TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 273, Number 1, September 1982

RATIONAL L.-S. CATEGORY AND ITS APPLICATIONS

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ABSTRACT. Let S be a 1-connected CW-complex of finite type and put $\operatorname{cat}_0(S) =$ Lusternik-Schnirelmann category of the localization $S_{\mathbf{Q}}$. This invariant is characterized in terms of the minimal model of S. It is shown that if $\phi\colon S \to T$ is injective on $\pi_* \otimes \mathbf{Q}$ then $\operatorname{cat}_0(S) \leqslant \operatorname{cat}_0(T)$, and this result is strengthened when ϕ is the fibre inclusion of a fibration. It is also shown that if $\dim H^*(S; \mathbf{Q}) < \infty$ then either $\dim \pi_*(S) \otimes \mathbf{Q} < \infty$ or the groups $\pi_k(S) \otimes \mathbf{Q}$ grow exponentially with k.

Introduction. In this paper *topological space* means a pointed, path-connected normal space S such that the inclusion of the basepoint is a cofibration and the rational singular homology is finite dimensional in each degree. All vector spaces and algebras have $\mathbf{Q}$ as ground field. Homology and cohomology of spaces is singular, with rational coefficients.

We shall be dealing with category of spaces ($\operatorname{cat}(S)$) in the sense of Lusternik and Schnirelmann [**18**]. A recent survey is given in [**15**] but our definition differs by one from that of [**15**], so that for us spheres have category one, not two.

Our purpose is to provide a computationally useful characterization of the *rational category* $\operatorname{cat}_0(S)$ of space S and to apply this in a variety of situations. The definition of $\operatorname{cat}_0(S)$ is sketched below and given precisely in §4. It satisfies $\operatorname{cat}_0(S) \leqslant \operatorname{cat}(S)$. When S is 1-connected $\operatorname{cat}_0(S)$ coincides with the category of the localization $S_{\mathbf{Q}}$, which has been studied by Toomer in [**26** and **27**] and Lemaire and Sigrist in [**17**]. The characterization problem was posed by Berstein [**26**].

Most of the applications depend on the following fundamental mapping theorem, which we derive as a consequence of our characterization.

THEOREM I. *Suppose $\phi\colon S \to T$ is a continuous map between 1-connected spaces, and assume that $\phi_\#\colon \pi_*(S) \otimes \mathbf{Q} \to \pi_*(T) \otimes \mathbf{Q}$ is injective. Then* $\operatorname{cat}_0(S) \leqslant \operatorname{cat}_0(T)$.

In [**16**] Lemaire considers a space $W = (\mathbf{C}P^2 \vee S^2) \cup_\omega e^7$, where $\omega = [\alpha, \beta]$ and $\alpha \in \pi_5(\mathbf{C}P^2)$, $\beta \in \pi_2(S^2)$ are the obvious basis elements. He announces a result of [**17**], namely that $\operatorname{cat}(W) = 3$. Using Theorem I we can recover this result and find (Example 5.9) that the n-fold Cartesian product W^n has category $3n$.

Theorem I can be essentially strengthened for the inclusion of the fibre in a Serre fibration. Indeed, suppose $\xi\colon F \xrightarrow{j} E \xrightarrow{\pi} B$ is a Serre fibration of simply connected

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Received by the editors November 11, 1980 and, in revised form, June 3, 1981.

1980 *Mathematics Subject Classification*. Primary 55P62, 55P50, 55M30, 55Q52.

Key words and phrases. Lusternik-Schnirelmann category, minimal model, rational homotopy, rank of a space.

[1]Chercheur qualifié au F.N.R.S.

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0002-9947/82/0000-0664/$09.50

spaces and consider $j_{\#}\colon \pi_*(F)\otimes \mathbf{Q}\to \pi_*(E)\otimes \mathbf{Q}$. Put

$$k_\xi = \begin{cases} \dim\ker j_{\#} & \text{if } j_{\#} \text{ is injective in even degrees,} \\ \infty & \text{otherwise.} \end{cases}$$

We have then

THEOREM II. *With the hypotheses and notation above,*

$$k_\xi \leqslant \mathrm{cat}_0(F) \leqslant \mathrm{cat}_0(E) + k_\xi.$$

Suppose $\phi\colon S\to T$ is a continuous map between 1-connected spaces inducing a surjection $\pi_2(S)\to\pi_2(T)$. If $\phi_{\#}\colon \pi_*(S)\otimes\mathbf{Q}\to\pi_*(T)\otimes\mathbf{Q}$ is not surjective in odd degrees or if $\operatorname{Im}\phi_{\#}$ has infinite codimension then the first inequality of Theorem II implies that the homotopy fibre of ϕ has infinite rational category and hence infinite category.

Perhaps more usefully, let $\psi\colon F\to E$ be a continuous map between simply connected spaces and recall Massey's problem 10 in **[19]**: When is ψ the inclusion of a fibre? Define $k(\psi)$ using $\psi_{\#}$ exactly the way we defined k_ξ using $j_{\#}$ and note that Theorem II gives $k(\psi)\leqslant \mathrm{cat}_0(F)\leqslant \mathrm{cat}_0(E)+k(\psi)$ as a necessary condition.

Next, fix a simply connected space F with $\mathrm{cat}_0(F)=m<\infty$. The first inequality of Theorem II asserts that for each Serre fibration

$$\xi\colon F\xrightarrow{j_\xi}E_\xi\xrightarrow{\pi_\xi}B_\xi$$

the space $\ker((j_\xi)_{\#}\colon \pi_*(F)\otimes\mathbf{Q}\to\pi_*(E_\xi)\otimes\mathbf{Q})$ is concentrated in odd degrees and has dimension at most m.

Consider the set $\bigcup_\xi \ker(j_\xi)_{\#}$ where ξ runs over all Serre fibrations with fibre F. This is easily identified with $G_*(F)\otimes\mathbf{Q}$, where $G_*(F)\subset\pi_*(F)$ is the graded subgroup defined by Gottlieb in **[9]**. We can improve Theorem II with

THEOREM III. *Let F be a simply connected space such that* $\mathrm{cat}_0(F)=m<\infty$. *Then $G_*(F)\otimes\mathbf{Q}$ is concentrated in odd degrees, and* $\dim G_*(F)\otimes\mathbf{Q}\leqslant m$.

Theorems II and III can be used to study the growth of the coefficients of the formal power series $f_\pi(S,t)=\sum_{k=2}^\infty \dim(\pi_k(S)\otimes\mathbf{Q})t^k$ of a simply connected space S. Recall that the coefficients of a power series $\sum a_k t^k$ *grow exponentially* if there are constants $C_2\geqslant C_1>1$ and an integer K such that

$$C_1^k\leqslant \sum_{p\leqslant k}|a_p|\leqslant C_2^k,\qquad k\geqslant K.$$

THEOREM IV. *Let S be a simply connected space such that* $\dim H^*(S)<\infty$. *Then either*

(i) $\dim\pi_*(S)\otimes\mathbf{Q}<\infty$, *or*

(ii) *the coefficients of $f_\pi(S;t)$ grow exponentially.*

DEFINITION. A simply connected space S such that $\dim H^*(S)<\infty$ will be called *rationally elliptic* if $\dim\pi_*(S)\otimes\mathbf{Q}<\infty$. Otherwise (by Theorem IV) the coefficients of $f_\pi(S;t)$ grow exponentially and S will be called *rationally hyperbolic.*

Rationally elliptic spaces are studied in **[6]** where they are called spaces of type F.

If S is rationally elliptic then

$$\dim(\pi_{\text{odd}}(S) \otimes \mathbf{Q})$$

is called the *rank of S*.

The coefficients in a formal power series $\Sigma a_k t^k$ *grow polynomially of order at most* r if for some $A > 0$ and all k, $|a_k| \leqslant Ak^r$. Writing $f_H(S; t) = \Sigma_{k=0}^{\infty} \dim H^k(S)t^k$ we have

THEOREM V. *Let S be a simply connected space such that* $\dim H^*(S) < \infty$. *Write* ΩS *for the loop space. If S is rationally elliptic the coefficients of* $f_H(\Omega S; t)$ *grow polynomially of order rank S. Otherwise they grow exponentially.*

Our remaining applications make use of another invariant, $e_0(S)$, introduced by Toomer [**26**] to approximate $\text{cat}_0(S)$ and denoted there by $f_0(S)$. This invariant is the largest integer k such that in the spectral sequence of Milnor and Moore, $E_\infty^{k,*} \neq 0$ (see §9 for more details).

It is always true that $\text{cat}_0(S) \geqslant e_0(S)$ and that equality holds when S is formal or π-formal (see §1 for the definitions). In [**26**] it is suggested that equality is always true, but Lemaire and Sigrist show in [**17**] that for $W = (\mathbf{C}P^2 \vee S^2) \cup_\omega e^7$, $\text{cat}_0(W) = 3$ and $e_0(W) = 2$. We shall show that the difference $\text{cat}_0 - e_0$ can be arbitrarily large by computing that for the n-fold Cartesian product $\text{cat}_0(W^n) = 3n$ and $e_0(W^n) = 2n$.

The invariant, $e_0(S)$, is an essential ingredient in the proofs of:

THEOREM VI. *Let S be a simply connected rationally hyperbolic space such that* $H^*(S)$ *is a Poincaré duality algebra. In the homotopy Lie algebra* $\pi_*(\Omega S) \otimes \mathbf{Q}$ *there are then homogeneous elements* α, β *such that the iterated brackets*

$$[\beta[\beta[\beta \cdots [\beta, \alpha]] \cdots] \qquad (k \textit{ factors } \beta,\ k = 1,2,3,\ldots)$$

are all nonzero. In particular, $\pi_*(\Omega S) \otimes \mathbf{Q}$ *is not nilpotent.*

THEOREM VII. *If S is simply connected and rationally elliptic then* $\text{cat}_0(S) \geqslant e_0(S) \geqslant \text{rank}(S)$. *If, in addition, S is* π*-formal then* $\text{cat}_0(S) = e_0(S) = \text{rank}(S)$.

Combining this with Theorem II we have the

COROLLARY. *If* $F \xrightarrow{j} E \xrightarrow{\pi} B$ *is a Serre fibration of simply connected spaces and if F is rationally elliptic then*

$$\text{cat}_0(E) \geqslant \dim j_\#(\pi_{\text{odd}}(F) \otimes \mathbf{Q}).$$

The results described above are based on our characterization of $\text{cat}_0(S)$. Recall that, by definition, $\text{cat}(S) \leqslant m$ if S can be covered by $m + 1$ open sets, each contractible in S. The least such m is the category of S; if there is no such cover, $\text{cat}(S) = \infty$.

Equivalently [**28**, **3**] $\text{cat}(S)$ is the least integer m such that the diagonal map of S into the $(m + 1)$-fold Cartesian product S^{m+1} can be deformed into the fat wedge, $T^{m+1}(S)$. (This is the subspace of those $(m + 1)$-tuples $(s_1, \ldots, s_{m+1})$ with at least one $s_i = *$.) Further, in [**7**] Ganea constructs spaces E_m from S and shows that $\text{cat}(S) \leqslant m$ if and only if S is a retract of E_m.

The second definition has an obvious analogue in Sullivan's theory of minimal models and we use this to define $\operatorname{cat}_0(S)$ (cf. §4): we set $\operatorname{cat}_0(S)$ to be the least integer m such that the Sullivan representative of the diagonal map $S \to S^{m+1}$ factors up to homotopy through the model of the fat wedge. It follows at once that $\operatorname{cat}_0(S) \leqslant \operatorname{cat}(S)$ and that if S is simply connected, $\operatorname{cat}_0(S) = \operatorname{cat}_0(S_{\mathbf{Q}}) = \operatorname{cat}(S_{\mathbf{Q}})$. Moreover if $H^p(S) = 0$, $p > m$, then $\operatorname{cat}_0(S) \leqslant m$.

To describe our characterization of $\operatorname{cat}_0(S)$ we recall first that the minimal model of S (in the sense of Sullivan) is, inter alia, a connected commutative graded differential algebra (c.g.d.a.) of the form $(\Lambda X, d)$, where

$$\Lambda X = \text{exterior algebra}\,(X^{\text{odd}}) \otimes \text{symmetric algebra}\,(X^{\text{even}})$$

is the free graded commutative algebra over X. This algebra carries a (second) *wedge gradation* $\Lambda X = \Sigma_{q \geqslant 0} \Lambda^q X$ with $\Lambda^q X$ the linear span of the products $x_1 \wedge \cdots \wedge x_q$, $x_i \in X$. The ideals $\Lambda^{>q} X$ are d-stable, and the filtration they define gives the Milnor-Moore spectral sequence.

Denote by $(\Lambda X_{(m)}, d)$ the minimal model of the quotient c.g.d.a. $(\Lambda X / \Lambda^{>m} X, d)$ and suppose $\phi_m \colon (\Lambda X, d) \to (\Lambda X_{(m)}, d)$ represents the projection $(\Lambda X, d) \to (\Lambda X/\Lambda^{>m} X, d)$. We have

THEOREM VIII. *For any space S, $\operatorname{cat}_0(S) \leqslant m$ if and only if there is a c.g.d.a. morphism $r \colon (\Lambda X_{(m)}, d) \to (\Lambda X, d)$ such that $r\phi_m = \mathrm{id}$.*

The first step in establishing Theorem VIII is to make an algebraic construction starting from $(\Lambda X, d)$, which produces a c.g.d.a. carrying the rational homotopy type of the fat wedge. The next is to construct c.g.d.a.'s (Γ_m, d) from $(\Lambda X, d)$ which are the analogues of Ganea's spaces E_m. Indeed, when S is simply connected, (Γ_m, d) carries the rational homotopy type of E_m (Proposition 2.7).

The main step is the analysis of the rational homotopy type of (Γ_m, d). We show that it has the same minimal model as a c.g.d.a. of the form $(\Lambda X/\Lambda^{>m} X) \oplus V$ in which V has trivial differential and $V \cdot (\Lambda^+ X/\Lambda^{>m} X \oplus V) = 0$. This yields

THEOREM IX. *Let S be a simply connected space with minimal model $(\Lambda X, d)$ and let W be a space whose rational homotopy type is represented by $(\Lambda X/\Lambda^{>m} X, d)$. Then Ganea's space E_m has the rational homotopy type of $W \vee \bigvee_\alpha S_\alpha$, where the S_α are spheres.*

This paper is organized as follows. In §1 we recall basic definitions, notation and results from Sullivan's theory of minimal models. In §2 we obtain a model for the fat wedge, construct (Γ_m, d) and show that it represents E_m when S is simply connected. The analysis of (Γ_m, d) is in §3, including the proof of Theorem IX, which appears there as Theorem 3.2.

Rational category is defined in §4, where also is established the characterization theorem (Theorem 4.7) which contains Theorem VIII. The mapping theorem (Theorem I) is proved in §5, while §6 contains the results on fibrations. In §7 we consider examples for which the rational Gottlieb groups vanish.

Theorems IV and V are in §8, while §9 contains the basic facts about $e_0(S)$ and §10 has the proofs of Theorems VI and VII. Finally, in §11 we list a number of open questions.

1. Minimal models. We recall here the basic facts and notation we shall need from Sullivan's theory of minimal models, for which the basic reference is **[25]**. A description can be found in **[13]** and complete details in **[12]**; the notation and terminology in these latter two references is identical with ours.

A *commutative graded differential algebra* (c.g.d.a.) (A, d_A) is a graded algebra $A = \Sigma_{p \geqslant 0} A^p$ with a derivation d_A of degree 1 such that $d_A^2 = 0$ and such that $ab = (-1)^{pq} ba$, $a \in A^p$, $b \in A^q$. The quotient algebra $\ker d_A / \operatorname{Im} d_A$ is written $H(A)$ and called the cohomology algebra. The algebra A is *connected* if $A^\circ = \mathbf{Q}.1$ and *c-connected* if $H(A)$ is connected. We write $(A, d_A)^{\otimes m}$ or simply $A^{\otimes m}$ for the m-fold tensor product in the category of c.g.d.a.'s.

A *morphism* ϕ: $(A, d_A) \to (B, d_B)$ is a homomorphism of graded algebras commuting with the differentials. It induces ϕ^*: $H(A) \to H(B)$. If ϕ^* is an isomorphism, ϕ is called a *quasi-isomorphism* or an elementary homotopy equivalence, and we write $\phi: (A, d_A) \xrightarrow{\simeq} (B, d_B)$. A sequence of elementary homotopy equivalences (in alternating directions) is called a *homotopy equivalence*.

If $(A, d_A) \to (C_i, d_i)$, $i = 1, 2$, are morphisms then $C_1 \otimes_A C_2$ is the quotient c.g.d.a. obtained from $(C_1, d_1) \otimes (C_2, d_2)$ by dividing by the ideal generated by $a \otimes 1 - 1 \otimes a$, $a \in A$.

If (A, d) and (B, d) are augmented c.g.d.a.'s with augmentation ideals I_A, I_B, the c.g.d.a. $\mathbf{Q} \oplus I_A \oplus I_B$ defined by $I_A \cdot I_B = 0$ is denoted $(A, d) \vee (B, d)$.

Finally we denote by $(v_1, v_2, \dots)$ the vector space W with basis $v_1, v_2, \dots$ and we write $\Lambda(v_1, v_2, \dots)$ for ΛW.

A *KS-complex* is a c.g.d.a. which can be written $(\Lambda X, d)$ and in which X admits a well-ordered homogeneous basis $\{x_\alpha\}$ with $dx_\alpha \in \Lambda X_{<\alpha}$, $X_{<\alpha}$ denoting the span of the $x_\beta < x_\alpha$. We call this a *KS-basis*. If the KS-basis can be chosen so that $x_\beta < x_\alpha \Rightarrow \deg x_\beta \leqslant \deg x_\alpha$ the KS-complex is called *minimal*. When ΛX is connected this is equivalent to d: $X \to \Lambda^{\geqslant 2} X$ [**12**, Chapter 2]. If in $(\Lambda X, d)$, $X = Y \oplus d(Y)$ and d: $Y \xrightarrow{\cong} d(Y)$ then the KS-complex is called *contractible*. An important example is $\Lambda(t, dt)$ in which $\deg t = 0$. Putting $t = 0, 1$ defines augmentations ρ_0, ρ_1: $\Lambda(t, dt) \to \mathbf{Q}$.

A *homotopy* between two morphisms ϕ_0, ϕ_1: $(\Lambda X, d) \to (A, d_A)$ from a KS-complex is a morphism $\phi: (\Lambda X, d) \to (A, d_A) \otimes \Lambda(t, dt)$ such that $\rho_i \phi = \phi_i$. We call ϕ_0 and ϕ_1 *homotopic* and write $\phi_0 \sim \phi_1$. The basic *lifting theorem* [**25**, Corollary 3.6; **12**, Theorem 5.19] asserts that given $\phi: (\Lambda X, d) \to (A, d_A)$ and $\psi: (C, d_C) \xrightarrow{\simeq} (A, d_A)$ there exists a unique homotopy class of morphisms $\chi: (\Lambda X, d) \to (C, d_C)$ such that $\psi \chi \sim \phi$.

A *KS-extension* is a sequence $\xi: (B, d_B) \xrightarrow{i} (C, d_C) \xrightarrow{\rho} (A, d_A)$ of morphisms in which we can identify $C = B \otimes A$, $i(b) = b \otimes 1$, $\rho = \varepsilon \otimes \mathrm{id}$ for some augmentation ε of B, and in which (A, d_A) is a KS-complex with KS-basis $\{x_\alpha\}$ such that $d_C(1 \otimes x_\alpha) \in B \otimes \Lambda X_{<\alpha}$. It is *minimal* if the KS-basis can be chosen so that $x_\alpha < x_\beta \Rightarrow \deg x_\alpha \leqslant \deg x_\beta$. If (B, d_B) is also a KS-complex we call ξ a Λ-*extension*; if B and ξ are both minimal we call ξ Λ-*minimal*. If ξ is a Λ-extension then (C, d_C) is a KS-complex.

