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THE ASYMPTOTIC EXPANSION FOR THE TRACE OF THE HEAT KERNEL ON A GENERALIZED SURFACE OF REVOLUTION

BY

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ABSTRACT. Let M be a smooth compact Riemannian manifold without boundary. Let I be an open interval. Let $h(r)$ be a smooth positive function. Let g be the metric on M. Consider the fundamental solution $E(x, y, r_1, r_2; t)$ of the heat equation on $M \times I$ with metric $h^2(r)g + dr \otimes dr$ (when E exists globally we call it the heat kernel on $M \times I$). The coefficients of the asymptotic expansion of the trace E are studied and expressed in terms of corresponding coefficients on the basis M. It is fulfilled by means of constructing a parametrix for E which is different from a parametrix in the standard form. One important result is that each of the former coefficients is a linear combination of the latter coefficients.

0. Introduction. We shall present here a relation between the coefficients of pointwise asymptotic expansion for the trace of the heat kernel on a generalized surface of revolution and the corresponding coefficients on its cross section. Let (M, g) be a compact Riemannian manifold with metric g and I be an open interval, $h(r)$: $I \to R^+$ a smooth positive function. Then we call $M \times I$ with metric $h^2(r)g + dr \otimes dr$ a generalized surface of revolution with section M. The asymptotic expansion for the heat kernel on M had been introduced and studied by Minakshisundaram and Pleijel [7]. Then the first author in a short paper [8] gave another proof and wrote the expansion in the following form:

$$(0.1) \qquad E(x, x, t) \underset{t \to 0}{\sim} \frac{1}{\left(2\sqrt{\pi}\right)^n} t^{-n/2}\left(1 + a_1(x)t + a_2(x)t^2 + \cdots\right).$$

Here $E(x, x, t)$ denotes the pointwise trace of the heat kernel on M^n. The heat kernel $E(x, y, t)$ can be written as $E(x, y, t) = \sum e^{-\lambda_i t}\phi_i(x)\phi_i(y)$ [1] where $0 \leq \lambda_0 \leq \lambda_1 \cdots \uparrow \infty$ are the eigenvalues of the Laplacian on M and $\{\phi_i\}$ are corresponding orthonormal eigenfunctions. If we integrate (0, 1) with respect to the volume form we obtain [1]

$$(0.2) \qquad \sum_{i=0}^{\infty} e^{-\lambda_i t} \underset{t \to 0}{\sim} (4\pi t)^{-n/2}\left(a_0 + a_1 t + a_2 t^2 + \cdots\right).$$

The coefficients $\{a_i\}$ have geometric meaning; for example a_0 is the volume of the manifold, and when $n = 2$, $a_1 = \pi\chi(M)/3$ (where $\chi(M^2)$ is the Euler characteristic of M^2). While it is known that the $\{a_i\}$ depend on the curvature tensor R and its

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successive covariant derivatives [**1**], it is very difficult to calculate them explicitly. In fact, only the first few of them have been calculated for general manifolds. For some special manifold, such as compact symmetric spaces of rank one, Cahn and Wolf [**2**] have found an explicit formula for all the coefficients.

The method used by Minakshisundaram-Pleijel [**7**] to study the asymptotic expansion was to construct a sufficiently good parametrix. The properties of the heat kernel at the diagonal are determined by the parametrix. Therefore it might appear that the most natural way of studying our problems would be to start with a parametrix of the following form:

(0.3)

$$\frac{1}{(4\pi t)^{(n+1)/2}} e^{\rho^2/4t}\left(\bar{u}_0(r_1, r_2, x, y) + \bar{u}_1(r_1, r_2, x, y)t + \bar{u}_2(r_1, r_2, x, y)t^2 + \cdots\right).$$

Here (x, y) denotes points on the cross section, $r_1, r_2 \in R$ and ρ denotes the distance between (r_1, x) and (r_2, y) on $M \times R$. But even though it is possible to express ρ in terms of the distance function on M and the warping function $h(r)$, this approach does not lead to a formula for the coefficients on $M \times I$ in terms of the corresponding coefficients on M (see §IV). Instead we construct a parametrix of nonstandard form. The idea of constructing such a parametrix was initially suggested by the following considerations. By using separation of variables we can express the heat kernel on $M \times R$ in the form $\sum_{i=0}^{\infty} f_i(r_1, r_2, t)\phi_i(x)\phi_i(y)$. Then the Laplace transform of each f_i will be a parametrix for the kernel of the resolvent of the associated 1-dimensional Sturm-Liouville problem. Gel′fand and Dikii have studied the asymptotic expansions for the trace of these kernels [**4**]. This leads to an asymptotic expansion for the f_i. The construction of our parametrix was motivated in part by a certain formal procedure for "adding" these asymptotic expansions. Here the following difficulty arises: the asymptotic expansion of each f_i starts with the term $t^{-1/2}$. But the asymptotic expansion we want must start with $t^{-(n+1)/2}$. In fact we have infinitely many asymptotic expansions starting with $t^{-1/2}$ and not holding uniformly. We want to "add" these in such a way to obtain an expansion starting with $t^{-(n+1)/2}$.

In concluding this introduction, the author would like to thank his advisor, Jeff Cheeger, who patiently guided the work.

I. Preliminaries.

I.1. *The Laplacian on $M \times I$.* Let $M \times I$ be the generalized surface of revolution as we defined it in the introduction. If we denote the Laplacians on $M \times I$ and M by Δ and $\underline{\Delta}$ respectively then a standard computation shows

I.1.1. PROPOSITION.

$$\Delta = -\left(\frac{\partial}{\partial r}\right)^2 - \frac{nh'(r)}{h(r)}\frac{\partial}{\partial r} + \frac{1}{h^2(r)}\underline{\Delta}.$$

I.2. *Asymptotic expansions.* We will use the following result.

I.2.1. THEOREM. *If $f(z)$ is analytic in $-\alpha < \theta < \alpha$, $0 < r < b$, $z = re^{i\theta}$ and if $f(z) \underset{z\to 0}{\sim} \sum_{k=0}^{\infty} a_k z^k$ uniformly in θ then $f'(z) \underset{z\to 0}{\sim} \sum_{k=1}^{\infty} k a_{k-1} z^{k-1}$ uniformly in any small sector $-\alpha < \alpha_1 < \theta < \alpha_2 < \alpha$.*

PROOF. See **[6]**.

I.2.2. COROLLARY. *If $\sum_{i=0}^{\infty} e^{-\lambda_i t} \underset{t\to 0}{\sim} t^{-n/2}(a_0 + a_1 t + \cdots)$ then*

$$\sum_{i=0}^{\infty} \lambda_i e^{-\lambda_i t} \underset{t\to 0}{\sim} \frac{n}{2} t^{-n/2-1} a_0 + \left(\frac{n}{2} - 1\right) t^{-n/2} a_1 + \cdots .$$

PROOF. $\sum_{i=0}^{\infty} e^{-\lambda_i t}$ is analytic in $-\alpha < \theta < \alpha$. $\alpha < \pi/2$ **[1]** so this follows from Theorem I.2.1.

