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q-EXTENSION OF THE p-ADIC GAMMA FUNCTION. II

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ABSTRACT. Taylor series and asymptotic expansions are developed for q-extensions of the p-adic psi (derivative of log-gamma) function "twisted" by roots of unity. Connections with p-adic L-functions and q-expansions of Eisenstein series are discussed. The p-adic series are compared with the analogous classical expansions.

Introduction. We shall study q-extensions of the ψ-function (derivative of log-gamma) and its "twists" (by roots of unity, Dirichlet characters, etc.) in the complex analytic, and especially the p-adic analytic cases. Using expressions for these functions as convolution transforms, we derive two types of expansions for them: Taylor expansions near $x = 0$ (or $x = 1$), and Stirling series for x large. For the usual type of ψ-function (which is the limit of the q-extension as $q \to 1$ in both the classical and p-adic cases), the coefficients in the Taylor and Stirling series are essentially the values of the Riemann zeta-function (or Dirichlet L-functions) at positive integers (Taylor series) and at negative integers (Stirling series). For the q-extensions, these coefficients involve Eisenstein series, as well as values of zeta- or L-functions; in this context, the kth normalized Eisenstein series G_k, for variable k, play the role of q-extensions of $\zeta(k)$.

In the complex analytic case this occurrence of G_k as Taylor coefficient, generalizing $\zeta(k)$, is related in a simple way to the appearance of $\frac{1}{2}\zeta(1-k)$ as the constant term in $G_k(q)$. Namely, a weight-k modular form $f(z)$, $q = e^{2\pi i z}$, satisfies $f(-1/z) = z^k f(z)$, so its behavior as $q = e^{2\pi i z} \to 1^-$ (i.e., as $z \to i0^+$) is directly determined by its behavior as $q \to 0^+$ (i.e., as $z \to i\infty$). In the p-adic case, the connection between the Taylor coefficient, which is a function of q near 1, and the q-expansion of the corresponding p-adic modular form, which is a function of q near 0, is more indirect. The connection is by p-adic analytic continuation, a purely analytic procedure which does not, so far as we know, admit any interpretation in terms of moduli of elliptic curves.

The p-adic construction of the convolution transform requires us to twist the p-adic ψ_q by a number z outside the unit disc around 1, e.g., $z \in \sqrt[d]{1}$, $z \neq 1$, $p \nmid d$. (The use of the letter z in this context should not be confused with its use in $q = e^{2\pi i z}$.) Such a twist, sometimes called "regularization", is routinely needed in order to make a bounded p-adic measure.

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In the complex analytic construction, one is not required to twist, and in fact, classical discussions of psi and q-psi functions generally treat the untwisted functions. However, the following classical examples illustrate that the twisted log-gamma and q-log-gamma functions have arisen naturally in the history of the subject.

EXAMPLE 1. Let $p > 2$ be a prime, let $S_n = \{j = 1, 2, \ldots, pn - 1 \mid (\frac{j}{p}) = 1\}$ be the set of numbers less than pn which are squares modulo p (here $(\frac{}{p})$ is the Legendre symbol), and let $NS_n = \{j = 1, 2, \ldots, pn - 1 \mid (\frac{j}{p}) = -1\}$ be the set of nonsquares.

Problem. Does the ratio

$$\prod_{j \in NS_n} j / \prod_{j \in S_n} j \tag{0.1}$$

approach a finite nonzero limit as $n \to \infty$, and, if so, what is it?

Solution. Using the formula

$$\Gamma(x) = \lim_{n \to \infty} \frac{(n-1)! n^x}{x(x+1) \cdots (x+n-1)}, \tag{0.2}$$

we see that n to the power $\sum_{j=1}^{p-1} (\frac{j}{p}) \frac{j}{p}$ times (0.1) approaches the limit

$$\prod_{j \in S_1} \Gamma(j/p) / \prod_{j \in NS_1} \Gamma(j/p) \tag{0.3}$$

as $n \to \infty$. The exponent $\sum (\frac{j}{p}) \frac{j}{p}$ is zero if and only if $p \equiv 1 \pmod 4$. If $p \equiv -1 \pmod 4$ and $p > 3$, then this exponent is a negative integer equal to minus the class number of $Q(\sqrt{-p})$, according to a well-known formula of Dirichlet. Thus, in the former case (0.1) approaches the limit (0.3), while in the latter case it diverges to $+\infty$.

Here the logarithm of (0.3) is the value at 0 of the log-gamma function twisted by the quadratic Dirichlet character $\chi = (\frac{}{p})$,

$$(\log \Gamma)_\chi(x) \underset{\text{def}}{=} \sum_{j=0}^{p-1} \chi(j) \log \Gamma((x+j)/p).$$

The derivative of this function is typical of the twisted ψ-functions whose q-extensions we shall be studying.

EXAMPLE 2. More generally, for any nontrivial Dirichlet character χ modulo d, the value $(\log \Gamma)_\chi(0)$ is essentially the derivative at 0 of the corresponding Dirichlet L-series (see [**23**, p. 271]):

$$L'(0, \chi) = B_{1,\chi} \log d + (\log \Gamma)_\chi(0),$$

where $B_{1,\chi} = \frac{1}{d} \sum_{j=0}^{d-1} j\chi(j)$. ($B_{1,\chi} = 0$ if and only if $\chi(-1) = 1$.)

EXAMPLE 3. Among the simplest of the Rogers-Ramanujan identities are the following power series identities:

$$\begin{aligned} \prod_{n=0}^{\infty} (1 - q^{5n+1})^{-1} (1 - q^{5n+4})^{-1} &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n}; \\ \prod_{n=0}^{\infty} (1 - q^{5n+2})^{-1} (1 - q^{5n+3})^{-1} &= \sum_{n=0}^{\infty} \frac{q^{n+n^2}}{(q)_n}, \end{aligned} \tag{0.4}$$

where

$$(q)_n = (q;q)_n \underset{\text{def}}{=} (1-q)(1-q^2)\cdots(1-q^n)$$

(more generally,

$$(x;q)_n \underset{\text{def}}{=} (1-x)(1-xq)\cdots\left(1-xq^{n-1}\right)\Big).$$

Both sides of (0.4) converge if q is a real or complex parameter with $|q|<1$. It can also be shown that the ratio of the first product to the second in (0.4) is equal to the continued fraction

$$1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cdots}}}.$$

The identities (0.4) are equivalent to a statement about partitions (see [**1**, Chapter 7] for a detailed discussion).

The logarithm of the ratio of the two products on the left in (0.4) is a twisted version of the q-gamma function

$$\text{(0.5)} \qquad \Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x} = \frac{(1-q)(1-q^2)\cdots}{(1-q^x)(1-q^{x+1})\cdots}(1-q)^{1-x}.$$

Namely,

$$\log\frac{\prod_{n=0}^{\infty}\left(1-q^{5n+1}\right)^{-1}\left(1-q^{5n+4}\right)^{-1}}{\prod_{n=0}^{\infty}\left(1-q^{5n+2}\right)^{-1}\left(1-q^{5n+3}\right)^{-1}} = \sum_{j=1}^{4}\left(\frac{j}{5}\right)\log\Gamma_{q^5}\left(\frac{j}{5}\right).$$

The right-hand side is the value at 0 of the q-log-gamma function twisted by the character $(\frac{\cdot}{5})$, where one defines

$$\text{(0.6)} \qquad \left(\log\Gamma_q\right)_\chi(x) \underset{\text{def}}{=} \sum_{j=0}^{d-1}\chi(j)\log\Gamma_{q^d}((x+j)/d),$$

for a character χ modulo d. It is the derivatives of functions like (0.6) which we shall be studying.