Given a KS-complex $(\Lambda X, d)$, the projection $\Lambda^+ X \to \Lambda^+ X/\Lambda^{\geq 2}X$ induces a differential $Q(d)$ in the quotient. The inclusion $X \to \Lambda^+ X$ gives an isomorphism $X \xrightarrow{\cong} \Lambda^+ X/\Lambda^{\geq 2}X$ so that we may and usually do regard $Q(d)$ as a differential in X. It is the linear part of d.

Suppose $(B, d_B) \xrightarrow{\phi} (E, d_E)$ is a morphism between c-connected c.g.d.a.'s with B augmented. A main theorem of minimal models [**25**, §5; **12**, Chapter 6] asserts the existence of a commutative diagram of morphisms

$$
\begin{tikzcd}
 & & E \\
B \arrow[urr, "\phi"] \arrow[r] & B \otimes \Lambda X \arrow[r] & \Lambda X \arrow[u, "\simeq"']
\end{tikzcd}
\tag{1.1}
$$

in which the bottom row is a minimal KS-extension. This extension is determined up to isomorphism by ϕ and (1.1) is called the *minimal model of* ϕ. When $B = \mathbf{Q}$ we refer to $\Lambda X \xrightarrow{\cong} E$ as the *minimal model of* E. A diagram of the form (1.1) where the extension is not required to be minimal is called simply a *model* for ϕ (or for E). If B is replaced by a model, (1.1) is called a Λ-*model.*

Fix a c-connected c.g.d.a. (A, d_A). The spaces $H(X, Q(d))$ as $(\Lambda X, d)$ runs through the models for A can be naturally identified [**12**, Chapter 8] and their common identification is denoted by $\pi_\psi^*(A, d_A)$, the ψ-*homotopy space*. A morphism $\phi: (A, d_A) \to (B, d_B)$ determines via the lifting theorem a unique homotopy class of morphisms $\psi: (\Lambda X, d) \to (\Lambda Y, d)$ between the minimal models. The linear part, $Q(\psi)$, of ψ is a linear map $\phi^\#: \pi_\psi^*(A, d_A) \to \pi_\psi^*(B, d_B)$.

1.2 Lemma. *Suppose* $(B, d) \xrightarrow{i} (B \otimes \Lambda X, d) \xrightarrow{\rho} (\Lambda X, \bar{d})$ *is a KS-extension in which* $H^0(B) = \mathbf{Q}$ *and* $H(X, Q(\bar{d})) = 0$. *Then* i^* *is an isomorphism and* $H(\Lambda X, \bar{d}) = \mathbf{Q}$.

Proof. In view of [**12**, Theorem 2.2] we need only prove that $H^0(B \otimes \Lambda X) = \mathbf{Q}$. Let $\{x_\alpha\}$ be a well-ordered basis of X for which $dx_\alpha \in B \otimes \Lambda X_{<\alpha}$. Choose the least α such that $H^0(B \otimes \Lambda X_{\leq\alpha}) \neq \mathbf{Q}$. Then $\deg x_\alpha = 0$ and there is a degree zero cocycle of the form $\Sigma_{k=0}^m \Phi_k x_\alpha^k$ with $\Phi_k \in (B \otimes \Lambda X_{<\alpha})^0$ and $\Phi_m \neq 0$.

Clearly $d\Phi_m = 0$ and $d\Phi_{m-1} + m\Phi_m dx_\alpha = 0$. The first equation implies (by hypothesis) that Φ_m is a scalar; we may suppose $\Phi_m = 1$. The second equation then shows that $x_\alpha + \frac{1}{m}\Phi_{m-1}$ is a cocycle. The composite projection $B \otimes \Lambda X \to \Lambda X \to \Lambda X/\Lambda^{\geq 2}X = X \oplus \mathbf{Q}$ maps this cocycle to $x_\alpha + y + \lambda$, $y \in X_{<\alpha}$, $\lambda \in \mathbf{Q}$, and $x_\alpha + y$ is necessarily a nonzero $Q(d)$ cocycle of degree zero, a contradiction. Q.E.D.

The connection of minimal models with topology is provided by Sullivan's functor $K \to A(K)$ from simplicial sets to c.g.d.a.'s [**25**, §7; **12**, Chapter 13]; $A(K)$ is the algebra of compatible rational polynomial differential forms on the simplices of K. Integration induces a natural algebra isomorphism $H(A(K)) \xrightarrow{\cong} H(K)$.

When this is composed with the functor "singular simplices" we obtain the functor $S \to A(S)$ from spaces to c.g.d.a.'s with $H(A(S))$ and $H^*(S)$ naturally isomorphic. A (minimal) model for $A(S)$ is called a (minimal) model for S and we write $\pi_\psi^*(S) = \pi_\psi^*(A(S))$. There is [**25**, Theorem 10.1] a natural isomorphism

$$\pi_\psi^*(S) \cong \mathrm{Hom}_{\mathbf{Z}}\big(\pi_*(S); \mathbf{Q}\big) \tag{1.3}$$

if S is simply connected. If $\phi: S \to T$ then $\phi^{\#}: \pi_\psi^*(T) \to \pi_\psi^*(S)$ is dual to $\phi_{\#}: \pi_*(S) \to \pi_*(T)$.

If $(\Lambda X, d)$ and $(\Lambda Y, d)$ are models for spaces S and T, a continuous map $\phi: S \to T$ determines via the lifting theorem a unique homotopy class of morphisms $(\Lambda Y, d) \to (\Lambda X, d)$. Any one of these is said to *represent* ϕ. If $\alpha: (A, d_A) \to (B, d_B)$ is a morphism, and $(\Lambda Y, d)$ and $(\Lambda X, d)$ are models for (A, d_A) and (B, d_B) and α lifts to a morphism representing ϕ we say also that α *represents* ϕ.

Finally a space or c.g.d.a. with cohomology H is called *formal* if its minimal model coincides with that of $(H, 0)$. It is called *π-formal* if its minimal model can be written in the form $(\Lambda X, d)$ with $d: X \to \Lambda^2 X$.

2. Models for the fat wedge and for Ganea's spaces E_m. In this section we first obtain a description of the model of the fat wedge, $T^{m+1}(S)$, in terms of a model

$$\gamma: \Lambda X \to A(S)$$

for S. A model for the $(m+1)$-fold Cartesian product, S^{m+1}, is given by $\gamma^{(m+1)}$: $(\Lambda X, d)^{\otimes m+1} \to A(S^{m+1})$ where, if π_i is the projection on the ith coordinate of S^{m+1},

$$\gamma^{(m+1)}(\Phi_1 \otimes \cdots \otimes \Phi_{m+1}) = A(\pi_1)\gamma\Phi_1 \wedge \cdots \wedge A(\pi_{m+1})\gamma\Phi_{m+1}.$$

With respect to these models the diagonal map $S \to S^{m+1}$ is represented by the multiplication map $\mu: (\Lambda X, d)^{\otimes m+1} \to (\Lambda X, d)$.

To analyse $A(T^{m+1}(S))$ we let $*$ be the basepoint of S and denote by $T_i^{m+1} \subset T^{m+1}(S)$ the subset of points whose ith coordinate is $*$. Thus $T^{m+1}(S) = \bigcup_{i=1}^{m+1} T_i^{m+1}$. The singular simplices of $T^{m+1}(S)$ whose image lies in some T_i^{m+1} form a subsimplicial set. The compatible differential forms on the simplices of this form [**25**, §7; **12**, §13.5] a c.g.d.a. $(\tilde{A}(T^{m+1}(S)), d)$ and restriction is a surjective morphism from $A(T^{m+1}(S))$ to $\tilde{A}(T^{m+1}(S))$.

The projections π_i determine morphisms

$$\lambda_i^{m+1}: A(S) \to A(S^{m+1}) \to A(T^{m+1}(S)) \to \tilde{A}(T^{m+1}(S)).$$

Multiplying these together yields a morphism

$$\lambda^{m+1}: (A(S), d)^{\otimes m+1} \to (\tilde{A}(T^{m+1}(S)), d).$$

If I_S denotes the kernel of the augmentation $A(S) \to \mathbf{Q}$ determined by $*$, then $I_S^{\otimes m+1} \subset \ker \lambda^{m+1}$. We thus obtain from λ^{m+1} a morphism

$$\phi^{m+1}: \frac{(A(S), d)^{\otimes m+1}}{I_S^{\otimes m+1}} \to \tilde{A}(T^{m+1}(S), d).$$

2.1 LEMMA. *The morphisms*

$$\frac{A(S)^{\otimes m+1}}{I_S^{\otimes m+1}} \xrightarrow{\phi^{m+1}} \tilde{A}(T^{m+1}(S)) \xleftarrow{\text{restriction}} A(T^{m+1}(S))$$

are quasi-isomorphisms.

PROOF. Because S is well pointed and normal the identity map of S can be deformed by a basepoint preserving homotopy to a map which contracts a neighbourhood U of $*$ onto $*$. The $(m+1)$-fold product of this map carries $S \times \cdots U \cdots \times S$ to $S \times \cdots * \cdots \times S$. It is straightforward to deduce from this that the inclusion of the subsimplicial set above into all the singular simplices induces a homology isomorphism. It follows [**12**, Theorem 14.18] that the restriction $A \to \tilde{A}$ is a quasi-isomorphism.

To prove that ϕ^{m+1} is a quasi-isomorphism we use induction on m. When $m = 0$, $\phi^1 = \mathrm{id}\colon \mathbf{Q} \xrightarrow{\cong} \mathbf{Q}$. Suppose the lemma holds for $m = n - 1$ and consider the case $m = n$. Write $T^{n+1}(S) = [S \times T^n(S)] \cup [* \times S^n]$ and observe that the intersection of these sets is $* \times T^n(S)$. Use the singular simplices whose image is in one of the T_i^{n+1} $(2 \leqslant i \leqslant n+1)$ to define a c.g.d.a. $(\tilde{A}(S \times T^n(S)), d)$; as above, restriction is a surjective quasi-isomorphism $A(S \times T^n(S)) \to \tilde{A}(S \times T^n(S))$. Use $\tilde{A}(T^n(S)) = \tilde{A}(* \times T^n(S))$ to denote the c.g.d.a. constructed via the sets $T_i^n \subset T^n(S)$.

The decomposition of $T^{n+1}(S)$ leads to the row-exact commutative diagram

$$\begin{array}{ccccccccc}
0 \to & \tilde{A}(T^{n+1}(S)) & \xrightarrow{(F_1, F_2)} & \tilde{A}(S \times T^n(S)) & \oplus & A(* \times S^n) & \xrightarrow{G_1 - G_2} & \tilde{A}(T^n(S)) & \to 0 \\
& \phi^{n+1} \uparrow & & \mathrm{id} \cdot \phi^n \uparrow & & \uparrow \gamma^{(n)} & & \uparrow \phi^n & \\
0 \to & \dfrac{A(S)^{\otimes n+1}}{I_S^{\otimes n+1}} & \underset{(f_1, f_2)}{\to} & \left[A(S) \otimes \dfrac{A(S)^{\otimes n}}{I_S^{\otimes n}} \right] & \oplus & A(S)^{\otimes n} & \underset{g_1 - g_2}{\to} & \dfrac{A(S)^{\otimes n}}{I_S^{\otimes n}} & \to 0
\end{array}$$

in which F_1, F_2, G_1, G_2 are the obvious restrictions, while f_1, f_2, g_1, g_2 are the algebraic analogues,

$$f_1 = \text{obvious projection}, \qquad f_2 = \varepsilon \otimes \mathrm{id},$$
$$g_1 = \varepsilon \otimes \mathrm{id}, \qquad g_2 = \text{obvious projection}.$$

Note that $(\mathrm{id} \cdot \phi^n)\Phi \otimes \Psi = A(\pi_1)\Phi \cdot A(\pi_2)\phi^n\Psi$ where $\pi_1\colon S \times T^n(S) \to S$, $\pi_2\colon S \times T^n(S) \to T^n(S)$ are the projections. The diagram

$$\begin{array}{ccccc}
& & A(S) \otimes A(T^n(S)) & \underset{\simeq}{\overset{\text{mult.}}{\to}} & A(S \times T^n(S)) \\
& & \downarrow \simeq & & \downarrow \simeq \\
\mathrm{id} \cdot \phi^n\colon A(S) \otimes \dfrac{A(S)^{\otimes n}}{I_S^{\otimes n}} & \underset{\mathrm{id} \otimes \phi^n}{\overset{\simeq}{\to}} & A(S) \otimes \tilde{A}(T^n(S)) & \underset{\text{mult.}}{\to} & \tilde{A}(S \times T^n(S))
\end{array}$$

shows that $\mathrm{id} \cdot \phi^n$ is a quasi-isomorphism. The induction is now completed by applying the five lemma to the long exact cohomology sequences arising from the earlier diagram. Q.E.D.

Denote by

$$\eta\colon (\Lambda X)^{\otimes m+1} \to (\Lambda X)^{\otimes m+1} / (\Lambda^+ X)^{\otimes m+1}$$

the projection, and use it to equip the quotient algebra with a differential. The model morphism $\gamma\colon \Lambda X \to A(S)$ then determines a morphism $\Gamma\colon (\Lambda X)^{\otimes m+1}/(\Lambda^+ X)^{\otimes m+1} \to A(S)^{\otimes m+1}/I_S^{\otimes m+1}$ which is a quasi-isomorphism. For simplicity we adopt the notation $\Lambda = (\Lambda X)^{\otimes m+1}$ and $I = (\Lambda^+ X)^{\otimes m+1}$. We have then the commutative diagram of morphisms

$$\begin{array}{ccccc} \Lambda & & \overset{\gamma^{(m+1)}}{\underset{\simeq}{\rightarrow}} & & A(S^{m+1}) \\ \eta\downarrow & & & & \downarrow \\ \Lambda/I & \underset{\Gamma}{\overset{\simeq}{\rightarrow}} \dfrac{A(S)^{\otimes m+1}}{I_S^{\otimes m+1}} & \underset{\phi^{m+1}}{\overset{\simeq}{\rightarrow}} & \tilde{A}(T^{m+1}(S)) \overset{\simeq}{\leftarrow} & A(T^{m+1}(S)). \end{array}$$

This yields

2.2 Proposition. *Let ν: $\Lambda Y \overset{\simeq}{\rightarrow} \Lambda/I$ be a model. Then it lifts to a model morphism $\Lambda Y \overset{\simeq}{\rightarrow} A(T^{m+1}(S))$. Moreover, if ξ: $\Lambda \to \Lambda Y$ represents η then it also represents the inclusion $T^{m+1}(S) \to S^{m+1}$.*

We turn our attention next to the space E_m defined in [7, 2.1] by Ganea. Gilbert [**8**, Proposition 3.3] has shown that if S is 1-connected, a space of the same homotopy type as E_m can be constructed by converting the inclusion $T^{m+1}(S) \to S^{m+1}$ into a fibration, and then restricting it to the diagonal. Alternatively, we can construct (up to homotopy equivalence) this space by converting the diagonal into a fibration, and restricting to the fat wedge.

This last construction has an exact analogue in the category of c-connected c.g.d.a.'s (A, d_A). Let $(\Lambda X, d)$ be the minimal model of one such. The minimal model of the multiplication map μ: $\Lambda X \otimes \Lambda X \to \Lambda X$ is necessarily of the form

$$\begin{tikzcd} & \Lambda X \\ \Lambda X \otimes \Lambda X \arrow[ur, "\mu"] \arrow[r] & \Lambda X \otimes \Lambda X \otimes \Lambda \bar{X}, \arrow[u, "\simeq", "\psi"'] \end{tikzcd}$$

in which the differential D in $\Lambda X \otimes \Lambda X \otimes \Lambda \bar{X}$ satisfies $Q(D)$: $\bar{X} \to X \oplus X$.

Evidently $Q(\psi) = Q(\mu)\colon (x \otimes 1 + 1 \otimes x') \mapsto x + x'$, and we may clearly suppose $Q(\psi) = 0$ in $\bar{X}$. Since [**12**, Theorem 7.3] $Q(\psi)^*$: $H(X \oplus X \oplus \bar{X}, Q(D)) \overset{\cong}{\rightarrow} X$ we may conclude that $\bar{X}^p = X^{p+1}$, and that $Q(D)\bar{x} = 1 \otimes x - x \otimes 1$, where the identification $\bar{X} = X$ is denoted by $\bar{x} \leftrightarrow x$.

Form the $(m+1)$-fold tensor product of this diagram with itself over the c.g.d.a. ΛX, adopting the convention that in $(\Lambda X \otimes \Lambda X) \otimes_{\Lambda X} (\Lambda X \otimes \Lambda X)$, ΛX acts on the immediately adjacent copies. The result is a diagram of the form

(2.3)
$$\begin{tikzcd} & \Lambda X \\ (\Lambda X)^{\otimes m+1} \arrow[ur, "\mu"] \arrow[r, "i"'] & (\Lambda X)^{\otimes m+1} \otimes (\Lambda \bar{X})^{\otimes m} \arrow[u, "\simeq", "\psi"'] \end{tikzcd}$$

in which i is the inclusion of a KS-extension.

If we adopt the further convention that $(\Lambda X)_i$, $1 \leqslant i \leqslant m+1$, and $(\Lambda \bar{X})_i$, $1 \leqslant i \leqslant m$, denote the respective copies of ΛX and $\Lambda \bar{X}$ in the ith position in $(\Lambda X)^{\otimes m+1} \otimes (\Lambda \bar{X})^{\otimes m}$, then it is immediate from the construction that

(2.4) for each $1 \leqslant i \leqslant m$, $(\Lambda X)_i \otimes (\Lambda X)_{i+1} \otimes (\Lambda \bar{X})_i$ is stable under D, and
(2.5) in the algebra of (2.4), $Q(D)\bar{x} = 1 \otimes x - x \otimes 1$.

Finally, let $(\Lambda X)^{\otimes m+1}$ act on $(\Lambda X)^{\otimes m+1}/(\Lambda^+ X)^{\otimes m+1}$ via the projection η and form the tensor product

$$\left(\frac{(\Lambda X)^{\otimes m+1}}{(\Lambda^+ X)^{\otimes m+1}} \otimes_{(\Lambda X)^{\otimes m+1}} (\Lambda X)^{\otimes m+1} \otimes (\Lambda \bar{X})^{\otimes m}, D\right) = \left(\frac{(\Lambda X)^{\otimes m+1}}{(\Lambda^+ X)^{\otimes m+1}} \otimes (\Lambda \bar{X})^{\otimes m}, D\right).$$

2.6 DEFINITION. The c.g.d.a.

$$(\Gamma_m(\Lambda X, d), D) = \left(\frac{(\Lambda X)^{\otimes m+1}}{(\Lambda^+ X)^{\otimes m+1}} \otimes (\Lambda \bar{X})^{\otimes m}, D\right)$$

will be called the *mth Ganea c.g.d.a. for* $(\Lambda X, d)$.

The analogy between E_m and Γ_m is made precise in

2.7 PROPOSITION. *Let $(\Lambda X, d)$ be the minimal model for a simply connected space S. Then $\Gamma_m(\Lambda X, d)$ represents the rational homotopy type of the Ganea space E_m.*

PROOF. The diagram (2.3) exhibits $(\Lambda X)^{\otimes m+1} \to (\Lambda X)^{\otimes m+1} \otimes (\Lambda \bar{X})^{\otimes m}$ as a model for the multiplication map μ. Since μ represents the diagonal map $S \to S^{m+1}$ it follows exactly as in [**10**, §2] that this model is a model for the fibration ξ, obtained from the diagonal map.

