TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 273, Number 1, September 1982

SIMPLE KNOTS IN COMPACT, ORIENTABLE 3-MANIFOLDS

BY

ROBERT MYERS[1]

ABSTRACT. A simple closed curve J in the interior of a compact, orientable 3-manifold M is called a simple knot if the closure of the complement of a regular neighborhood of J in M is irreducible and boundary-irreducible and contains no non-boundary-parallel, properly embedded, incompressible annuli or tori. In this paper it is shown that every compact, orientable 3-manifold M such that ∂M contains no 2-spheres contains a simple knot (and thus, from work of Thurston, a knot whose complement is hyperbolic). This result is used to prove that such a 3-manifold is completely determined by its set $\mathcal{K}(M)$ of knot groups, i.e, the set of groups $\pi_1(M - J)$ as J ranges over all the simple closed curves in M. In addition, it is proven that a closed 3-manifold M is homeomorphic to S^3 if and only if every simple closed curve in M shrinks to a point inside a connected sum of graph manifolds and 3-cells.

1. Introduction. The topology of a 3-manifold is closely related to the type of "knot theory" it supports. This was demonstrated by Bing [1], who proved that a closed 3-manifold M is homeomorphic to S^3 if every knot in M can be shrunk to a point inside a 3-cell. McMillan [10] then proved that M is homeomorphic to S^3 if every knot in M can be shrunk to a point inside a solid torus. In another direction Jaco and Myers [7] and Row [14], inspired by earlier work of Fox [2], have shown that closed, orientable 3-manifolds are completely determined by their knot groups: If $\mathcal{K}(M)$ denotes the set of isomorphism classes of the groups $\pi_1(M - J)$ as J ranges over all the knots of M, then two closed, orientable 3-manifolds M and N are homeomorphic if and only if $\mathcal{K}(M) = \mathcal{K}(N)$.

Each of these results depends on the existence of certain "nice knots" in the 3-manifold whose exteriors share certain properties with those of nontrivial knots in S^3, such as irreducibility and boundary-irreducibility. It is to be expected that the existence of knots with nicer properties would lead to stronger theorems about their ambient 3-manifolds.

One very nice class of knots in S^3 is the class of simple knots. A simple knot in S^3 is classically defined [15] as a knot which has no nontrivial companions. This property is equivalent to the assertion that either the knot is a torus knot or every incompressible annulus and torus in its exterior is boundary-parallel. This latter

Received by the editors September 10, 1980 and, in revised form, May 19, 1981.

1980 *Mathematics Subject Classification*. Primary 57N10; Secondary 57M40, 57M25.

Key words and phrases. 3-manifold, knot, simple knot, simple 3-manifold, semisimple 3-manifold, hyperbolic 3-manifold, knot group, Poincaré Conjecture.

[1] This research was partially supported by NSF Grant MCS-8002524.

0002-9947/81/0000-0792/$05.25

property of the exterior has in recent years been taken over as the definition of a simple 3-manifold [**8**, **9**]. It is in this sense that we shall use the term "simple". Thus, a knot in a 3-manifold is simple if its exterior is irreducible and boundary-irreducible and contains no non-boundary-parallel incompressible annuli or tori. (Note that under this definition torus knots are not simple because their exteriors contain non-boundary-parallel incompressible annuli, although they do have the virtue of containing no non-boundary-parallel incompressible tori. Perhaps such knots and such 3-manifolds should be called "pseudo-simple".) One particularly nice feature of simple knots is that they are hyperbolic, i.e. their exteriors admit complete hyperbolic structures. This follows from Thurston's theorem that simple Haken manifolds admit hyperbolic structures.

In this paper it is proven that every compact, orientable 3-manifold whose boundary contains no 2-spheres contains a simple knot (Theorem 6.1), and thus a hyperbolic knot. Two applications of this theorem are given. First, the theorem of Jaco and Myers [**7**] and Row [**14**] is extended to 3-manifolds with boundary: Two compact, orientable 3-manifolds M and N, whose boundaries contain no 2-spheres, are homeomorphic if and only if $\mathcal{K}(M) = \mathcal{K}(N)$ (Theorem 8.1). Second, the characterization of S^3 due to McMillan [**10**] is generalized: A closed 3-manifold M is S^3 if every knot in M can be shrunk to a point inside a connected sum of graph manifolds and 3-cells (Corollary 9.2).

A larger class of "nice knots" in S^3 is formed by the prime, nontorus, noncabled knots. The exteriors of these knots are "semisimple" in the sense that they contain no non-boundary-parallel incompressible annuli. Johannson [**9**] has shown that semisimple Haken manifolds are completely determined by their fundamental groups. This one of the main tools used to prove Theorem 8.1.

The general strategy in proving Theorem 6.1 is similar to that of Bing [**1**], McMillan [**10**], and Row [**14**]. It consists in approximating the dual 1-skeleton of a triangulation of the manifold by a simple closed curve, replacing vertices by suitably chosen "tangles".

The main novelty consists in expressing the exterior of the resulting knot as the union of two 3-manifolds along an incompressible 2-manifold in their boundaries in such a way that it is seen to be simple. Lemmas 3.1–3.3 give fairly general conditions under which the union of two compact, orientable 3-manifolds along an incompressible 2-manifold in their boundaries is semisimple or simple. These results may therefore be of independent interest.

2. Preliminaries. We shall work throughout in the PL category.

A *knot* J in a 3-manifold M is a simple closed curve in the interior of M. The *exterior* of J is the closure of the complement of a regular neighborhood of J in M.

We refer to [**3**, **4**, **8**, and **19**] for the definitions of incompressible and boundary-incompressible surfaces and of irreducible, boundary-irreducible, and sufficiently large 3-manifolds. The meaning of the terms "compressing disk" and "boundary-compressing disk" will be apparent from these definitions. These references are also cited for the notions of parallel surfaces, boundary-parallel surfaces, and parallelisms in a 3-manifold. The expressions "surface in a 3-manifold" and "surface in the boundary of a 3-manifold" are used as in [**19**].

A compact, orientable, irreducible, boundary-irreducible, sufficiently large 3-manifold is called a *Haken manifold.*

An irreducible, boundary-irreducible 3-manifold is called *semisimple* if every incompressible annulus in M is boundary-parallel. A semisimple 3-manifold M is called *simple* if every incompressible torus in M is boundary-parallel.

A compact 3-manifold M is *hyperbolic* if the complement of the torus components of ∂M has a complete Riemannian metric with finite volume and constant negative sectional curvature with respect to which the nontorus components of ∂M are totally geodesic.

A knot in a 3-manifold is called semisimple, simple, or hyperbolic if its exterior is, respectively, semisimple, simple, or hyperbolic.

A *3-manifold pair* (M, F) consists of a 3-manifold M and a 2-manifold F in ∂M. (M, F) is an *irreducible* 3-manifold pair if M is irreducible and F is incompressible.

Let (M, F) be a 3-manifold pair. A surface G in M with ∂G in F is *F-compressible* if there is a boundary-compressing disk D for G such that ∂D lies in $F \cup G$. G is *F-parallel* if it is parallel to a surface in F.

A properly embedded, non-boundary-parallel arc in an annulus is called a *spanning arc.*

Let (M, F) be a 3-manifold pair. Let G and H be surfaces in M with ∂G and ∂H in F. H is *minimal with respect to* G if H and G are in general position and among all surfaces in M which are isotopic to H, have their boundaries in F, and meet G in general position, $H \cap G$ has a minimal number of components.

The following two lemmas are straightforward consequences of the definitions. Their proofs are left to the reader.