Although the classical and p-adic cases (the first and second sections of the present paper) are logically independent, there is a striking similarity between them. In fact, they are formally the same in the following sense. If the root of unity z used in the twist is replaced by z in the open unit disc about zero, then the same power series (in $\mathbf{Z}[[z,q]]$) occurs as Taylor coefficient classically and p-adically (except for "removal of the p-Euler factor" in the p-adic case). With z in this open disc the measures in the convolution transform are given by essentially the same formula in the classical and p-adic situations. Of course, the theory only has arithmetic or modular meaning when z is a root of unity; hence, the relation between the complex and p-adic functions is formal and indirect.

1. The classical case.

1. *Taylor series*. Let $\varphi(-x)$ be defined by convolution of a function $g(u)$ with a Stieltjes measure $df(u)$ on R^+:

$$\varphi(x) = \int_0^\infty g(u+x)\, df(u). \tag{1.1}$$

We shall suppose in what follows that f and g are such that the integrals below converge.

Expanding

$$g(u+x) = \sum_{j=0}^\infty \frac{g^{(j)}(u)}{j!} x^j \tag{1.2}$$

(or, alternately, noting that $\varphi^{(j)}(0) = \int_0^\infty g^{(j)}(u)\, df(u)$), we have for x small:

$$\varphi(x) = \sum_{j=0}^\infty \frac{x^j}{j!} \int_0^\infty g^{(j)}(u)\, df(u). \tag{1.3}$$

The ψ-function, twisted ψ-functions, q-extension of the ψ-function, and q-extensions of twisted ψ-functions are special cases of (1.1) with rather simple choices of f and g.

EXAMPLES. I. Let $f(u) = u - [u] - \frac{1}{2}$ be the first Bernoulli polynomial (made periodic with period 1), and let $g(u) = -1/u$. Start with the formula [**23**, p.261]

$$\log \frac{\Gamma(x)}{\sqrt{2\pi}} = \left(x - \frac{1}{2}\right) \log x - x - \int_0^\infty \frac{f(u)\, du}{u+x}. \tag{1.4}$$

Differentiating and replacing x by $x+1$, we obtain

$$\psi(1+x) = \log(1+x) - \frac{1}{2(1+x)} + \int_1^\infty \frac{f(u)\, du}{(u+x)^2}.$$

Integrating by parts, we have

$$\begin{aligned} \psi(1+x) &= \log(1+x) - \frac{1}{2(1+x)} + \left.\frac{f(u)}{u+x}\right|_{u=1^-} + \int_{1^-}^\infty \frac{df(u)}{u+x} \\ &= \log(1+x) + \int_{1^-}^\infty \frac{df(u)}{u+x}. \end{aligned} \tag{1.5}$$

To find the Taylor coefficients in the expansion of $\psi(1+x)$, we have $\psi(1) = -\gamma$, and for $j \geqslant 1$,

$$\begin{aligned} \frac{1}{j!}\psi^{(j)}(1) &= \frac{(-1)^{j-1}}{j} + \int_{1^-}^\infty (-1)^j \frac{df(u)}{u^{j+1}} \\ &= \frac{(-1)^{j-1}}{j} + (-1)^j \int_{1^-}^\infty \left(\frac{du}{u^{j+1}} - \frac{d[u]}{u^{j+1}} \right) \\ &= (-1)^{j-1} \int_{1^-}^\infty \frac{d[u]}{u^{j+1}}. \end{aligned}$$

But

$$\int_{1^-}^{\infty} \frac{d[u]}{u^{j+1}} = \sum_{n=1}^{\infty} \frac{1}{n^{j+1}} = \zeta(j+1).$$

This gives the expansion

$$\psi(1+x) = -\gamma + \sum_{j=2}^{\infty} (-1)^j \zeta(j) x^{j-1}. \tag{1.6}$$

Alternately, in the integral in (1.5) we could have expanded

$$g(u+x) = -1/(u+x) = \sum_{j=1}^{\infty} (-1)^j x^{j-1}/u^j$$

for $|x| < 1 \leqslant u$ as in (1.2).

Our reason for belaboring this derivation of a well-known expansion (see, e.g., [**23**, p. 241]) is that it is the prototype for the examples that follow and for our discussion of the p-adic case in §2.

II. Let ρ: $\mathbf{Z} \to \mathbf{C}$ be a periodic function of period d such that $\sum_{a=0}^{d-1}\rho(a) = 0$. The key examples are: (1) $\rho(j) = \chi(j)$, a nontrivial Dirichlet character modulo d; and (2) $\rho(j) = z^j$, for $z \neq 1$ a dth root of unity. Let $g(u) = -1/u$ be the same as in Example I, and set $f_\rho(u) = \sum_{a=0}^{[u]}\rho(a)$.

We define the twisted ψ-function as follows:

$$\psi_\rho(x) \underset{\text{def}}{=} \frac{d}{dx} \sum_{a=0}^{d-1} \rho(a) \log \Gamma\left(\frac{x+a}{d}\right). \tag{1.7}$$

Using the formula (0.2) for $\Gamma(x)$, we easily see that

$$\psi_\rho(x) = -\int_{0^-}^{\infty} \frac{df_\rho(u)}{u+x}. \tag{1.8}$$

If, for example, $\rho = \chi$ is a nontrivial Dirichlet character, so that, in particular, $\rho(0) = 0$, then the integral really goes from 1^- to ∞, and for $|x| < 1 \leqslant u$ we have

$$\psi_\rho(x) = \sum_{j=1}^{\infty} (-1)^j x^{j-1} \int_{1^-}^{\infty} \frac{df_\chi(u)}{u^j} = \sum_{j=1}^{\infty} (-1)^j x^{j-1} L(j,\chi), \tag{1.9}$$

since

$$\int_{1^-}^{\infty} u^{-j}\, df_\chi(u) = \sum_{n=1}^{\infty} \chi(n) n^{-j} = L(j,\chi).$$

In the example $\rho(j) = z^j$, we have $df_z(u-1) = z^{-1}\, df_z(u)$ for $u \geqslant 1$ (we use the notation $f_z = f_\rho$, $\psi_z = \psi_\rho$ here), so that by (1.8),

$$\begin{aligned} \psi_z(x+1) &= -\frac{1}{z}\int_{1^-}^{\infty} \frac{df_z(u)}{u+x} \\ &= \frac{1}{z}\sum_{j=1}^{\infty} (-1)^j x^{j-1} \int_{1^-}^{\infty} \frac{df_z(u)}{u^j} \\ &= \frac{1}{z}\sum_{j=1}^{\infty} (-1)^j x^{j-1} L(j,z), \end{aligned} \tag{1.10}$$

where

$$L(j,z) \underset{\text{def}}{=} \sum_{n=1}^{\infty} \frac{z^n}{n^j}.$$

Note that (1.8), combined with the rule $df_z(u-1) = z^{-1}\,df_z(u)$, gives the functional relation

$$z\psi_z(x+1) - \psi_z(x) = 1/x, \tag{1.11}$$

which generalizes the relation $\psi(x+1) - \psi(x) = 1/x$ for the usual ψ-function. Combining (1.11) with the above Taylor series for $\psi_z(x+1)$ gives

$$\psi_z(x) = -\frac{1}{x} + \sum_{j=1}^{\infty} (-1)^j x^{j-1} L(j,z). \tag{1.12}$$

III. Let $0 < |q| < 1$, and take $f(u) = [u]$, $g(u) = (\log q)(q^u/(1-q^u))$. The q-gamma function defined by (0.5) has the properties:

(1) $\Gamma_q(x+1) = ((1-q^x)/(1-q))\Gamma_q(x)$,

(2) $\Gamma_q(x) \to \Gamma(x)$ as $q \to 1^-$.