On the other hand, by Proposition 2.2 we can think of η as representing the inclusion $T^{m+1}(S) \hookrightarrow S^{m+1}$. It follows from [**12**, 20.6] that Γ_m represents the rational homotopy type of the pullback of the fibration ξ to $T^{m+1}(S)$. But, as we observed above, this pullback is E_m. Q.E.D.

3. The rational homotopy type of Γ_m. Let $(\Lambda X, d)$ be the minimal model of a c-connected c.g.d.a., and fix $m \geqslant 0$. As in §2 let

$$\eta\colon (\Lambda X)^{\otimes m+1} \to (\Lambda X)^{\otimes m+1}/(\Lambda^+ X)^{\otimes m+1}$$

denote the projection, and let (Γ_m, D) be the mth Ganea c.g.d.a. In this section we shall establish

3.1 THEOREM. *There is a homotopy equivalence*

$$\Gamma_m \simeq [\Lambda X/\Lambda^{>m}X] \oplus V$$

in which the right-hand c.g.d.a. is defined by: $\Lambda X/\Lambda^{>m}X$ is a sub-c.g.d.a., $V = \sum_{p\geqslant 1} V^p$, the differential in V is zero, and

$$V \cdot (\Lambda X/\Lambda^{>m}X \oplus V)^+ = 0.$$

The homotopy equivalence will be constructed explicitly as a sequence of three quasi-isomorphisms $\alpha_1, \alpha_2, \alpha_3$.

As an immediate corollary of Theorem 3.1 and Proposition 2.7 we have

3.2 THEOREM. *Let S be a simply connected space with minimal model $(\Lambda X, d)$, and let W be a space whose rational homotopy type is represented by $\Lambda X/\Lambda^{>m}X$. Then the space E_m of Ganea has the rational homotopy type of $W \vee \bigvee_\alpha S_\alpha$, where the S_α are spheres.*

We now proceed to the construction of $\alpha_1, \alpha_2, \alpha_3$. As a first step we consider the c.g.d.a.

$$\left((\Lambda X)^{\otimes m+1} \otimes (\Lambda \bar{X})^{\otimes m}, D\right),$$

which we denote (A, D) for brevity, and collect some necessary results. Let $\varepsilon\colon \Lambda X \to \mathbf{Q}$ be the projection, and consider the KS-extension

$$\Lambda X \underset{\simeq}{\overset{\lambda_i}{\to}} A \overset{\rho_i}{\to} (\Lambda X)^{\otimes m} \otimes (\Lambda \bar{X})^{\otimes m},$$

where λ_i is the inclusion onto the ith factor, and $\rho_i = \varepsilon$ in the ith factor ΛX and is the identity in the other factors.

A simple calculation using (2.5) shows that the differential $\bar{D}$ induced in $(\Lambda X)^{\otimes m} \otimes (\Lambda \bar{X})^{\otimes m}$ (which depends, of course, on i) satisfies $Q(\bar{D})\colon \bar{X} \oplus \cdots \oplus \bar{X} \overset{\cong}{\to} X \oplus \cdots \oplus X$. We can thus apply Lemma 1.2 to obtain

$$H\left((\Lambda X)^{\otimes m} \otimes (\Lambda \bar{X})^{\otimes m}, \bar{D}\right) = \mathbf{Q}. \tag{3.3}$$

The fact that $Q(\bar{D})$ is the isomorphism above, together with (3.3), implies that the differential $\bar{\bar{D}}$ induced in $(\Lambda \bar{X})^{\otimes m}$ is zero;

$$\bar{\bar{D}} = 0, \tag{3.4}$$

as follows from [**13**, Theorem 5.2].

Now bigrade A by putting

$$A^{p,q} = \sum_{p_1 + \cdots + p_{m+1} = p} \left[\Lambda^{p_1} X \otimes \cdots \otimes \Lambda^{p_{m+1}} X \otimes (\Lambda \bar{X})^{\otimes m}\right]^{p+q}.$$

Since $(\Lambda X, d)$ is minimal, $d\colon X \to \Lambda^{\geq 2} X$. This, together with (3.4), implies that $D\colon A^{p,*} \to \Sigma_{i>p} A^{i,*}$. Thus we have

$$D = \sum_{i=1}^{\infty} D_i, \qquad D_i \text{ homogeneous of bidegree } (i, 1-i). \tag{3.5}$$

With respect to the induced bigradation in $(\Lambda X)^{\otimes m} \otimes (\Lambda \bar{X})^{\otimes m}$ it follows that $\bar{D} = \Sigma_{i \geq 1} \bar{D}_i$. Evidently $Q(\bar{D}_1) = Q(\bar{D})$ so that as above

$$Q(\bar{D}_1)\colon \bar{X} \oplus \cdots \oplus \bar{X} \overset{\cong}{\to} X \oplus \cdots \oplus X. \tag{3.6}$$

Lemma 1.2 again applies and yields

$$H\left((\Lambda X)^{\otimes m} \otimes (\Lambda \bar{X})^{\otimes m}, \bar{D}_1\right) = \mathbf{Q}. \tag{3.7}$$

Observe next that η defines a projection $(A, D) \to (\Gamma_m, D)$ and that the bigradation projects to a natural bigradation of Γ_m. In the sequel we will construct a number of other c.g.d.a.'s from (A, D) by a sequence of ideals and quotients. In each case the newly constructed object will carry a unique (natural) bigradation with respect to which the defining map is homogeneous of bidegree $(0,0)$. In each case, therefore, (3.5) will apply.

Filter Γ_m by the ideals $F^p = \sum_{j\geq p} \Gamma_m^{p,*}$. This defines a spectral sequence (1st quadrant, converging to $H(\Gamma_m)$), which will be denoted $({}_\Gamma E_i^{p,q}, d_i)$. From (3.5) we see that

$$ {}_\Gamma E_0 = {}_\Gamma E_1 = \Gamma_m \quad \text{and} \quad d_1 = D_1. $$

There will be analogously defined spectral sequences with the same properties for the other c.g.d.a.'s we construct.

In particular, we are now ready to construct the quasi-isomorphism α_1. Consider the short exact sequence

$$ 0 \to \frac{(\Lambda X)^{\otimes m} \otimes \Lambda^+ X}{(\Lambda^+ X)^{\otimes m} \otimes \Lambda^+ X} \otimes (\Lambda \bar{X})^{\otimes m} \to \Gamma_m \xrightarrow{\bar{\rho}_{m+1}} (\Lambda X)^{\otimes m} \otimes (\Lambda \bar{X})^{\otimes m} \to 0 $$

where $\bar{\rho}_{m+1}$ is induced from the projection ρ_{m+1} above. Set

$$ \Delta_m = \mathbf{Q} \oplus \left\{ \frac{(\Lambda X)^{\otimes m} \otimes \Lambda^+ X}{(\Lambda^+ X)^{\otimes m} \otimes \Lambda^+ X} \otimes (\Lambda \bar{X})^{\otimes m} \right\} $$

$$ = \mathbf{Q} \oplus \left\{ \frac{(\Lambda X)^{\otimes m}}{(\Lambda^+ X)^{\otimes m}} \otimes \Lambda^+ X \otimes (\Lambda \bar{X})^{\otimes m} \right\} $$

and denote its differential again by D.

Denote the spectral sequence for (Δ_m, D) by ${}_\Delta E_i^{p,q}$. The inclusion

$$ \alpha_1 \colon (\Delta_m, D) \to (\Gamma_m, D) $$

induces an isomorphism at the E_2-level of the spectral sequences, as follows from (3.7) and the short exact sequence above. Hence (or directly from (3.3)) α_1 is a quasi-isomorphism.

3.8 PROPOSITION. ${}_\Gamma E_2^{p,*} = {}_\Delta E_2^{p,*} = 0$, $p > m$.

PROOF. Since α_1 gives an isomorphism ${}_\Gamma E_2 \cong {}_\Delta E_2$ we have only to show that $H^{p,*}(\Delta_m, D_1) = 0$, $p > m$. Permute the factors of Δ_m to write

$$ \Delta_m = \mathbf{Q} \oplus \left\{ \frac{(\Lambda X)^{\otimes m}}{(\Lambda^+ X)^{\otimes m}} \otimes (\Lambda \bar{X})^{\otimes m-1} \right\} \otimes \Lambda^+ X \otimes \Lambda \bar{X} $$

where we have included the first $m - 1$ copies of $\Lambda \bar{X}$ inside the brackets.

It follows from (2.4) that the algebra in the brackets is D-stable (hence also D_1-stable); moreover it clearly coincides with Γ_{m-1}. Thus we have

$$ \Delta_m = \mathbf{Q} \oplus \Gamma_{m-1} \otimes \Lambda^+ X \otimes \Lambda \bar{X}. \tag{3.9} $$

On the other hand, by dividing Γ_m by the ideal generated by $(\Lambda^+ X)^{\otimes m} \otimes \Lambda X$ we obtain a quotient c.g.d.a. Permuting the factors of this as we did for Δ_m, we can write this quotient in the form $(\Gamma_{m-1} \otimes \Lambda X \otimes \Lambda \bar{X}, D)$. Evidently

$$ 0 \to (\Delta_m^+, D_1) \to (\Gamma_{m-1} \otimes \Lambda X \otimes \Lambda \bar{X}, D_1) \to (\Gamma_{m-1} \otimes \Lambda \bar{X}, \bar{D}_1) \to 0 $$

is a short exact sequence of bigraded c.g.d.a.'s.

Since α_1 gives an isomorphism ${}_{\Delta}E_2 \cong {}_{\Gamma}E_2$ and $\Delta_m^0 = \mathbf{Q}$ (all m), $H^0(\Gamma_m) = \mathbf{Q} = H^0(\Gamma_m, D_1)$, all m. Thus Lemma 1.2 implies (in view of (3.6)) that

$$H^{p,q}(\Gamma_{m-1}, D_1) \cong H^{p,q}\big(\Gamma_{m-1} \otimes \Lambda X \otimes \Lambda \overline{X}, D_1\big). \tag{3.10}$$

We argue now by induction on m. When $m = 1$, $\Gamma_0 = \mathbf{Q}$ and (3.10) shows that

$$H^+(\Delta_1, D_1) \cong D_1(1 \otimes \Lambda \overline{X}) \subset X \otimes \Lambda \overline{X},$$

which establishes the proposition in this case.

Assume the proposition to hold for $m - 1$. In view of the short exact sequence above and (3.10) it will be sufficient to prove that

$$H^{p,*}(\Gamma_{m-1} \otimes \Lambda \overline{X}, \overline{D}_1) = 0, \qquad p \geqslant m. \tag{3.11}$$

By the very definition of a KS-extension, $\overline{X}$ admits a well-ordered basis $\{\overline{x}_\alpha\}$ such that $\overline{D}_1 \overline{x}_\alpha \in \Gamma_{m-1} \otimes \Lambda \overline{X}_{<\alpha}$ where $\overline{X}_{<\alpha}$ is the span of the $\overline{x}_\beta$, $\beta < \alpha$.

Suppose we have established

$$H^{p,*}\big(\Gamma_{m-1} \otimes \Lambda \overline{X}_{\leqslant\beta}, \overline{D}_1\big) = 0, \qquad p \geqslant m, \tag{3.12}$$

for all $\beta < \alpha$. (This is true by our earlier inductive hypothesis when α is the initial element.)

Then

$$H^{p,*}\big(\Gamma_{m-1} \otimes \Lambda \overline{X}_{<\alpha}, \overline{D}_1\big) = \varinjlim_{\beta<\alpha} H^{p,*}\big(\Gamma_{m-1} \otimes \Lambda \overline{X}_{\leqslant\beta}, D_1\big) = 0, \qquad p \geqslant m.$$

There are obvious short exact sequences

$$0 \to \Gamma_{m-1} \otimes \Lambda \overline{X}_{<\alpha} \otimes \Lambda^{\leqslant k} \overline{x}_\alpha \to \Gamma_{m-1} \otimes \Lambda \overline{X}_{<\alpha} \otimes \Lambda^{\leqslant k+1} \overline{x}_\alpha \to \Gamma_{m-1} \otimes \Lambda \overline{X}_{<\alpha} \to 0$$

($k = 0$ if $\deg \overline{x}_\alpha$ is odd; $k = 0, 1, 2, \ldots$ if $\deg \overline{x}_\alpha$ is even). These imply (3.12) with β replaced by α. A final limit argument completes the proof of (3.11) and hence closes the induction. Q.E.D.

We are now ready to complete the proof of Theorem 3.1 by constructing α_2 and α_3. In $\Delta_m^{m,*}$ choose a graded complement U for the subspace $\Delta_m^{m,*} \cap \ker D_1$. Put

$$J = \Delta_m^{>m,*} \oplus U.$$

Clearly J is a bigraded, D-stable subspace. Moreover, because $\Delta_m^+ \subset \Delta_m^{+,*}$, J is an ideal.

Let

$$\alpha_2 \colon (\Delta_m, D) \to (\Delta_m/J, D)$$

be the projection. Because of Proposition 3.8,

$$H^{p,*}(J, D_1) = \begin{cases} 0, & p \leqslant m, \\ H^{p,*}(\Delta_m, D_1) = 0, & p > m. \end{cases}$$

It follows that α_2 induces an isomorphism at the E_2-level of the spectral sequences. In particular, α_2 is a quasi-isomorphism.

Note also that because $\Delta_m^+ \subset \Delta_m^{+,*}$, Δ_m/J satisfies $(\Delta_m/J)^+ \cdot \ldots \cdot (\Delta_m/J)^+ = 0$ ($m + 1$ factors).

We now construct our final quasi-isomorphism

$$\alpha_3\colon \Lambda X/\Lambda^{>m}X \oplus V \xrightarrow{\simeq} \Delta_m/J.$$

The inclusion

$$\Lambda^+ X \to 1 \otimes \Lambda^+ X \otimes 1 \hookrightarrow \frac{(\Lambda X)^{\otimes m}}{(\Lambda^+ X)^{\otimes m}} \otimes \Lambda^+ X \otimes (\Lambda \bar{X})^{\otimes m}$$

defines an inclusion $\xi\colon (\Lambda X, d) \to (\Delta_m, D)$, which factors to yield a morphism $\bar{\xi}\colon \Lambda X/\Lambda^{>m}X \to \Delta_m/J$. Bigrade ΛX by putting $(\Lambda X)^{p,q} = (\Lambda^p X)^{p+q}$; then $d = \Sigma_{i\geqslant 1} d_i$, d_i homogeneous of bidegree $(i, 1-i)$, and ξ and $\bar{\xi}$ are homogeneous of bidegree $(0,0)$.

Choose $V \subset (\Delta_m/J)^{m,*}$ to be a graded subspace such that

$$H^{m,*}(\Delta_m/J, D_1) = \bar{\xi}^*\big(H^{m,*}(\Lambda X/\Lambda^{>m}X, d_1)\big) \oplus V.$$

Then $\bar{\xi}$, together with the inclusion of V, defines a morphism

$$\alpha_3\colon \Lambda X/\Lambda^{>m}X \oplus V \to \Delta_m/J.$$

We show that α_3, too, induces an isomorphism at the E_2-level of the spectral sequences. For this it is sufficient to prove that

$$\bar{\xi}^*\colon H^{p,*}(\Lambda X/\Lambda^{>m}X, d_1) \to H^{p,*}(\Delta_m/J, D_1)$$

is an isomorphism for $p < m$ and injective for $p = m$.

For this, consider the sub-c.g.d.a.

$$\mathbf{Q} \oplus \left\{(\Lambda X)^{\otimes m} \otimes \Lambda^+ X \otimes (\Lambda \bar{X})^{\otimes m}\right\} \hookrightarrow A$$

and note (via (3.7)) that the inclusion induces an isomorphism of spectral sequences at the E_2-level. Next observe that the projection $\eta \otimes \mathrm{id}\colon A \to \Gamma_m$ restricts to a projection from this subalgebra to Δ_m and that this restriction factors to produce a commutative diagram:

$$\begin{tikzcd}
 & \mathbf{Q} \oplus \left\{(\Lambda X)^{\otimes m} \otimes \dfrac{\Lambda^+ X}{\Lambda^{>m}X} \otimes (\Lambda \bar{X})^{\otimes m}\right\} \arrow[d, "\phi"] \\
\Lambda X/\Lambda^{>m}X \arrow[ur, "\text{inclusion}"] \arrow[r, "\bar{\xi}"'] & \Delta_m/J
\end{tikzcd}$$

Using Lemma 1.2 we see that the arrow marked inclusion is an isomorphism at the E_2-level. But ϕ itself is an isomorphism in bidegrees (p, q) with $p < m$, and injective in $(\ker D_1)^{m,*}$. It follows that ϕ (and hence $\bar{\xi}$) induces an isomorphism at the E_2-level in bidegrees (p, q) with $p < m$ and an injection in bidegree (m, q). This completes the proof that α_3 induces an isomorphism at the E_2-level.

The quasi-isomorphisms

$$(\Lambda X/\Lambda^{>m}X \oplus V) \underset{\simeq}{\xrightarrow{\alpha_3}} \Delta_m/J \underset{\simeq}{\xleftarrow{\alpha_2}} \Delta_m \underset{\simeq}{\xrightarrow{\alpha_1}} \Gamma_m$$

establish Theorem 3.1.

4. Rational category. Let S be a space with minimal model $\gamma\colon \Lambda X \to A(S)$. Recall from §2 that $\gamma^{(m+1)}\colon (\Lambda X)^{\otimes m+1} \to A(S^{m+1})$ is a model for S^{m+1} and that the multiplication map $\mu\colon (\Lambda X)^{\otimes m+1} \to \Lambda X$ represents the diagonal map. Let $\Lambda Y \to A(T^{m+1}(S))$ be a model for the fat wedge, and let $\xi\colon (\Lambda X)^{\otimes m+1} \to \Lambda Y$ represent the inclusion of $T^{m+1}(S)$ in S^{m+1}.

4.1 Definition. The rational category of S, $\operatorname{cat}_0(S)$, is the least integer $m \geq 0$ such that there is a morphism $\rho\colon \Lambda Y \to \Lambda X$ for which

$$\begin{tikzcd} & (\Lambda X)^{\otimes m+1} \arrow[d, "\xi"] \\ \Lambda X \arrow[ur, "\mu"] & \Lambda Y \arrow[l, "\rho"] \end{tikzcd}$$

is homotopy commutative.

4.2 Remarks. (1) It is trivial that the definition is independent of the various choices.

(2) A deformation of the diagonal into the fat wedge induces such a diagram, and so $\operatorname{cat}_0(S) \leq \operatorname{cat} S$. If S is simply connected then such a diagram induces a deformation of the diagonal map of the localization $S_{\mathbf{Q}}$ into the fat wedge $T^{m+1}(S_{\mathbf{Q}})$. Thus in this case $\operatorname{cat}_0(S) = \operatorname{cat}(S_{\mathbf{Q}})$.

In view of Proposition 2.2, $\xi\colon (\Lambda X)^{\otimes m+1} \to \Lambda Y$ fits into a homotopy commutative diagram

$$\begin{tikzcd} & (\Lambda X)^{\otimes m+1} \arrow[dl, "\xi"'] \arrow[d, "\eta"] \\ \Lambda Y \arrow[r, "\simeq", "\nu"'] & (\Lambda X)^{\otimes m+1}/(\Lambda^+ X)^{\otimes m+1}. \end{tikzcd}$$

Of course ΛY and ξ are completely determined (up to homotopy) by this diagram.