We also need

I.2.3. THEOREM. *Suppose $F(s)$ is the Laplace transform of $f(t)$ and $F(s)$ has a half-plane of convergence. If $f(t) \underset{t\to 0}{\sim} \sum_{\nu=0}^{\infty} c_\nu t^{\lambda_\nu}$ (where $-1 < R\lambda_0 < R\lambda_1 \cdots$) then*

$$F(s) \underset{s\to\infty}{\sim} \sum_{\nu=0}^{\infty} c_\nu \frac{\Gamma(\lambda_\nu + 1)}{s^{\lambda_{\nu+1}}}.$$

$R\lambda_i$ denotes the real part of λ_i.

PROOF. See **[9]**.

I.3. *Heat kernels and their parametrices.*

I.3.1. DEFINITION. A fundamental solution of the heat equation on a smooth Riemannian manifold N is defined as a function F on $N \times N \times R_+^*$ which satisfies the following:

(1) F is C^0 in the three variables, C^2 in the second variable, C^1 in the third variable.

(2) $(\Delta_2 + \partial/\partial t)F = 0$. Here Δ_2 is the Laplacian applied to the second variable.

(3) $\lim_{t\to 0} F(x, \cdot, t) = \delta_x$ for any $x \in N$.[1]

I.3.2. DEFINITION. When N is compact the fundamental solution is unique. We call it the heat kernel of N.

REMARK. Since we are interested in the case of pointwise asymptotic expansion, we can regard $M \times I$ as a portion of a larger compact smooth manifold.

I.3.3. DEFINITION. We call H a parametrix of $\square = \Delta + \partial/\partial t$ if it satisfies:

(i) $H \in C^\infty(N \times N \times R_+^*)$.

(ii) $\square H$ can be extended to become a function in $C^0(N \times N \times R_+)$.

(iii) $\lim_{t\to 0} H(x, \cdot, t) = \delta_x$ for all $x \in N$.

In the definition, R_+ denotes the nonnegative real numbers, R_+^* denotes the positive reals and δ_x denotes the Dirac function at the point x.

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[1]For R_+^*, δ_x see Definition I.3.3.

II. Construction of a parametrix. We will construct a parametrix of the following nonstandard form:

$$(2.1)\quad \frac{1}{2\sqrt{\pi}}\sum_{i=0}^{\infty} h^{-n/2}(r_1)h^{-n/2}(r_2)\exp\left(-\frac{(r_1-r_2)^2}{4t}\right)\exp\left(-\frac{\lambda_i t}{h(r_1)h(r_2)}\right) \cdot \sum_{j=0}^{\infty} a_j(r_1, r_2, \lambda_i)\phi_i(x)\phi_i(y)t^{j-1/2}.$$

Apply the heat operator $\square_2 = \Delta_2 + \partial/\partial t$ to the above expansion with respect to the second variable (r_2, y). Collect those terms with the same power of t and set the resulting expression equal to 0. In particular by setting the coefficient of $t^{j-1/2}$ equal to zero we get

$$(2.2)\quad \left(u_i(r_2) - \frac{\lambda_i}{h(r_1)h(r_2)}\right)a_{j-1} - \left(\frac{\partial}{\partial r_2}\right)^2 a_{j-1} - (r_1-r_2)\frac{\lambda_i h'(r_2)}{h(r_1)h^2(r_2)}a_{j-1}$$
$$+ja_j - (r_1-r_2)\frac{\partial}{\partial r_2}a_j - \frac{\lambda_i(h(r_2)h''(r_2) - 2(h'(r_2))^2)}{h(r_1)h^3(r_2)}a_{j-2}$$
$$-2\frac{\lambda_i h'(r_2)}{h(r_1)h^2(r_2)}\frac{\partial}{\partial r_2}a_{j-2} - \left(\frac{\lambda_i h'(r)}{h(r_1)h^2(r)}\right)^2 a_{j-3} = 0$$

where $j \geqslant 0$, $a_{-3} = a_{-2} = a_{-1} = 0$.

$$(2.3)\quad u_i(r) = \frac{n}{2}\left(\frac{n}{2}-1\right)\frac{h'(r)^2}{h^2(r)} + \frac{n}{2}\frac{h''(r)}{h(r)} + \frac{\lambda_i}{h^2(r)}.$$

We can then solve a_j successively: $a_0 = \text{const}$. Here we use the following normalizing conditions; let $a_0 = 1$.

$$(2.4)\quad a_1 = \frac{1}{r_1-r_2}\int_{r_1}^{r_2} u_i(r)\,dr + \frac{\lambda_i}{h(r_1)h(r_2)}.$$

(2.5)

$$a_2 = \frac{1}{2(r_1-r_2)^2}\left(\int_{r_1}^{r_2} u_i(r)\,dr\right)^2 + \frac{1}{r_1-r_2}\left(\int_{r_1}^{r_2} u_i(r)\,dr\right)\frac{\lambda_i}{h(r_1)h(r_2)}$$
$$-\frac{2}{(r_1-r_2)^3}\int_{r_1}^{r_2} u_i(r)\,dr - \frac{1}{(r_1-r_2)^2}(u_i(r_1)+u_i(r_2)) + \frac{1}{2}\frac{\lambda_i}{h(r_1)h(r_2)}.$$

The lower limit is so chosen in order to avoid the singularities which would arise when $r_1 = r_2$ if we did otherwise. In general,

(2.6)

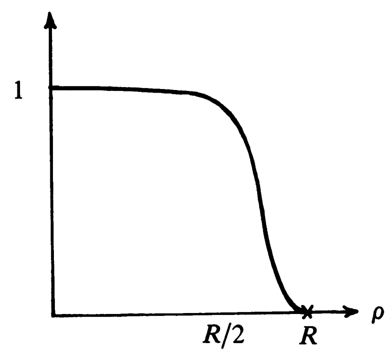
$$a_{j+1} = \frac{1}{(r_1 - r_2)^{j+1}} \int_{r_1}^{r_2} \left\{ -(r_1 - r)^j \left(\frac{\lambda_i h'(r)}{h(r_1)h(r)} \right)^2 - 2(r_1 - r)^j \frac{\lambda_i h'(r)}{h(r_1)h^2(r)} \right.$$
$$\cdot \frac{\partial}{\partial r_2} a_{j-1} - (r_1 - r)^j \frac{\lambda_i (h(r)h''(r) - 2h'(r))^2}{h(r_1)h^3(r)} a_{j-1}$$
$$- (r_1 - r)^j \frac{\partial^2}{\partial r_2^2} a_j + (r_1 - r)^j \left(u_i(r) - \frac{\lambda_i}{h(r_1)h(r)} \right) a_j$$
$$\left. - (r_1 - r)^{j+1} \frac{\lambda_i h'(r_2)}{h(r_1)h^2(r)} a_j \right\} dr.$$

Let H_k denote the expression (2.1) with the second summation replaced by the finite sum $\Sigma_{j=0}^{k}$.