2.1. LEMMA. *Let (M, F) be an irreducible 3-manifold pair. Then every F-compressible annulus in M is F-parallel.*

2.2. LEMMA. *Let (M, F) be an irreducible 3-manifold pair. Let G be an incompressible surface in M with ∂G in F. Suppose G is not F-compressible.*

(1) *If A is an incompressible annulus in M with ∂A in F such that A is minimal with respect to G, then $A \cap G$ consists at most of spanning arcs or of noncontractible simple closed curves in A.*

(2) *If T is an incompressible torus in M which is minimal with respect to G, then $T \cap G$ consists at most of noncontractible simple closed curves in T.*

3. Gluing lemmas. In this section sufficient conditions are given for the union of two compact, orientable 3-manifolds along a compact, orientable, incompressible 2-manifold to be semisimple or simple. Let (M, F) be a compact orientable 3-manifold pair.

(M, F) has *Property* A if

(1) (M, F) and $(M, \overline{\partial M - F})$ are irreducible 3-manifold pairs,

(2) no component of F is a disk or 2-sphere, and

(3) every disk D in M with $D \cap F$ a single arc is boundary-parallel.

(M, F) has *Property* B if

(1) (M, F) has Property A,

(2) no component of F is an annulus, and

(3) every incompressible annulus A in M with $\partial A \cap \partial F = \varnothing$ is boundary-parallel.

(M, F) has *Property* C if

(1) (M, F) has Property B, and

(2) every disk D in M with $D \cap F$ a pair of disjoint arcs is boundary-parallel.

(M, F) has *Property* B′ (respectively *Property* C′) if

(1) (M, F) has Property B (respectively Property C),

(2) no component of F is a torus, and

(3) every incompressible torus in M is boundary-parallel.

Now suppose $M = M_0 \cup M_1$, where M_0 and M_1 are compact, orientable 3-manifolds and $F = M_0 \cap M_1 = \partial M_0 \cap \partial M_1$ is a compact 2-manifold.

3.1. LEMMA. *If (M_0, F) and (M_1, F) have Property* A, *then M is irreducible and boundary-irreducible and F is incompressible and boundary-incompressible.*

PROOF. The incompressibility and boundary-incompressibility of F are obvious. See [**18**] for a proof of the irreducibility of M.

Suppose D is a compressing disk for ∂M which is minimal with respect to F. If $D \cap F = \varnothing$, the result follows from the incompressibility of $\overline{\partial M_i - F}$ and the fact that no component of F is a disk. If $D \cap F \neq \varnothing$, then minimality and irreducibility imply that no component of $D \cap F$ is a simple closed curve. Thus, some component α of $D \cap F$ is an arc which cobounds a disk E in D with an arc β in ∂D such that $E \cap F = \alpha$. E is parallel in some M_i to a disk E' in ∂M_i. The component F' of $E' \cap F$ containing α must be a disk. Isotop D in M so as to move E across the 3-cell in M_i bounded by $E \cup E'$ to F' and then into $M - M_i$. This removes at least one component from $D \cap F$, thereby contradicting minimality.

3.2. LEMMA. *If (M_0, F) has Property* B *and (M_1, F) has Property* C, *then M is semisimple.*

PROOF. Suppose A is an incompressible annulus in M which is minimal with respect to F.

If $A \cap F = \varnothing$, then A is boundary-parallel in some M_i. Since no component of F is a disk or annulus, A is boundary-parallel in M.

If $A \cap F$ consists of noncontractible simple closed curves, then some component α of $A \cap F$ cobounds a subannulus A' of A with a component β of ∂A such that A' lies in some M_i. A' is parallel in M_i to an annulus A'' in ∂M_i. Since no component of F is a disk or annulus, $F' = F \cap A''$ is an annulus. Isotop A in M so that A' is moved across the parallelism between A' and A'' to F' and then into $M - M_i$. This removes at least one component of $A \cap F$, contradicting minimality.

If $A \cap F$ consists of spanning arcs, then a component of $A \cap M_1$ is a disk E such that $E \cap F$ consists of two disjoint arcs α and β. Let γ and δ be the components of $E \cap (\overline{\partial M_1 - F})$. E is parallel in M_1 to a disk E' in ∂M_1. There are two possibilities for $E' \cap F$.

Case 1. $E' \cap F$ consists of two disks, each of which meets F in one component of $E \cap F$. In this case A is boundary-compressible in M and hence boundary-parallel in M.

Case 2. $E' \cap F$ consists of a single disk F'. In this case isotop A in M so that E is moved across the parallelism between E and E' to F' and then into M_0. This removes at least two components from $A \cap F$, contradicting minimality.

3.3. LEMMA. *If (M_0, F) has Property* B′ *and (M_1, F) has Property* C′, *then M is simple.*

PROOF. Suppose T is an incompressible torus in M which is minimal with respect to F.

If $T \cap F = \varnothing$, then T is boundary-parallel in some M_i. Since no component of F is a disk, annulus, or torus, T is boundary-parallel in M.

If $T \cap F \neq \varnothing$, then a component of $T \cap M_0$ is an annulus A_0 which is parallel in M_0 to an annulus A_0' in ∂M_0. There are two possibilities for $A_0' \cap F$.

Case 1. $A_0' \cap F = A_0'$. Isotop T in M so that A_0 is moved across the parallelism between A_0 and A_0' to A_0' and then into M_1. This removes at least two components of $T \cap F$, contradicting minimality.

Case 2. $A_0' \cap F$ consists of two annuli F_1 and F_2. Let $G_0' = \overline{(A_0' - (F_1 \cup F_2))}$. Then the annulus $G_0 = A_0 \cup F_1 \cup F_2$ is parallel in M_0 to G_0'. Let $A = \overline{T - A_0}$ and $G = A \cup F_1 \cup F_2$. G is incompressible in M and hence is parallel in M to an annulus G' in ∂M. If $G_0' = G'$, then T is compressible in M, a contradiction. If $G_0' \neq G'$, then T is parallel in M to the torus $G_0' \cup G'$ in ∂M.

4. Atoroidal tangles. A *tangle* is a pair (λ', λ'') of disjoint, properly embedded arcs in a 3-cell B. A *tangle space* is the closure of the complement of a regular neighborhood of a tangle in B. A tangle is *atoroidal* if its tangle space is simple.

The tangle in Figure 1 is called the true lover's tangle.

4.1. PROPOSITION. *The true lover's tangle is atoroidal.*

This proposition is presented as Exercise IX.23 on p. 194 of Jaco's book [**6**]. The following proof is presented for the convenience of both the reader and the author.

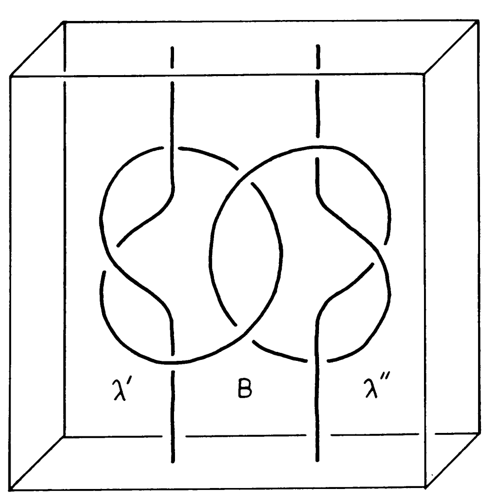

FIGURE 1

The lemmas constituting the proof are used in a sequel [**13**] to this paper dealing with homology cobordisms.

PROOF. We express the true lover's tangle space as the union of three cubes-with-handles, as illustrated in Figure 2. Let $F = F_1 \cup F_2$. The proof consists in showing that $(P_1 \cup P_2, F)$ has Property B′ and (P, F) has Property C′. To prove the former it is sufficient to prove that (P_1, F_1) has Property B′.