The logarithmic derivative of $\Gamma_q(x)$ is, by (0.5),

$$\psi_q(x) = -\log(1-q) + \log q \sum_{j=0}^{\infty} \frac{q^{x+j}}{1-q^{x+j}}. \tag{1.13}$$

Hence,

$$\begin{aligned}\psi_q(1+x) &= -\log(1-q) + \log q \sum_{j=1}^{\infty} \frac{q^{x+j}}{1-q^{x+j}} \\ &= -\log(1-q) + \int_{1^-}^{\infty} g(u+x)\,df(u).\end{aligned} \tag{1.14}$$

Note that for $q < 1$ the integral always converges, whereas in the limit when $q \to 1^-$ and $g(u)$ becomes $-1/u$ (Example I) one must modify $f(u)$ to make the integral converge.

Now define

$$P_k(q) = \left(q\frac{d}{dq}\right)^{k-1} \frac{q}{1-q} = \sum_{n=1}^{\infty} n^{k-1} q^n, \tag{1.15}$$

so that $(1-q)^k P_k(q) \in \mathbf{Z}[q]$. Since $g(u) = (\log q)(q^u/(1-q^u))$, we have $g^{(k-1)}(u) = (\log q)^k P_k(q^u)$, so that for $k \geqslant 2$,

$$\frac{\psi_q^{(k-1)}(1)}{(k-1)!} = \frac{(\log q)^k}{(k-1)!} \int_{1^-}^{\infty} P_k(q^u)\,df(u);$$

this integral is equal to $\sum_{j=1}^{\infty} P_k(q^j) = \sum_{m=1}^{\infty} \sigma_{k-1}(m) q^m$, where

$$\sigma_{k-1}(m) \underset{\text{def}}{=} \sum_{d|m} d^{k-1}.$$

It is common to set $\psi_q(1) = -\gamma_q$ and call γ_q the q-Euler constant. Thus

$$\gamma_q = \log(1-q) - \log q \sum_{j=1}^{\infty} \frac{q^j}{1-q^j}.$$

Now for $k \geqslant 4$ an even integer, the weight-k Eisenstein series

$$E_k(z) = \sum_{\substack{m,n \in \mathbf{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(mz+n)^k} \tag{1.16}$$

has the following expression in terms of $q = e^{2\pi i z}$ [22]:

$$E_k(z) = \frac{2(2\pi i)^k}{(k-1)!} G_k(q), \tag{1.17}$$

where

$$G_k(q) \underset{\text{def}}{=} \frac{1}{2}\zeta(1-k) + \sum_{m=1}^{\infty} \sigma_{k-1}(m) q^m = -\frac{B_k}{2k} + \sum_{m=1}^{\infty} \sigma_{k-1}(m) q^m, \tag{1.18}$$

in which B_k is the kth Bernoulli number:

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}. \tag{1.19}$$

The definition (1.18) makes sense for any $k \neq 0$ (since $\zeta(1-k)$ and $\sigma_{k-1}(m) = \sum_{d|m} d^{k-1}$ make sense), but the relation to modular forms for $SL(2, \mathbf{Z})$ only holds for $k = 4, 6, 8, \ldots$.

Returning to ψ_q, we thus have

$$\psi_q(1+x) = -\gamma_q + \sum_{k=2}^{\infty} \frac{x^{k-1}}{(k-1)!} (\log q)^k \left(G_k(q) + \frac{B_k}{2k} \right).$$

Using (1.19) with t replaced by $x \log q$, we can rewrite this in the form

$$\psi_q(1+x) = -\gamma_q + \frac{1}{4}\log q - \frac{1}{2x} + \frac{1}{2}\frac{\log q}{q^x - 1} + \sum_{k=2}^{\infty} \frac{(\log q)^k}{(k-1)!} G_k(q) x^{k-1}. \tag{1.20}$$

Comparing with the Taylor expansion (1.6) for the usual ψ-function, we see that

$$\frac{(-\log q)^k}{(k-1)!} G_k(q) \quad \text{corresponds to } \zeta(k); \tag{1.21}$$

in particular,

$$\lim_{q \to 1^-} \frac{(-\log q)^k}{(k-1)!} G_k(q) = \zeta(k). \tag{1.22}$$

(Thus one might want to take $(-\log q)^s G_s(q)/\Gamma(s)$ as a q-extension of $\zeta(s)$.)

REMARKS. 1. For even $k \geqslant 4$, the relationship (1.22) follows from the fact that $G_k(0) = \frac{1}{2}\zeta(1-k)$ (see (1.18)). Namely, for $q = e^{2\pi i z}$, we have

$$\frac{(-\log q)^k}{(k-1)!} G_k(q) = \frac{(-2\pi i z)^k}{(k-1)!} G_k(q) = \frac{z^k}{2} E_k(z) = \frac{1}{2} E_k\left(-\frac{1}{z}\right)$$

by the modular property of E_k. But as $-1/z \to i\infty$ along the positive imaginary axis (i.e., as $q \to 1^-$), this gives

$$\frac{1}{2} E_k(i\infty) = \frac{1}{2} \sum_{n \in \mathbf{Z}, n \neq 0} \frac{1}{n^k} = \zeta(k).$$

2. If one takes $(d^2/dx^2)\log$ of both sides of the relation $\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x$, one obtains

$$\psi'(x) + \psi'(1-x) = \pi^2/\sin^2 \pi x. \tag{1.23}$$

The right side has the partial fraction decomposition

$$\frac{\pi^2}{\sin^2 \pi x} = (2\pi i)^2 \frac{e^{2\pi i x}}{(1-e^{2\pi i x})^2} = \sum_{n\in\mathbf{Z}} \frac{1}{(x+n)^2}, \tag{1.24}$$

while $\psi'(x)$ is equal to $\sum_{n=0}^{\infty} 1/(x+n)^2$.Thus, (1.23) expresses the fact that the sum for $\psi'(x)$ is "half" of the sum for $\pi^2 \csc^2 \pi x$.