We can use this formulation to extend the definition of cat_0 to the category of c-connected c.g.d.a.'s (A, d_A). Indeed, let $(\Lambda X, d)$ be the minimal model for one such and *define* ΛY and ξ by a diagram of the above form. Then we have the

4.3 Definition. The *rational category*, $\operatorname{cat}_0(A, d_A)$, of (A, d_A) is the least integer $m \geq 0$ such that there is a morphism $\rho\colon \Lambda Y \to \Lambda X$ for which $\rho \circ \xi \sim \mu$. If there is no such m, $\operatorname{cat}_0(A, d_A) = \infty$.

4.4 Remarks. (1) Evidently $\operatorname{cat}_0(S) = \operatorname{cat}_0(A(S))$ for any space S.

(2) If $\lambda_i\colon \Lambda X \to (\Lambda X)^{\otimes m+1}$ is the inclusion of the ith factor then $\mu \circ \lambda_i = \mathrm{id}$ and so $\rho \circ \xi \circ \lambda_i = \psi_i \sim \mathrm{id}$. It follows from [**12**, Theorem 7.2] that (because ΛX is minimal) ψ_i is an isomorphism. We may replace ξ by $\xi' = \xi \circ (\psi_1^{-1} \otimes \cdots \otimes \psi_{m+1}^{-1})$ without changing the homotopy class of ξ; i.e., we may assume $\rho \circ \xi \circ \lambda_i = \mathrm{id}$.

4.5 Definition. An augmented algebra with augmentation ideal K has *product length* m if m is the least integer such that $K \cdot \ldots \cdot K = 0$ ($m + 1$ factors).

4.6 Definition. A morphism $\phi\colon A \to B$ of c-connected c.g.d.a.'s makes A into a *retract* of B if there are morphisms

$$\Lambda X \xrightarrow{\alpha} \Lambda Y \xrightarrow{\beta} \Lambda X$$

(ΛX a model for A, ΛY a model for B) such that α represents ϕ and $\beta\alpha \sim \mathrm{id}$. (If ΛX is minimal we can always modify β so that $\beta\alpha = \mathrm{id}$.)

The main result of this section is the

4.7 THEOREM. *Let $(\Lambda X, d)$ be the minimal model for a c-connected c.g.d.a., (A, d_A). The following conditions are then equivalent:*

(i) $\mathrm{cat}_0(A, d_A) \leqslant m$.

(ii) $(\Lambda X, d)$ *is a retract of* $\Gamma_m(\Lambda X, d)$.

(iii) $(\Lambda X, d)$ *is a retract of an augmented c.g.d.a. of product length* $\leqslant m$.

(iv) *The projection* $(\Lambda X, d) \to (\Lambda X/\Lambda^{>m}X, d)$ *makes* ΛX *into a retract of* $\Lambda X/\Lambda^{>m}X$.

4.8 COROLLARY. *If* (A, d_A) *is a retract of* (B, d_B) *then* $\mathrm{cat}_0(A, d_A) \leqslant \mathrm{cat}_0(B, d_B)$.

4.9 COROLLARY. *The c.g.d.a.* (A, d_A) *has* $\mathrm{cat}_0(A, d_A) = 1$ *if and only if there is a quasi-isomorphism* $\phi\colon (\Lambda X, d) \to (V \oplus \mathbf{Q}, 0)$, *where* $(\Lambda X, d)$ *is the minimal model for* (A, d_A) *and the multiplication in V is trivial.*

4.10 COROLLARY. *For any c-connected* (A, d_A), $\mathrm{cat}_0(A, d_A) \geqslant$ *product length of* $H(A)$. *If* (A, d_A) *is formal (i.e., its minimal model is isomorphic to the model for* $(H(A), 0)$) *then* $\mathrm{cat}_0(A, d_A) =$ *product length of* $H(A)$.

4.11 REMARKS. (1) When $A = A(S)$ and S is a simply connected rational space then $\mathrm{cat}_0(A, d_A) = \mathrm{cat}\, S$ (cf. Remark 4.2(2)). Given Proposition 2.7, the implication (i) $\Leftrightarrow$ (ii) coincides in this case with a result [7, Proposition 2.2] of Ganea.

(2) Theorem 4.7 clearly implies Theorem VIII of the introduction.

(3) For simply connected spaces S, Corollary 4.9 is simply the well-known fact that $\mathrm{cat}_0(S) = 1$ if and only if S has the rational homotopy type of a wedge of spheres.

PROOF OF THEOREM 4.7. (i) $\Rightarrow$ (ii) As in §2 denote $(\Lambda X)^{\otimes m+1}$ by Λ and $(\Lambda^+ X)^{\otimes m+1}$ by I. Denote $(\Lambda \overline{X})^{\otimes m}$ by $\overline{\Lambda}$. By the definition we have a diagram

$$\begin{tikzcd}
\Lambda X & \Lambda \arrow[l, "\mu"'] \arrow[dl, "\xi"] \arrow[d, "\eta"] \\
\Lambda Y \arrow[u, "\rho"] \arrow[r, "\simeq", "\nu"'] & \Lambda/I
\end{tikzcd}$$

in which the lower triangle is homotopy commutative, while the upper is strictly commutative (Remark 4.4(2)).

Now consider the KS-extension

$$\Lambda \xrightarrow{i} \Lambda \otimes \overline{\Lambda} \to \overline{\Lambda}$$

defined in diagram (2.3). Let Λ act on ΛY via ξ and form the tensor product

$$\Lambda Y \otimes_\Lambda \Lambda \otimes \overline{\Lambda} = \Lambda Y \otimes \overline{\Lambda}.$$

The morphism ν extends to a morphism

$$\Lambda Y \otimes \overline{\Lambda} \to \Lambda/I \otimes_{\Lambda Y} \Lambda Y \otimes \overline{\Lambda} = \Lambda/I \otimes \overline{\Lambda} \tag{4.12}$$

which is a quasi-isomorphism because ν is [**12**, Lemma 1.9]. In particular it exhibits $\Lambda Y \otimes \overline{\Lambda}$ as a model for $\Lambda/I \otimes \overline{\Lambda}$.

Next, consider the morphism $\psi\colon \Lambda \otimes \overline{\Lambda} \xrightarrow{\simeq} \Lambda X$ of (2.3) and observe (because $\rho \circ \xi = \mathrm{id}$) that $\beta = \rho \otimes \psi\colon \Lambda Y \otimes \overline{\Lambda} \to \Lambda X$ is a morphism. On the other hand, any of the inclusions $\Lambda X \to \Lambda$, when composed with ξ and the inclusion $\Lambda Y \to \Lambda Y \otimes \overline{\Lambda}$, yields a morphism $\alpha\colon \Lambda X \to \Lambda Y \otimes \overline{\Lambda}$ such that $\beta \circ \alpha = \mathrm{id}$. We thus exhibit ΛX as a retract of $\Lambda Y \otimes \overline{\Lambda}$.

Finally, observe that the c.g.d.a. $\Lambda/I \otimes \overline{\Lambda}$ of (4.12) can be written

$$\Lambda/I \otimes \overline{\Lambda} = \Lambda/I \otimes_\Lambda \Lambda \otimes \overline{\Lambda},$$

where Λ acts on Λ/I via $\nu \circ \xi$. Because $\nu \circ \xi \sim \eta$ there is a c.g.d.a. morphism $\Phi\colon \Lambda \to \Lambda/I \otimes \Lambda(t, dt)$ with $\rho_0\Phi = \nu \circ \xi$ and $\rho_1\Phi = \eta$ (cf. §1). Use Φ, ρ_0, ρ_1 to define

$$\Lambda/I \otimes_\Lambda \Lambda \otimes \overline{\Lambda} \xleftarrow{\simeq} \Lambda/I \otimes \Lambda(t, dt) \otimes_\Lambda \Lambda \otimes \overline{\Lambda} \xrightarrow{\simeq} \Lambda/I \otimes_\Lambda \Lambda \otimes \overline{\Lambda},$$

in which Λ acts on Λ/I on one side via $\nu \circ \xi$ and on the other via η. In particular, one of these c.g.d.a.'s is $\Gamma_m(\Lambda X)$, and so they are all c-connected.

It now follows from (4.12) that there is a quasi-isomorphism $\Lambda Y \otimes \overline{\Lambda} \xrightarrow{\simeq} \Gamma_m(\Lambda X)$, which exhibits $\Lambda Y \otimes \overline{\Lambda}$ as a model for Γ_m and ΛX as a retract of Γ_m.

(ii) $\Rightarrow$ (iii) This follows from Theorem 3.1.

(iii) $\Rightarrow$ (iv) Suppose ΛX is a retract of a c.g.d.a. B with augmentation ideal K such that $K \cdot \ldots \cdot K = 0$ ($m + 1$ factors). Let $\Lambda Y \xrightarrow{\pi} B$ be a model, and let $i\colon \Lambda X \to \Lambda Y$, $r\colon \Lambda Y \to \Lambda X$ be morphisms such that $ri = \mathrm{id}$.

Clearly $\pi \circ i$ maps $\Lambda^{>m}X$ to zero, and so factors over the projection $\Lambda X \to \Lambda X/\Lambda^{>m}X$. If $\Lambda Z \to \Lambda X/\Lambda^{>m}X$ is a model we can thus find morphisms

$$\Lambda X \xrightarrow{i_1} \Lambda Z \xrightarrow{i_2} \Lambda Y$$

whose composite is homotopic to i. Set $r_1 = r \circ i_2$; then $r_1 i_1 \sim \mathrm{id}$ and (iv) is established.

(iv) $\Rightarrow$ (i) It is a matter of trivial diagram chasing to verify that if A is a retract of B and $\mathrm{cat}_0(B) \leq m$ then $\mathrm{cat}_0(A) \leq m$. We need only verify, therefore, that $B = \Lambda X/\Lambda^{>m}X$ satisfies $\mathrm{cat}_0(B) \leq m$. Let K be the maximal ideal B^+ and notice that multiplication $B^{\otimes m+1} \to B$ factors over the projection $B^{\otimes m+1} \to B^{\otimes m+1}/K^{\otimes m+1}$ to give a morphism from this latter c.g.d.a. to B.

Some more elementary diagram chasing completes the proof. Q.E.D.

5. The mapping theorem.

5.1 THEOREM. *Suppose $\phi\colon (A, d_A) \to (B, d_B)$ is a morphism of c-connected c.g.d.a.'s such that $\phi^\#\colon \pi_\psi^*(A, d_A) \to \pi_\psi^*(B, d_B)$ is surjective. Then*

$$\mathrm{cat}_0(A, d_A) \geq \mathrm{cat}_0(B, d_B).$$

PROOF. Let $(\Lambda X, d) \xrightarrow{i} (\Lambda X \otimes \Lambda Y, d) \xrightarrow{\rho} (\Lambda Y, \bar{d})$ be a Λ-minimal Λ-model for ϕ, and write $\mathrm{cat}_0(A, d_A) = m$. Then by Theorem 4.7, the projection $(\Lambda X, d) \to (\Lambda X/\Lambda^{>m}X, d)$ makes ΛX into a retract of $\Lambda X/\Lambda^{>m}X$. It follows that $\Lambda X \otimes \Lambda Y$

is a retract of $\Lambda X/\Lambda^{>m}X \otimes \Lambda Y$, so that we need only prove

$$\mathrm{cat}_0(\Lambda X/\Lambda^{>m}X \otimes \Lambda Y, d) \leq m$$

(cf. Corollary 4.8).

Our hypothesis on $\phi^{\#}$ implies that $Q(d)\colon Y \to X$ is injective. It follows from [**13**, Theorem 5.2] that $\bar{d} = 0$ and so $d\colon \Lambda Y \to \Lambda^+ X \otimes \Lambda Y$. Now consider the projection $p\colon \Lambda X/\Lambda^{>m}X \to X \oplus \mathbf{Q}$ and use

$$p \otimes \mathrm{id}\colon \Lambda X/\Lambda^{>m}X \otimes \Lambda Y \to (X \oplus \mathbf{Q}) \otimes \Lambda Y$$

to induce a differential, D, in $(X \oplus \mathbf{Q}) \otimes \Lambda Y$.

Since $\bar{d} = 0$, $\Lambda Y \xrightarrow{D} X \otimes \Lambda Y \xrightarrow{D} 0$. A second derivation, D_1, also of this form in $(X \oplus \mathbf{Q}) \otimes \Lambda Y$, is defined by $D_1 y = Q(d)y \otimes 1$, $y \in Y$. Since $[d - Q(d)]\colon Y \to \Lambda^{\geq 2}(X \oplus Y)$ we have

$$(5.2) \qquad D_1\colon \Lambda^m Y \to X \otimes \Lambda^{m-1}Y, \qquad (D - D_1)\colon \Lambda^m Y \to X \otimes \Lambda^{\geq m}Y.$$

Since $Q(d)\colon Y \to X$ is injective so is $D_1\colon \Lambda^+ X \to X \otimes \Lambda Y$; in view of (5.2), $D\colon \Lambda^+ Y \to X \otimes \Lambda Y$ is also injective.

Let $U \subset X \otimes \Lambda Y$ be a graded complement for $D(\Lambda Y)$ and

$$p'\colon (X \otimes \Lambda Y) \oplus \Lambda Y \to (X \otimes \Lambda Y/U) \oplus \Lambda Y$$

be the projection. Then $p' \circ (p \otimes \mathrm{id})$ projects $\Lambda X/\Lambda^{>m}X \otimes \Lambda Y$ onto an acyclic c.g.d.a. Hence if $K = \ker(p' \circ (p \otimes \mathrm{id}))$ then $K \oplus \mathbf{Q}$ is a c.g.d.a. and the inclusion

$$K \oplus \mathbf{Q} \hookrightarrow \Lambda X/\Lambda^{>m}X \otimes \Lambda Y$$

is a quasi-isomorphism.

By inspection, $K \subset \Lambda^+ X/\Lambda^{>m}X \otimes \Lambda Y$ and so K has product length m. Now Theorem 4.7(iii) implies that

$$\mathrm{cat}_0(\Lambda X/\Lambda^{>m}X \otimes \Lambda Y, d) = \mathrm{cat}_0(K \oplus \mathbf{Q}, d) \leq m. \qquad \text{Q.E.D.}$$

5.3 Proof of Theorem I. Given Remark 4.4(1) and the natural isomorphisms $\pi_\psi^*(A(S)) \cong \mathrm{Hom}_{\mathbf{Z}}(\pi_*(S); \mathbf{Q})$, Theorem I is a corollary of Theorem 5.1. Q.E.D.

5.4 Remarks. 1. The main theorem of [**13**] asserts (with the same hypothesis on π_ψ^* as in Theorem 5.1) that if $H^p(A) = 0$, $p > n$, then $H(B)$ has product length $\leq n$. Since then $\mathrm{cat}_0(A) \leq n$, and product length $H(B) \leq \mathrm{cat}_0(B)$, the present result is a substantially stronger theorem.

2. Theorem 5.1 immediately implies the well-known fact that a sub-Lie algebra L of a free graded Lie algebra $\mathfrak{L}_V$ on a positively graded space of finite type is again free. Indeed, if C^* denotes Koszul's cochain functor to c.g.d.a.'s it transforms the inclusion $L \to \mathfrak{L}_V$ into a surjective c.g.d.a. morphism $C^*(\mathfrak{L}_V) \to C^*(L)$ in which both c.g.d.a.'s are minimal models.

On the other hand, it is easy to see that the obvious projection $C^*(\mathfrak{L}_V) \to \mathfrak{L}_V^* \oplus \mathbf{Q} \to V \oplus \mathbf{Q}$ is a quasi-isomorphism, whence (Corollary 4.9 and Theorem 5.1) $\mathrm{cat}_0(C^*(L)) \leq 1$. Apply Corollary 4.9 again to finish.

5.5 Example. Let W be the space $(\mathbf{C}P^2 \vee S^2) \cup_\omega e^7$, where the seven-cell e^7 is attached by $[\alpha, \beta] \in \pi_6(\mathbf{C}P^2 \vee S^2)$, and $\alpha \in \pi_5(\mathbf{C}P^2)$ and $\beta \in \pi_2(S^2)$ are the obvious basis elements. We shall show that

$$(5.6) \qquad \mathrm{cat}_0(W) = 3,$$

and, more generally for the n-fold Cartesian product,

$$\operatorname{cat}_0(W^n) = 3n. \tag{5.7}$$

First observe that $H^*(W) \cong H^*(\mathbf{C}P^2 \vee S^2 \vee S^7)$ as graded algebras so that a model of W can be obtained by perturbing a model of the formal space $\mathbf{C}P^2 \vee S^2 \vee S^7$ as described in [**14**, §4]. The latter has the form $(\Lambda Z, d)$ in which Z carries a second, lower, grading $Z = \Sigma_{p \geq 0} Z_p$, d is homogeneous of lower degree -1 and $H_+(\Lambda Z) = 0$.

Explicit bases are given by (subscripts denote degrees)

$$Z_0\colon x_2, x_2', x_7, \qquad Z_1\colon y_3, y_3', y_5, y_8, y_8', \qquad Z_2\colon v_4, v_6, \ldots$$

and $d(Z_0) = 0$ while

$$\begin{aligned} dy_3 &= x_2 x_2', & dy_8 &= x_2 x_7, & dv_4 &= y_3 x_2' - y_3' x_2, \\ dy_3' &= (x_2')^2, & dy_8' &= x_2' x_7, & dv_6 &= y_5 x_2' - y_3 x_2^2, \\ dy_5 &= x_2^3. \end{aligned}$$

The model for W is then of the form $(\Lambda Z, D)$ with $D - d\colon Z_p \to (\Lambda Z)_{\leq p-2}$. Necessarily $D = d$ in Z_0 and Z_1 and in $Z^{<6}$. The choice of ω forces $Dv_6 = dv_6 - x_7$ and $D = d$ in $Z^6_{>2}$.

In ΛZ the ideal generated by $Z_{\geq 2}$, y_3, y_3', y_8, y_8', x_2^4, $y_5 x_2$, $(x_2')^2$, $x_2 x_2'$ and $x_7 - y_5 x_2' + y_3 x_2^2$ is D-stable. The quotient c.g.d.a. has the form $A = \Lambda(x_2, x_2', y_5)/I$, where I is the ideal generated by $(x_2')^2$, $x_2 x_2'$, x_2^4 and $y_5 x_2$. The differential d_A is given by $d_A x_2 = d_A x_2' = 0$ $d_A y_5 = x_2^3$.

The elements

$$\begin{matrix} 1, x_2, x_2^2, x_2^3 \\ x_2', y_5, y_5 x_2' \end{matrix} \qquad \text{and} \qquad \begin{matrix} 1, x_2, x_2^2 \\ x_2', y_5 x_2' \end{matrix}$$

represent, respectively, a basis for A and a basis for $H(A)$. It follows that $(\Lambda Z, D) \xrightarrow{\simeq} d_A)$. Since A has product length 3, Theorem 4.7 shows that $\operatorname{cat}_0(S) \leq 3$.

On the other hand a second D-stable ideal in ΛZ is generated by x_2', x_2^3, $x_7 + y_3 x_2^2$, y_3', y_5, y_8, y_8' and $Z_{\geq 2}$. Its factor algebra is $\Lambda(x_2, y_3)/I$ where I is the ideal generated by x_2^3. The projection lifts to a c.g.d.a. morphism $\phi\colon (\Lambda Z, D) \to (\Lambda(x_2, y_3, y_5), \bar{D})$ where $\bar{D} x_2 = \bar{D} y_3 = 0$ and $\bar{D} y_5 = x_2^3$. Moreover $\phi x_2 = x_2$, $\phi y_3 = y_3$ and $\phi y_5 = y_5$.