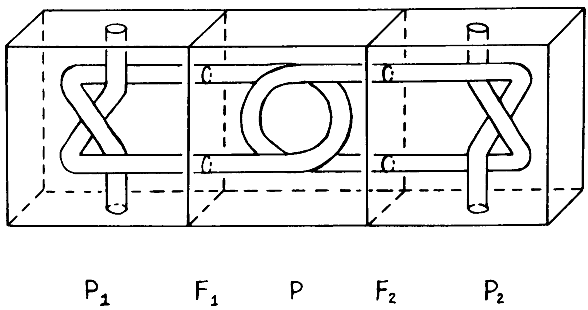

FIGURE 2

∂P_1 is the union of the planar surfaces F_1 and G_1 and the annuli U_1' and U_1'' shown in Figure 3. $\pi_1(P_1)$ is free on x and y, $\pi_1(F_1)$ is free on y and z, and $\pi_1(G_1)$ is free on x and w. The following relations hold: $z = x^{-1}y^{-1}x^{-1}yx$, $w = y^{-1}x^{-1}y^{-1}xy$, $r = x^{-1}yx$, $s = y^{-1}xy$, $t = xyx$. Note that any reduced word $W'(y, z)$ is a reduced word $W(x, y)$, and any reduced word $V'(x, w)$ is a reduced word $V(x, y)$. This shows that F_1 and G_1 are incompressible in P_1. P_1 is a cube-with-handles and is thus irreducible.

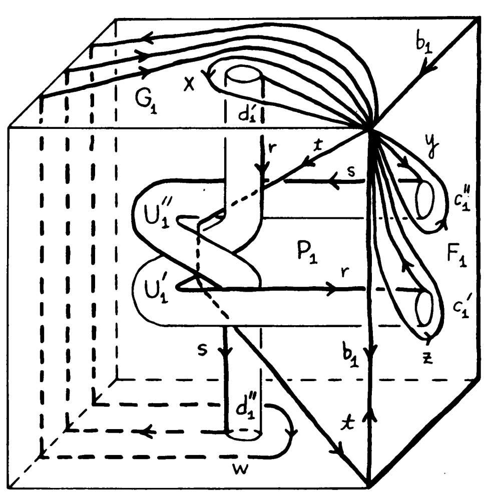

FIGURE 3

4.2. LEMMA. $\pi_1(F_1) \cap \pi_1(G_1) = gp(b_1)$.

PROOF. Since $b_1 = zy = (wx)^{-1}$, $gp(b_1) \subset \pi_1(F_1) \cap \pi_1(G_1)$. Suppose $g \in \pi_1(F_1) \cap \pi_1(G_1)$. Clearly g is not a power of y, z, w, or x unless $g = 1$. If $g = y^{\varepsilon_1}z^{\delta_1}\ldots = y^{\varepsilon_1}x^{-1}y^{-1}x^{-\delta_1}yx\ldots$, then $g = w^{\alpha_1}x^{\beta_1}\ldots = y^{-1}x^{-1}y^{-\alpha_1}xyx^{\beta_1}\ldots$. Hence $g = y^{-1}x^{-1}y^{-1}xyxh = y^{-1}z^{-1}h = b_1^{-1}h$, where $h \in \pi_1(F_1) \cap \pi_1(G_1)$. If $g = z^{\varepsilon_1}y^{\delta_1}\ldots$, a similar argument shows that $g = b_1 h$, where $h \in \pi_1(F_1) \cap \pi_1(G_1)$. The result follows by induction on the letter length of g in x and y.

4.3. LEMMA. *$\pi_1(F_1)s \cap \pi_1(G_1) = \varnothing$, and $\pi_1(F_1)r^{-1} \cap \pi_1(G_1) = \varnothing$.*

PROOF. Suppose $g \in \pi_1(F_1)s \cap \pi_1(G_1)$. Then $g = W'(y, z)(y^{-1}xy) = V'(w, x)$, where W' and V' are reduced words in y, z and w, x, respectively.

Case 1. $W(x, y)(y^{-1}xy)$ is reduced in x and y. Then

$$g = (\dots z^m)(y^{-1}xy) = (\dots x^{-1}y^{-1}x^{-m}yxy^{-1}xy).$$

So $g = (\dots w^n) = (\dots y^{-1}x^{-1}y^{-1}xy)$, which is impossible.

Case 2. $W(x, y)(y^{-1}xy)$ is not reduced in x and y. If $g = y^m s = y^{m-1}xy$, then $g = (\dots w^n) = (\dots y^{-1}x^{-1}y^{-n}xy)$, which is impossible. If

$$g = (\dots z^p y^m)s = (\dots x^{-1}y^{-1}x^{-p}yxy^{m-1}xy),$$

then $g = (\dots x^{-1}y^{-1}x^{-p}yx^2y)$ or $(\dots x^{-1}y^{-1}x^{-p}yxy^{m-1}xy)$ is reduced, and $g = (\dots w^n) = (\dots y^{-1}x^{-1}y^{-n}xy)$, which is impossible.

The proof that $\pi_1(F_1)r^{-1} \cap \pi_1(G_1) = \varnothing$ is similar.

4.4. LEMMA. *$\pi_1(F_1) \cap s\pi_1(G_1)s^{-1} = gp(y)$, and $r\pi_1(F_1)r^{-1} \cap \pi_1(G_1) = gp(x)$.*

PROOF. Suppose $g \in s^{-1}\pi_1(F_1)s \cap \pi_1(G_1)$. Then

$$g = s^{-1}W'(y, z)s = V'(w, x),$$

so $g = (y^{-1}x^{-1}y)W(x, y)(y^{-1}xy) = V(x, y)$, where W and V are reduced.

Case 1. $(y^{-1}x^{-1}y)W(x, y)(y^{-1}xy)$ is reduced. Then

$$g = s^{-1}(z^m\dots)s = (y^{-1}x^{-1}y)(x^{-1}y^{-1}x^{-m}yx\dots)(y^{-1}xy).$$

So $g = (w^n\dots) = (y^{-1}x^{-1}y^{-n}xy\dots)$, which is impossible.

Case 2. $(y^{-1}x^{-1}y)W(x, y)(y^{-1}xy)$ is not reduced.

If $W'(y, z) = (y^m\dots z^n)$, then

$$g = (y^{-1}x^{-1}y)(y^m\dots x^{-1}y^{-1}x^{-n}yx)(y^{-1}xy),$$

which has reduction $y^{-1}x^{-1}y^{m+1}\dots x^{-1}y^{-1}x^{-n}yxy^{-1}xy$ or $y^{-1}x^{-2}\dots y^{-1}x^{-n}yxy^{-1}xy$. This implies $g = (\dots w^p) = (\dots y^{-1}x^{-1}y^{-p}xy)$, which is impossible.

If $W'(y, z) = (z^m\dots y^n)$, then

$$g = (y^{-1}x^{-1}y)(x^{-1}y^{-1}x^{-m}yx\dots y^n)(y^{-1}xy),$$

which has reduction $y^{-1}x^{-1}yx^{-1}y^{-1}x^{-m}yx\dots y^{n-1}xy$ or $y^{-1}x^{-1}yx^{-1}y^{-1}x^{-m}y\dots x^2y$. So $g = (w^p\dots) = (y^{-1}x^{-1}y^{-p}xy\dots)$, which is impossible.

