdots \circ h_1(a_i) \in B$, take $h_{2i} = \mathrm{id}$. Otherwise, choose a neighborhood U_{2i} of $h_{2i-1} \circ \cdots \circ h_1(a_i)$ disjoint from the finite set $\{b_1,\dots,b_{i-1}\} \cup h_{2i-1} \circ \cdots \circ h_1(\{a_1,\dots,a_{i-1}\})$, and take h_{2i} to be a homeomorphism of X supported on U_{2i} and such that $h_{2i} \circ \cdots \circ h_1(a_i) \in B$.

If $(h_{2i} \circ \cdots \circ h_1)^{-1}(b_i) \in A$, take $h_{2i+1} = \mathrm{id}$. Otherwise, choose a neighborhood U_{2i+1} of b_i disjoint from the finite set $\{b_1,\dots,b_{i-1}\} \cup h_{2i} \circ \cdots \circ h_1(\{a_1,\dots,a_i\})$, and take h_{2i+1} to be a homeomorphism of X supported on U_{2i+1} and such that $h_{2i+1}^{-1}(b_i) \in (h_{2i} \circ \cdots \circ h_1)^{-1}(A)$. If the neighborhoods U_{2i} and U_{2i+1} are chosen small enough, the conditions of the inductive convergence procedure are satisfied.

5.3. THEOREM. *Let Y be a topologically complete separable metric space which is strongly locally homogeneous, and let D be a countable dense subset. Then $Y \setminus D$ is also strongly locally homogeneous, and every countable set in $Y \setminus D$ is strongly negligible.*

PROOF. Let U be a neighborhood of a point x in $Y \setminus D$. Then there exists an open neighborhood V of x in Y with $V \setminus D \subset U$ and such that, for every $y \in V \setminus D$, there exists a homeomorphism h_y of Y supported on V and taking x to y. Then a homeomorphism $h = \lim_{n\to\infty}(h_n \circ \cdots \circ h_1)$ of Y may be constructed as in the preceding lemma such that $h(h_y(D)) = D$, and each h_n is supported on V and leaves y fixed. The restriction of $h \circ h_y$ to $Y \setminus D$ takes x to y and is supported on $V \setminus D$. Thus $Y \setminus D$ is strongly locally homogeneous.

Let $G \subset Y \setminus D$ be countable, and $\mathcal{U}$ an open cover of $Y \setminus D$. Then there is a countable collection $\mathcal{V}$ of pairwise disjoint open sets in Y covering G, such that for each $V \in \mathcal{V}$, $V \setminus D$ lies in a member of $\mathcal{U}$. The proof of 5.2 shows that there exists a homeomorphism h of Y limited by the cover $\mathcal{V} \cup \{\{x\}: x \in Y\}$ and taking D onto $D \cup G$. Then the restriction of h to $Y \setminus D$ is a homeomorphism onto $(Y \setminus D) \setminus G$ which is limited by $\mathcal{U}$.

Applying 5.3, we conclude that if D is any countable dense subset of the space $Y = Q \setminus \bigcup_1^\infty W_i$, then the space $X = Y \setminus D$ has the homogeneity property (5) and the negligibility property (6) of §1. And by the theorems of §3, X retains the properties (1) through (4). We show in the next section that X has the nonnegligibility property (7).

6. Nonnegligibility of Cantor sets.

6.1. THEOREM. *No Cantor set is negligible in the space $X = Q \setminus (\bigcup_1^\infty W_i \cup D)$.*

PROOF. Suppose $X \approx X \setminus K$ for some Cantor set K in X. Then by Lemma 3.6 there exist a compact space M and monotone surjections $\Pi_1: M \to Q$ and $\Pi_2: M \to Q$ such that $\Pi_1^{-1}(\bigcup_1^\infty W_i \cup D) = \Pi_2^{-1}(\bigcup_1^\infty W_i \cup D \cup K)$. For each $x \in K$, consider the continuum $\Pi_2^{-1}(x)$. Since $\Pi_1^{-1}(\bigcup W_i \cup D)$ is a countable disjoint union of compacta, and since no continuum is the countable infinite union of disjoint nonempty compacta (Sierpinski's Theorem), we must have $\Pi_2^{-1}(x) \subset \Pi_1^{-1}(C)$ for some $C \in \{W_i\colon i = 1, 2, \dots\} \cup D$. Since K is uncountable, there exist distinct points $x, y \in K$ such that $\Pi_2^{-1}(x) \cup \Pi_2^{-1}(y) \subset \Pi_1^{-1}(C)$ for some continuum C as above. Since $\Pi_1^{-1}(C)$ is a continuum, $\Pi_2(\Pi_1^{-1}(C)) \subset Q$ is a continuum, and another application of Sierpinski's Theorem shows that $\Pi_2(\Pi_1^{-1}(C)) \subset K$. Thus $\Pi_2(\Pi_1^{-1}(C))$ is a point, but $\{x, y\} \subset \Pi_2(\Pi_1^{-1}(C))$, a contradiction.

Of course, the same argument applies as well to the space $Y = Q \setminus \bigcup_1^\infty W_i$. A similar argument shows that in Y, no countable infinite subset is negligible. In particular, $X = Y \setminus D \not\approx Y$.