This implies that $\phi^\#\colon H(Z, Q(D)) \to (x_2, y_3, y_5)$ is surjective and we may thus apply Theorem 5.1. Since $(\Lambda(x_2, y_3, y_5), \bar{D})$ is the model for the formal space $\mathbf{C}P^2 \times S^3$, and since $H(\mathbf{C}P^2 \times S^3)$ has product length three, we conclude that

$$\operatorname{cat}_0(S) \geq \operatorname{cat}_0(\mathbf{C}P^2 \times S^3) = 3.$$

This finishes the proof of (5.6). Formula (5.7) is proved in the identical way, except that all c.g.d.a.'s must be replaced by their n-fold tensor products.

5.8 Example. We show now that

$$\operatorname{cat}(\mathbf{C}P^2 \vee S^2 \cup_\omega e^7) = 3 \quad \text{and} \quad \operatorname{cat}(\mathbf{C}P^2 \vee S^2 \cup_\omega e^7)^n = 3n.$$

Indeed, write

$$\mathbf{C}P^2 \vee S^2 \cup_\omega e^7 = S^2 \vee S^2 \cup e^4 \cup e^7$$

and conclude from [**15**, §2] that $\operatorname{cat}(\mathbf{C}P^2 \vee S^2 \cup_\omega e^7) \leq 3$. Since $\operatorname{cat}_0 \leq \operatorname{cat}$ (Remark 4.2(2)) we get equality from (5.6).

Use [**15**, Proposition 2.3] to find $\operatorname{cat}(\mathbf{C}P^2 \vee S^2 \cup_\omega e^7)^n \leq 3n$, and conclude equality from (5.7).

6. Fibrations. Recall the restrictions on topological spaces imposed at the start of the introduction. Given a sequence ξ: $F \xrightarrow{j} E \xrightarrow{\pi} B$ of base point preserving continuous maps between such spaces, there is a commutative diagram of augmented c.g.d.a. morphisms

$$
\text{(6.1)} \qquad
\begin{array}{ccccc}
A(B) & \xrightarrow{A(\pi)} & A(E) & \xrightarrow{A(j)} & A(F) \\
m_B \uparrow\simeq & & \uparrow\simeq & & \uparrow\alpha \\
(\Lambda Y, D) & \xrightarrow[i]{} & (\Lambda Y \otimes \Lambda X, D) & \longrightarrow & (\Lambda X, d)
\end{array}
$$

in which m_B is the minimal model for B and the bottom row is the Λ-minimal Λ-extension which models $A(\pi) \circ m_B$.

If α^*: $H(\Lambda X) \to H(F)$ is an isomorphism (so that α is the minimal model for F) then ξ is called a *rational fibration* [**13**]. If ξ is a Serre fibration in which $\pi_1(B)$ acts nilpotently in each $H^p(F)$ then ξ is a rational fibration [**12**, Theorem 20.3]. This includes the Serre fibrations of Theorem II in the introduction.

With each rational fibration ξ is associated [**13**, (4.9)] a canonical long exact sequence

$$
\text{(6.2)} \qquad \to \pi_\psi^p(B) \xrightarrow{\pi^\#} \pi_\psi^p(E) \xrightarrow{j^\#} \pi_\psi^p(F) \xrightarrow{\partial^*} \pi_\psi^{p+1}(B) \to ,
$$

which, when ξ is a Serre fibration of simply connected spaces, is dual to the standard long exact homotopy sequence.

We can derive (6.2) from (6.1) by recalling that $Y = \pi_\psi^*(G)$, $X = \pi_\psi^*(F)$ and, because $(\Lambda Y \otimes \Lambda X, D)$ is a model for E, $H(Y \oplus X, Q(D)) = \pi_\psi^*(E)$. Now (6.2) is the sequence arising from the short exact sequence

$$(Y, 0) \rightarrowtail (Y \oplus X, Q(D)) \twoheadrightarrow (X, 0).$$

Note that $Q(D)$ maps X into Y and Y to 0 and that $\partial^* = Q(D)$: $X \to Y$.

Given a rational fibration ξ, put

$$
k_\xi = \begin{cases} \dim \operatorname{Coker} j^\#, & \text{if } (j^\#)^{\text{even}} \text{ is surjective}, \\ \infty, & \text{otherwise}. \end{cases}
$$

(Even if $(j^\#)^{\text{even}}$ is surjective it may well happen that $k_\xi = \infty$.) Note also that $k_\xi = \infty$ if ∂^* is not zero in $\pi_\psi^{\text{even}}(F)$. If it is zero there then by the exactness of (6.2),

$$
\text{(6.3)} \qquad k_\xi = \dim \operatorname{Im} \partial^* = \dim \operatorname{Ker} \pi^\# = \dim \operatorname{Coker} j^\#.
$$

6.4 THEOREM. *Let ξ be a rational fibration. Then*

$$k_\xi \leq \operatorname{cat}_0(F) \leq \operatorname{cat}_0(E) + k_\xi.$$

6.5 REMARKS. Because the fibrations of Theorem II of the introduction are rational fibrations, that theorem follows from this. We also recover Theorem 4.15(ii) and (iii) of **[13]**.

PROOF. We show first that $k_\xi \leqslant \mathrm{cat}_0(F)$. Consider (6.1) and let $\Lambda Y \to \Lambda Y \otimes \Lambda \overline{Y} \to \Lambda \overline{Y}$ be the Λ-minimal Λ-extension in which $(\Lambda Y \otimes \Lambda \overline{Y}, D_1)$ is acyclic. Then $Q(D_1)$: $\overline{Y} \xrightarrow{\cong} Y$ and hence by [**13**, Theorem 5.2] the differential $\overline{D}_1$ induced in $\Lambda \overline{Y}$ is zero.

Tensor $(\Lambda Y \otimes \Lambda \overline{Y}, D_1)$ with $(\Lambda Y \otimes \Lambda X, D)$ over $(\Lambda Y, D)$ to obtain a c.g.d.a. $\Lambda Y \otimes \Lambda \overline{Y} \otimes \Lambda X$ together with c.g.d.a. morphisms

$$\phi_1\colon \Lambda Y \otimes \Lambda \overline{Y} \otimes \Lambda X \to \Lambda X, \qquad \phi_2\colon \Lambda Y \otimes \Lambda \overline{Y} \otimes \Lambda X \to \Lambda \overline{Y}.$$

Because $\Lambda Y \otimes \Lambda \overline{Y}$ is acyclic, ϕ_1^* is an isomorphism. Since ΛX is a KS-complex there is a c.g.d.a. morphism ψ: $\Lambda X \to \Lambda Y \otimes \Lambda \overline{Y} \otimes \Lambda X$ with $\phi_1 \psi = \mathrm{id}$.

It follows that the linear part of ψ, $Q(\psi)$: $X \to Y \oplus \overline{Y} \oplus X$, has the form $x \mapsto (\alpha(x), -Q(D_1)^{-1} Q(D)x, x)$ for some $\alpha(x)$.

Now let $\overline{Y}_1 \subset \overline{Y}$ correspond under $Q(D_1)$ to $\mathrm{Im}(Q(D)\colon X \to Y)$, and let p: $\Lambda \overline{Y} \to \Lambda \overline{Y}_1$ extend the identity in $\Lambda \overline{Y}_1$. Then $Q(p\phi_2\psi)$ is surjective and so by Theorem 5.1, $\mathrm{cat}_0(\Lambda \overline{Y}_1, 0) \leqslant \mathrm{cat}_0(\Lambda X, d) = \mathrm{cat}_0(F)$.

On the other hand, by Corollary 4.10, $\mathrm{cat}_0(\Lambda \overline{Y}_1, 0)$ is the product length of $\Lambda \overline{Y}_1$. This is ∞ unless $\overline{Y}_1$ is oddly graded, in which case it is $\dim \overline{Y}_1$. Since $\overline{Y}_1 \cong \mathrm{Coker}\, j^\#$ the inequality $k_\xi \leqslant \mathrm{cat}_0(F)$ follows.

To establish $\mathrm{cat}_0(F) \leqslant \mathrm{cat}_0(E) + k_\xi$, we need the

6.6 LEMMA. *If $\Lambda Z \to \Lambda Z \otimes \Lambda u \to \Lambda u$ is a connected Λ-extension in which u has odd degree then* $\mathrm{cat}_0(\Lambda Z \otimes \Lambda u, d) \leqslant \mathrm{cat}_0(\Lambda Z, d) + 1$.

PROOF. We may clearly suppose $\mathrm{cat}_0(\Lambda Z, d) = m < \infty$. Then by Theorem 4.7, $(\Lambda Z, d)$ is a retract of $(\Lambda Z/\Lambda^{>m} Z, d)$ and it follows that $(\Lambda Z \otimes \Lambda u, d)$ is a retract of $(\Lambda Z/\Lambda^{>m} Z \otimes \Lambda u, d)$. This has product length $m+1$ and we apply Theorem 4.7 again to finish. Q.E.D.

Now we show $\mathrm{cat}_0(F) \leqslant \mathrm{cat}_0(E) + k_\xi$ by induction on k_ξ. (When $k_\xi = \infty$ there is nothing to prove.) When $k_\xi = 0$ this is just Theorem 5.1; assume now that $k_\xi = m$ and that the inequality holds for η with $k_\eta = m - 1$.

In the c.g.d.a. $(\Lambda Y \otimes \Lambda X, D)$ of (6.1) choose a well-ordered basis $\{x_\alpha\}$ of X such that $Dx_\alpha \in \Lambda Y \otimes \Lambda X_{<\alpha}$ and let x_α be the first basis vector for which $Q(D)x_\alpha = y \neq 0$. Set $n = \deg y$; because $k_\xi < \infty$, n is even.

Divide by $\Lambda Y^{<n}$ to obtain a c.g.d.a. $(\Lambda Y^{\geqslant n} \otimes \Lambda X, \overline{D})$ and note (by Theorem 5.1) that $\mathrm{cat}_0(\Lambda Y^{\geqslant n} \otimes \Lambda X, \overline{D}) \leqslant \mathrm{cat}_0(\Lambda Y \otimes \Lambda X, D) = \mathrm{cat}_0(E)$. Define a c.g.d.a. $(\Lambda Y^{\geqslant n} \otimes \Lambda X \otimes \Lambda u, \overline{D})$ extending this, by putting $\overline{D}u = y$. Lemma 6.6 implies that $\mathrm{cat}_0(\Lambda Y^{\geqslant n} \otimes \Lambda X \otimes \Lambda u, \overline{D}) \leqslant \mathrm{cat}_0(\Lambda Y^{\geqslant n} \otimes \Lambda X, \overline{D}) + 1$.

Divide now by y and by u to obtain a quasi-isomorphism

$$(\Lambda Y^{\geqslant n} \otimes \Lambda X \otimes \Lambda u, \overline{D}) \xrightarrow{\simeq} (\Lambda(Y^{\geqslant n}/y) \otimes \Lambda X, D').$$

In the Λ-extension η: $\Lambda(Y^{\geqslant n}/y) \to \Lambda(Y^{\geqslant n}/y) \otimes \Lambda X \to \Lambda X$ we have $k_\eta = k_\xi - 1$. With the aid of the inequalities just derived we can now close the induction. Q.E.D.

6.7 REMARK. Note that the theorem holds for all connected minimal Λ-extensions, and not merely those arising from rational fibrations.

Fix a connected minimal KS-complex $(\Lambda X, d)$ and suppose

$$\xi\colon \Lambda Y \xrightarrow{i_\xi} \Lambda Y \otimes \Lambda X \xrightarrow{\rho_\xi} \Lambda X$$

is a connected Λ-extension with $(\Lambda X, d)$ as fibre. The first inequality of Theorem 6.4 asserts (in view of (6.7)) that if $\mathrm{cat}_0(\Lambda X, d) = m < \infty$ then $(\rho_\xi^\#)^{\mathrm{even}}$ is surjective and $\mathrm{Im}(\rho_\xi^\#)^{\mathrm{odd}}$ has codimension at most m in X^{odd}.

Define a graded subspace $G_\psi^*(\Lambda X, d) \subset X$ by

$$G_\psi^*(\Lambda X, d) = \bigcap_\xi \mathrm{Im}\, \rho_\xi^\#, \tag{6.8}$$

where ξ runs over all connected Λ-extensions with $(\Lambda X, d)$ as fibre. If $(\Lambda X, d)$ is the minimal model of a space F then we write $G_\psi^*(F)$ and note that it can be canonically identified as a subspace of $\pi_\psi^*(F)$.

6.9 THEOREM. *Suppose $(\Lambda X, d)$ is a connected minimal KS-complex with $\mathrm{cat}_0(\Lambda X, d) = m < \infty$. Then*

(i) $G_\psi^{\mathrm{even}}(\Lambda X, d) = X^{\mathrm{even}}$, and

(ii) $G_\psi^{\mathrm{odd}}(\Lambda X, d)$ has codimension at most m in X^{odd}.

6.10 COROLLARY. *Suppose $\mathrm{cat}_0(\Lambda X, d) < \infty$. There is then an integer N such that for any connected Λ-extension ξ with fibre ΛX, $(\rho_\xi^\#)^n$ is surjective, $n \geqslant N$.*

6.11 COROLLARY. *If F is a topological space and $\mathrm{cat}_0(F) = m < \infty$ then*

(i) $G_\psi^{\mathrm{even}}(F) = \pi_\psi^{\mathrm{even}}(F)$.

(ii) $G_\psi^{\mathrm{odd}}(F)$ has codimension at most m in $\pi_\psi^{\mathrm{odd}}(F)$.

(iii) For any rational fibration $F \xrightarrow{j} E \xrightarrow{\pi} B$, $\mathrm{Im}\, j^\# \supset G_\psi^*(F)$.

Let $G_n^\psi(\Lambda X, d) \subset \mathrm{Hom}(X^n; \mathbf{Q})$ be the space of linear functions $f\colon X^n \to \mathbf{Q}$ which extend to derivations θ_f of degree $-n$ in ΛX such that $\theta_f d - (-1)^n d\theta_f = 0$. It is easy to see that

$$G_\psi^n(\Lambda X, d) = \left\{x \in X^n \mid f(x) = 0, f \in G_n^\psi(\Lambda X, d)\right\}.$$

We thus obtain

6.12 COROLLARY. *If $\mathrm{cat}_0(\Lambda X, d) = m < \infty$, then $G_*^\psi(\Lambda X, d)$ is concentrated in odd degrees and has dimension at most m.*

6.13 REMARK. Let S be a simply connected space with minimal model $(\Lambda X, d)$. Write $G_n^\psi(\Lambda X, d) = G_n^\psi(S)$. Then $\dim X^n < \infty$ each n, and so $G_n^\psi(S) \subset \pi_n(S) \otimes \mathbf{Q} = \pi_n(S_{\mathbf{Q}})$.

If $f \in G_n^\psi(S)$, extend it to θ_f and define a c.g.d.a. morphism $(\Lambda X, d) \to H(S^n) \otimes (\Lambda X, d)$ by $x \to 1 \otimes x + \alpha \otimes \theta_f(x)$, where α is a fixed basis of $H^n(S^n)$. The resulting continuous map $S^n \times S_{\mathbf{Q}} \to S_{\mathbf{Q}}$ exhibits f as an element of the Gottlieb subgroup $G_n(S_{\mathbf{Q}}) \subset \pi_n(S_{\mathbf{Q}})$. It follows easily that

$$G_n(S) \otimes \mathbf{Q} \subset G_n(S_{\mathbf{Q}}) = G_n^\psi(S).$$

If S has finite rational category m we may conclude from Corollary 6.12 that $G_*(S) \otimes \mathbf{Q}$ is concentrated in odd degrees and has dimension at most m. This proves Theorem III of the introduction.

PROOF OF THEOREM 6.9. Assertion (i) is immediate from Theorem 6.4. Were (ii) to fail we could find connected Λ-minimal Λ-extensions $\xi_1, \ldots, \xi_r$ with fibre $(\Lambda X, d)$ and such that

$$\bigcap_1^r \operatorname{Im}\left(\rho_{\xi_i}^{\#}\right) = \bigcap_1^r \ker \partial_{\xi_i}^*$$

had codimension $> m$ in X.

Write ξ_i: $\Lambda Y_i \to \Lambda Y_i \otimes \Lambda X \to \Lambda X$ and divide by $\Lambda^{\geqslant 2} Y_i$ to produce a KS-extension $(Y_i \oplus \mathbf{Q}) \to (Y_i \oplus \mathbf{Q}) \otimes \Lambda X \to \Lambda X$. The differential D_i is given by $D_i(Y_i) = 0$ and $D_i(x) = dx + \delta_i x$, $\delta_i(x) \in Y_i \otimes \Lambda X$.

Let $A = \mathbf{Q} \oplus \bigoplus_{i=1}^r Y_i$, with trivial multiplication and differential, and define a KS-extension $(A, 0) \to (A \otimes \Lambda X, D) \to (\Lambda X, d)$ by putting $Dx = dx + \Sigma_{i=1}^r \delta_i x$. Construct a commutative diagram of c.g.d.a. morphisms

$$\begin{array}{ccccc}
(A,0) & \to & (A \otimes \Lambda X, D) & \to & (\Lambda X, d) \\
\uparrow\simeq & & \uparrow\simeq & & \uparrow \alpha \\
(\Lambda Y, D) & \underset{i_\xi}{\to} & (\Lambda Y \otimes \Lambda Z, D) & \underset{\rho_\xi}{\to} & (\Lambda Z, d)
\end{array}$$

in which the bottom row is a Λ-minimal Λ-extension. By [12, Chapter 6] α is an isomorphism. We may thus identify $(\Lambda Z, d) = (\Lambda X, d)$ and put $\alpha = \mathrm{id}$.

Evidently β restricts to a linear map $Y \to \bigoplus_i Y_i$. A simple calculation shows that $\beta \partial_\xi^* x = (\partial_{\xi_1}^* x, \ldots, \partial_{\xi_r}^* x)$, $x \in X$. It follows that $\ker \partial_\xi^* \subset \bigcap_i \ker \partial_{\xi_i}^*$ has codimension $> m$. This contradicts Theorem 6.4. Q.E.D.

7. Examples.

7.1 *The wedge* $F_1 \vee F_2$. Let $(\Lambda X_1, d)$ and $(\Lambda X_2, d)$ be the minimal models of spaces F_1 and F_2. It is then easy [12, Chapter 15] to see that the minimal models of $F_1 \vee F_2$ and $(\Lambda X_1, d) \vee (\Lambda X_2, d)$ are the same. In particular (use Theorem 4.7)

$$\operatorname{cat}_0(F_1 \vee F_2) = \max\{\operatorname{cat}_0(F_1), \operatorname{cat}_0(F_2)\}.$$

7.2 PROPOSITION. *Let* ξ: $F_1 \vee F_2 \xrightarrow{j} E \xrightarrow{\pi} B$ *be a rational fibration of simply connected spaces in which* F_1 *and* F_2 *have finite rational category and nontrivial rational homology. Then* $j_\#: \pi_*(F_1 \vee F_2) \otimes \mathbf{Q} \to \pi_*(E) \otimes \mathbf{Q}$ *is injective.*

PROOF. Since $\operatorname{cat}_0(F_1 \vee F_2) < \infty$, $\dim \ker j_\# < \infty$ (Theorem 6.4). With a shift of degrees down by one, we may identify this with

$$\ker\left((\Omega j)_\#: \pi_*(\Omega(F_1 \vee F_2)) \otimes \mathbf{Q} \to \pi_*(\Omega E) \otimes \mathbf{Q}\right).$$

This latter space is thus a finite dimensional ideal in the graded Lie algebra $\pi_*(\Omega(F_1 \vee F_2)) \otimes \mathbf{Q}$ and we must show it vanishes.