If $W'(y, z) = (y^m z^n\dots y^p)$, then

$$g = (y^{-1}x^{-1}y)(y^m x^{-1}y^{-1}x^{-n}yx\dots y^p)(y^{-1}xy),$$

which has reduction $y^{-1}x^{-1}y^{m+1}x^{-1}y^{-1}x^{-n}yx\dots y^{p-1}xy$ or $y^{-1}x^{-2}y^{-1}x^{-n}yx\dots y^{p-1}xy$ or $y^{-1}x^{-1}y^{m+1}x^{-1}y^{-1}x^{-n}y\dots x^2y$ or $y^{-1}x^{-2}y^{-1}x^{-n}y\dots x^2y$. So $g = (w^q\dots) = (y^{-1}x^{-1}y^{-q}xy\dots)$, which is impossible.

The only remaining possibility is $W'(y, z) = y^m$, as desired. The reverse inclusion is easily checked.

The proof that $r\pi_1(F_1)r^{-1} \cap \pi_1(G_1) = gp(x)$ is similar.

4.5. LEMMA. *(P_1, F_1) has Property* A.

PROOF. Suppose D is a disk in P_1 which meets F_1 in an arc α and $\overline{\partial P_1 - F_1}$ in an arc β. Let $J = \partial D$.

Case 1. α joins b_1 to itself. Isotop D and orient J so that the initial point of α is the basepoint. Let γ be an arc in b_1 running from the terminal point of α to the basepoint. Then $[J] = [\alpha\gamma][\gamma^{-1}\beta] = fg$, where $f \in \pi(F_1)$ and $g \in \pi_1(G_1)$. Since $[J] = 1, f = g^{-1}$, hence by Lemma 4.2 $f = (b_1)^k$. Since f is represented by a simple closed curve, $k = 0$ or ± 1. It follows that α is parallel in F_1 to γ or $\overline{b_1 - \gamma}$. Thus, D can be isotoped so that J lies in G_1. By the incompressibility of G_1, D is boundary-parallel.

Case 2. α joins b_1 to c_1''. Isotop D and orient J so that α runs from the basepoint to $s \cap c_1''$ and $\beta \cap U_1'' = s \cap U_1''$. Let $\gamma = s \cap F_1$, $\delta = s \cap G_1$, and $\varepsilon = \beta \cap G_1$. Then $[J] = [\alpha\gamma^{-1}]s[\delta^{-1}\varepsilon] = fsg$, where $f \in \pi_1(F_1)$ and $g \in \pi_1(G_1)$. Since $[J] = 1$, $fs = g^{-1}$. However, this is impossible by the first part of Lemma 4.3.

Case 3. α joins b_1 to c_1'. Using the second part of Lemma 4.3, the proof is similar to that of the previous case.

Case 4. α joins c_1'' to itself. Isotop D and orient J so that α ends at $s \cap c_1''$ and $\beta \cap U_1''$ has two components, one of which is $s \cap U_1''$. Let $\gamma = s \cap F_1$, $\delta = s \cap G_1$, and $\varepsilon = \beta \cap G_1$. Let ξ be an arc in c_1'' joining the points of $\partial\alpha$, and let η be an arc in d_1'' joining the points of $\partial\varepsilon$. Then, referring J to the basepoint via γ and orienting the arcs properly, we have

$$[J] = \left[\gamma\xi\alpha\gamma^{-1}\right]s\left[\delta^{-1}\varepsilon\eta\delta\right]s^{-1} = fsgs^{-1},$$

where $f \in \pi_1(F_1)$ and $g \in \pi_1(G_1)$. Since $[J] = 1$, $f = sg^{-1}s^{-1}$. By the first part of Lemma 4.4, $f = y^m$. Since f is represented by a simple closed curve, $m = 0$ or ± 1. It follows that α is parallel in F_1 to ξ or $\overline{c_1'' - \xi}$. Thus, D can be isotoped so that J lies in G_1 and the result follows.

Case 5. α joins c_1' to itself. Using the second part of Lemma 4.4, the proof is similar to that of the previous case.

Case 6. α joins c_1' and c_1''. Isotop D so that J misses a collar C on b_1 in F_1. Regard P_1 as embedded in S^3 in the manner shown in Figure 3. Let E be a 3-cell in S^3 such that $E \cap P_1 = C$. Then $E \cup P_1$ is the exterior of a trefoil knot in S^3 and J is a simple closed curve in $\partial(E \cup P_1)$ which meets a meridian of the knot transversely in a single point. Thus, $[J] \neq 1$ in $E \cup P_1$ and, hence, $[J] \neq 1$ in P_1, a contradiction.

4.6. LEMMA. *(P_1, F_1) has Property* B$'$.

PROOF. Clearly, P_1 contains no incompressible tori.

Let E be the disk in Figure 4. Regard E as $[0,2] \times [0,1]$, where $\{0\} \times [0,1] = E \cap U_1'$, $\{2\} \times [0,1] = E \cap U_1''$, and $([0,1] \times \{0\}) \cup ([1,2] \times \{1\}) = E \cap F_1$. Suppose A is an incompressible annulus in P_1 with $\partial A \cap \partial F_1 = \varnothing$ which is minimal with respect to E. We may assume that $A \cap (U_1' \cup U_1'') = \varnothing$. Then $A \cap E$ consists of spanning arcs of A each of which is either parallel to an arc in $[0,2] \times \{0,1\}$ or joins the components of $[0,2] \times \{0,1\}$. These arcs will be called, respectively, end-parallel and spanning arcs of E.

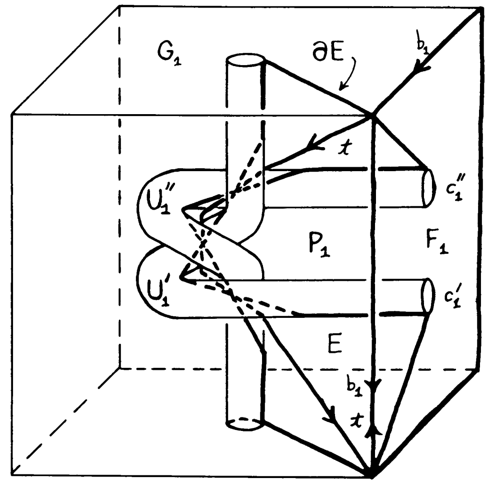

FIGURE 4

Case 1. ∂A lies in F_1.

Suppose $\partial A \cap ([0,1] \times \{0\}) = \varnothing$. Then every component of $A \cap E$ is parallel in E to an arc in $[1,2] \times \{1\}$. It follows that A is F_1-compressible and, hence F_1-parallel. A similar argument handles the case $\partial A \cap ([1,2] \times \{1\}) = \varnothing$.

Suppose A meets both components of $E \cap F_1$. For homological reasons, the components of ∂A are parallel in F_1 to b_1. Hence, $\partial A = \partial A'$, where A' is an annulus in F_1. Since A is minimal with respect to E, $A' \cap E$ consists of two arcs. Thus, $A \cap E$ consists of two arcs which are either both end-parallel or both spanning in E. If both are end-parallel, then A is F_1-compressible and, hence F_1-parallel. If both are spanning, then consider the torus $T = A \cup A'$. Referring loops to the basepoint, we have that a component of ∂A is homotopic to b_1. The existence of a spanning arc in $A \cap E$ implies that of a simple closed curve J in T which is homotopic to t. Hence t and b_1 commute. But this is easily seen to be impossible.

Case 2. ∂A lies in G_1.

By a proof similar to that of the previous case, A is G_1-parallel.

Case 3. One component α of ∂A lies in F_1; the other component β lies in G_1.

Suppose α is parallel to c_1' in F_1. Then β is parallel to d_1' in G_1. Let C and D be the respective parallelisms. By minimality, $A \cap E$ is a single spanning arc γ. Let E' be the disk in E joining γ with U_1'. Then E' is a $(U_1' \cup C \cup D)$-compressing disk for A. It follows that A is parallel to U_1'.

If α is parallel to c_1'' in F_1, then a similar argument shows that A is parallel to U_1''.