II.1. MAIN THEOREM. *H_k is a parametrix for $\square$ if: $[\frac{k}{3}] > \frac{n}{2} + \frac{3}{2}$. More precisely we multiply H_k by a C^∞ cut off function*

$$\eta_R(\rho((r_1, x), (r_2, y_2))) = \begin{cases} 1, & \rho \leq R/2, \\ 0, & \rho > R, \end{cases}$$

in order to get a parametrix in the sense we have defined.



In order to prove the theorem we need to check the conditions for a parametrix (I.3.3). For the first condition we need the following lemma.

II.2. LEMMA. *$a_{j+1}(r_1, r_2, \lambda_i)$ is C^∞ in the variables r_1, r_2.*

PROOF. We prove it by induction. Write

$$a_{j+1} = \frac{1}{(r_1 - r_2)^{j+1}} \int_{r_1}^{r_2} (r_1 - r)^j \mathcal{Q}_{j+1}(r_1, r, \lambda_i)\, dr.$$

Here $\mathcal{Q}_{j+1}$ is C^∞ in r_1, r_2 from the induction hypothesis. In order to show that a_{j+1} is C^∞ it is then sufficient to check that $(\partial/\partial r_1)^k(\partial/\partial r_2)^l a_{j+1}$ exists for $r_1 = r_2$. By Taylor's theorem

$$\mathcal{Q}_{j+1}(r_1, r, \lambda_i) = \mathcal{Q}_{j+1}(r_1, r_1, \lambda_i) - (\partial/\partial r_2)\mathcal{Q}_{j+1}(r_1, r_1, \lambda_i)(r_1 - r) + \cdots + \left((-1)^{k+l}/(k+l)!\right)(\partial/\partial r_2)^{k+l}\mathcal{Q}_{j+1}(r_1, \xi, \lambda_i)(r_1 - r)^{k+l},$$

ξ is a number between r_1 and r. Therefore,

$$a_{j+1} = \frac{\mathcal{Q}_{j+1}(r_1, r_1, \lambda_i)}{j+1} - \frac{(\partial/\partial r_2)\mathcal{Q}_{j+1}(r_1, r_1, \lambda_i)(r_1 - r_2)}{j+2} + \cdots + \frac{(-1)^{k+l}}{(r_1 - r_2)^{j+1}} \int_{r_1}^{r_2} \left(\frac{\partial}{\partial r_2}\right)^{k+l} \mathcal{Q}_{j+1}(r_1, \xi, \lambda_i)(r_1 - r)^{j+l+k}\, dr.$$

Each term of the right-hand side of the above expression is C^∞ in r_1, r_2 except possibly the last term. But, after applying $(\partial/\partial r_1)^k(\partial/\partial r_2)^l$ to it then taking the limit $r_2 \to r_1$ we will get const $\cdot\, (\partial/\partial r_2)^{k+l}\mathcal{Q}_{j+1}$ which shows $(\partial/\partial r_1)^k(\partial/\partial r_2)^l\mathcal{Q}_{j+1}$ exists for $r_1 = r_2$. This completes the proof.

For condition (iii) we need the following lemma. From (2.6) the degree of λ_i in a_{j+1} increases at most with each increase of index. Therefore $\deg \lambda_i a_{j+1} \leqslant j + 1$, which we will need in the proof.

II.3. LEMMA. $\lim_{t\to 0} H_k(r_1, x; \cdot, t) = \delta_{(r,x)}$.

PROOF. Let $f(r_2, y)$ be a smooth function with compact support which contains the point (r_1, x). Then

$$\begin{aligned}
&\lim_{t\to 0} \int_{M\times I} H_k(r_1, x, r_2, y; t) f(r_2, y)\, dV \\
&= \lim_{t\to 0} \int_{M\times I} H_k(r_1, x, r_2, y; t) f(r_2, y) h^n(r_2)\, dr_2\, dy \\
&= \lim_{t\to 0} \int_I \Bigg\{ \frac{1}{2\sqrt{\pi}} h^{-n/2}(r_1) h^{-n/2}(r_2) \exp\left(-\frac{(r_1 - r_2)^2}{4t}\right) t^{-1/2} \\
&\qquad \cdot \sum_{j=0}^{k} \left[\int_M \sum_{i=0}^{\infty} \exp\left(-\frac{\lambda_i t}{h(r_1)h(r_2)}\right) \phi_i(x)\phi_i(y) f(r_2, y) \right. \\
&\qquad\qquad \left. \cdot\, a_j(r_1, r_2, \lambda_i) t^j\, dy \right] h^n(r_2) \Bigg\}\, dr_2 \qquad (2.7) \\
&= \lim_{t\to 0} \sum_{j=0}^{k} \int_M \sum_{i=0}^{\infty} \exp\left(-\frac{\lambda_i t}{h^2(r_1)}\right) \phi_i(x)\phi_i(y) f(r_1, y) a_j(r_1, r_1, \lambda_i) t^j\, dy \\
&= f(r_1, x).
\end{aligned}$$

The second from the last equality holds since $\deg_{\lambda_i} a_j(r_1, r_2, \lambda_i)$ is finite. In fact, when $j \neq 0$

$$\lim_{t\to 0}\left[\int_M \sum_{i=0}^{\infty} \exp\left(-\frac{\lambda_i t}{h_2(r_1)}\right)\lambda_i^s\phi_i(x)\phi_i(y)f(r_1, y)\,dy\right]t^j = \lim_{t\to 0}\Delta^s f(r_1, x)t^j = 0.$$

Hence only the term containing $a_0 = 1$ remains. Q.E.D.

For condition (ii) we need the following:

II.4. LEMMA. *For given $T > 0$*

$$|\,\square H_k| \leqslant \text{const}\; t^{[k/3]-n/2-3/2}, \qquad t < T.$$

The constant in the above estimate depends on $h(r)$. For the proof of Lemma II.4 some inequalities about the degree of λ_i in a_l are essential. The need to control the degree arises from the fact that each increase in the degree of λ_i in the expression $\sum_{i=0}^{\infty}\lambda_i^s e^{-\lambda_i t}$ will lower by 1 the degree of t in the asymptotic expression. This follows easily from Theorem I.2.1.

II.5. LEMMA. (i) $\deg_{\lambda_i} a_l(r, r, \lambda_i) \leqslant [2l/3]$.
(ii) $\deg_{\lambda_i}((\partial/\partial r_2)^k a_l)(r, r, \lambda_i) \leqslant [(2l + k - 1)/3]$, $0 < k \leqslant l$.
(iii) $\deg_{\lambda_i}((\partial/\partial r_2)^k a_l)(r, r, \lambda_i) \leqslant l$, $k \geqslant l$.