The situation with the q-extension $\psi_q'(x)$ is similar. For simplicity, we take one more derivative, and we use (1.13) and (1.24) to write

$$\begin{aligned} \psi_q''(x) &= \frac{d}{dx}(\log q)^2 \sum_{j=0}^{\infty} \frac{q^{x+j}}{(1-q^{x+j})^2} \\ &= (2\pi i z)^2 \sum_{j=0}^{\infty} \frac{d}{dx} \frac{e^{2\pi i z(x+j)}}{(1-e^{2\pi i z(x+j)})^2} \\ &= z^2 \sum_{j=0}^{\infty} \frac{d}{dx} \sum_{n\in\mathbf{Z}} \frac{1}{(z(x+j)+n)^2} \\ &= -2 \sum_{j\geqslant 0,\, n\in\mathbf{Z}} \frac{1}{(x+j+n(-1/z))^3}; \end{aligned} \tag{1.25}$$

$$\psi_q''(1-x) = -2 \sum_{j<0,\, n\in\mathbf{Z}} -\frac{1}{(x+j+n(-1/z))^3}. \tag{1.26}$$

So ψ_q'' is "half" of the sum for $\wp'(x, -1/z)$ (where $\wp'$ denotes the derivative of the Weierstrass $\wp$-function); that is, subtracting (1.26) from (1.25) gives

$$\frac{d^3}{dx^3}\log \Gamma_q(x)\Gamma_q(1-x) = -2 \sum_{j,n\in\mathbf{Z}} \frac{1}{(x+j+n(-1/z))^3} = \wp'\left(x, -\frac{1}{z}\right).$$

Thus, the expansion (1.20) for $\psi_q(x+1)$ is closely related to the expansion

$$\begin{aligned} \wp\left(x, -\frac{1}{z}\right) &= \frac{1}{x^2} + \sum_{k=4}^{\infty} E_k\left(-\frac{1}{z}\right)(k-1)x^{k-2} \\ &= \frac{1}{x^2} + \sum_{\substack{k\geqslant 4 \\ k \text{ even}}} \frac{2(\log q)^k}{(k-1)!} G_k(q)(k-1)x^{k-2}. \end{aligned} \tag{1.27}$$

Namely, if one adds (1.20) plus $(\log q)\,(q^x/(1-q^x))$ (which will give $\psi_q(x)$ on the left) to (1.20), with x replaced by $-x$, then the odd terms cancel in the sum on the right, and the derivative of the resulting expression is (up to a constant not containing x) equal to (1.27).

IV. Let ρ and f_ρ be as in Example II, and let $g(u) = (\log q)(q^u/(1-q^u))$ as in Example III. We define the q-extension of the twisted ψ-function (1.7) as follows:

$$\psi_{q,\rho}(x) \underset{\text{def}}{=} \frac{1}{d} \sum_{a=0}^{d-1} \rho(a) \psi_{q^d}\left(\frac{x+a}{d}\right). \tag{1.28}$$

By (1.13),

$$\psi_{q,\rho}(x) = \log q \sum_{j=0}^{\infty} \frac{\rho(j)q^{x+j}}{1-q^{x+j}} = \int_{0^-}^{\infty} g(u+x)\,df_\rho(u). \tag{1.29}$$

If $\rho = \chi$ is a nontrivial Dirichlet character, then for $|x| < 1 \leq u$,

$$\psi_{q,\chi}(x) = \sum_{k=1}^{\infty} x^{k-1} \frac{(\log q)^k}{(k-1)!} \int_{1^-}^{\infty} P_k(q^u)\,df_\chi(u),$$

where P_k is defined by (1.15). The integral here is equal to $\sum_{j=1}^{\infty}\rho(j)P_k(q^j) = \sum_{m=1}^{\infty}\sigma_{k-1,\chi}(m)q^m$, with

$$\sigma_{k-1,\chi}(m) \underset{\text{def}}{=} \sum_{d|m} \chi\left(\frac{m}{d}\right) d^{k-1}.$$

If $\rho(j) = z^j$, where $z \neq 1$ is a dth root of unity, then

$$\begin{aligned}\psi_{q,z}(x+1) &= \frac{1}{z}\int_{1^-}^{\infty} g(u+x)\,df_z(u)\\ &= \frac{1}{z}\sum_{k=1}^{\infty} x^{k-1}\frac{(\log q)^k}{(k-1)!}\int_{1^-}^{\infty} P_k(q^u)\,df_z(u),\end{aligned}$$

in which the integral is equal to

$$\sum_{m=1}^{\infty} \sigma_{k-1,z}(m)q^m, \quad \text{with } \sigma_{k-1,z}(m) \underset{\text{def}}{=} \sum_{d|m} z^{m/d}d^{k-1}. \tag{1.30}$$

In analogy to (1.11), $\psi_{q,z}$ satisfies the relationship $z\psi_{q,z}(x+1) - \psi_{q,z}(x) = -(\log q)(q^x/(1-q^x))$, which combines with the above Taylor series for $\psi_{q,z}(x+1)$ to give

$$\psi_{q,z}(x) = \log q \frac{q^x}{1-q^x} + \sum_{k=1}^{\infty} x^{k-1}\frac{(\log q)^k}{(k-1)!}\int_{1^-}^{\infty} P_k(q^u)\,df_z(u). \tag{1.31}$$

2. *Stirling series.* As before, let $\varphi(x)$ be of the form $\int_0^\infty g(u+x)\,df(u)$. Heuristically, we would like to consider x to be large, u to be small compared to x; then expand $g(u+x)$ with x (rather than u) as the center: $g(u+x) = \sum(g^{(j)}(x)/j!)u^j$; and finally write

$$\varphi(x) \sim \sum_{j=0}^{\infty} \frac{g^{(j)}(x)}{j!}\int_0^{\infty} u^j\,df(u). \tag{1.32}$$

Unfortunately, for positive j, the integrals $\int_0^\infty u^j\,df(u)$ diverge in the examples I–IV above, although heuristically we might take, for example, $\int_0^\infty u^j d[u] = \sum n^j = \zeta(-j)$. Because of the divergence, we have to proceed somewhat differently to obtain the asymptotic series in the classical case; but we shall see that in the p-adic case there will be no convergence problem, and the p-adic version of (1.32) will be literally correct.

We now make the additional assumption that $f(u)$ is periodic of period d and

$$\int_0^d f(u)\,du = 0. \tag{1.33}$$

We write f as a Fourier series

$$f(u)=\sum_{0\neq n\in\mathbf{Z}} a_n e^{2\pi i n u/d}.$$

Let $f^{(-1)}$ denote the integral of f normalized so that $\int_0^d f^{(-1)}(u)\,du=0$, and define $f^{(-j)}$ inductively as $(f^{(-j+1)})^{(-1)}, j=2,3,\ldots$. Then

$$f^{(-j)}(u)=\left(\frac{d}{2\pi i}\right)^j \sum_{0\neq n\in\mathbf{Z}} \frac{a_n}{n^j} e^{2\pi i n u/d}.$$

In particular, $f^{(-j)}(0)$ is the value of the corresponding Dirichlet series:

$$f^{(-j)}(0)=\left(\frac{d}{2\pi i}\right)^j \sum_{0\neq n\in\mathbf{Z}} \frac{a_n}{n^j}. \tag{1.34}$$

EXAMPLES REVISITED. I. If $f(u)=u-[u]-\frac{1}{2}=B_1(u-[u])$, then $f^{(-j)}(u)=(1/(j+1)!)B_{j+1}(u-[u])$ are the successive Bernoulli polynomials, and

$$f^{(-j)}(0)=\frac{1}{(j+1)!}B_{j+1}=-\frac{1}{j!}\zeta(-j). \tag{1.35}$$

(Alternately, we have $f(u)=-\frac{1}{2\pi i}\sum\frac{1}{n}e^{2\pi i n u}$, and so by (1.34):

$$f^{(-j)}(0)=-\frac{1}{(2\pi i)^{j+1}}\sum_{0\neq n\in\mathbf{Z}}\frac{1}{n^{j+1}}=-\frac{1}{j!}\zeta(j)$$

by the functional equation for $\zeta(s)$.)