Suppose α is parallel to b_1 in F_1. Then β is parallel to b_1 in G_1. Let A' be the annulus in ∂P_1 bounded by ∂A. If $A \cap E$ contains a spanning arc of E, then as in Case 1, t and b_1 commute, which is impossible. Hence, $A \cap E$ consists of end-parallel arcs, and thus A is A'-compressible and so A'-parallel. This completes the proof.

Now consider (P, F), shown in Figure 5. Note that $P = H \times [1,2]$, where H is a punctured torus and $H \times \{i\} = (F_i \cup U_i)$, $i = 1, 2$. This implies that $F_i \cup U_i$ is incompressible in P and hence that F_i, U_i, and G are incompressible in P. Clearly P is irreducible and has no incompressible tori.

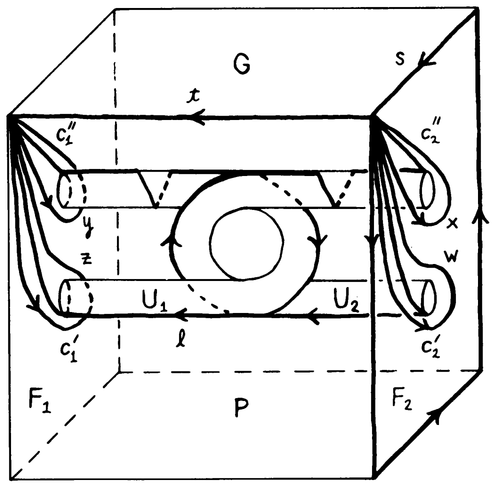

FIGURE 5

4.7. LEMMA. (P, F) *has Property* A.

PROOF. Suppose D is a disk in P which meets F in the arc α and $\overline{\partial P - F}$ in the arc β. We may assume α lies in F_1. By the incompressibility of $F_1 \cup U_1$, β must be a boundary-parallel arc in U_1 or G. Thus D can be isotoped so that ∂D lies in F_1. The result now follows from the incompressibility of F_1.

4.8. LEMMA. (P, F) *has Property* B$'$.

PROOF. Suppose A is an incompressible annulus in P such that $\partial A \cap \partial F = \varnothing$. Isotop A so that ∂A lies in F. For homological reasons both components of ∂A lie in the same component of F, say F_1. Thus, ∂A lies in $H \times \{1\}$, and hence, by Corollary 3.2 of **[15]**, A is parallel to an annulus in $H \times \{1\}$.

4.9. LEMMA. (P, F) *has Property* C$'$.

PROOF. Suppose D is a disk in P which meets F in the disjoint arcs α_1 and α_2.

If $\partial D \cap F_2 = \varnothing$ or $\partial D \cap F_1 = \varnothing$, then D can be isotoped so that ∂D lies in $F_i \cup U_i$ for some i. The incompressibility of $F_i \cup U_i$ implies that D is boundary-parallel.

Suppose α_i lies in F_i, $i = 1, 2$. We may assume neither α_i is boundary-parallel in F_i. But this implies that ∂D is homologous in P to $c_1' \pm c_2'$, which is absurd.

5. Special handle decompositions. For the definition of a *handle decomposition* of a compact 3-manifold we refer to **[19]**. A handle of index k is denoted by h_i^k. Note that our 0-handles, 1-handles, and 2-handles are, respectively, the balls, beams, and plates of **[19]**. Let H' be the union of the 0-handles and 1-handles, and let H'' be the union of the 2-handles and 3-handles.

A handle decomposition $\{h_i^k\}$ of a compact, orientable 3-manifold N is *special* if:

(1) The intersection of any handle with any other handle or with ∂N is either empty or connected.

(2) Each 0-handle meets exactly four 1-handles and six 2-handles.

(3) Each 1-handle meets exactly two 0-handles and three 2-handles.

(4) Each pair of 2-handles either

(a) meets no common 0-handle or 1-handle, or

(b) meets exactly one common 0-handle and no common 1-handle, or

(c) meets exactly one common 1-handle and two common 0-handles.

(5) The complement of any 0-handle in H' is connected.

(6) The union of any 0-handle with H'' is a cube-with-handles which meets ∂N in a disjoint collection of disks.

5.1. LEMMA. *Every compact orientable* 3-*manifold N has a special handle decomposition.*

PROOF. Let (K_0, L_0) be a triangulation of $(N, \partial N)$. Let (K, L) be the second barycentric subdivision of (K_0, L_0). Note that L is a full subcomplex of K and hence that each 3-simplex of K meets L in at most one 2-simplex. Let (K^*, L^*) be the dual cell complex of (K, L). A k-cell of K^* is the dual σ_k^* of either a $3 - k$ simplex σ^{3-k} in K or a $2 - k$ simplex σ^{2-k} in L, i.e. $\sigma_k^* = \bigcap_v \operatorname{st}(v)$, where v ranges over the vertices of σ^{3-k} (respectively σ^{2-k}) and $\operatorname{st}(v)$ is the star of v in the first barycentric subdivision of K (respectively L).

We now associate a handle decomposition to (K^*, L^*) in the usual way: the 0-handles are regular neighborhoods of the 0-cells, the 1-handles are regular neighborhoods of the intersections of the 1-cells with the closure of the complement of the 0-handles, and so on.

Properties (1)–(4) are easily checked.

Since L is a full subcomplex of K, the complement of any 3-simplex of K is connected. This implies (5).

Let G be the graph $\overline{(K^{(1)} - L^{(1)})}$, where $K^{(1)}$ and $L^{(1)}$ are the 1-skeleta of K and L, respectively. Then H'' is a regular neighborhood of G in N and hence is a cube-with-handles. The core of a 0-handle h_i^0 is the barycenter of some k-simplex σ^k, where either $k = 3$ and $\sigma^k \in K$ or $k = 2$ and $\sigma^k \in L$. Let G' be the graph obtained from G by replacing $G \cap \sigma^k$ with the star on the vertices of σ^k from the barycenter of σ^k. Then $H'' \cup h_i^0$ is a regular neighborhood of G' in N and hence is a cube-with-handles. Its intersection with ∂N clearly consists of disks. This establishes (6) and completes the proof.

Now let M be a compact, orientable 3-manifold and let C be a collar on ∂M. Let $N = \overline{M - C}$. Choose a special handle decomposition for N, with H' and H'' as above. Let $Z = H'' \cup C$. For each 0-handle h_i^0, let $R_i = h_i^0 \cap Z$. Let $R = \bigcup_i R_i$. For each 1-handle h_j^1, let $S_j = h_j^1 \cap Z$. Let $S = \bigcup_j S_j$.

5.2. LEMMA. *If* ∂M *contains no* 2-*spheres, then* (Z, R) *has Property* A.

PROOF. Since H'' is a cube-with-handles, $C = (\partial M) \times I$, no component of ∂M is a 2-sphere, and $H'' \cap C$ consists of disks, Z is irreducible and ∂M is incompressible in Z.

Suppose D is a compressing disk for R_i in Z. Then $\partial D = \partial D'$ for a disk D' in ∂h_i^0. $H'' \cup h_i^0$ is a cube-with-handles which meets C in disks. Therefore, $Z \cup h_i^0$ is irreducible. Thus, $D \cup D'$ bounds a 3-cell B in M. Since the complement of h_i^0 in H'

is connected, it lies either in B or $\overline{M-B}$. This implies that $\overline{(\partial h_i^0 - R_i)}$ lies either in D' or in $\overline{(\partial h_i^0 - D')}$. In either case ∂D bounds a disk in R_i. Thus R_i is incompressible in Z.

The incompressibility of S follows from that of R.