PROOF. By induction on l, suppose the above inequalities hold for all $l' < l$. Then

$$\text{(2.8)} \quad \begin{aligned} \deg_{\lambda_i} a_l(r, r, \lambda_i) &\leqslant \max\left\{\left[\frac{2\cdot(l-1)}{3}\right], 1 + \left[\frac{2(l-2)}{3}\right],\right. \\ &\qquad \left. 1 + \left[\frac{2(l-2)+1-1}{3}\right], 2 + \left[\frac{2(l-3)}{3}\right]\right\} \\ &= \left[\frac{2l}{3}\right] \end{aligned}$$

which proves the first inequality. Consider

$$\text{(2.9)} \quad \begin{aligned} &\left[\left(\frac{\partial}{\partial r_2}\right)^k\left(la_l - (r_1 - r_2)\frac{\partial}{\partial r_2}a_l\right)\right]_{r_1=r_2} \\ &= \left[l\left(\frac{\partial}{\partial r_2}\right)^k a_l + \binom{k}{1}\left(\frac{\partial}{\partial r_2}\right)^k a_l - (r_1 - r_2)^{k+1}a_l\right]_{r_1=r_2} \\ &= \left((l+k)\left(\frac{\partial}{\partial r_2}\right)^k a_l\right)(r, r, \lambda_i) \end{aligned}$$

and

(2.10)

$$
\begin{aligned}
la_l - (r_1 - r_2)\frac{\partial}{\partial r_2}a_l = &-\left(u_i(r_2) - \frac{\lambda_i}{h(r_1)h(r_2)}\right)a_{l-1} + \left(\frac{\partial}{\partial r_2}\right)^2 a_{l-1} \\
&+ (r_1 - r_2)\frac{\lambda_i h'(r_2)}{h(r_1)h(r_2)}a_{l-1} + \frac{\lambda_i\left(h(r_2)h''(r_2) - 2(h'(r_2))^2\right)}{h(r_1)h^3(r_2)}a_{l-2} \\
&+2\frac{\lambda_i h'(r_2)}{h(r_1)h(r_2)^2}\frac{\partial}{\partial r_2}a_{l-2} + \left(\frac{\lambda_i h'(r_2)}{h(r_1)h(r_2)}\right)^2 a_{l-3}.
\end{aligned}
$$

If we want to know the degree of λ_i in $((\partial/\partial r_2)^k a_l)(r, r, \lambda_i)$ we need only know the degree of the kth derivation w.r.t. r_2 of the r.h.s. of the above expression:

(2.11)

$$
\begin{aligned}
\left(\frac{\partial}{\partial r_2}\right)^k\left[\left(u_i(r_2) - \frac{\lambda_i}{h(r_1)h(r_2)}\right)a_{l-1}\right] = &\left(u_i(r_2) - \frac{\lambda_i}{h(r_1)h(r_2)}\right)\left(\frac{\partial}{\partial r_2}\right)^k a_{l-1} \\
&+ \binom{k}{1}\frac{\partial}{\partial r_2}\left(u_i(r_2) - \frac{\lambda_i}{h(r_1)h(r_2)}\right)\left(\frac{\partial}{\partial r_2}\right)^{k-1} a_{l-1} \\
&+ \cdots + \binom{k}{m}\left(\frac{\partial}{\partial r_2}\right)^m\left(u_i(r_2) - \frac{\lambda_i}{h(r_1)h(r_2)}\right)\left(\frac{\partial}{\partial r_2}\right)^{k-m} a_{l-1} \\
&+ \cdots + \left(\frac{\partial}{\partial r_2}\right)^k\left(u_i(r_2) - \frac{\lambda_i}{h(r_1)h(r_2)}\right)a_{l-1}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\deg_{\lambda_i}\left(\frac{\partial}{\partial r_2}\right)^k\left[\left(u_i(r_2) - \frac{\lambda_i}{h(r_1)h(r_2)}\right)a_{l-1}\right]_{r_1=r_2} \\
(2.12)\quad &\leqslant \max\left\{1, +\left[\frac{2(l-1)+k-1+1}{3}\right], \ldots, 1 + \left[\frac{2(l-1)+k-m-1}{3}\right], \ldots, 1 + \left[\frac{2(l-1)}{3}\right]\right\} \\
&= \left[\frac{2l+k-1}{3}\right].
\end{aligned}
$$

Similarly we have

(2.13)
$$\deg_{\lambda_i}\left(\left(\frac{\partial}{\partial r_2}\right)^k\left(\frac{\partial}{\partial r_2}\right)a_{l-1}\right)_{r_1=r_2}\leqslant\left[\frac{2l+k-1}{3}\right],$$

$$\deg_{\lambda_i}\left\{\left(\frac{\partial}{\partial r_2}\right)^k\left[(r_1-r_2)\frac{\lambda_i h'(r_2)}{h(r_1)h(r_2)}a_{l-1}\right]\right\}_{r_1=r_2}\leqslant\left[\frac{2l+k-1}{3}\right],$$

$$\deg_{\lambda_i}\left\{\left(\frac{\partial}{\partial r_2}\right)^k\left[\frac{\lambda_i\left(h(r_1)h''(r_2)-2(h'(r_2))^2\right)}{h(r_1)^3(r_2)}a_{l-2}\right]\right\}_{r_1=r_2}\leqslant\left[\frac{2l+k-1}{3}\right],$$

$$\deg_{\lambda_i}\left\{\left(\frac{\partial}{\partial r_2}\right)^k\left(\left(\frac{\lambda_i h'(r_2)}{h(r_1)h^2(r_2)}\right)\frac{\partial}{\partial r_2}a_{l-2}\right)\right\}_{r_1=r_2}\leqslant\left[\frac{2l+k-1}{3}\right],$$

$$\deg_{\lambda_i}\left\{\left(\frac{\partial}{\partial r_2}\right)\left(\left(\frac{\lambda_i h'(r_2)}{h(r_1)h^2(r_2)}\right)^2a_{l-3}\right)\right\}_{r_1=r_2}\leqslant\left[\frac{2l+k-1}{3}\right].$$

Combining the above inequalities we get

(2.14)
$$\deg_{\lambda_i}\left(\left(\frac{\partial}{\partial r_2}\right)^k a_l\right)(r,r,\lambda_i)\leqslant\left[\frac{2l+k-1}{3}\right]$$

so we have proved (ii). (iii) is trivial since $\deg_{\lambda_i}a_l(r_1,r_2,\lambda_i)$ is never greater than l. Q.E.D.