For this, observe that an element of maximum degree in $\ker(\Omega j)_\#$ is in the centre of $\pi_*(\Omega(F_1 \vee F_2)) \otimes \mathbf{Q}$ and hence in the centre of its universal enveloping algebra U. If follows easily from the work of Quillen [24] and Baues-Lemaire [2] that

$U = U_1 * U_2$ (coproduct in the category of associative algebras) where U_i is the universal enveloping algebra of $\pi_*(\Omega F_i) \otimes \mathbf{Q}$.

Thus U^+ is the direct sum of the vector spaces $U_{i_1}^+ \otimes \cdots \otimes U_{i_p}^+$, $p \geq 1$, with each $i_\nu = 1$ or 2 and $i_\nu \neq i_{\nu+1}$. Multiplication is given by $(\cdots \otimes a) \cdot (a' \otimes \cdots) = \cdots \otimes aa' \otimes \cdots$ and $(\cdots \otimes a) \cdot (b \otimes \cdots) = (\cdots \otimes a \otimes b \otimes \cdots)$, with a, a' in one of U_1^+ or U_2^+ and b in the other. An elementary calculation shows that if U has a nontrivial centre then $\dim U_1^+ = \dim U_2^+ = 1$. It follows that $\pi_*(\Omega F_i) \otimes \mathbf{Q}$ is a one-dimensional space of odd degree.

Hence $X_i = (x_i)$, x_i of even degree, and $H(F_i) = \Lambda(x_i)$ is a polynomial algebra, which contradicts $\mathrm{cat}_0(F_i) < \infty$. Q.E.D.

7.3 *Connected sum*. Suppose M, N are compact oriented connected n-manifolds with $H^1(M) = H^1(N) = 0$ ($n \geq 3$), and minimal models $\Lambda X_M \to A(M)$, $\Lambda X_N \to A(N)$. Let $\alpha_M \in \Lambda^n X_M$ and $\alpha_N \in \Lambda^n X_N$ represent the fundamental classes of M and N and define a c.g.d.a. $(\Lambda X_M \vee \Lambda X_N \otimes \Lambda u, D)$ extending $\Lambda X_M \vee \Lambda X_N$ by $Du = \alpha_M - \alpha_N$.

Let

$$(\Lambda X_M \vee \Lambda X_N) \otimes \Lambda u \xrightarrow{i} (\Lambda X_M \vee \Lambda X_N) \otimes \Lambda u \otimes \Lambda X \xrightarrow{\rho} \Lambda X$$

be a minimal KS-extension such that i^* is an isomorphism in cohomology in degrees $\leq n$, and such that $H^p((\Lambda X_M \vee \Lambda X_N) \otimes \Lambda u \otimes \Lambda X) = 0$ for $p > n$. In particular X is concentrated in degrees $\geq n$.

7.4 Proposition. *The minimal model of $M \# N$ is the minimal model for the c.g.d.a. $((\Lambda X_M \vee \Lambda X_N) \otimes \Lambda u \otimes \Lambda X, D)$. In particular, if M and N are formal, so is $M \# N$.*

Proof. Write

$$M \# N = (M - \mathring{D}^n) \cup_{S^{n-1}} (N - \mathring{D}^n)$$

and let K be the set of singular simplices for $M \# N$ whose image is in one of $M - \mathring{D}^n$, $N - \mathring{D}^n$. Restriction defines a quasi-isomorphism $A(M \# N) \xrightarrow{\simeq} A(K)$.

The model for M can be chosen so that $\Lambda^+ X_M$ maps to forms which restrict to zero in D^n. These forms, when restricted to $M - \mathring{D}^n$, extend by zero to elements of $A(K)$. The same construction applies to N and yields a morphism $\phi\colon \Lambda X_M \vee \Lambda X_N \to A(K)$ such that $\phi(\alpha_M, -\alpha_N) = d\Phi$. Extend ϕ to $(\Lambda X_M \vee \Lambda X_N) \otimes \Lambda u$ by putting $\phi u = \Phi$. The new ϕ is a cohomology isomorphism in degrees $\leq n$; since $H^{>n}(M \# N) = 0$ it extends automatically to a quasi-isomorphism

$$\phi\colon (\Lambda X_M \vee \Lambda X_N) \otimes \Lambda u \otimes \Lambda X \xrightarrow{\simeq} A(K).$$

To prove the second part, suppose $\psi_M\colon \Lambda X_M \xrightarrow{\simeq} H(M)$ and $\psi_N\colon \Lambda X_N \xrightarrow{\simeq} H(N)$ induce the identity in cohomology and extend these to a quasi-isomorphism

$$\psi\colon (\Lambda X_M \vee \Lambda X_N) \otimes \Lambda u \otimes \Lambda X \xrightarrow{\simeq} \frac{H(M) \vee H(N)}{\alpha_M - \alpha_N} = H(M \# N)$$

by $\psi u = \psi(X) = 0$. Q.E.D.

7.5 *The manifold $M_g^{2n} = (S^n \times S^n) \# \cdots \# (S^n \times S^n)$*. Fix an odd integer n and let M_g^{2n} denote the connected sum of g copies of $S^n \times S^n$. When $n = 1$ this is a

Riemann surface and hence formal by [**4**; **14**, Corollary 6.9]. For $n \geqslant 3$ it is formal by Proposition 7.4.

We establish now properties of the model $(\Lambda X, d)$ for M_g^{2n}, beginning with $n = 1$. (See [**25**, §5] for a sketch of the answer here.)

7.6 LEMMA. *When $n = 1$, $X = X^1$.*

PROOF. Choose $x \in X^1$, write $\Lambda X = \Lambda x \otimes \Lambda Y$ and observe that $H(\Lambda Y) = H(\Lambda X \otimes \Lambda u, d)$, where $du = x$. Filter $\Lambda X \otimes \Lambda u$ by putting $F^{-p} = \Lambda X \otimes \Lambda^{\leqslant p} u$. The resulting spectral sequence is convergent and has $E_2 \cong H(H(\Lambda X) \otimes \Lambda u, d)$ with $du = [x]$.

Now $H(\Lambda X) = H(M_g^2)$, so that an elementary calculation gives E_2. Since u has degree zero one sees that (aside from $\mathbf{Q}$) E_2 is concentrated in degree one. This is then true for $H(\Lambda Y)$ and the argument in [**14**, Theorem 7.10] shows now that $Y = Y^1$. Q.E.D.

Because M_g^2 is formal its minimal model carries [**14**, §3] an additional (lower) grading: $X = \Sigma_{q \geqslant 0} \Lambda X_q^1$ with respect to which d is homogeneous of lower degree -1 and $H_+(\Lambda X) = 0$. Because $X = X^1$ we have therefore that $d\colon X_p^1 \to (\Lambda^2 X^1)_{p-1}$. If $x, y \in X_0^1$ satisfy $xy = dx_1$ then there are elements $x_p \in X_p^1$ with $dx_p = x_{p-1} y$. Thus for $g \geqslant 2$ no X_p vanishes.

We turn now to the general case.

7.7 PROPOSITION. *The minimal model of M_g^{2n} has the form $(\Lambda Z, d)$ in which $Z = \Sigma_{p \geqslant 0} Z_p^{(p+1)(n-1)+1}$, $d\colon Z_p \to (\Lambda^2 Z)_{p-1}$ and $H_+(\Lambda Z, d) = 0$.*

7.8 COROLLARY. *If $n \geqslant 3$ and $g \geqslant 2$ then $\pi_*(M_g^{2n}) \otimes \mathbf{Q}$ is zero except in (the arithmetic progression of) the degrees $1 + q(n-1)$, $q \geqslant 1$. In these degrees it is always nonzero.*

7.9 COROLLARY. *M_g^{2n} is intrinsically formal and π-formal.*

PROOF OF 7.7. Put $Z_p^{(p+1)(n-1)+1} = X_p^1$. Because n is odd so is $(p+1)(n-1)+1$ and so we can identify $\Lambda Z_* = \Lambda X_*$ as algebras. Use this to transport the differential in ΛX to a differential, d, in ΛZ. A direct computation shows that $d\colon Z_p^k \to (\Lambda^2 Z)_{p-1}^{k+1}$ (of course $k = (p+1)(n-1)+1$).

Moreover $H_+(\Lambda Z, d) = 0$ so that $(\Lambda Z, d)$ is a model for $H_0(\Lambda Z, d) \cong H(M_g^{2n})$. Since M_g^{2n} is formal we are done. Q.E.D.

Proposition 7.7 shows that in the minimal model $(\Lambda Z, d)$ for M_g^{2n} $(n \geqslant 3, g \geqslant 2)$:

$$(7.10) \qquad \begin{cases} Z \text{ is concentrated in odd degrees.} \\ d\colon Z \to \Lambda^2 Z. \\ \dim H(\Lambda Z) < \infty \text{ and } \Sigma(-1)^p \dim H^p(\Lambda Z) \neq 0. \end{cases}$$

7.11 PROPOSITION. *Let $F \xrightarrow{j} E \xrightarrow{\pi} B$ be a rational fibration of simply connected spaces in which the minimal model of F satisfies* (7.10) *and* $\operatorname{cat}_0(F) < \infty$. *Then $j_\#\colon \pi_*(F) \otimes \mathbf{Q} \to \pi_*(E) \otimes \mathbf{Q}$ is injective.*

PROOF. If the proposition fails at all it does so when B is an even sphere S^{2k}. In this case we get a c.g.d.a. of the form $((1 \oplus a_{2k}) \otimes \Lambda Z, D)$ representing E with

$Dz = dz + a \otimes \theta z$, where θ is a derivation of odd degree in ΛZ, anticommuting with d. The failure of the proposition translates to the supposition that $\theta\colon Z^{2k-1} \to \mathbf{Q}$ is nonzero.

Extend this linear function to a derivation θ_0 in ΛZ^{2k-1} and extend θ_0 to ΛZ by zero in Z^i, $i \neq 2k-1$. Then $\theta - \theta_0$ preserves $\Lambda^+ Z$ while $\theta_0\colon \Lambda^i Z \to \Lambda^{i-1} Z$, and it follows from (7.10) that $\theta_0 d + d\theta_0 = 0$.

Set $Y = \ker \theta_0 \cap Z$ and choose $z \in Z^{2k-1}$ so that $\theta_0(z) = 1$. Then $\Lambda Z = \Lambda Y \otimes \Lambda z$ and hence $\Lambda Y = \ker \theta_0$. Thus ΛY is d-stable and contains dz. Let τ denote multiplication by $[dz]$ in $H(\Lambda Y)$. The inclusion $\Lambda Y \to \Lambda Z$ induces an inclusion $\operatorname{Coker} \tau \to H(\Lambda Z)$, so that by (7.10), $\dim \operatorname{Coker} \tau < \infty$.

On the other hand, by (7.10), Y is concentrated in odd degrees, and it follows that for some p, $(dz)^p = 0$. The same is thus true of $[dz]$. Since $H(\Lambda Y)$ is generated as an algebra by a subspace isomorphic to $\operatorname{Coker} \tau$, together with $[dz]$, we conclude that $\dim H(\Lambda Y) < \infty$.

Finally, filtering by ΛY gives a spectral sequence converging from $H(\Lambda Y) \otimes \Lambda z$ to $H(\Lambda Z)$ and it follows that $H(\Lambda Z)$ has zero Euler characteristic, in contradiction with the final hypothesis of (7.10). Q.E.D.

8. Rational homotopy.

8.1 THEOREM. *Let $(\Lambda X, d)$ be the minimal model of a c-connected c.g.d.a. of rational category m such that $H^p(\Lambda X) = 0$, $p > n$. Assume that* $\dim X^{>1} = \infty$. *Then*

(i) *there is an infinite sequence of integers $r_1 < r_2 < \cdots$ with $r_i < n + r_{i-1}$, such that $X^{r_i} \neq 0$, $i = 1, 2, \ldots$,*

(ii) *there is an infinite sequence of integers $q_1 < q_2 < \cdots$ with $q_i < (m+1)q_{i-1}$, and there is a constant $C > 1$ such that* $\dim X^{q_i} \geq C^{q_i}$, $i = 1, 2, \ldots$.

PROOF. *Step* I. *Proof of* (i). We note first that $\dim X^p \neq 0$ for infinitely many p. Otherwise $\dim X^p = \infty$ for some largest $p \geq 2$ and $\dim X^{>p} < \infty$. Since $d(X^{>p})$ can then only involve a finite-dimensional subspace $Y \subset X^p$ there is a surjection $(\Lambda X, d) \to (\Lambda X^p/Y, 0)$. By Theorem 5.1, $\dim X^p/Y \leq \operatorname{cat}_0(\Lambda X^p/Y, 0) \leq m$, which contradicts $\dim X^p = \infty$.

Next, by Corollary 6.12, there is an integer N such that $G_r^\psi(\Lambda X, d) = 0$, $r \geq N$. Let $r_1 < r_2 < \cdots$ be the infinite sequence of integers $r \geq N$ such that $X^r \neq 0$. We show that $r_i - r_{i-1} < n$.

If not there is some $r \geq N$ such that $X^r \neq 0$ and $X^i = 0$, $r < i < n + r$. Fix a nonzero function $f\colon X^r \to \mathbf{Q}$. We shall show it is in $G_r^\psi(\Lambda X, d)$ by extending it to a derivation θ_f of degree $-r$ with $\theta_f d - (-1)^r d\theta_f = 0$.

First put $\theta_f(x) = 0$, $x \in X^{<r}$, and $\theta_f(x) = f(x)$, $x \in X^r$. Then $\theta_f d - (-1)^r d\theta_f = 0$ in $\Lambda X^{\leq r}$. Choose a well-ordered basis $\{x_\alpha\}$ of $X^{\geq n+r}$ such that $dx_\alpha \in \Lambda X^{\leq r} \otimes \Lambda X^{\geq n+r}_{<\alpha}$. Suppose θ_f extended to $\Lambda X^{\leq r} \otimes \Lambda X^{\geq n+r}_{<\alpha}$ with $\theta_f d - (-1)^r d\theta_f = 0$.

Then $d\theta_f dx_\alpha = (-1)^r \theta_f d^2 x_\alpha = 0$. Thus $\theta_f dx_\alpha$ is a cocycle in ΛX of degree $\geq n+1$. Thus we can solve $\theta_f dx_\alpha = (-1)^r dy$ for y. Put $y = \theta_f x_\alpha$. Continue in this way to construct θ_f and achieve a contradiction.

Step II. We show that either $\dim X^p = \infty$ for infinitely many p or that for those p with $\dim X^p < \infty$ the integers $\dim X^p$ are unbounded. Suppose the first condition to fail and let r be one of the integers r_i of (i), chosen so that $\dim X^p < \infty$, $p \geq r$. Put $Y = \sum_{j=r}^{2r-2} X^j$ and $Z = X^{>2r-2}$. Write $k = \dim Y$ and note from (i) that

(8.2) $$k \geq (r-2)/(n-1).$$

Moreover, in the quotient c.g.d.a. $(\Lambda X^{\geq r}, \bar{d}) = (\Lambda Y \otimes \Lambda Z, \bar{d})$ we have $\bar{d} = 0$ in ΛY. By Theorem 5.1, $\operatorname{cat}_0(\Lambda X^{\geq r}, \bar{d}) \leq \operatorname{cat}_0(\Lambda X, d) = m$. Hence $\Lambda^{m+1} Y \subset \operatorname{Im} \bar{d}$. Define $\alpha\colon Z \to \Lambda Y$ of degree 1 by $\bar{d}z - \alpha z \in \Lambda Y \otimes \Lambda^+ Z$. Then $\alpha(Z) \cdot \Lambda Y \supset \Lambda^{m+1} Y$.

Because $\deg \alpha = 1$ and elements in $\Lambda^{m+1} Y$ have degree at most $2(r-1)(m+1)$ this yields $\alpha(Z^{<2(r-1)(m+1)}) \cdot \Lambda Y \supset \Lambda^{m+1} Y$. Because $(\Lambda X, d)$ is minimal, $\alpha(Z) \subset \Lambda^{\geq 2} Y$. It follows that in the quotient algebra $\Lambda Y / \Lambda^{>m+1} Y$,

$$\alpha(Z^{<2(r-1)(m+1)}) \cdot \left(\sum_{j=0}^{m-1} \Lambda^j Y \right) \supset \Lambda^{m+1} Y.$$

By (8.2) we may suppose (for sufficiently large r) that $k \geq 2m+2$. Then

$$\binom{k}{m+1} \geq \left(\frac{k}{2(m+1)} \right)^{m+1},$$

while

$$\binom{k+j-1}{j} \leq (2k)^j \leq (2k)^{m-1}, \qquad j \leq m-1.$$

Because $\binom{k}{j} \leq \dim \Lambda^j Y \leq \binom{k+j-1}{j}$ it follows that $\dim Z^{<2(r-1)(m+1)} \geq \lambda k^2$, where λ is a constant dependent only on m.

Again by (8.2) $Z^{<2(r-1)(m+1)} \subset Z^{\leq \mu k}$ where μ is a constant dependent only on n and m. It follows that for some $p \geq r$, $\dim X^p = \dim Z^p \geq \lambda k / \mu$. Thus as $r \to \infty$ the integers $\dim X^p$ are unbounded.

Step III. *Proof of* (ii). Set $e = (1/4(m+1))^{m+1}$. By Step II we may choose p so that $N = \dim X^p$ satisfies $Ne > 1$. The argument of Step II, applied to $(\Lambda X^p \otimes \Lambda X^{>p}, \bar{d})$, yields a degree 1 linear map $\alpha\colon X^{>p} \to \Lambda^{\geq 2} X^p$ such that $\alpha(X^{>p}) \cdot \Lambda X^p \supset \Lambda^{m+1} X^p$.

Now $\alpha(X^q) = 0$ unless $q = lp - 1$, $l \geq 2$, and $\alpha(X^{lp-1}) \subset \Lambda^l X^p$. It follows that

$$\Lambda^{m+1} X^p = \sum_{l=2}^{m+1} \Lambda^{m+1-l} X^p \cdot \alpha(X^{lp-1}).$$

The inequalities of Step II then yield (because $N \geq 1/e \geq 2m+2$)

$$\left(\frac{N}{2(m+1)} \right)^{m+1} \leq \sum_{l=2}^{m+1} (2N)^{m+1-l} \dim X^{lp-1}.$$

This implies that for some $l \in [2, m+1]$, $\dim X^{lp-1} \geq eN^l$.

Since $eeN^l \geq (eN)^2 > 1$ the procedure can be iterated to yield a sequence $p = p_1 < p_2 < \cdots$ with $p_{i+1} = l_i p_i - 1$ for some integer $l_i \in [2, m+1]$ and such that $\dim X^{p_{i+1}} \geq e(\dim X^{p_i})^{l_i}$. Because $1 + l_i + l_i l_{i-1} + \cdots + l_i \cdots l_2 \leq l_i \cdots l_1$ (since $l_j \geq 2$), it follows that

$$(\dim X^{p_{i+1}})^{1/p_{i+1}} \geq (e \dim X^p)^{1/p}.$$

Now set $q_i = p_{i+j}$. Q.E.D.

The next theorem implies Theorem IV of the introduction.