Let D be a disk in Z such that $D \cap R_i$ is a single arc α. Let $\beta = \overline{\partial D - \alpha}$. Then α meets exactly one 1-handle h_j^1, and β is boundary-parallel in S_j. Isotop D so that ∂D lies in R_i. By the incompressibility of R_i, D is boundary-parallel.

5.3. LEMMA. *If ∂M contains no 2-spheres, then (Z, R) has Property* B$'$.

PROOF. Let F be any incompressible surface in Z such that either $\partial F = \varnothing$ or ∂F lies in ∂M. We may assume F is minimal with respect to $H'' \cap C$. Then F lies in C and so, by Corollary 3.2 of **[19]**, is parallel to a surface in ∂M. Thus, every incompressible torus in Z and every incompressible annulus A in Z with $\partial A \cap H' = \varnothing$ is boundary-parallel.

Let A be an incompressible annulus in Z such that $\partial A \cap \partial R = \varnothing$ and $\partial A \cap \partial H' \neq \varnothing$. Then at least one component α of ∂A lies in R or S. We may assume α lies in some R_i. Let β be the other component of ∂A.

Let C_k^2 denote the core of the 2-handle h_k^2. We may assume that A is minimal with respect to $C^2 = \bigcup_k C_k^2$. It follows from the fact that R is incompressible and Z is irreducible that $A \cap C^2$ consists of spanning arcs of A.

Case 1. α is boundary-parallel in R_i.

Then α is parallel in R_i to $\partial(h_i^0 \cap h_j^1)$ for some j. Let h_k^2 and h_l^2 be two distinct 2-handles meeting h_j^1. Then $A \cap C_k^2 \neq \varnothing$. Let γ be a component of $A \cap C_k^2$. γ is a spanning arc of A which runs from R_i to some R_m or S_m. Thus β lies in R_m or S_m. We may assume β lies in R_m.

If $R_i = R_m$, then γ is parallel in C_k^2 to an arc in $R_i \cap C_k^2$. Thus A is R_i-compressible and hence R_i-parallel in Z.

If $R_i \neq R_m$, let δ be a component of $A \cap C_l^2$. δ is a spanning arc of A which runs from R_i to R_m. Thus, h_k^2 and h_l^2 meet the common 0-handle R_m. This implies that h_j^1 joins h_i^0 and h_m^0. Suppose β is not parallel in R_m to $\partial(h_m^0 \cap h_j^1)$. Then A meets some C_n^2 which does not meet h_j^1. Let ε be a component of $A \cap C_n^2$. ε is a spanning arc of A which joins R_m to R_p, where we may assume $R_m \neq R_p$. But since C_n^2 does not meet h_j^1, $R_p \neq R_i$, hence α does not lie in R_i, a contradiction. Therefore, β is parallel in R_m to $\partial(h_m^0 \cap h_j^1)$. Thus A is S_j'-compressible, where S_j' is the union of S_j with the boundary-parallelisms of α and β in R_i and R_m, respectively. Therefore A is boundary-parallel.

Case 2. α is not boundary-parallel in R_i.

Then α partitions ∂R_i into two groups of two components each. It follows that α must meet the cores of at least four distinct 2-handles which meet h_i^0. Let $\gamma_1, \gamma_2, \gamma_3$, and γ_4 be arcs in the intersection of A with the cores of these 2-handles. They are all spanning arcs of A. Each γ_q joins R_i with the same R_m or S_m. So β lies in R_m or S_m. We may assume β lies in R_m. If $R_i = R_m$, then A is R_i-compressible and hence R_i-parallel. If $R_i \neq R_m$, then each of the four distinct 2-handles meets both h_i^0 and h_m^0. Therefore, h_i^0 and h_m^0 are joined by a 1-handle h_j^1 and each of the four distinct 2-handles meets h_j^1. This contradicts the fact that each 1-handle meets exactly three distinct 2-handles.

6. Simple knots.

6.1. THEOREM. *Let M be a compact, orientable 3-manifold whose boundary contains no 2-spheres. Then M contains a simple knot.*

PROOF. Let $(\lambda_i', \lambda_i'')$ be a copy of the true lover's tangle in a 3-cell B_i. Let Q_i be the associated tangle space. Let C be a collar on ∂M. Choose a special handle decomposition of $N = \overline{M - C}$. Let Z and R be as in Lemma 5.2.

Identify each 0-handle h_i^0 with B_i in such a way that $\partial(\lambda_i' \cup \lambda_i'')$ is identified with the intersection of h_i^0 with the cores C_j^1 of the four 1-handles meeting h_i^0. Do this in such a manner that $J = \bigcup_i(\lambda_i' \cup \lambda_i'') \cup \bigcup_j C_j^1$ is a simple closed curve, where the unions are taken over all the 0-handles and 1-handles of the handle decomposition.

Let $Q = \bigcup_i Q_i$. Then the exterior X of J is $Q \cup Z$. Note that $Q \cap Z = R$. By Proposition 4.1 Q is irreducible, boundary-irreducible, and simple. Since R and $\overline{\partial Q - R}$ are clearly incompressible in Q, (Q, R) has Property C′. By Lemma 5.3 (Z, R) has Property B′. Thus by Lemma 3.3 X, and hence J, is simple. This completes the proof.

We now quote

6.2. THURSTON'S THEOREM. *Every simple Haken manifold is hyperbolic.*

in order to prove

6.3. COROLLARY. *Let M be a compact, orientable 3-manifold whose boundary contains no 2-spheres. Then M contains a hyperbolic knot.*

7. Some technical lemmas. Let (V, V') be the pair of solid tori in Figure 6. Let L be the core of V'. Let $T = \partial V$, $T' = \partial V'$, and $W = \overline{V - V'}$. Note that W is the exterior of the Whitehead link in S^3. Let τ be a simple twist of V, i.e. τ is a self-homeomorphism of V such that $\tau(s)$ is homologous to s and $\tau(t)$ is homologous to $s + t$ in T.

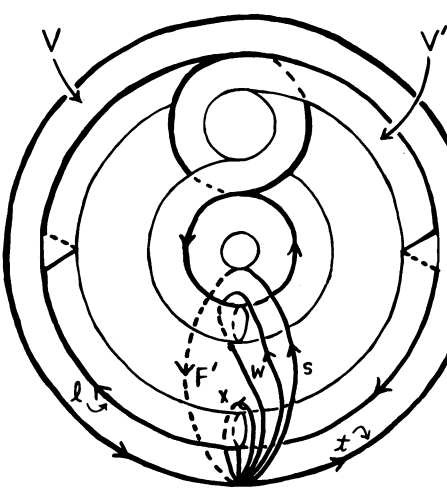

FIGURE 6

W can be obtained from the cube-with-handles P in Figure 5 by identifying F_1 and F_2 to obtain the surface F' in Figure 6. Since $\pi_1(P)$ is free on x and y (the basepoint is on F_2), $\pi_1(W)$ has the presentation

$$\langle x, y, t\colon txt^{-1} = y, twt^{-1} = z \rangle.$$

Noting that $w = xy^{-1}x^{-1}yx^{-1}$ and $z = xy^{-1}x^{-1}$, we use the first relation to eliminate the generator y and obtain the presentation

$$\langle x, t\colon ts = st \rangle,$$

where $s = wx = [x, t][x^{-1}, t]$. ($[a, b] = aba^{-1}b^{-1}$.)

Now let $W'(p, q)$ be the manifold obtained from W by adding a 2-handle to T along a simple closed curve homotopic to $t^p s^q$, where $(p, q) = 1$. Then $G_{p,q} = \pi_1(W'(p, q))$ has the presentation

$$\langle x, t\colon ts = st, t^p s^q = 1 \rangle.$$

Let l be the oriented longitude of V', referred to the basepoint as shown in Figure 6. Then $l = [x, t^{-1}][x, t]$.