Differentiating (1.4) and successively integrating by parts, we have

$$\begin{aligned}\psi(x)&=\log x-\frac{1}{2x}+\int_0^\infty\frac{f(u)}{(u+x)^2}du\\&=\log x-\frac{1}{2x}-\frac{f^{(-1)}(0)}{x^2}+2\int_0^\infty\frac{f^{(-1)}(u)}{(u+x)^3}du\\&=\cdots=\log x-\frac{1}{2x}-\frac{f^{(-1)}(0)}{x^2}-\frac{2f^{(-2)}(0)}{x^3}\\&\quad-\cdots-\frac{j!f^{(-j)}(0)}{x^{j+1}}+(j+1)!\int_0^\infty\frac{f^{(-j)}(u)}{(u+x)^{j+2}}du.\end{aligned}$$

Making obvious estimates for the integral and using (1.35), we see that

$$\log x+\sum_{j=0}^{\infty}\frac{\zeta(-j)}{(-x)^{j+1}} \tag{1.36}$$

is an asymptotic series for $\psi(x)$.

II. Let ρ be as before, but now modify f_ρ by setting $f_\rho(u)=B_{1,\rho}+\sum_{a=0}^{[u]}\rho(a)$, where $B_{1,\rho}=(1/d)\sum_{a=0}^{d-1}a\rho(a)$; in that way we have (1.33). As in Example I, we first rewrite $\int g(u+x)\,df(u)$ as $\int g'(u+x)f(u)\,du$ and then continue integrating by parts j more times:

$$\psi_\rho(x) = -\int_{0^-}^{\infty} \frac{df_\rho(u)}{u+x} = \frac{B_{1,\rho}}{x} - \int_0^\infty \frac{f_\rho(u)}{(u+x)^2}\,du$$

$$= \frac{B_{1,\rho}}{x} + \frac{f_\rho^{(-1)}(0)}{x^2} + \cdots + \frac{j!f_\rho^{(-j)}(0)}{x^{j+1}} - (j+1)!\int_0^\infty \frac{f_\rho^{(-j)}(u)}{(u+x)^{j+2}}\,du.$$

We find, as in Example I, that $\psi_\rho(x)$ has asymptotic series

$$\psi_\rho(x) \sim \sum_{j=0}^{\infty} \frac{L(-j,\rho)}{(-x)^{j+1}}. \tag{1.37}$$

III. Let $f(u) = u - [u] - \frac{1}{2}$ be as in Example I, and let

$$g(u) = \log q \frac{q^u}{1-q^u},$$

where q is a parameter, $|q| < 1$. By (1.13) we have

$$\begin{aligned} \psi_q(x) &= -\log(1-q) + \int_{0^-}^{\infty} g(u+x)\,d[u] \\ &= -\log(1-q) - \int_{0^-}^{\infty} g(u+x)\,df(u) + \int_0^\infty g(u+x)\,du \\ &= -\log(1-q) + \log(1-q^x) - \int_{0^-}^{\infty} g(u+x)\,df(u) \\ &= \log\frac{1-q^x}{1-q} + \frac{1}{2}g(x) + \int_0^\infty g'(u+x)f(u)\,du. \end{aligned}$$

Proceeding just as in Example I, we obtain

$$\psi_q(x) \sim \log\frac{1-q^x}{1-q} + \sum_{j=0}^{\infty} \frac{\zeta(-j)}{j!} g^{(j)}(x), \tag{1.38}$$

where

$$g^{(j)}(x) = (\log q)^{j+1}\left(t\frac{d}{dt}\right)^j \frac{t}{1-t}\bigg|_{t=q^x} = (\log q)^{j+1} P_{j+1}(q^x).$$

This is D. Moak's **[20]** asymptotic series for ψ_q. (Note that our use of the notation P_j is slightly different from Moak's.)

IV. Let $f(u) = f_\rho(u)$ be as in Example II, and let $g(u) = (\log q)(q^u/(1-q^u))$. Following the procedure in Example III, we find the asymptotic series,

$$\psi_{q,\rho}(x) \sim \sum_{j=0}^{\infty} \frac{L(-j,\rho)}{j!} g^{(j)}(x). \tag{1.39}$$

Note that in Examples III and IV, as $q \to 1^-$ we have

$$\frac{1}{j!}g^{(j)}(x) \to \frac{1}{j!}\frac{d^j}{dx^j}\left(-\frac{1}{x}\right) = \frac{1}{(-x)^{j+1}},$$

so we obtain the asymptotic series in Examples I and II, as expected.

2. The p-adic case. Let $\mathbf{Q}_p$ denote the field of p-adic numbers, $\mathbf{Z}_p$ denote the ring of p-adic integers, $\mathbf{Z}_p^* = \mathbf{Z}_p - p\mathbf{Z}_p$, $a + (p^N) = \{x \in \mathbf{Z}_p \mid x \equiv a \bmod p^N\}$, and $\mathbf{C}_p$ denote the completion of the algebraic closure of $\mathbf{Q}_p$. The p-adic absolute value $| \ |_p$ on $\mathbf{C}_p$ is normalized so that $|p|_p = 1/p$.

The compact-open subsets of $\mathbf{Z}_p$ are finite disjoint unions of sets of the form $a + (p^N)$. A $\mathbf{C}_p$-valued measure μ on $\mathbf{Z}_p$ is a bounded, finitely additive map from compact-open subsets to $\mathbf{C}_p$. The integral of a continuous function g: $\mathbf{Z}_p \to \mathbf{C}_p$ with respect to μ, defined as the usual limit of Riemann sums, always exists. These facts are easy to prove; for details, see, for example, [**12**, Chapter 2].

Let $z \in \mathbf{C}_p$ be outside the open unit disc about 1: $|z - 1|_p \geqslant 1$. Define

$$\mu_z\big(a + (p^N)\big) = z^a / \left(1 - z^{p^N}\right),$$

where a is the least nonnegative integer representative: $0 \leqslant a < p^N$. Then it is easy to prove [**14**, Chapter 2] that:

(1) μ_z extends to a measure on $\mathbf{Z}_p$, with $|\mu_z(U)|_p \leqslant 1$ for all compact-open U.