Now let us return to the proof of Lemma II.4. Consider that

(2.15)
$$\begin{aligned}\square_2 H_k=&\sum_{i=0}^{\infty}\left[(r_1-r_2)\frac{\partial}{\partial r_2}a_{k+1}(r_1,r_2,\lambda_i)-(k+1)a_{k+1}(r_1,r_2,\lambda_i)\right]G_it^{k-1/2}\\&+\sum_{i=0}^{\infty}\left[\frac{\lambda_i\left(h(r_2)h''(r_2)-2(h'(r_2))^2\right)}{h(r_1)h(r_2)^3}a_k(r_1,r_2,\lambda_i)\right.\\&\qquad-2\frac{\lambda_i h'(r_2)}{h(r_1)h^2(r_2)}\frac{\partial}{\partial r_2}a_k(r_1,r_2,\lambda_i)-\left(\frac{\lambda_i h'(r_2)}{h(r_1)h^2(r_2)}\right)^2\\&\qquad\left.\cdot a_{k-1}(r_1,r_2,\lambda_i)\right]G_it^{k+1/2}\\&+\sum_{i=0}^{\infty}\left[-\left(\frac{\lambda_i h'(r_2)}{h(r_1)h^2(r_2)}\right)^2a_k(r_1,r_2,\lambda_i)\right]G_it^{k+3/2}\end{aligned}$$

where

$$G_i=\frac{1}{2\sqrt{\pi}}h^{-n/2}(r_1)h^{-n/2}(r_2)\exp\left(-\frac{(r_1-r_2)^2}{4t}\right)\exp\left(-\frac{\lambda_i t}{h(r_1)h(r_2)}\right)\phi_i(x)\phi_i(y).$$

Since we know that

$$(2.16)\qquad \sum_{i=0}^{\infty} e^{-\lambda_i t}\phi_i(x)\phi_i(y) \leqslant \text{const}\cdot t^{-n/2}, \qquad t<T,$$

$$(2.17)^*\qquad \sum_{i=0}^{\infty} \lambda_i^s e^{-\lambda_i t}\phi_i(x)\phi_i(y) \leqslant \text{const}\cdot t^{-n/2-s}, \qquad t<T,$$

and

$$(2.18)\qquad \left|\exp\left(\frac{(r_1-r_2)^2}{4t}\right)(r_2-r_1)^k\right| \leqslant \text{const}\cdot t^{-k/2}, \qquad k\neq 0,$$

it is easy to see that

$$(2.19)\qquad \left|\sum_{i=0}^{\infty}\exp\left(-\frac{(r_1-r_2)^2}{4t}\right)\frac{(r_2-r_1)^k}{k!}\left(\frac{\partial^k}{\partial r_2{}^k}a_l(r_1,r_2,\lambda_i)\right)_{r_1=r_2}\cdot\exp\left(-\frac{\lambda_i t}{h(r_1)h(r_2)}\right)\phi_i(x)\phi_i(y)\right|$$
$$\leqslant \text{const}\cdot t^{n/2+k/2-[(2l+k-1)/3]}.$$

Now, we expand $a_l(r_1, r_2, \lambda_i)$ into a Taylor series at $r_2 = r_1$ with r_2 as variable. If we denote the sum of all terms with power of $(r_2 - r_1)$ greater than l by $\bar{a}_l(r_1, r_2, \lambda_i)$ then

$$(2.20)\qquad a_l(r_1,r_2,\lambda_i) = a_l(r_1,r_1,\lambda_i) + (r_2-r_1)\left(\frac{\partial}{\partial r_2}a_l(r_1,r_2,\lambda_i)\right)_{r_1=r_2}$$
$$+\cdots+\frac{(r_2-r_1)^k}{k!}\left[\left(\frac{\partial}{\partial r_2}\right)^k a_l(r_1,r_2,\lambda_i)\right]_{r_1=r_2}$$
$$+\cdots+\frac{(r_2-r_1)^l}{l}\bar{a}_l(r_1,r_2,\lambda_i).$$

Note that $\deg_{\lambda_i}\bar{a}_l(r_1, r_2, \lambda_i) \leqslant l$; we have pointed this out in the remark following II.2. In order to estimate G_i, we multiply each term of the Taylor series expansion by G_i, and then use the above inequalities to obtain

$$(2.21)\qquad |a_l(r_1,r_2,\lambda_i)G_i| \leqslant \text{const}\cdot t^{-n/2-[2l/3]}.$$

See Corollary I.2.2. Similarly we can estimate $\square_2 H_k$ since we can estimate each term of it.

(2.22)

$$|\square_2 H_k| \leqslant \text{const}\cdot t^{k-1/2-n/2-[2(k+1)/3]} + \text{const}\cdot t^{k+1/2-n/2-n/2-[2k/3]-1}$$
$$+\ \text{const}\cdot t^{k+1/2-n/2-[2(k-1)/3]-2} + \text{const}\cdot t^{k+3/2-n/2-[2k/3]-2}$$
$$\leqslant \text{const}\cdot t^{[k/3]-n/2-3/2}.$$

This completes the proof of Lemma II.4 and shows that H_k does represent a parametrix for the heat kernel on $M\times I$ when $[\frac{k}{3}] > \frac{n}{2} + \frac{3}{2}$. Since the heat kernel is symmetric w.r.t. its variables (r_1, x), (r_2, y), if we represent it as $\Sigma_{i=0}^{\infty} f_i(r_1, r_2, t)\phi_i(x)\phi_i(y)$ we can see that $f_i(r_1, r_2, t)$ is also symmetric w.r.t. r_1 and r_2. For some special functions $h(r)$ (for example; $h(r) = \sin r$),

$$f_i(r_1, r_2, t)\phi_i(x)\phi_i(x)\phi_i(y) = \sum_{j=0}^{\infty} a_j(r_1, r_2, \lambda_i)G_i t^{j-1/2}$$

is convergent for each i. Since G_i is symmetric w.r.t. r_1 and r_2, in these special cases $A_j(r_1, r_2, \lambda_i)$ is also symmetric w.r.t. r_1 and r_2. But if we look at the construction of a_j more carefully we find that the form of a_j in terms of $h(r)$ is universal[2] so that we get the following

II.6. COROLLARY. *$a_j(r_1, r_2, \lambda_i)$ is symmetric w.r.t. r_1, r_2.*

In proving H_k represents a parametrix we also get the following asymptotic expansion:

II.7. COROLLARY.

$$\text{(2.23)} \quad \text{Trace } E \underset{t\to 0}{\sim} \sum_{i=0}^{\infty} \frac{1}{2\sqrt{\pi}} h^{-n}(r) \sum_{i=0}^{\infty} \exp\left(-\frac{\lambda_i t}{h^2(r)}\right) a_j(r, r, \lambda_i)\phi_i^2(x)t^{j-1/2}.$$

III. An explicit formula for the asymptotic expansion. In this section we relate the result of §II to the result of Gel'fand and Dikii **[4]** in order to get an explicit formula. They have studied the asymptotic expansion of the resolvent kernel for the Sturm-Liouville equation and obtained recursive formulas for the coefficients of the expansion. Moreover they also have a generating function for the recursive formulas. Let us start with some formal considerations.