7.1. LEMMA. *T' is compressible in $W'(p, q)$ if and only if $p = \pm 1$ and $q = 0$.*

PROOF. Suppose $p = \pm 1$ and $q = 0$. Then clearly there is a compressing disk D for T' in $W'(\pm 1, 0)$ with ∂D homotopic to l.

Suppose T' is compressible in $W'(p, q)$. Let D be a compressing disk. Then ∂D is homotopic to l because every other noncontractible simple closed curve on T' is homologically nontrivial in $W'(p, q)$. We first show that $p = \pm 1$ by proving that otherwise $l \neq 1$.

Assume $p \neq \pm 1$. We may assume $p \geqslant 0$.

Case 1. $p = 2m$, $m \geqslant 0$.

Define $\phi\colon G_{p,q} \to S_3$ by setting $\phi(x) = (1,2,3)$ and $\phi(t) = (1,2)$. Then $\phi([x, t]) = \phi([x, t^{-1}]) = (1,3,2)$ and $\phi([x^{-1}, t]) = (1,2,3)$. This implies $\phi(s) = \phi(t^2) = 1$, so ϕ is well defined. Since $\phi(l) = (1,2,3)$, $l \neq 1$.

Case 2. $p = 2m - 1$, $m \geqslant 2$.

Define $\phi\colon G_{p,q} \to S_{2m}$ by setting $\phi(x) = (1, m+1)(2, m+2)\ldots(m, 2m)$ and $\phi(t) = (1, 2, \ldots, 2m-1)$. Then $\phi([x, t]) = \phi([x^{-1}, t]) = (m-1, m)(2m-1, 2m)$ and $\phi([x, t^{-1}]) = (1, 2m)(m, m+1)$. This implies $\phi(s) = 1$ and $\phi(t^p) = 1$, so ϕ is well defined. Since $\phi(l) = (1,3)(2,4)$ for $m = 2$ and

$$\phi(l) = (1, 2m-1, 2m)(m-1, m, m+1)$$

for $m > 2$, $l \neq 1$.

Thus $p = \pm 1$. Suppose $q \neq 0$. Then $W'(p, q)$ is homeomorphic to the manifold obtained from $(V - \tau^{-q}(V'))$ by attaching a 2-handle to T along a simple closed curve homotopic to t. But $\tau^{-q}(T')$ is the boundary of a regular neighborhood of a nontrivial twist knot $\tau^{-q}(L)$ in the 3-cell formed by the union of V and the 2-handle. Therefore T' is incompressible, a contradiction.

Now let $W(p, q)$ be the 3-manifold obtained from W by adding a 2-handle to T' along a simple closed curve homotopic to $x^p l^q$.

7.2. LEMMA. *T is compressible in $W(p, q)$ if and only if $p = \pm 1$ and $q = 0$.*

PROOF. This follows from Lemma 7.1 and the fact that the Whitehead link is interchangeable: there is a self-homeomorphism of W which interchanges T and T', interchanging l with s^{-1} and x with t^{-1}.

7.3. LEMMA. *W is simple.*

PROOF. Following Whitten [20], this can be deduced from work of Seifert [16] and Schubert [15] on doubled knots. Alternatively, it follows from the fact, due to Thurston [17], that W is hyperbolic. However, it also can easily be deduced from Lemmas 3.1–3.3, as follows.

Regard W as the union of the cube-with-handles P in Figure 5 and a regular neighborhood $N = F' \times [1,2]$ of the planar surface F' in Figure 6, where F_i is identified with $F' \times \{i\}$, $i = 1, 2$. By Lemma 4.9, (P, F) has Property C′. (N, F) has Property A by the incompressibility of F in N. Property B′ of (N, F) follows from Corollary 3.2 and Lemma 3.4 of [19].

8. An algebraic determination of compact, orientable 3-manifolds.

8.1. THEOREM. *Let M and N be compact, orientable 3-manifolds whose boundaries contain no 2-spheres. Then $\mathcal{K}(M) = \mathcal{K}(N)$ if and only if M is homeomorphic to N.*

The main tool used to prove this theorem will be

8.2. JOHANNSON'S THEOREM [9]. *Let X and Y be Haken manifolds. Suppose X is semisimple. If $\pi_1(X)$ and $\pi_1(Y)$ are isomorphic, then X and Y are homeomorphic.*

PROOF OF THEOREM 8.1. Suppose $\mathcal{K}(M) = \mathcal{K}(N)$. By Theorem 6.1 M contains a simple knot K. Let U be a regular neighborhood of K in M, and let $M' = \overline{M - U}$. Let n be the torus Haken number [3] of N, i.e. n is a positive integer such that if $\{T_1, \ldots, T_m\}$ is a collection of disjoint, incompressible tori in N or ∂N and $m > n$, then at least two of the tori are parallel in N.

Let (V_i, V_i'), $1 \leqslant i \leqslant n + 2$, be copies of (V, V'). Identify V_1 with U. Identify V_2 with V_1' so that s_2 is homotopic to x_1 and t_2 is homotopic to $x_1^q l_1$, where $q \neq 0$. For $3 \leqslant i \leqslant n + 2$, identify V_i with V_{i-1}' so that s_i is homotopic to x_{i-1} and t_i is homotopic to l_{i-1}. Let $T_0 = \partial V_1$ and $T_i = \partial V_i'$, $1 \leqslant i \leqslant n + 2$.

Let J be the core of V_{n+2}'. Let X be the exterior of J in M. $X = M' \cup \bigcup_{i=1}^{n+2} W_i$. Since M' and the W_i are simple, (M', T_0), (W_i, T_{i-1}), and (W_i, T_i) all have Property C. So by Lemma 3.2, X is semisimple.

By hypothesis there is a knot $\tilde{J}$ in N such that $\pi_1(M - J)$ is isomorphic to $\pi_1(N - \tilde{J})$. Let $\tilde{Y}$ be the exterior of $\tilde{J}$ in N. Let $\tilde{T}$ be the component of $\partial\tilde{Y}$ which bounds a regular neighborhood of $\tilde{J}$ in N. $\pi_1(X)$ is isomorphic to $\pi_1(\tilde{Y})$. Since X is irreducible and boundary-irreducible, $\pi_1(X)$ is not an infinite cyclic group or a nontrivial free product [4, 5]. It follows that $\tilde{Y} = Y \# \Sigma$, where Y is a Haken manifold and Σ a homotopy 3-sphere [4]. Let $\tilde{N}$ be the union of Y with a regular neighborhood of $\tilde{J}$. By Johannson's theorem there is a homeomorphism $f\colon X \to Y$. Let $\tilde{T}_i = f(T_i)$ and $\tilde{W}_i = f(W_i)$.

Suppose $\tilde{T}_{n+2} \neq \tilde{T}$. Since the W_i are simple, none of the $\tilde{T}_i$ are parallel in N. Therefore, some of the $\tilde{T}_i$ are compressible in N. Since the W_i are boundary irreducible, $\tilde{T}_0$, $\tilde{T}_1$, and $\tilde{T}_2$ are compressible in N. Let $\tilde{E}_0$ be a regular neighborhood of a compressing disk $\tilde{D}_0$ for $\tilde{T}_0$. Since $\tilde{T}_1$ is compressible in N and $\partial(\tilde{W}_1 \cup \tilde{E}_0) - \tilde{T}_1$ is a 2-sphere, $\tilde{T}_1$ is compressible in $\tilde{W}_1 \cup \tilde{E}_0$. By Lemma 7.1 $\partial \tilde{D}_0$ is homotopic in $\tilde{T}_0$ to $\tilde{t}^{\pm 1}$. Moreover, if $\tilde{E}_1$ is a regular neighborhood of a compressing disk $\tilde{D}_1$ for $\tilde{T}_1$, then $\partial \tilde{D}_1 = \tilde{l}_1$. But since $q \neq 0$, $\tilde{W}_2 \cup \tilde{E}_1$ is homeomorphic to the exterior of a nontrivial twist knot in a 3-cell. Therefore $\tilde{T}_2$ is incompressible in $\tilde{W}_2 \cup \tilde{E}_1$ and hence in N, a contradiction.