(2) If z is a root of unity, $\rho(j) = z^j$ (see Examples II and IV in §1), and $L(s, z) = L(s, \rho) = \sum_{n=1}^{\infty} z^n/n^s$ (continued analytically onto the complex s-plane), then the number $L(-k, z) \in \mathbf{Q}(z)$ is given by the p-adic formula

$$L(-k, z) = \int_{\mathbf{Z}_p} x^k \, d\mu_z(x). \tag{2.1}$$

(3) If ω: $\mathbf{Z}_p^* \to 1^{1/(p-1)}$ is the Teichmüller character (considered to be either $\mathbf{C}$-valued or $\mathbf{C}_p$-valued; we suppose we have fixed imbeddings of the algebraic numbers in both $\mathbf{C}$ and $\mathbf{C}_p$), $\langle x \rangle = x/\omega(x)$ for $x \in \mathbf{Z}_p^*$, and

$$L^*(s, \omega^k, z) \underset{\text{def}}{=} \sum_{n \geqslant 1,\, p \nmid n} \omega^k(n) \frac{z^n}{n^s},$$

then the p-adic function

$$L_p(s, \omega^j, z) \underset{\text{def}}{=} \int_{\mathbf{Z}_p^*} \langle x \rangle^{-s} \omega^{j-1}(x) \, d\mu_z(x), \tag{2.2}$$

$|z - 1|_p \geqslant 1$, $s \in \mathbf{Z}_p$, when z is a root of unity interpolates the algebraic values

$$L^*(-k, \omega^{j-k-1}, z) = \int_{\mathbf{Z}_p^*} x^k \omega^{j-k-1}(x) \, d\mu_z(x).$$

(4) If a function φ: $\mathbf{C}_p \to \mathbf{C}_p$ is defined by $\varphi(x) = \int_{\mathbf{Z}_p} g(u + x) \, d\mu_z(u)$ (where g: $\mathbf{C}_p \to \mathbf{C}_p$ is continuous; in examples, φ and g might only be defined on a subset of $\mathbf{C}_p$), then φ satisfies

$$z\varphi(x + 1) - \varphi(x) = -g(x). \tag{2.3}$$

(5) If $\varphi^*(x)$ is defined by $\varphi^*(x) = \int_{\mathbf{Z}_p^*} g(u + x) \, d\mu_z(u)$, then

$$z^p \varphi^*(x + p) - \varphi^*(x) = -\sum_{a=1}^{p-1} z^a g(x + a). \tag{2.4}$$

A basic example of the construction of φ and φ^* occurs when $g(u) = -1/u$. For any $z \in \mathbf{C}_p$ with $|z-1|_p \geqslant 1$, we define (see [**13** or **14**])

$$\psi_{p,z}(x) = -\int_{\mathbf{Z}_p} \frac{1}{u+x} d\mu_z(u), \qquad x \in \mathbf{C}_p - \mathbf{Z}_p; \tag{2.5}$$

$$\psi^*_{p,z}(x) = -\int_{\mathbf{Z}_p^*} \frac{1}{u+x} d\mu_z(u), \qquad x \in \mathbf{C}_p - \mathbf{Z}_p^*. \tag{2.6}$$

By (2.3), $\psi_{p,z}$ satisfies the relationship

$$z\psi_{p,z}(x+1) - \psi_{p,z}(x) = 1/x \tag{2.7}$$

(compare with (1.11)). It is also easy to check that $\psi_{p,z}$ satisfies the "Gauss multiplication formula"

$$\psi_{p,z}(x) = \frac{1}{m} \sum_{h=0}^{m-1} z^h \psi_{p,z^m}\left(\frac{x+h}{m}\right), \tag{2.8}$$

for any positive integer m, and the "Euler parity relation"

$$\psi_{p,z}(x) = z^{-1}\psi_{p,z^{-1}}(1-x). \tag{2.9}$$

The function $\psi^*_{p,z}(x)$ relates to $\psi_{p,z}(x)$ as follows ("removing the p-Euler factor"):

$$\psi^*_{p,z}(x) = \psi_{p,z}(x) - \frac{1}{p}\psi_{p,z^p}\left(\frac{x}{p}\right).$$

REMARK. Compare with Example II of §1 when $\rho(z) = z^j$. In the classical case the measure $df_z(u)$ gives the integer $u = a$ the point mass z^a. In the p-adic case, if we take $|z|_p < 1$, then as $N \to \infty$, the measure of the interval $a + (p^N)$ around a has measure $\mu_z(a + (p^N)) = z^a/(1 - z^{p^N})$, which approaches z^a; that is, formally the measure $\mu_z(u)$ also gives $u = a$ point mass z^a. Thus, for $|z|_p < 1$ the p-adic construction is formally the same as the classical one: $\psi_z(x)$ and $\psi_{p,z}(x)$ are given by the same series $-\sum_{a=0}^{\infty} z^a/(x+a)$. However, the functions ψ_z and $\psi_{p,z}$ are of arithmetic interest (relate to Dirichlet L-series or modular forms, see below) only when z is a root of unity. In other words, we must extend analytically beyond the disc where ψ_z and $\psi_{p,z}$ are formally the same in order to reach values of z for which the functions are of number-theoretic interest. Note that $\psi_{p,z}$ and ψ_z satisfy the same relations (2.7) and (2.8).

Now let $q \in \mathbf{C}_p$ be a parameter with $|q-1|_p < 1$. We want the function q^u to make sense for certain $u \in \mathbf{C}_p$. If $u \in \mathbf{Z}_p$, then q^u converges for any $|q-1|_p < 1$. More generally, q^u can be defined as $\exp(u \log_p q)$ for any $u \in \mathbf{C}_p$ with $|u|_p < r_q$, where $r_q = |\log_p q|_p^{-1} p^{-1/(p-1)}$. We shall usually assume that $|q-1|_p < p^{-1/(p-1)}$, in which case

$$r_q = \frac{1}{|q-1|_p p^{1/(p-1)}} > 1.$$

Let

$$g(u) = \log_p q \frac{q^u}{1-q^u} = -\frac{d}{du} \log_p \frac{1-q^u}{1-q}.$$

Note that $\lim_{q\to 1} g(u) = -1/u$. Also define

$$g^*(u) = g(u) - g(pu) = \log_p q \left(\frac{q^u}{1-q^u} - \frac{q^{pu}}{1-q^{pu}} \right).$$

THEOREM 1. *Let* $|z-1|_p \geq 1$, $|q-1|_p < 1$. *The functions*

$$\psi_{p,q,z}(x) \underset{\text{def}}{=} \int_{\mathbf{Z}_p} g(u+x)\, d\mu_z(u)$$

$$= \log_p q \int_{\mathbf{Z}_p} \frac{q^{u+x}}{1-q^{u+x}} d\mu_z(u), \qquad |x|_p < r_q,\ x \notin \mathbf{Z}_p;$$

$$\psi^*_{p,q,z}(x) \underset{\text{def}}{=} \int_{\mathbf{Z}_p^*} g(u+x)\, d\mu_z(u)$$

$$= \log_p q \int_{\mathbf{Z}_p^*} \frac{q^{u+x}}{1-q^{u+x}} d\mu_z(u), \qquad |x|_p < r_q,\ x \notin \mathbf{Z}_p^*;$$

$$\psi^{**}_{p,q,z}(x) \underset{\text{def}}{=} \int_{\mathbf{Z}_p^*} g^*(u+x)\, d\mu_z(u)$$

$$= \log_p q \int_{\mathbf{Z}_p^*} \left(\frac{q^{(u+x)}}{1-q^{u+x}} - \frac{q^{p(u+x)}}{1-q^{p(u+x)}} \right) d\mu_z(u), \quad |x|_p < r_q,\ x \notin \mathbf{Z}_p^*,$$

satisfy the following relations:

$$(2.10) \qquad z\psi_{p,q,z}(x+1) - \psi_{p,q,z}(x) = -\log_p q \frac{q^x}{1-q^x} = \frac{d}{dx} \log_p \frac{1-q^x}{1-q};$$

$$(2.11) \qquad \psi_{p,q,z}(x) = \frac{1}{m} \sum_{h=0}^{m-1} z^h \psi_{p,q^m,z^m}\left(\frac{x+h}{m} \right), \quad \text{for } m = 1, 2, 3, \ldots;$$

$$(2.12) \qquad \psi_{p,q,z}(x) = z^{-1} \psi_{p,q^{-1},z^{-1}}(1-x);$$

$$(2.13) \qquad \psi_{p,q^{-1},z}(x) = \psi_{p,q,z}(x) + \frac{\log_p q}{1-z};$$

$$(2.14) \qquad \psi^*_{p,q,z}(x) = \psi_{p,q,z}(x) - \frac{1}{p} \psi_{p,q^p,z^p}(x/p);$$

$$(2.15) \qquad z^p \psi^*_{p,q,z}(x+p) - \psi^*_{p,q,z}(x) = -\sum_{a=1}^{p-1} z^a \log_p q \frac{q^{x+a}}{1-q^{x+a}};$$

$$(2.16) \qquad \psi^{**}_{p,q,z}(x) = \psi^*_{p,q,z}(x) - \frac{1}{p} \psi^*_{p,q^p,z^p}(x).$$