Consider $E(r_1, r_2, x, y, t) = \Sigma_{i=0}^{\infty} f_i(r_1, r_2, t)\phi_i(x)\phi_i(y)$ since the Laplace transform of the heat kernel is the kernel of the resolvent of Δ **[5]**.

$$\text{(3.1)} \qquad \int_0^{\infty} e^{-\Delta t} e^{-st}\, dt = (\Delta + s)^{-1},$$

the Laplace transform of each f_i, should satisfy the following differential equation:

$$\text{(3.2)} \qquad \left[-\left(\frac{\partial}{\partial r_2}\right)^2 - \frac{nh'(r_2)}{h(r_2)}\frac{\partial}{\partial r_2} + \frac{\lambda_i}{h^2(r_2)} + s\right]\hat{f}_i(r_1, r_2, s) = 0$$

or

(3.3)

$$\left[-\left(\frac{\partial}{\partial r_2}\right)^2 + \frac{n}{2}\left(\frac{n}{2} - 1\right)\frac{h''(r_2)^2}{h^2(r_2)} + \frac{n}{2}\frac{h''(r_2)}{h(r_2)} + \frac{\lambda_i}{h^2(r_2)}\right]h^{n/2}(r_2)\hat{f}(r_1, r_2, s)$$
$$= -sh^{n/2}(r_2)\hat{f}(r_1, r_2, s).$$

[2] a_j is the summation of $(r_1 - r_2)^{-1}$ to some power multiplied by the integral of a rational function of $h(r), h'(r), \ldots$; the coefficients of the denominator and numerator of the rational function are independent of $h(r)$. See (2.5) for $j = 2$.

This in turn suggests that $h^{n/2}(r_1)h^{n/2}(r_2)\hat{f}_i(r_1, r_2, s)$ is the kernel of the resolvent of $-d^2/dx^2 + (u_i(r) + s)$ where $u_i(r)$ is defined as in (2.3).

Applying the result of Gel'fand and Dikii [4] we have

$$h^n(r)\hat{f}_i(r, r, s) \underset{s\to\infty}{\sim} \sum_{l=0}^{\infty} \frac{R_l[u_i]}{s^{l+1/2}}$$

where $R_l = R_l[u, u', u'', \ldots]$ is a polynomial in $u, u', u'', \ldots$ and satisfies

$$R_0 = \tfrac{1}{2}, \quad R_1 = -\tfrac{1}{4}u,$$

$$4R_{l+1} = 2\sum_{k=0}^{l-1} R_k R''_{l-k} - \sum_{k=1}^{l-1} R'_k R'_{l-k} \tag{3.4}$$

$$-4u\sum_{k=0}^{l} R_k R_{l-k} - 4\sum_{k=1}^{l} R_k R_{l-k+1}.$$

From I.2.3 we see that if f_i has an asymptotic expansion it is in the following form:

$$h^n(r)f_i(r, r, t) \underset{t\to 0}{\sim} \sum_{l=0}^{\infty} \frac{R_l[u_i]}{\Gamma(l+\frac{1}{2})} t^{l-1/2}. \tag{3.5}$$

The following proposition tells us that it is possible to factor $e^{-u_i t}$ out of the above asymptotic expansion.

III.1. Proposition. *Summing up those terms in $\sum_{l=0}^{\infty}(R_l[u_i]/\Gamma(l+\frac{1}{2}))t^{l-1/2}$ which contain a fixed monomial of derivatives of u_i of the form $u_i^{(k_1)}u_i^{(k_2)} \cdots u_i^{(k_j)}$, $k_j \neq 0$, gives the sum as $A_{k_1\cdots k_j}e^{-u_i t}u_i \cdots u_i^{(k_j)}t^k$, $A_{k_1\cdots k_j}$ is a constant, k is the power of t in the first monomial of such a form.*

Proof. From (3.4) a straightforward proof by induction shows (cf. [3]) $(\partial/\partial u_i)R_l[u_i] = -(l-\frac{1}{2})R_{l-1}[u_i]$ which in turn implies

$$\frac{\partial}{\partial u_i}\frac{R_l[u_i]}{\Gamma(l+\frac{1}{2})} = -\frac{R_{l-1}[u_i]}{\Gamma(l-1+\frac{1}{2})}. \quad \text{Q.E.D.} \tag{3.6}$$

If we compare the expression after factoring $e^{-u_i t}$ out of (3.5) with the asymptotic expansion in the end of the last section, we find that we should multiply some part of $e^{-u_i t}$ back into the expression. Split u_i into the following two parts:

$$u_i(r) = q(r) + \lambda_i/h^2(r).$$

Then multiplying $e^{-q(r)t}$ back into the expression, we obtain

$$f_i(r, r, t) \underset{t\to 0}{\sim} \frac{1}{2\sqrt{\pi}} h^{-n/2}(r)\exp\left(-\frac{\lambda_i t}{h^2(r)}\right) \sum_{j=0}^{\infty} \tilde{a}_j(r, \Lambda_i)t^{j-1/2}. \tag{3.7}$$

Therefore $\tilde{a}_j(r, \lambda_i)$ is nothing but $(R_j[q, u_i', u_i'', \ldots]/\Gamma(j+\frac{1}{2})) \cdot 2\sqrt{\pi}$, i.e. replace u_i (but not its derivatives) in R_j by q.

The following theorem is essential for the purpose of this section.

III.2. Theorem. $a_j(r, r, \lambda_i) = \tilde{a}_j(r, \lambda_i)$.

PROOF. Consider the expression

(3.8)

$$K_1(r_1, r_2, \lambda_i) = u_i(r_2)\Bigg[a_{l-1} - a_{l-2}\frac{\lambda_i}{h(r_1)h(r_2)} + \frac{1}{2!}a_{l-3}\left(\frac{\lambda_i}{h(r_1)h(r_2)}\right)^2 + \cdots + (-1)^{l-1}\frac{1}{(l-1)!}\left(\frac{\lambda_i}{h(r_1)h(r_2)}\right)^{l-1} a_0\Bigg]$$
$$+ l\left[a_l - a_{l-1}\frac{\lambda_i}{h(r_1)h(r_2)} + \cdots + (-1)^l\frac{1}{l!}\left(\frac{\lambda_i}{h(r_1)h(r_2)}\right)^l a_0\right]$$
$$+ (r_1 - r_2)\frac{\partial}{\partial r_2}\Bigg[a_l - a_{l-1}\frac{\lambda_i}{h(r_1)h(r_2)} + \cdots + (-1)^l\frac{1}{l!}\left(\frac{\lambda_i}{h(r_1)h(r_2)}\right)^l a_0\Bigg]$$
$$+ \frac{\partial}{\partial r_2}\Bigg[a_{l-1} - a_{l-2}\left(\frac{\lambda_i}{h(r_1)h(r_2)}\right) + \cdots + (-1)^l\frac{1}{(l-1)!}\left(\frac{\lambda_i}{h(r_1)h(r_2)}\right)^{l-1} a_0\Bigg].$$

We can express the right-hand side of the above as the sum of terms of the following form:

(3.9)

$$(-1)^k\frac{1}{k!}\left(\frac{\lambda_i}{h(r_1)h(r_2)}\right)^k\Bigg[\left(u_i(r_2) - \frac{\lambda_i}{h(r_1)h(r_2)}\right)a_{l-k-1} - \left(\frac{\partial}{\partial r_2}\right)^2 a_{l-k-1}$$
$$-(r_1 - r_2)\frac{\lambda_i}{h(r_1)}\left(\frac{1}{h(r_2)}\right)' a_{l-k-1} + (l-k)a_{l-k}$$
$$-(r_1 - r_2)\frac{\partial}{\partial r_2}a_{l-k} + \frac{\lambda_i}{h(r_2)}\left(\frac{\lambda_i}{h(r_2)}\right)'' a_{l-k-2}$$
$$+\frac{2\lambda_i i}{h(r_1)}\left(\frac{1}{h(r_2)}\right)'\frac{\partial}{\partial r_2}a_{l-k-2} - \frac{\lambda_i}{h(r_1)}\left(\left(\frac{1}{h(r_2)}\right)'\right)^2 a_{l-k-3}\Bigg].$$

Since a_l satisfies (2.2) of the last section, each term of this form must be zero and so is their sum, i.e. $K_1(r_1, r_2, \lambda_i) = 0$. Similarly, if we denote by $K_2(r_1, r_2, \lambda_i)$ the expression which comes from $K_1(r_1, r_2, \lambda_i)$ by interchanging the roles played by r_1

and r_2, then we also have $K_2(r_1, r_2, \lambda_i) = 0$. Moreover,

$$\text{(3.10)} \qquad \frac{\partial}{\partial r_2} K_1(r_1, r_2, \lambda_i) = 0, \qquad 3\frac{\partial}{\partial r_1} K_1(r_1, r_2, \lambda_i) = 0,$$
$$3\frac{\partial}{\partial r_2} K_2(r_1, r_2, \lambda_i) = 0, \qquad \frac{\partial}{\partial r_1} K_2(r_1, r_2, \lambda_i) = 0.$$

Combining these identities, and letting $r_1 = r_2$, we get

$$\text{(3.11)} \quad \left(l - \frac{1}{2}\right)\left[a_l - \frac{\lambda_i}{h^2} a_{l-1} + \frac{1}{2}\left(\frac{\lambda_i}{h^2}\right)^2 a_{l-2} + \cdots\right]'$$
$$= \frac{1}{4}\left[a_{l-1} - a_{l-1}\frac{\lambda_i}{h^2} + \frac{1}{2!}\left(\frac{i}{h^2}\right)^2 a_{l-3} + \cdots\right]'''$$
$$-\frac{1}{2}u_i'\left[a_{l-1} - a_{l-2} - \frac{\lambda_i}{h^2} + \ldots\right] - u_i\left[a_{l-1} - a_{l-2}\frac{\lambda_i}{h^2} + \cdots\right].$$

Now, since by construction of $\tilde{a}_l$ we have

$$\text{(3.12)} \qquad \left[\tilde{a} - \tilde{a}_{l-1}\frac{\lambda_i}{h^2} + \cdots + \frac{(-1)^l}{l!} a_0 \left(\frac{\lambda_i}{h^2}\right)^l\right] = \frac{R_l}{\Gamma(l + \frac{1}{2})} 2\sqrt{\pi}$$

and from (3.4) by induction we have $R_l' = \frac{1}{4}R_{l-1} - uR_{l-1}' - \frac{1}{2}u'R_{l-1}$, then by substitution we will see that a_l satisfies the same differential equation as above. Since both of them satisfy the same differential equation of first order they may differ by a constant. But because this constant is universal,[3] it is sufficient to check the special case h = const. Then we will get that it must be zero, so that $\tilde{a}_j(r, r, \lambda_i) = \tilde{a}_j(r, \lambda_i)$. Q.E.D.

III.3. COROLLARY.

$$\text{(3.13)} \quad \operatorname{Trace} E \underset{t\to 0}{\sim} \sum_{j=0}^{\infty} \frac{1}{\sqrt{4\pi}} h^{-n}(r) \sum_{i=0}^{\infty} \exp\left(-\frac{\lambda_i t}{h^2(r)}\right) \tilde{a}_j(r, \lambda_i)\phi_i^2(x) t^{j-1/2}.$$

This proves the legitimacy of the addition of the infinitely many asymptotic expansions to form the asymptotic expansion we wanted.

Now we are ready to calculate the coefficients. Now if we let

$$\text{(3.14)} \quad \sum_{i=0}^{\infty} e^{-\lambda_i t}\phi_i^2(x) \underset{t\to 0}{\sim} \frac{1}{\sqrt{4\pi}^{\,n}}\left(C_0 t^{-n/2} + C_1 t^{-n/2}\right.$$
$$\left. + \cdots + C_s t^{-n/2} + \cdots + C_s t^{-n/2+s} + \cdots\right)$$

[3] From the constructions of both a_l and $\tilde{a}_l$ we see that the coefficients involved in them are independent of $h(r)$.

then

(3.15)

$$\sum_{i=0}^{\infty} \lambda_i^l e^{-\lambda_i t}\phi_i^2(x) \underset{t\to 0}{\sim} \frac{1}{\sqrt{4\pi}^n}\Big[C_0\Big(\frac{n}{2}\Big)\Big(\frac{n}{2}+1\Big)\cdots\Big(\frac{n}{2}+l-1\Big)t^{-n/2-l}$$

$$+C_1\Big(\frac{n}{2}-1\Big)\frac{n}{2}\cdots\Big(\frac{n}{2}+l-2\Big)t^{-n/2-l+1}$$

$$+\cdots+C_s\Big(\frac{n}{2}-s\Big)\cdots\Big(\frac{n}{2}+l-s\Big)t^{-n/2-l+s}+\cdots\Big]$$

where $l \geqslant 1$ and

(3.16) $$\sum_{i=0}^{\infty} \lambda_i \exp\left(-\frac{\lambda_i t}{h^2(r)}\right)\phi_i^2(x)$$

$$\underset{t\to 0}{\sim} \frac{1}{\sqrt{4\pi}^n}\left[C_0\Big(\frac{n}{2}\Big)\Big(\frac{n}{2}+1\Big)\cdots\Big(\frac{n}{2}+l-s\Big)\left(\frac{t}{h^2(r)}\right)^{-n/2-l}\right.$$

$$+\cdots+C_s\Big(\frac{n}{2}-s\Big)\cdots\Big(\frac{n}{2}+l-s\Big)$$

$$\left.\cdot\left(\frac{t}{h^2(r)}\right)^{-n/2-l+s}+\cdots\right].$$

Suppose we denote the coefficients of λ_i^l in $\tilde{a}_j^l(r, \lambda_i)$ by $a_j^l(r)$, and then consider those terms containing $t^{-(n+1)/2+k}$ which come from $\tilde{a}_j(r, \lambda_i)$.