Thus, $\tilde{T}_{n+2} = \tilde{T}$. Let $\tilde{x}$ be a meridian of a regular neighborhood of $\tilde{J}$.

Note that both $\tilde{T}_{n+1}$ and $\tilde{T}_{n+2}$ are compressible in N. By Lemma 7.2 this implies that $\tilde{x} = \tilde{x}_{n+2}^{\pm 1}$.

Thus f carries a meridian of a regular neighborhood of J to a meridian of a regular neighborhood of $\tilde{J}$. Therefore, f can be extended to a homeomorphism f: $M \to \tilde{N}$.

We therefore have that N is homeomorphic to $M \# \Sigma$. By a symmetrical argument we have that M is homeomorphic to $N \# \Sigma'$. The uniqueness theorem for connected sums of compact, orientable 3-manifolds [**4**, **11**] now implies that $\Sigma = \Sigma' = S^3$ and thus that M is homeomorphic to N.

9. A characterization of S^3. Recall that a graph manifold [**18**] is a compact, orientable 3-manifold containing a disjoint collection of tori, the closures of whose complementary domains are S^1-bundles. Clearly any finite union of graph manifolds along their boundaries is again a graph manifold.

9.1. THEOREM. *Let M be a closed, orientable 3-manifold such that every knot in M shrinks to a point inside a connected sum of graph manifolds and 3-cells. Then M is homeomorphic to S^3.*

PROOF. By Theorem 6.1 M contains a simple knot J. By hypothesis J shrinks to a point inside a connected sum X of graph manifolds and 3-cells. Assume that among all such manifolds ∂X has a minimal number of components. Let V be a regular neighborhood of J in X. Let $X' = \overline{X - V}$ and $M' = \overline{M - V}$.

If ∂X contains a 2-sphere S, then S bounds a 3-cell B in $\overline{M - X}$. Replacing X by $X \cup B$ produces a connected sum of graph manifolds and 3-cells in which J shrinks to a point and which has fewer boundary components, contradicting minimality. Therefore ∂X contains no 2-spheres.

If ∂X contains a torus T, then either T is compressible in M' or T is boundary-parallel in M'. Suppose T is compressible in M', with compressing disk D. We may assume that D lies either in X' or $\overline{M - X'}$.

Suppose D lies in X'. Let E be a regular neighborhood of D in X', and let S be the 2-sphere $\overline{T - \partial E} \cup \overline{\partial E - T}$. S bounds a 3-cell B in $\overline{M - X} \cup E$. It follows from Satz 6.3 and Lemma 7.2 of [**18**] that $\overline{X - E}$ is the connected sum of some graph manifolds and one 3-cell. Thus, $\overline{X - E} \cup B$ is a connected sum of graph manifolds which contains J and has fewer boundary components than did X. This contradicts minimality, so D must lie in $\overline{M - X}$.

Let E be a regular neighborhood of D in $\overline{M - X}$. Let S be the 2-sphere $\overline{T - \partial E} \cup \overline{\partial E - T}$ Then S bounds a 3-cell B in $\overline{M - (X \cup E)}$. Therefore, $E \cup B$ is a solid torus. Replacing X by $X \cup E \cup B$ gives a connected sum of graph manifolds which contains J and has fewer boundary components. This contradicts minimality, so T is incompressible in M'.

Thus T is parallel to ∂V in M'. Since ∂X contains no 2-spheres or compressible tori it follows from Corollary 3.2 of **[19]** that T is parallel to ∂V in X'. However, this implies that X is a solid torus with core J, contradicting the fact that J shrinks to a point in X.

Therefore $\partial X = \varnothing$ and so M is a connected sum of graph manifolds. The conclusion now follows from the theorem of Montesinos **[12]** that every simply connected graph manifold is homeomorphic to S^3.

9.2. COROLLARY (MCMILLAN **[10]**). *Let M be a closed 3-manifold such that every knot in M shrinks to a point inside a connected sum of solid tori and 3-cells. Then M is homeomorphic to S^3.*

REFERENCES

1. R. H. Bing, *Necessary and sufficient conditions that a 3-manifold be S^3*, Ann. of Math. **68** (1958), 17–37.

2. R. H. Fox, *Recent development of knot theory at Princeton*, Proc. Internat. Congress Math., Cambridge, Mass., 1950, Amer. Math. Soc., Providence, R. I., pp. 453–457.

3. W. Haken, *Some results on surfaces in 3-manifolds*, Studies in Modern Topology, Prentice-Hall, Englewood Cliffs, N. J., 1968, pp. 39–98.

4. J. Hempel, *3-manifolds*, Ann. of Math. Studies, no. 86, Princeton Univ. Press, Princeton, N. J., 1976.

5. W. Jaco, *Three-manifolds with fundamental group a free product*, Bull. Amer. Math. Soc. **75** (1969), 972–977.

6. ______, *Lectures on three-manifold topology*, CBMS Regional Conference Series in Math. No. 43, Amer. Math. Soc., Providence, R. I., 1980.

7. W. Jaco and R. Myers, *An algebraic determination of closed, orientable 3-manifolds*, Trans. Amer. Math. Soc. **253** (1979), 149–170.

8. W. Jaco and P. Shalen, *Seifert fibered spaces in 3-manifolds*, Mem. Amer. Math. Soc. No. 220 (1979).

9. K. Johannson, *Homotopy equivalences of 3-manifolds with boundaries*, Lecture Notes in Math., vol. 761, Springer-Verlag, Berlin and New York, 1979.

10. D. R. McMillan, *On homologically trivial 3-manifolds*, Trans. Amer. Math. Soc., **98** (1961), 350–367.

11. J. Milnor, *A unique factorization theorem for 3-manifolds*, Amer. J. Math. **84** (1962), 1–7.

12. J. M. Montesinos, *Surgery on links and double branched covers of S^3*, Knots, Groups, and 3-Manifolds, Ann. of Math. Studies, no. 84, Princeton Univ. Press, Princeton, N. J., 1975, pp. 227–260.

13. R. Myers, *Homology cobordisms, link concordances, and hyperbolic 3-manifolds* (preprint).

14. W. H. Row, *An algebraic characterization of connected sum factors of closed 3-manifolds*, Trans. Amer. Math. Soc. **250** (1979), 347–356.

15. H. Schubert, *Knoten und Vollringe*, Acta. Math. **90** (1953), 131–286.

16. H. Seifert, *Schlingknoten*, Math. Z. **52** (1949), 62–80.

17. W. Thurston, *The geometry and topology of 3-manifolds*, lecture notes, Princeton.

18. F. Waldhausen, *Eine Klasse von 3-dimensionalen Mannigfaltigkeiten*. I, II, Invent. Math. **3** (1967), 308–333; ibid **4** (1967), 87–117.

19. ______, *On irreducible 3-manifolds which are sufficiently large*, Ann. of Math. (2) **87** (1968), 56–88.

20. W. Whitten, *Groups and manifolds characterizing links*, Knots, Groups, and 3-Manifolds, Ann. of Math. Studies, no. 84, Princeton Univ. Press, Princeton, N. J., 1975, pp. 63–86.

DEPARTMENT OF MATHEMATICS, OKLAHOMA STATE UNIVERSITY, STILLWATER, OKLAHOMA 74078