6.2. Theorem. *No countable infinite subset is negligible in the space Y.*

Proof. Suppose $Y \approx Y \setminus E$, where E is countable infinite. There exist a compact space M and monotone surjections $\Pi_1\colon M \to Q$ and $\Pi_2\colon M \to Q$ with $\Pi_1^{-1}(\bigcup_1^\infty W_i) = \Pi_2^{-1}(\bigcup_1^\infty W_i \cup E)$. For each $x \in E$, $\Pi_2^{-1}(x) \subset \Pi_1^{-1}(W_i)$ for some i. Since $\Pi_2(\Pi_1^{-1}(W_i))$ is a continuum, $\Pi_2(\Pi_1^{-1}(W_i)) = \{x\}$. Thus $\Pi_2^{-1}(x) = \Pi_1^{-1}(W_i)$. Consider the infinite subcollection $\mathcal{W} = \{W_i\colon \Pi_1^{-1}(W_i) = \Pi_2^{-1}(x) \text{ for some } x \in E\}$. Then $\bigcup \mathcal{W}$ is connected,

$$\Pi_1^{-1}(\cup \mathcal{W}) = \cup \left\{\Pi_1^{-1}(W_i)\colon W_i \in \mathcal{W}\right\} = \cup \left\{\Pi_2^{-1}(x)\colon x \in E\right\} = \Pi_2^{-1}(E)$$

is connected, and $E = \Pi_2(\Pi_2^{-1}(E))$ is connected, a contradiction.

7. A fake capset. As previously remarked, various characterizations of boundary sets (dense σ-Z-sets in Q whose complements are homeomorphic to s) are given in [**8**]. One of these takes the following form.

7.1. Theorem. *A dense σ-Z-set $B \subset Q$ is a boundary set if and only if*

> *for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every compact set $K \subset B$ with $\operatorname{diam} K < \delta$, there exists a compact set $\tilde{K} \subset B$ with $\operatorname{diam} \tilde{K} < \varepsilon$ such that K contracts to a point in every neighborhood of $\tilde{K}$ in Q.*

Without loss of generality we may assume that $\tilde{K}$ is a continuum containing K. Thus, every boundary set is continuum-connected and locally continuum-connected.

This property of boundary sets provides another way of seeing that the spaces $Y = Q \setminus \bigcup_1^\infty W_i$ and $X = Q \setminus (\bigcup_1^\infty W_i \cup D)$ are not homeomorphic to l_2, since by Sierpinski's Theorem the σ-Z-sets $\bigcup_1^\infty W_i$ and $\bigcup_1^\infty W_i \cup D$ are not continuum-connected.

However, starting with the set $\bigcup_1^\infty W_i$, we may add a null sequence of arcs $\{\alpha_i\}$ in Q to obtain a σ-Z-set $\bigcup_1^\infty W_i \cup (\bigcup_1^\infty \alpha_i)$ which *is* continuum-connected, but not locally continuum-connected. Taking products with copies of Q, we obtain the following.

7.2. Example. There exists a tower of compacta $B_1 \subset B_2 \subset \dots$ in Q such that

(1) each $B_n \approx Q$;

(2) each B_n is a Z-set in Q;

(3) each B_n is a Z-set in B_{n+1};

(4) for every $\varepsilon > 0$ there exists for some n a map η: $Q \to B_n$ with $d(\eta, \mathrm{id}) < \varepsilon$;

(5) $B = \bigcup_1^\infty B_n$ is not locally continuum-connected, and therefore $Q \setminus B \not\approx s$.

Note. The set B is a counterexample to the Capset Characterization Lemma 1.1 of **[13]**, which claimed that conditions (1) through (4) imply that B is a *capset* (i.e., there exists a homeomorphism of Q taking B onto the pseudo-boundary $Q \setminus s$). The argument given there breaks down at the attempted application of the Anderson-Barit estimated homeomorphism extension theorem (every homeomorphism h between Z-sets in Q with $d(h, \mathrm{id}) < \varepsilon$ can be extended to a homeomorphism H of Q with $d(H, \mathrm{id}) < \varepsilon$). Stated in this form, the extension theorem is valid only with respect to one of the standard convex metrics on Q. However, the application is attempted for the copies B_n of Q, using the restrictions of a metric on Q, and these restrictions may be highly nonconvex.

The capset characterization theorem has been widely used by the authors, and others. Fortunately, in all applications of which we are aware, the mapping condition (4) can be replaced by the stronger condition

(4*) there exists a deformation α: $Q \times [0,1] \to Q$ with $\alpha(q, 0) = q$ and such that, for every $t > 0$, $\alpha(Q \times [t, 1]) \subset B_n$ for some n.

It is shown in **[8]** that if conditions (1) through (4*) are met, then $B = \bigcup_1^\infty B_n$ *is* a capset.

Construction of example. We first construct a tower $A_1 \subset A_2 \subset \ldots$ of compact AR's in Q satisfying the conditions (4) and (5), and then take $B_n = A_n \times Q \times I^n \times \{(1, 1, \ldots)\} \subset Q \times Q \times Q$. By Edward's product theorem (see **[6]**), each $B_n \approx Q$.

For each $i = 1, 2, \ldots,$ let

$$\alpha_i = \left(\{1\}_i \times [1 - 2^{-i}, 1]_{i+1} \cup [1 - 2^{-(i+1)}, 1]_i \times \{1\}_{i+1}\right) \times \prod_{j \neq i, i+1} \{0\}_j \subset Q.$$

Then $\{\alpha_i\}$ is a null sequence of disjoint arcs converging to the point $(0, 0, \ldots)$; each α_i connects W_i and W_{i+1}; and $\alpha_i \cap (\bigcup_1^\infty W_j)$ contains only the endpoints of α_i. Take $A_n = \bigcup_1^n (W_i \cup \alpha_i)$. Clearly, $\bigcup_1^\infty A_n$ is not locally continuum-connected.

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