PROOF. (2.10) is an immediate consequence of (2.3). To prove the "Gauss multiplication formula" (2.11), we write the right side as

$$\lim_{N\to\infty} \frac{1}{m} \sum_{h=0}^{m-1} z^h \log_p q \sum_{j=0}^{p^N-1} \frac{q^{m(j+(x+h)/m)}}{1-q^{m(j+(x+h)/m)}} \frac{z^{mj}}{1-z^{mp^N}}$$

$$= \lim_{N\to\infty} \log_p q \sum_{h,j} \frac{q^{mj+h+x}}{1-q^{mj+h+x}} \frac{z^{mj+h}}{1-z^{mp^N}}.$$

If we write $mj + h = p^N k + l$, $k = 0, 1, \ldots, m-1$, $l = 0, 1, \ldots, p^N - 1$; and if we use the fact that $q^{p^N k + l} \to q^l$ as $N \to \infty$, we find that the last limit is equal to

$$\lim_{N\to\infty} \log_p q \sum_{l=0}^{p^N-1} \frac{q^{l+x}}{1-q^{l+x}} z^l \sum_{k=0}^{m-1} \frac{z^{p^N k}}{1-z^{mp^N}} = \lim_{N\to\infty} \log_p q \sum_{l=0}^{p^N-1} \frac{q^{l+x}}{1-q^{l+x}} \frac{z^l}{1-z^{p^N}}$$
$$= \log_p q \int_{\mathbf{Z}_p} \frac{q^{u+x}}{1-q^{u+x}} d\mu_z(u),$$

as desired. The proofs of (2.12)–(2.16) are equally straightforward. Q.E.D.

REMARK. The "multiplication formula" (2.11) and the parity relation (2.12) can be combined into the following more general identity:

$$\psi_{p,q,z}(x) = \frac{1}{|m|} \sum z^h \psi_{p,q^m,z^m}\left(\frac{x+h}{m}\right),$$

where the summation is over $\min(m,0) \leqslant h < \max(m,0)$ and m is any positive or negative integer.

Returning to the general situation, $\varphi(x) = \int_{\mathbf{Z}_p} g(u+x)\, d\mu(u)$, we note that the following two expansions are possible:

$$\text{(2.17)} \qquad \text{for } x \text{ small,} \quad g(u+x) = \sum_{j=0}^{\infty} \frac{x^j}{j!} g^{(j)}(u)$$
$$\Rightarrow \quad \varphi(x) = \sum_{j=0}^{\infty} \frac{x^j}{j!} \int_{\mathbf{Z}_p} g^{(j)}(u)\, d\mu(u);$$

$$\text{(2.18)} \qquad \text{for } x \text{ large,} \quad g(u+x) = \sum_{j=0}^{\infty} \frac{u^j}{j!} g^{(j)}(x)$$
$$\Rightarrow \quad \varphi(x) = \sum_{j=0}^{\infty} \frac{g^{(j)}(x)}{j!} \int_{\mathbf{Z}_p} u^j\, d\mu(u).$$

The second of these expansions holds whenever the infinite sum converges; since the integral is bounded, we have convergence if $|g^{(j)}(x)/j!|_p \to 0$. The first expansion holds for $|x|_p < p^{-1/(p-1)}$ provided that the $g^{(j)}(u)$ are defined, continuous, and bounded (uniformly in j) on the support of μ. We can apply the second expansion for $\mu = \mu_z$ and $\mu = \mu_z|_{\mathbf{Z}_p^*}$ and for $g(u) = -1/u$ and $g(u) = (\log_p q)(q^u/(1-q^u))$. We can apply the first expansion for $\mu = \mu_z|_{\mathbf{Z}_p^*}$ and for $g(u) = -1/u$, $g(u) = (\log_p q)(q^u/(1-q^u))$ and

$$g^*(u) = \log_p q \left(\frac{q^u}{1-q^u} - \frac{q^{pu}}{1-q^{pu}} \right).$$

THEOREM 2. *Let $\psi_{p,z}(x)$, $\psi^*_{p,z}(x)$, $\psi_{p,q,z}(x)$, $\psi^*_{p,q,z}(x)$ and $\psi^{**}_{p,q,z}(x)$ be the functions defined in* (2.5)–(2.6) *and in Theorem* 1. *In* (2) *below we suppose that $|q-1|_p < p^{-1/(p-1)}$, and in* (4) *we suppose that $|q-1|_p < p^{-2/(p-1)}$ (i.e., $r_q > p^{1/(p-1)}$). Then*

(1) *for $|x|_p < 1$,*

$$\psi^*_{p,z}(x) = \sum_{j=1}^{\infty} (-1)^j x^{j-1} L_p(j, \omega^{1-j}, z)$$

(*compare with* (1.12));

(2) *for* $|x|_p < p^{-1/(p-1)}$,

$$\psi^*_{p,q,z}(x) = \sum_{j=1}^{\infty} x^{j-1} \frac{(\log_p q)^j}{(j-1)!} \int_{\mathbf{Z}_p^*} P_j(q^u)\, d\mu_z(u),$$

$$\psi^{**}_{p,q,z}(x) = \sum_{j=1}^{\infty} x^{j-1} \frac{(\log_p q)^j}{(j-1)!} \int_{\mathbf{Z}_p^*} P_j^*(q^u)\, d\mu_z(u),$$

where P_j *was defined in* (1.15) *and*

$$P_j^*(q) \underset{\text{def}}{=} P_j(q) - p^{j-1} P_j(q^p) = \left(q\frac{d}{dq}\right)^{j-1}\left(\frac{q}{1-q} - \frac{q^p}{1-q^p}\right)$$

(*compare with* (1.31));

(3) *for* $|x|_p > 1$,

$$\psi_{p,z}(x) = \sum_{j=0}^{\infty} \frac{L(-j, z)}{(-x)^{j+1}}, \quad \psi^*_{p,z}(x) = \sum_{j=0}^{\infty} \frac{L_p(-j, \omega^{j+1}, z)}{(-x)^{j+1}}$$

(*compare with* (1.37)). *Here in the first sum* $L(-j, z) \underset{\text{def}}{=} \int_{\mathbf{Z}_p} u^j\, d\mu_z(u)$, *which coincides with the classical value* $L(-j, z)$ *when* z *is a root of unity*;

(4) *for* $p^{1/(p-1)} < |x|_p < r_q$,

$$\psi_{p,q,z}(x) = \sum_{j=0}^{\infty} \frac{L(-j, z)}{j!} (\log_p q)^{j+1} P_{j+1}(q^x),$$

$$\psi^*_{p,q,z}(x) = \sum_{j=0}^{\infty} \frac{L_p(-j, \omega^{j+1}, z)}{j!} (\log_p q)^{j+1} P_{j+1}(q^x),$$

$$\psi^{**}_{p,q,z,}(x) = \sum_{j=0}^{\infty} \frac{L_p(-j, \omega^{j+1}, z)}{j!} (\log_p q)^{j+1} P^*_{j+1}(q^x)$$

(*compare with* (1.39)).