(3.17) $$\frac{1}{\sqrt{4\pi}}h^{-n}(r)\frac{1}{\sqrt{4\pi}^n}a_j^l(r)C_s^l\Big(\frac{n}{2}-s\Big)\cdots\Big(\frac{n}{2}+l-s\Big)\left(\frac{t}{h^2(r)}\right)^{n/2+s-l}t^{j-1/2}$$

with $-l+s+j=k$. But since $\deg_{\lambda_i}\tilde{a}_j = \deg_{\lambda_i} a_j(r, r, \lambda_i) \leqslant [2j/3]$ and $s \geqslant 0$ then $j \leqslant 3k$. Now let j vary and collect all those terms containing $t^{-(n+1)/2+k}$. We will have

(3.18) $$\sum_{j=0}^{[2j/3]}\sum_{j=0}^{3k}\frac{1}{\sqrt{4\pi t}^{n+1}}a_j^l C_{k-j+1}\Big(\frac{n}{2}-k+j-l\Big)\cdots$$

$$\Big(\frac{n}{2}-k+j-1\Big)\left(\frac{t}{h^2(r)}\right)^k h^{2j}(r).$$

If we denote

(3.19) $$d_k = \sum_{l=0}^{[2j/3]}\sum_{j=0}^{3k} a_j^l C_{k-j+1}\Big(\frac{n}{2}-k+j-l\Big)\cdots\Big(\frac{n}{2}-k+j-1\Big)h^{2j}(r)$$

then

$$\text{Trace}\, E \underset{t\to 0}{\sim} \sum_{k=0}^{\infty} \frac{1}{\sqrt{4\pi t}^{\,n+1}} d_k \left(\frac{1}{h^2(r)}\right)^k. \tag{3.20}$$

REMARK. When $l = 0$, $(\frac{n}{2} - k + j - l) \cdots (\frac{n}{2} - k + j - 1)$ will be replaced by 1.

IV. The significance of the approach and applications of the result. In the end of the last section we got a formula for the coefficients d_k on $M \times I$ in terms of coefficients c_j and a_j^l where a_j^l are the coefficients of λ_i^l. When we write a_j into the polynomial of λ_i, we should know a_j. Although the a_j can be constructed by the recursive formula which we have derived in §II, a generating function for them will be more helpful. Gel′fand and Dikii have a generating function for R_l which can be exploited to construct a_l as a polynomial in u, u', u'',... and then to translate R_l into the so-called symbolic polynomial, which in turn has a generating function. Judging from the sophisticated way they derived the function, we expect that a generating or even a more directly recursive formula for d_k than the one we got is unlikely.

Now we will give some arguments to show why we cannot start from (0.3), the standard parametrix, and get a reasonable result. Let us consider first the case of the metric cone, $h(r) = r$. Since the first coefficient of the asymptotic expansion of the heat kernel is the reciprocal of the square root of the determinant of the exponential map, we should find the relation between the determinant of the exponential map on the cone and the determinant of the exponential map on the base.[4]

A straightforward calculation leads us to

$$\theta((p,\tau),(Q,\kappa)) = \underline{\theta}(P,Q)\left(\frac{l}{\sin l}\right)^{n-1} \tag{1}$$

where $\theta((p,\tau),(Q,\kappa))$ denotes the determinant of the exponential map from tangent space $M \times I_{(p,\tau)}$ to $M \times I$ evaluated at $(\exp_{(p,\tau)})^{-1}(Q,\kappa)$, $\underline{\theta}(P,Q)$ denotes the corresponding notion on M, and l denotes the distance between P and Q on M. In order to find the coefficients of the asymptotic expansion of the heat kernel we should apply Δ to $\theta^{-1/2}$ [**1**, p. 208]. From the relation $\Delta = \underline{\Delta}/r^2 - (n/r)(\partial/\partial r) - \partial^2/\partial r^2$ and the fact that the right-hand side of the above relation is independent of r we have

$$\Delta^k \theta^{-1/2} = \frac{\underline{\Delta}}{r^{2k}}(\underline{\Delta} + 2n - 2.3)(\underline{\Delta} + 4n - 4.5) \cdots (\underline{\Delta} + (2k-2)n - (2k-2)(2k-1))\theta^{-1/2}\left(\frac{l}{\sin l}\right)^{-(n-1)/2}.$$

This formula shows the complication of a direct computation. There is the difficulty not only of writing out explicitly the operator in the right-hand side into a

[4] More generally, the kth coefficients are something like $\Delta^k\theta^{-1/2}$ on the trace (i.e. $r_1 = r_2$, $x = y$) [**1**, p. 208].

polynomial of $\underline{\Delta}$ but also of applying each term to a product of two functions. Moreover the expression does not lead immediately to the formula of correct type, i.e. coefficients on $M \times I$ are a linear combination of coefficients on base M. Thus our result implies some interesting cancellation. In the case of more general $h(r)$ it becomes even more hopeless since the relation between the determinant of the exponential maps must involve r. This involvement of r will make the application of Δ to the determinant even more complicated and it will be unlikely there is any expression as the above one.

For the first application, if we set $h(r) = r$ and $d_k = 0$ in formula (3.19) this will give us a recursive formula for C_j which is just the coefficient of the asymptotic expansion of the trace of the heat kernel on S^n. The formula so obtained is no more complicated than the corresponding formula obtained by Cahn and Wolf [**2**]. In their formula they have two cases, when n is even or odd. We have a single formula for all n. However their approach is more general and they apply it to the cases of compact symmetric spaces of rank one.

For the second application, let us consider the special case when M is flat. In this case all the coefficients c_i in the expansion for trace of the heat kernel of M except c_0 are zero. Then

$$d_k = a^l_{k+l}\left(\frac{n}{2}\right)\left(\frac{n}{2}+1\right)\cdots\left(\frac{n}{2}+l-1\right)h^{2(k+l)}(r).$$

Here, as before $(\frac{n}{2})(\frac{n}{2}+1)\cdots(\frac{n}{2}+l-1)$ will be repaced by 1 when $l = 0$.

Let us list below the first two nontrivial coefficients:

$$\begin{aligned}
d_1 &= -\frac{n(n-1)}{6}h'(r)^2 - \frac{n}{3}h(r)h''(r),\\
d_2 &= \frac{n(n-1)(5n^2+17n-66)}{360}h'(r)^4\\
&\quad -\frac{n(n^2-36n+41)}{90}h(r)h'(r)^2h'(r)\\
&\quad -\frac{n(2n-3)}{15}h^2(r)h'(r)h'''(r) - \frac{n}{15}h^3(r)h^{Iv}(r)\\
&\quad -\frac{n(n-9)}{60}h^2(r)^2h''(r)^2.
\end{aligned}$$

In general d_k is a polynomial of $h(r)$ and its derivatives with homogeneous differential order $2k$.

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