8.3 THEOREM. *Let* $(\Lambda X, d)$ *be the minimal model of a c-connected c.g.d.a. such that* $\dim H(\Lambda X) < \infty$ *and* $\dim X^p < \infty, p = 1, 2, \ldots$. *Then either*

(i) $\dim X < \infty$, *or*

(ii) *the coefficients of* $f_X(t) = \sum_{p \geq 1} \dim X^p t^p$ *grow exponentially.*

PROOF. Because $\dim X^1 < \infty$, if $\dim X = \infty$ then $\dim X^{>1} = \infty$. By Theorem 8.1 there is then a sequence $q_1 < q_2 < \cdots$ and a constant $C > 1$ such that $\dim X^{q_i} \geq C^{q_i}$ and $q_{i+1} \leq (m+1)q_i$. ($m = \operatorname{cat}_0(\Lambda X, d)$.) For any $k \geq q_1$ we have $q_i \leq k < q_{i+1}$, some i, and so

$$\sum_{p=1}^{k} \dim X^p \geq \dim X^{q_i} \geq C^{q_i} \geq \left(C^{1/(m+1)}\right)^k.$$

On the other hand, Lemma 8.5 below shows that for some n, $H^p(\Lambda X^{>1}, d) = 0$, $p > n$. Since $\Lambda X^{>1}$ has finite type this implies $\dim H(\Lambda X^{>1}, d) < \infty$. Let $f(t)$ be the polynomial $\sum_{p=1}^{n-1} \dim H^{p+1}(\Lambda X^{>1}) t^p$.

Define a graded space $\bar{X}$ by $\bar{X}^p = X^{p+1}$. In [14, §7] is constructed a spectral sequence converging to $\Lambda \bar{X}^{\geq 1}$ with E_1-term isomorphic as a graded space with the tensor algebra $\otimes V$, where $V^p = H^{p+1}(\Lambda X^{>1}), p \geq 1$. It follows that $\dim X^{p+1} \leq a_p$, where

$$\sum a_p t^p = \frac{1}{1 - f(t)}.$$

In particular

$$\sum_{p=2}^{k} \dim X^p \leq \sum_{p=1}^{k-1} a_p \leq \sum_{p=0}^{k-1} f(1)^p \leq f(1)^k. \quad \text{Q.E.D.}$$

8.4 REMARK. If S is a simply connected space, the path fibration $\Omega S \to PS \to S$ is a rational fibration. Since the model for the augmentation $(\Lambda X, d) \to \mathbf{Q}$ of the minimal model of S is of the form $(\Lambda X \otimes \Lambda \bar{X}, d)$ with $Q(d)\colon \bar{X} \xrightarrow{\cong} X$, we may identify $\Lambda \bar{X} = H(\Omega S)$.

Recall that Theorem V of the introduction reads

THEOREM V. *Let* S *be a simply connected space such that* $\dim H^*(S) < \infty$. *Write* ΩS *for the loop space. If* S *is rationally elliptic the coefficients of* $f_H(\Omega S; t)$ *grow polynomially of order rank* S. *Otherwise they grow exponentially.*

In view of 8.4, this theorem follows at once from Theorem 8.3 and its proof.

8.5 LEMMA. *Let* $(\Lambda u \otimes \Lambda X, d)$ *be a KS-complex in which* u *is a degree* 1 *cocycle and* $H^p(\Lambda u \otimes \Lambda X) = 0, p > n$. *Then* $H^p(\Lambda X, \bar{d}) = 0, p > n$.

PROOF. Extend the KS-complex to $(\Lambda u \otimes \Lambda X \otimes \Lambda v, d)$ with $\deg v = 0$ and $dv = u$. This has the homotopy type of $(\Lambda X, \bar{d})$. Filter by putting $F^{-p} = \Lambda u \otimes \Lambda X \otimes \Lambda^{\leq p} v$ and note that $E_1 \cong H(\Lambda u \otimes \Lambda X) \otimes \Lambda v$. Q.E.D.

9. The invariant e_0. Let S be a simply connected space. There is a standard spectral sequence converging to $H^*(S)$ (usually the dual, homology spectral sequence is considered) due to Milnor and Moore (cf. [**20**, **21**, **22**]). It is based on considering S as a classifying space for ΩS.

One description (of many) is to form the bar construction $BC^*(S)$ on the augmented cochains for S and then form the cobar construction on this differential coalgebra to obtain a differential algebra $\mathcal{G}BC^*(S)$. If $V = B^+C^*(S)$ is the canonical complement to $\mathbf{Q}$ then $\mathcal{G}BC^*(S)$ is the tensor algebra over V. Filtering by the ideals $\Sigma_{j\geqslant p} \otimes^j V$ yields the spectral sequence.

Clearly any homomorphism $\phi\colon A_1 \to A_2$ of augmented graded differential algebras yields a spectral sequence homomorphism, which is an isomorphism from E_1 on if $\phi^*\colon H(A_1) \xrightarrow{\cong} H(A_2)$. According to [**12**, Theorem 14.18] there are such homomorphisms $C^*(S) \to C \leftarrow A(S)$ for a certain g.d.a. C. We may thus replace $C^*(S)$ by $A(S)$ and hence by the minimal model $(\Lambda X, d)$ of S to compute the spectral sequence.

On the other hand the projection $B^+(\Lambda X) \to \Lambda^+ X \to X$ extends to a g.d.a. homomorphism $\mathcal{G}B(\Lambda X) \to \Lambda X$. If we filter $(\Lambda X, d)$ by the ideals $\Lambda^{>j}X$ then this homomorphism is filtration preserving, and it is an easy exercise to see that it gives a spectral sequence isomorphism from E_2 on. This proves

9.1 PROPOSITION. *Let $(\Lambda X, d)$ be the minimal model of the simply connected space S. Then the Milnor-Moore spectral sequence for S can be identified from E_2 on with the spectral sequence arising from the filtration of ΛX by the ideals $\Lambda^{\geqslant j}X$.*

Henceforth if $(\Lambda X, d)$ is the minimal model of any space S or c-connected c.g.d.a. (A, d_A) we call the spectral sequence arising from the ideals $\Lambda^{\geqslant j}X$ the *Milnor-Moore spectral sequence* for S (or (A, d_A)).

9.2 DEFINITION. Let $E_i^{p,q}$ be the Milnor-Moore spectral sequence for a space S (or c.g.d.a. (A, d_A)). The largest integer k such that $E_\infty^{k,*} \neq 0$ will be written $e_0(S)$ (or $e_0(A, d_A)$). If there is no such integer we put $e_0 = \infty$.

9.3 REMARKS. 1. If $(\Lambda X, d)$ is any KS-complex then $e_0(\Lambda X, d)$ is the largest integer k such that some nontrivial class in $H(\Lambda X)$ is represented by a cocycle in $\Lambda^{\geqslant k}X$. Equivalently it is the least integer such that the projection $\Lambda X \to \Lambda X/\Lambda^{>k}X$ induces an injection of cohomology algebras.

In view of Theorem 4.7 we recover the inequality $\operatorname{cat}_0(\Lambda X, d) \geqslant e_0(\Lambda X, d)$ (cf. [**26**, Theorem II B2; **17**]).

2. If S_1 and S_2 are spaces then the minimal model of $S_1 \times S_2$ is $(\Lambda X_1, d) \otimes (\Lambda X_2, d)$ where $(\Lambda X_i, d)$ is the model for S_i. The spectral sequence for $S_1 \times S_2$ is then the tensor product of the spectral sequences for S_1 and S_2, and we recover [**26**, Theorem II C4]:

$$e_0(S_1 \times S_2) = e_0(S_1) + e_0(S_2).$$

3. Evidently if $n_0(\Lambda X, d)$ is the product length of $H(\Lambda X)$ then $n_0(\Lambda X, d) \leqslant e_0(\Lambda X, d)$.

4. [17] If $(\Lambda X, d)$ is formal (i.e., it is the model of $(H(\Lambda X), 0)$, then by Theorem 4.7, $n_0(\Lambda X, d) = \mathrm{cat}_0(\Lambda X, d)$ and so

$$n_0(\Lambda X, d) = e_0(\Lambda X, d) = \mathrm{cat}_0(\Lambda X, d).$$

5. [17] If $(\Lambda X, d)$ is π-formal then $d\colon X \to \Lambda^2 X$ and in this case it is immediate from Theorem 4.7 that $e_0(\Lambda X, d) = \mathrm{cat}_0(\Lambda X, d)$.

6. Suppose $e_0(\Lambda X, d) = 1$. Then $\Lambda X \to \Lambda X/\Lambda^{\geqslant 2} X = X \oplus \mathbf{Q}$ is injective in cohomology. Since X is equipped with the zero differential and product we may divide X by a subspace V to obtain a quasi-isomorphism $\Lambda X \xrightarrow{\cong} X/V \oplus \mathbf{Q}$. We thus recover a result of Toomer [26] which asserts that a simply connected space S with $e_0(S) = 1$ has the rational homotopy type of a wedge of spheres.

9.4 EXAMPLES. 1. Let $W = (\mathbf{C}P^2 \vee S^2) \cup_\omega e^7$ be the space of Example 5.5. It is shown there that the minimal model of W maps to c.g.d.a. (A, d_A) by a quasi-isomorphism and, by construction, $(A^+)^4 = 0$. It follows, moreover, from 5.8 that $A \to A/(A^+)^3$ induces an injection at the level of cohomology. Hence the model $\Lambda X \to A$ satisfies $\Lambda X \to \Lambda X/\Lambda^{\geqslant 3} X$ is injective in cohomology, and so $e_0(W) \leqslant 2$.

In view of 9.3.6 we obtain Lemaire's result [16] that $e_0(W) = 2$. Hence $e_0(W^n) = 2n$, and (cf. (5.7)) $\mathrm{cat}_0(W^n) - e_0(W^n) = 3n - 2n = n$.

2. We show that Theorem I has no analogue for e_0. Indeed, in Example 5.5 we constructed a c.g.d.a. morphism $\psi\colon (\Lambda Z, D) \to (\Lambda(x_3, y_3, y_5), \overline{D})$ which defines a continuous map

$$f\colon (\mathbf{C}P^2 \times S^3)_{\mathbf{Q}} \to W_{\mathbf{Q}}$$

and f is injective at the level of rational homotopy groups.

Because $\mathbf{C}P^2$ and S^3 are formal the remarks above yield $e_0(\mathbf{C}P^2 \times S^3)_{\mathbf{Q}} = 2 + 1 = 3$, whereas we have just seen in Example 9.4.1 that $e_0(W_{\mathbf{Q}}) = 2$.

10. Poincaré duality and rationally elliptic spaces. A c.g.a. H will be said to have *formal dimension* n if $H^p = 0$, $p > n$, and $H^n \neq 0$. If in addition $\dim H < \infty$, $\dim H^n = 1$, and multiplication $H^p \otimes H^{n-p} \to H^n \cong \mathbf{Q}$ is a nondegenerate bilinear form for each p, then H is called a *Poincaré duality algebra* (*P.d.a*). An element $0 \neq \omega \in H^n$ is called a *fundamental class*.

Our aim is to establish Theorems VI and VII of the introduction.

10.1 LEMMA. *Let $(\Lambda X, d)$ be a connected KS-complex such that $H(\Lambda X)$ is a P.d.a. and suppose ω is a fundamental class. Then* $e_0(\Lambda X, d) = \sup\{k \mid \omega$ *can be represented by a cocycle in* $\Lambda^{\geqslant k} X\}$.

PROOF. Observe that any algebra homomorphism ϕ with domain $H(\Lambda X)$ is injective if and only if $\phi\omega \neq 0$. Q.E.D.

10.2 LEMMA. *Let $(\Lambda X \otimes \Lambda u, d)$ be a connected KS-complex in which $(\Lambda X, d)$ is a sub-KS-complex,* $\dim H(\Lambda X) < \infty$, *$H(\Lambda X)$ has formal dimension r, u has odd degree q and $du \in \Lambda X$. Then* $\dim H(\Lambda X \otimes \Lambda u) < \infty$ *and $H(\Lambda X \otimes \Lambda u)$ has formal dimension $q + r$.*

If in addition $H(\Lambda X)$ is a P.d.a. then so is $H(\Lambda X \otimes \Lambda u)$ and in this case, $e_0(\Lambda X \otimes \Lambda u, d) \leqslant e_0(\Lambda X, d) + 1$.

PROOF. Filter $\Lambda X \otimes \Lambda u$ by the degree of ΛX and use a standard spectral sequence argument for the assertions on dimension, formal dimension and Poincaré duality. If $e_0(\Lambda X \otimes \Lambda u, d) = k$ then Lemma 10.1 gives a representative $\Phi \otimes u + \Psi \in \Lambda^{\geqslant k}(X \oplus u)$ for the fundamental class. Since $\Phi \in \Lambda^{\geqslant k-1}X$ represents the fundamental class of $H(\Lambda X)$, $e_0(\Lambda X, d) \geqslant k - 1$. Q.E.D.

Fix now a Λ-extension

$$(10.3) \qquad (\Lambda y, 0) \to (\Lambda y \otimes \Lambda X, D) \xrightarrow{\rho} (\Lambda X, d)$$

in which $(\Lambda y \otimes \Lambda X, D)$ is a connected minimal KS-complex. The differential D can be written $D = D_2 + D_3 + \cdots$, D_i raising wedge degree by $i - 1$. Write

$$D_2(1 \otimes \Phi) = 1 \otimes d_2\Phi + y \otimes \gamma\Phi, \qquad \Phi \in \Lambda X;$$

then γ is a derivation of ΛX, homogeneous of wedge degree zero. The main step in proving Theorems VI and VII is

10.4 THEOREM. *Suppose in* (10.3) *that* $\dim H(\Lambda y \otimes \Lambda X) < \infty$. *Write* $q = \deg y$ *and* $n =$ *formal dimension* $H(\Lambda y \otimes \Lambda X)$.

(i) *If* $D\colon X \to \Lambda^{\geqslant 3}(y \oplus X)$ *then* $\dim H(\Lambda X) < \infty$.

(ii) *If there is a decomposition* $X = V \oplus \gamma(X)$ (*for some graded subspace* V) *such that vectors of the form* $\gamma^k v$, $v \in V$, $k = 0, 1, \ldots,$ *span* X, *and if* $X^1 = 0$ *and* $q > 1$, *then* $\dim H(\Lambda X) < \infty$.

(iii) *If* $\dim H(\Lambda X) < \infty$ *and* $H(\Lambda X)$ *has formal dimension* r *then*

$$r = \begin{cases} n - q & \text{if } q \text{ is odd}, \\ n + q - 1 & \text{if } q \text{ is even}. \end{cases}$$

(iv) *If* $H(\Lambda y \otimes \Lambda X)$ *is a P.d.a. and* $\dim H(\Lambda X) < \infty$, *then* $H(\Lambda X)$ *is a P.d.a. In this case*

$$e_0(\Lambda X, d) \leqslant \begin{cases} e_0(\Lambda y \otimes \Lambda X, D) - 1 & \text{if } q \text{ is odd}, \\ e_0(\Lambda y \otimes \Lambda X, D) & \text{if } q \text{ is even}. \end{cases}$$

10.5 COROLLARY. *Suppose* $(\Lambda X, d)$ *is a connected minimal KS-complex such that* $d\colon X \to \Lambda^{\geqslant 3}X$ *and* $H(\Lambda X)$ *is a P.d.a. Then* $\dim X < \infty$.

PROOF. Let $x_1, x_2, \ldots$ be linearly independent elements of X^1 such that $dx_i \in \Lambda(x_1, \ldots, x_{i-1})$. Parts (i) to (iv) of the theorem imply that if $(\Lambda X_i, d)$ is the quotient c.g.d.a. obtained by putting $x_1, \ldots, x_i = 0$ then $H(\Lambda X_i)$ is a P.d.a. of formal dimension $n - i$ ($n =$ formal dimension $H(\Lambda X)$).

It follows that $\dim X^1 < \infty$ and that the quotient c.g.d.a. $(\Lambda X^{>1}, d)$ satisfies the hypothesis of the corollary. Since $\dim H(\Lambda X^{>1}) < \infty$ we find $\dim X^p < \infty$, each p. Use part (iv) of the theorem to conclude that $\dim X^{\mathrm{odd}} < \infty$. Hence for some p, $X^{\geqslant p}$ is concentrated in even degrees, and so the induced differential in $\Lambda X^{\geqslant p}$ is zero.

By Theorem 5.1, $\operatorname{cat}_0(\Lambda X^{\geqslant p}, 0) < \infty$. Thus $X^{\geqslant p} = 0$. Q.E.D.

Recall that Theorem VI of the introduction reads

THEOREM VI. *Let* S *be a simply connected rationally hyperbolic space such that* $H^*(S)$ *is a Poincaré duality algebra. In the homotopy Lie algebra* $\pi_*(\Omega S) \otimes \mathbf{Q}$ *there* *are then homogeneous elements α, β such that the iterated brackets*

$$[\beta[\beta[\beta\cdots[\beta,\alpha]]\cdots] \qquad (k \text{ factors } \beta,\ k=1,2,3,\ldots)$$

are all nonzero. In particular, $\pi_*(\Omega S)\otimes\mathbf{Q}$ is not nilpotent.

PROOF. Let $(\Lambda X, d)$ be the minimal model of S and write $d = d_2 + d_3 + \cdots$ where d_p: $\Lambda^q X \to \Lambda^{q+p-1}X$. Because S is simply connected, $X^1 = 0$. Choose a basis $x_1,\ldots$ of X with $dx_i \in \Lambda(x_1,\ldots,x_{i-1})$. Since $\dim X = \infty$ we conclude from Theorem 10.4(iv) (as in Corollary 10.5) that for some i, $\dim H(\Lambda(x_i,\ldots)) < \infty$ but $\dim H(\Lambda(x_{i+1},\ldots)) = \infty$.

If $\bar{d}$ is the differential in $\Lambda(x_i,\ldots)$ we can now apply part (ii) of Theorem 10.4 and deduce the existence of an infinite sequence of vectors y_1, $y_2,\ldots$ in $(x_{i+1},\ldots)$ such that $\bar{d}_2 y_j - x_i \otimes y_{j-1} \in \Lambda(x_{i+1},\ldots)$.

Now $(\Lambda X, d_2)$ is the Koszul construction **[24, 2, 5]** of the cochain algebra on a graded Lie algebra $L = \Sigma_{p\geqslant 1} L_p$ with $L_p = (X^{p+1})^*$. If $\alpha, \beta \in L$ satisfy $\langle \alpha, x_i\rangle = 1$, $\langle \beta, y_1\rangle = 1$, $\langle \beta, x_i\rangle = 0$ then our conclusion above shows that

$$\langle \underbrace{[\beta[\beta[\cdots[\beta}_{j \text{ factors}},\alpha]\cdots], y_{j+1}\rangle \neq 0,$$

and so these iterated brackets are nonzero.

Finally, the results of **[14**, §7**]** imply that L can be identified with the Lie algebra of primitive elements in the Hopf algebra dual to $H(B(\Lambda X, d))$, where $B(\Lambda X, d)$ denotes the bar construction. The techniques of **[2]** thus identify $L \cong \pi_*(\Omega S)\otimes\mathbf{Q}$ as graded Lie algebras. Q.E.D.

The next result contains Theorem VII.

10.6 PROPOSITION. *Let $(\Lambda X, d)$ be a connected minimal KS-complex such that $\dim X < \infty$ and $\dim H(\Lambda X) < \infty$. Then $\operatorname{cat}_0(\Lambda X, d) \geqslant e_0(\Lambda X, d) \geqslant \dim X^{\text{odd}}$. If, moreover, $X^1 = 0$ and d: $X \to \Lambda^2 X$ then $\operatorname{cat}_0(\Lambda X, d) = e_0(\Lambda X, d) = \dim X^{\text{odd}}$.*

PROOF. Recall from **[11**, Theorem 3**]** that $H(\Lambda X)$ is a P.d.a. Choose a basis $x_1,\ldots,x_n$ of X such that $dx_i \in \Lambda(x_1,\ldots,x_{i-1})$. The quotient c.g.d.a.'s $(\Lambda(x_i,\ldots,x_n), \bar{d})$ all have finite rational category by Theorem 5.1. Since $\dim \Lambda(x_i,\ldots,x_n)/\Lambda^{\geqslant r}(x_i,\ldots,x_n)$ is finite this implies $\dim H(\Lambda(x_i,\ldots,x_n)) < \infty$ for all i. Apply part (iv) of Theorem 10.4 to each $\Lambda x_i \otimes \Lambda(x_{i+1},\ldots,x_n)$ to obtain the inequality $e_0(\Lambda X, d) \geqslant \dim X^{\text{odd}}$.