The proof of Theorem 2 follows immediately from (2.17), (2.18), (2.1), (2.2) and the following observations concerning convergence: in (1), (3) and (4), $L_p(j, \omega^{1-j}, z)$, $L(-j, z)$, $L_p(-j, \omega^{j+1}, z)$ are bounded with respect to j; in (2), $(\log_p q)^j P_j(q^u)$ and $(\log_p q)^j P_j^*(q^u)$ are bounded over $u \in \mathbf{Z}_p^*$ uniformly with respect to j; and in (4), we have

$$|(\log_p q)^{j+1} P_{j+1}(q^x)|_p \leqslant \left|\frac{\log_p q}{1-q^x}\right|_p^{j+1} = \left|\frac{1}{x}\right|_p^{j+1},$$

and the same relation for $|(\log_p q)^{j+1} P^*_{j+1}(q^x)|_p$.

REMARK. The integrals $a_j = a_{j,q,z}$ in the coefficients of the Taylor series for $\psi^{**}_{p,q,z}(x)$ (part (2) of Theorem 2) are related to p-adic Eisenstein series, but more indirectly than in the classical case. To see this connection, first suppose that

$|z|_p < 1$. In that case

$$a_j = a_{j,q,z} = \int_{\mathbf{Z}_p^*} P_j^*(q^u)\, d\mu_z(u) = \sum_{m \geq 1,\, p \nmid m} P_j^*(q^m) z^m.$$

If we also suppose that $|q|_p < 1$, then we have

$$P_j^*(q) = \sum_{n \geq 1,\, p \nmid n} n^{j-1} q^n$$

and

$$\sum_{m \geq 1,\, p \nmid m} P_j^*(q^m) z^m = \sum_{m \geq 1,\, p \nmid m} q^m \sum_{dd' = m} d^{j-1} z^{d'}.$$

Now set z equal to a nontrivial Dth root of unity, and define f on $\mathbf{Z}/D\mathbf{Z} \times \mathbf{Z}/D\mathbf{Z}$ by $f(u, v) = z^v$ (f does not depend on u). Then

$$a_{j,q,z} + (-1)^j a_{j,q,z^{-1}} = \sum_{m \geq 1,\, p \nmid m} q^m \sum_{dd' = m} d^{j-1}\big(f(d, d') + (-1)^j f(-d, -d')\big)$$

is the series Katz denotes $2\Phi^*_{j-1,0,f}$ (see (6.4.1) of **[8]**; note that the definitions and results generalize trivially to $\mathbf{Z}_p[z]$-valued functions f). It is a p-adic modular form for the congruence-subgroup $\Gamma(D)$ (see **[8]**).

In other words, we can consider the power series

$$a_j = \sum_{m \geq 1,\, p \nmid m} q^m \sum_{dd' = m} d^{j-1} z^{d'} \in \mathbf{Z}[z][[q]] \subset \mathbf{Z}[[z, q]]$$

as a p-adic analytic function in the two variables z, q for $|z|_p \leq 1$, $|q|_p < 1$. When $z^D = 1$ we essentially have a p-adic Eisenstein series. On the other hand, for any fixed z with $|z|_p < 1$, the same series a_j has a unique analytic continuation

$$a_j = \sum_{m \geq 1,\, p \nmid m} P_j^*(q^m) z^m$$

to the region $\{q \in \mathbf{C}_p \mid |q|_p \leq 1,\ q^n \neq 1 \text{ for all } n,\ p^2 \nmid n\}$. This p-adic analytic function is given on the region $0 < |q - 1|_p < p^{-1/(p-1)}$ by the integral $\int_{\mathbf{Z}_p^*} P_j^*(q^u)\, d\mu_z(u)$. Finally, for any fixed q with $0 < |q - 1|_p < p^{-1/(p-1)}$, this integral gives a unique analytic continuation (in z) to the region $|z - 1|_p \geq 1$. In particular, for z a Dth root of unity we obtain essentially the jth Taylor expansion coefficient of the twisted psi-function $\psi^{**}_{p,q,z}$. This is a p-adic analogy to the occurrence discussed in §1 of classical Eisenstein series in the expansion of $\psi_q(x + 1)$ and $\psi_{q,z}(x + 1)$.

Finally, we discuss the relationship between $\psi_{p,q,z}$ and the q-extensions defined in **[15]** for J. Diamond's p-adic log-gamma function **[5]**. We temporarily use $\tilde{\psi}$ to denote the functions defined in **[15]** by means of limits of the form

$$\lim_{N \to \infty} \frac{1}{dp^N} \sum_{0 \leq j < dp^N} g(x + j).$$

In what follows Σ' means that indices divisible by p are omitted. Thus for $z^d = 1$ (with $z \neq 1, p \nmid d$) and for $|x|_p < r_q$, $x \notin \mathbf{Z}_p$,

$$\tilde{\psi}_{p,q,z}(x) \underset{\text{def}}{=} \lim_{N\to\infty} \frac{1}{dp^N} \sum_{0\leq j<dp^N} z^j \log_p \frac{1-q^{x+j}}{1-q};$$

$$\tilde{\psi}^*_{p,q,z}(x) \underset{\text{def}}{=} \lim_{N\to\infty} \frac{1}{dp^N} \sum_{0\leq j<dp^N}{}' z^j \log_p \frac{1-q^{x+j}}{1-q}$$

$$= \tilde{\psi}_{p,q,z}(x) - \frac{1}{p}\tilde{\psi}_{p,q^p,z^p}(x/p).$$

THEOREM 3. (1) $\tilde{\psi}_{p,q,z}(x) = \psi_{p,q,z}(x)$ *for* $|x|_p < r_q$, $x \notin \mathbf{Z}_p$.
(2) $\tilde{\psi}^*_{p,q,z}(x) = \psi^*_{p,q,z}(x)$ *for* $|x|_p < r_q$, $x \notin \mathbf{Z}^*_p$.

The proof is similar to the proof that $\tilde{G}_{p,z}(x) = G_{p,z}(x)$ in [**14**, p. 51]. Namely, one first shows that (2.11) also holds for $\tilde{\psi}$. Then we can use (2.11) with $m = p^n$ for both ψ and $\tilde{\psi}$ to see that it suffices to prove that

$$\tilde{\psi}_{p,q^{p^n},z^{p^n}}\left(\frac{x+h}{p^n}\right) = \psi_{p,q^{p^n},z^{p^n}}\left(\frac{x+h}{p^n}\right),$$

i.e., to prove that $\tilde{\psi}_{p,q,z}(x) = \psi_{p,q,z}(x)$ for $|x|_p$ large. But this is true because both sides have the same Stirling series. The proof that $\tilde{\psi}^* = \psi^*$ is similar.

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