Finally, suppose d: $X \to \Lambda^2 X$ and put $e = e_0(\Lambda X, d)$. Bigrade ΛX by putting $(\Lambda X)^{p,q} = (\Lambda^p X)^{p+q}$; then d is homogeneous of bidegree $(1,0)$. Since $X^1 = 0$, $(\Lambda X)^{+,0} = 0$ and hence $H^{+,0} = 0$. By Poincaré duality $H^{*,q} = 0$, $q > n - e$, and $H^{*,n-e} = H^{e,n-e}$. Thus the formal power series

$$\sum_{p,q} (-1)^p \dim(\Lambda X)^{p,q} t^q = \sum_{p,q} (-1)^p \dim H^{p,q} t^q$$

is a polynomial of degree $n - e$.

On the other hand the left-hand side can be written in the form $\Pi_r(1 + (-1)^r t^{r-1})^{a_r}$, where $a_r = (-1)^{r-1}\dim X^r$. Since this is a polynomial of degree $n - e$ we find

$$n - e = \sum_{r \text{ odd}} (r-1)\dim X^r - \sum_{r \text{ even}} (r-1)\dim X^r.$$

But according to [**11**, Theorem 3]

$$n = \sum_{r\,\text{odd}} r \dim X^r - \sum_{r\,\text{even}} (r-1)\dim X^r$$

and we conclude $e = \dim X^{\text{odd}}$. Q.E.D.

PROOF OF THEOREM 10.4. *Case* I. *q is odd*. In this case $D\Phi = d\Phi + y \otimes \theta\Phi$, $\Phi \in \Lambda X$, where θ is a derivation of even degree $1-q$ in ΛX, commuting with d. A short exact sequence of differential spaces $(\Lambda X, d) \xrightarrow{j} (\Lambda y \otimes \Lambda X, D) \xrightarrow{\rho} (\Lambda X, d)$ is given by $j\Phi = (-1)^{\deg \Phi} y \otimes \Phi$. The connecting map in the resulting long exact cohomology sequence is (up to sign) the derivation θ^* induced by θ in $H(\Lambda X)$. Thus $\ker j^* = \operatorname{Im} \theta^*$, $\operatorname{Im} \rho^* = \ker \theta^*$ and j^* factors over the projection $H(\Lambda X) \to \operatorname{Coker} \theta^*$ to yield the short exact sequence

$$\operatorname{Coker} \theta^* \overset{j'}{\rightarrowtail} H(\Lambda y \otimes \Lambda X) \overset{\rho^*}{\twoheadrightarrow} \ker \theta^*. \tag{10.7}$$

We now consider each part of the theorem in turn.

(i) If $D\colon X \to \Lambda^{\geqslant 3}(y \oplus X)$ then $\theta\colon \Lambda^p X \to \Lambda^{\geqslant p+1} X$. Using Theorem 5.1 we conclude that $e_0(\Lambda X, d) \leqslant \operatorname{cat}_0(\Lambda X, d) \leqslant \operatorname{cat}_0(\Lambda y \otimes \Lambda X, D) \leqslant n$. Hence $(\theta^*)^n = 0$ and it follows that $H(\Lambda X)$ is spanned linearly by vectors of the form $(\theta^*)^p v$, $0 \leqslant p < n$, $v \in W$, where W is any subspace complementing $\operatorname{Im} \theta^*$. Since j^* maps W injectively it follows that $\dim H(\Lambda X) < \infty$.

(ii) Suppose $\dim H(\Lambda X) = \infty$ and choose $0 \neq \alpha_0 \in H^p(\Lambda X)$ with $p > n$. Because j' is injective $\alpha_0 = \theta^* \alpha_1$ with $\deg \alpha_1 = \deg \alpha_0 + q - 1 > \deg \alpha_0$. Continue to find an infinite sequence (α_i) with $\theta^* \alpha_i = \alpha_{i-1}$.

As in (i), Theorem 5.1 gives $e_0(\Lambda X, d) \leqslant n$. Let $n_i \leqslant n$ be the greatest integer such that α_i admits a representing cocycle in $\Lambda^{\geqslant n_i} X$. Thus $n_0 \geqslant n_1 \geqslant \cdots$. We establish a contradiction by showing that for any j there is some $i > j$ such that $n_j > n_i$. It is of course sufficient to do this when $j = 0$.

The hypotheses $X^1 = 0$, $\deg y > 1$, $\dim H(\Lambda y \otimes \Lambda X) < \infty$ together imply $\dim X^q < \infty$, all q. The existence of V thus implies a decomposition $X = Y \oplus Z$ into γ-stable graded spaces with $X^{\leqslant p} \subset Y \subset X^{\leqslant r}$.

Choose i so that $\deg \alpha_i > nr$. Let $\Phi \in \Lambda^{\geqslant n_i} X$ represent α_i, and write $\Phi = \Phi_1 + \Phi_2$, $\Phi_1 \in \Lambda^{n_i} X$ and $\Phi_2 \in \Lambda^{> n_i} X$. Since $n_i \leqslant n$, Φ_1 is in the ideal generated by Z. Hence so is $\gamma^i \Phi_1$. But $\deg \gamma^i \Phi_1 = \deg \alpha_0 = p$ and $Z = Z^{>p}$.

We conclude $\gamma^i \Phi_1 = 0$ and hence $\theta^i \Phi \in \Lambda^{> n_i} X$, whence $n_0 > n_i$.

(iii) From the fact that (10.3) is a Λ-extension we may deduce $\theta^N \Phi = 0$, $\Phi \in \Lambda X$ where N may depend on Φ. With $\dim H(\Lambda X) < \infty$ it follows that for some fixed N, $(\theta^*)^N = 0$. Because $\deg \theta^* \leqslant 0$ it follows that the graded space $\operatorname{Coker} \theta^*$ has the same formal dimension r as $H(\Lambda X)$. Apply (10.7) to find $n = q + r$.

(iv) Suppose $H(\Lambda y \otimes \Lambda X)$ is a P.d.a. with fundamental class ω. Define a nondegenerate bilinear form $\langle\, ,\rangle$ in $H(\Lambda y \otimes \Lambda X)$ by $\langle H^p, H^s \rangle = 0$ if $p + s \neq n$ and $\alpha \cdot \beta = \langle \alpha, \beta \rangle \omega$ if $\deg \alpha + \deg \beta = n$. Since $\operatorname{Im} j^* \cdot \operatorname{Im} j^* = 0$, $\langle\, ,\rangle$ factors to give a bilinear form $\operatorname{Im} j^* \times H(\Lambda y \otimes \Lambda X)/\operatorname{Im} j^* \to \mathbf{Q}$.

In view of (10.7) this can be identified with a bilinear form $((\, ,))$: $\operatorname{Coker} \theta^* \times \ker \theta^* \to \mathbf{Q}$ and because $\dim \operatorname{Coker} \theta^* = \dim \ker \theta^*$ and $\langle\, ,\rangle$ is nondegenerate, so is $((\, ,))$. We show next that $H^r(\Lambda X) = (\operatorname{Coker} \theta^*)^r$ is one-dimensional.

Because $j'\colon (\operatorname{Coker}\theta^*)^r \xrightarrow{\cong} H^n(\Lambda y \otimes \Lambda X)$, $\dim(\operatorname{Coker}\theta^*)^r = 1$. If $\deg\theta^* < 0$ then $(\operatorname{Coker}\theta^*)^r = H^r(\Lambda X)$. If $\deg\theta^* = 0$ then $H^r(\Lambda X)$ has a basis of the form ω', $\theta^*\omega', \ldots, (\theta^*)^p\omega'$, and $(\theta^*)^{p+1}\omega' = 0$. Since $(\theta^*)^p\omega' \in \ker\theta^*$ and $((\,,))$ is nondegenerate, it follows that

$$j^*\big((\theta^*)^p\omega'\big) = \big((1, (\theta^*)^p\omega')\big)\omega \neq 0.$$

Hence $p = 0$ and $\dim H^r(\Lambda X) = 1$ in this case as well.

Fix $\omega' \in H^r(\Lambda X)$ so that $j^*\omega' = \omega$. Define a bilinear form $(\,,)$ in $H(\Lambda X)$ by $(H^p, H^s) = 0$ if $p + s \neq r$ and $\alpha \cdot \beta = (\alpha, \beta)\omega'$ if $\deg\alpha + \deg\beta = r$. Recall that $\deg\theta^* \leqslant 0$ and that if $\deg\theta^* = 0$ then $\theta^*\omega' = 0$. It follows that $(\theta^*\alpha, \beta) = -(\alpha, \theta^*\beta)$, $\alpha, \beta \in H(\Lambda X)$. In particular $(\operatorname{Im}\theta^*, \ker\theta^*) = 0$ and so $(\,,)$ induces a bilinear form $\operatorname{Coker}\theta^* \times \ker\theta^* \to \mathbf{Q}$. This coincides up to sign with $((\,,))$ and hence is nondegenerate.

Let $N \subset H$ be the subspace of those α such that $(\alpha, H) = 0$. We have just observed that $N \subset \operatorname{Im}\theta^*$. Suppose inductively that $N \subset \operatorname{Im}(\theta^*)^p$, some $p \geqslant 1$, and let $\alpha \in N$. Then $\alpha = \theta^*\beta$ and $(\operatorname{Im}\theta^*, \beta) = -(H, \alpha) = \pm(\alpha, H) = 0$. Since $((\,,))$ is nondegenerate there is some $\gamma \in \ker\theta^*$ such that $\beta - \gamma \in N$. Hence $\alpha = \theta^*\beta = \theta^*(\beta - \gamma) \in \theta^*(N) \subset \operatorname{Im}(\theta^*)^{p+1}$. It follows by induction that $N \subset \operatorname{Im}(\theta^*)^p$ for all p and so $N = 0$. Thus $H(\Lambda X)$ is a P.d.a.

Suppose finally that $e_0(\Lambda X, d) = k$; by Lemma 10.1, ω' can be represented by $\Phi \in \Lambda^{\geqslant k}X$. Then $y \otimes \Phi$ represents ω and so $e_0(\Lambda y \otimes \Lambda X, D) \geqslant k + 1$.

Case II. *q is even*. Extend $(\Lambda y \otimes \Lambda X, D)$ to a minimal KS-complex $(\Lambda y \otimes \Lambda X \otimes \Lambda u, D)$ by putting $Du = y^2$. Apply Lemma 10.2 to find $\dim H(\Lambda y \otimes \Lambda X \otimes \Lambda u) < \infty$, formal dimension

$$H(\Lambda y \otimes \Lambda X \otimes \Lambda u) = n + 2q - 1.$$

If, moreover, $H(\Lambda y \otimes \Lambda X)$ is a P.d.a. so is $H(\Lambda y \otimes \Lambda X \otimes \Lambda u)$ and then

$$e_0(\Lambda y \otimes \Lambda X \otimes \Lambda u, D) \leqslant e_0(\Lambda y \otimes \Lambda X, D) + 1.$$

Define a quasi-isomorphism $\pi\colon (\Lambda y \otimes \Lambda X \otimes \Lambda u, D) \xrightarrow{\simeq} ((\Lambda y/y^2) \otimes \Lambda X, \overline{D})$ by $\pi y^2 = \pi u = 0$. Now exactly the same argument as in Case I establishes (i), (ii), (iii) and the assertion on Poincaré duality in (iv). It remains to prove the assertion on e_0, assuming $H(\Lambda y \otimes \Lambda X)$ a P.d.a.

Write $\Lambda y \otimes \Lambda X \otimes \Lambda u = \Lambda(y, u) \otimes \Lambda X$ and let $I \subset \Lambda(y, u)$ be the ideal generated by y^2 and u. Thus $\ker\pi = I \otimes \Lambda X$. Define $\sigma\colon I \to I$ by $\sigma(y^k u) = 0$ and $\sigma(y^k) = y^{k-2}u$; then $\sigma D + D\sigma = \mathrm{id}$. Put $\psi = D \circ (\sigma \otimes \mathrm{id}) + (\sigma \otimes \mathrm{id}) \circ D\colon I \otimes \Lambda X \to I \otimes \Lambda X$.

Claim. For each $\Phi \in I \otimes \Lambda X$ there is an N such that $(\psi - \mathrm{id})^N\Phi = 0$.

Indeed by hypothesis there is a well-ordered basis x_α of X such that $Dx_\alpha \in \Lambda(y, u) \otimes \Lambda X_{<\alpha}$. In the quotient space

$$\frac{I \otimes \Lambda X_{<\alpha} \otimes \Lambda^{\leqslant k}x_\alpha}{I \otimes \Lambda X_{<\alpha} \otimes \Lambda^{<k}x_\alpha} = I \otimes \Lambda X_{<\alpha} \otimes x_\alpha^k,$$

the map induced by ψ coincides with $\bar{\psi} \otimes \mathrm{id}$, where $\bar{\psi}$ is the restriction of ψ to $I \otimes \Lambda X_{<\alpha}$. The claim follows by a straightforward induction.

Write $Z = (y, u) \oplus X$ so that $I \otimes \Lambda X \subset \Lambda Z$. If $\Omega \in I \otimes \Lambda X \cap \Lambda^{\geqslant k} Z$ then the same is true for $\psi\Omega$, because D increases wedge degree by at least one, while σ decreases wedge degree by exactly one.

Now choose $\Phi \in \Lambda^{\geqslant p} X$ to represent the fundamental class. As in Case I, $y \otimes \Phi$ represents a nonzero class in $H((y \oplus 1) \otimes \Lambda X, \bar{D})$. In particular $D(y \otimes \Phi) \in I \otimes \Lambda X$. By our claim above $(\psi - \mathrm{id})^N D(y \otimes \Phi) = 0$ and hence

$$\begin{aligned} D(y \otimes \Phi) &= \sum_{k=1}^{N} a_k \psi^k D(y \otimes \Phi) \\ &= \sum_{k=1}^{N} a_k [D \circ (\sigma \otimes \mathrm{id})]^k D(y \otimes \Phi) \\ &= D \circ (\sigma \otimes \mathrm{id}) \circ \sum_{k=1}^{N} a_k \psi^{k-1} \circ D(y \otimes \Phi). \end{aligned}$$

Put $\Psi = (\sigma \otimes \mathrm{id}) \sum a_k \psi^{k-1} D(y \otimes \Phi)$. Then $\Psi \in \Lambda^{\geqslant p} Z$. Evidently $y \otimes \Phi - D\Psi$ is a cocycle in $\Lambda^{\geqslant p+1} Z$ such that $\pi(y \otimes \Phi - \Psi) = y \otimes \Phi$ represents a nonzero class. Hence $e_0(\Lambda(y, u) \otimes \Lambda X, D) \geqslant p + 1$.

It follows by Lemma 10.1 that

$$e_0(\Lambda X, d) \leqslant e_0(\Lambda(y, u) \otimes \Lambda X, D) - 1 \leqslant e_0(\Lambda y \otimes \Lambda X, D). \quad \text{Q.E.D.}$$

11. Some open problems and remarks. We collect here some unresolved questions. The first is classical.

1. *The product formula.* It is evident from Theorem 4.7 that $\mathrm{cat}_0(S \times T) \leqslant \mathrm{cat}_0(S) + \mathrm{cat}_0(T)$. Does equality hold? Does equality hold at least when one space is a sphere? an odd sphere?

2. *Nilpotence* [**17**]. Is there a c.g.d.a. or space with rational category m whose minimal model does not have the homotopy type of a c.g.d.a. of product length m? (Theorem 4.7 asserts only that the model is a retract of such a c.g.d.a.)

3. *Ganea's spaces.* Theorem 3.2 shows that Ganea's space E_m (for a simply connected S) has the rational homotopy type of a space W wedged with spheres. Give a geometric construction of W and determine the spheres.

4. Is there an analogue of Theorem I for category?

5. *Residual category.* Denote by S_n the n-connected Postnikov fibre of a simply connected space S. By Theorem 5.1, $\mathrm{cat}_0(S) \geqslant \mathrm{cat}_0(S_2) \geqslant \mathrm{cat}_0(S_3) \geqslant \cdots$. We call $\lim_{n \to \infty} \mathrm{cat}_0(S_n)$ the *residual rational category of* S, and write it $\mathrm{resid\,cat}_0(S)$.

Conjecture. If S is 1-connected and $\dim H^*(S; \mathbf{Q}) < \infty$, then $\mathrm{resid\,cat}_0(S) \leqslant 1$.

6. *Gottlieb groups.* If S is simply connected of formal dimension $\leqslant n$, Theorem III implies that $G_*(S) \otimes \mathbf{Q}$ is finite-dimensional. Is there a function $N(n)$ such that $G_p(S) \otimes \mathbf{Q} = 0$, $p > N(n)$? (Clearly $N(n) = 2n - 1$ would be best possible. Does it in fact work?)

Is there a relation between $e_0(S)$ and $\dim G_*(S) \otimes \mathbf{Q}$?

7. *The homotopy Lie algebra.* Can the rational homotopy Lie algebra of a finite simply connected CW-complex be nilpotent? abelian? (i.e., can the hypothesis of Poincaré duality in Theorem VI be weakened to $\dim H^* < \infty$?). What are the constraints on the Lie structure?

8. *Growth of* $\pi_*(S) \otimes \mathbf{Q}$. Suppose S simply connected, $\dim H^*(S) < \infty$ and $H^p(S) = 0$, $p > n$. If S is rationally hyperbolic then for $k \geqslant K$ and some $C > 1$, $\Sigma_{p \leqslant k} \dim H^p(\Omega S) \geqslant C^k$. It is easy to see that $C \leqslant \lim_{j \to \infty} \sup_{k \geqslant j} (\dim H^k(\Omega S))^{1/k}$. If $f(z) = \Sigma_{p=1}^{n-1} \dim H^{p+1}(S) z^p$ and z_0 is the complex root of $f(z) - 1$ of smallest modulus then it follows from the proof of Theorem 8.3 that $C \leqslant 1/|z_0|$.

Since the coefficients of f are nonnegative, $|f(z)| \leqslant f(|z|)$. It follows that $|f(z)| < 1$ if $|z| < x_0$, where x_0 is the unique positive real root of $f(x) - 1 = 0$. Hence $|z_0| = x_0 \leqslant (\dim H^*(S) - 1)^{1-n}$. Thus

$$C \leqslant (\dim H^*(S) - 1)^{n-1}$$

and equality is achieved for a wedge of n-spheres.

Is there a lower bound for C of the form $1 < M \leqslant C$ where M depends only on $H^*(S)$?

9. *The invariant* e_0. Are there spaces for which $e_0(S) < \infty$ but $\operatorname{cat}_0(S) = \infty$? Can $\operatorname{cat}_0(S) > e_0(S)$ for rationally elliptic spaces?

10. *Category for other fields.* Let $k \supset \mathbf{Q}$ be a field and define $\operatorname{cat}_k(S)$ exactly as $\operatorname{cat}_0(S)$ was defined, but using k as ground field. Then $\operatorname{cat}_k(S) \leqslant \operatorname{cat}_0(S)$. Does equality hold?

11. *Cocategory.* In **[7]** Ganea discusses the dual notion of cocategory, and the rational analogue is considered by Toomer in **[27]**. Which results of this paper dualize?

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